ACTS Topoi Notes

September 2021

Contents

Chapter 1

Day 1: Elementary Topoi

1.1 Introduction

Our plan is to spend the day understanding the statement of the following theorem, with an emphasis of reviewing our categorical terminology from last semester:

Theorem 1.1.1. *There exists a Boolean topos satisfying the axiom of choice in which the continuum hypothesis fails.*

The first thing we dissect is the **continuum hypothesis**:

Definition 1.1.1. If there is a sequence of monomorphisms

$$N \hookrightarrow A \hookrightarrow \Omega^N$$

then either there exists a monomorphism $A \hookrightarrow N$ or $\Omega^N \hookrightarrow A$.

Let the definition bashing begin!

1.2 Categorical Notions

1.2.1 Morphisms

Definition 1.2.1. $\mathrm{Hom}_{\mathcal{C}}(A, B) := \{f : A \to B\}$

Definition 1.2.2. Recall that a **monomorphism** m is such that for any diagram of the form

$$\bullet \overset{f}{\underset{g}{\rightrightarrows}} \bullet \overset{m}{\hookrightarrow} \bullet$$

we have $m \circ g = m \circ f$ implies $f = g$.

Similarly, $e \in \mathrm{Hom}(A, B)$ is an **epimorphism** provided for any object Z, $f, g \in \mathrm{Hom}(B, Z)$ we have $f \circ e = g \circ e$ implies $f = g$.

We should note that in the category *Set*, a morphism is a monomorphism if and only if it is injective, and a morphism is an epimorphism if and only if it is surjective, so a monomorphism that is also an epimorphism is an isomorphism (bijection) in *Set*.
We now state the categorical axiom of choice.

Definition 1.2.3. The **axiom of choice** states that every epimorphism $f : A \to B$ has a section, that is a function $\sigma : B \to A$ such that $f \circ g = id_B$.

Theorem 1.2.1 (Cantor-Schroder-Bernstein)**.** *For two objects A, B in Set, if there are monomorphisms from one into the other:*

$$\begin{tikzcd} A \arrow[r, hook, shift right, "m_a"'] & B \arrow[l, hook', shift right, "m_b"'] \end{tikzcd}$$

then A and B are isomorphic.

Proof. https://web.williams.edu/Mathematics/lg5/CanBer.pdf □

This explains why we are looking for a monomorphism in the statement of the hypothesis. In the traditional hypothesis, the goal is to show that there is a set between the natural numbers and the real numbers with different cardinality from both. This motivates categorifying the concept of natural numbers.

1.2.2 Natural Numbers Object

Definition 1.2.4. An object 1 in a category $\mathcal{C}$ is **final** if $\mathrm{Hom}(C, 1) = \{*\}$. In words, there is a unique arrow from any object to 1.

What is the final object in *Set*?

Definition 1.2.5. We say a topos E satisfies the **axiom of infinity** if it contains a natural numbers object.

Definition 1.2.6. A natural numbers object N in a topos (or category) $\mathcal{E}$ is a object N of $\mathcal{E}$ along with maps $s : N \to N$ and $0 : 1 \to N$ where 1 is the final object of $\mathcal{E}$

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

satisfying a universal property. For any diagram of the form

$$1 \xrightarrow{x} X \xrightarrow{f} X$$

there is a unique map h making the following diagram commute:

$$\begin{tikzcd} 1 \arrow[r, "0"] & N \arrow[r, "s"] \arrow[d, dashed, "h"] & N \arrow[d, dashed, "h"] \\ 1 \arrow[r, "x"] & X \arrow[r, "f"] & X \end{tikzcd}$$

What is the natural numbers object in *Set*?

1.2.3 Limits

For the sake of brevity, we will contain our examples of limits and colimits to the categories Set, $R-mod$, and $\mathcal{C}$ where the objects of $\mathcal{C}$ are natural numbers and there is an arrow from $n \to m$ if and only if $m \leq n$.

$$\begin{tikzcd}
0 \arrow[loop below] & 1 \arrow[l, bend left] \arrow[l, bend right] \arrow[loop below] & 2 \arrow[l, bend left] \arrow[l, bend right] \arrow[ll, bend right] \arrow[loop below] & \ldots \arrow[l]
\end{tikzcd}$$

which is less offensively characterized by the following commutative diagram:

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow ...$$

We begin by defining a categorical limit.

Definition 1.2.7. Provided a small index category $\mathcal{I}$, an I shaped diagram in $\mathcal{C}$ is the image of a functor $F : I \to \mathcal{C}$.

$$\begin{tikzcd}
 & \mathcal{I} & & \mathcal{C} & \\
A \arrow[dr, "f"] & & & FA \arrow[dr, "Ff"] & \\
B & D & & FB & FD
\end{tikzcd}$$

Definition 1.2.8. A **limit** of an $\mathcal{I}$ shaped diagram in $\mathcal{C}$ is an object $L \in \mathcal{C}$ with maps to every object in $\{FI\}_{I \in \mathcal{I}}$ that commute with all maps Ff for f a morphism in I that is universal with respect to this property. For any L' with maps to all $\{FI\}_{I \in \mathcal{I}}$, there exists a unique map $L' \to L$.

$$\begin{tikzcd}
 & & L' \arrow[dl, dashed] \arrow[ddll, bend right] \arrow[ddl] \arrow[dd, bend left] \\
 & L \arrow[dl] \arrow[d] \arrow[dr] & \\
FB & FA \arrow[r, "Ff"] & FD
\end{tikzcd}$$

Example 1.2.1 (Products)**.** Consider the index category $I = 2$ where 2 is the category with two objects and where the only morphisms are the identity arrows. Then a functor $F : \mathcal{I} \to Set$ picks out two sets, and the limit L of this $\mathcal{I}$ shaped diagram is a diagram

$$\begin{tikzcd}
 & L \arrow[dl] \arrow[dr] & \\
S_1 & & S_2
\end{tikzcd}$$

such that for any other such L', we have a universal function f making the following commute:

$$\begin{tikzcd}
& L' \arrow[d, dashed, "f"] \arrow[ddl] \arrow[ddr] & \\
& L \arrow[dl] \arrow[dr] & \\
S_1 & & S_2
\end{tikzcd}$$

What set is L?

What is L in the category of $R - mod$?

What is L in the category $\mathcal{C}$ defined above?

Example 1.2.2 (Equalizers)**.**

Definition 1.2.9. An **equalizer** of two arrows $f, g : X \to Y$ is an object E and a map $e : E \to X$ such that $f \circ e = g \circ e$ and for any object Z and morphism $z : Z \to X$ with the property that $f \circ z = g \circ z$, there is a unique map $e' : Z \to E$ making the following diagram commute:

$$\begin{tikzcd}
Z \arrow[d, dashed, "e'"] \arrow[dr, "z"] & & \\
E \arrow[r, "e"] & X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y
\end{tikzcd}$$

What is an equalizer of two functions in Set?
We can think of an equalizer as a limit of the diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y
\end{tikzcd}$$

What is the index category?

Since we have

$$\begin{tikzcd}
L' \arrow[dr, dashed] \arrow[ddr] \arrow[ddrr] & & \\
& L \arrow[d, "l_x"] \arrow[dr, "l_y"] & \\
& X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y
\end{tikzcd}$$

and we must have $l_y = g \circ l_x$ and $l)y = f \circ l_x$ to commute, so we can simplify this diagram to the same diagram as the equalizer

$$\begin{tikzcd}
L' \arrow[dr, dashed] \arrow[ddr] & & \\
& L \arrow[d, "l_x"] & \\
& X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y
\end{tikzcd}$$

Definition 1.2.10. A **pullback** is the limit of a diagram of the following form:

$$\begin{tikzcd}
& Z \arrow[d, "f"] \\
Y \arrow[r, "g"] & X
\end{tikzcd}$$

so is L with the two dashed arrows in the following diagram

$$\begin{tikzcd}
L \arrow[r, dashed] \arrow[d, dashed] & Z \arrow[d, "f"] \\
Y \arrow[r, "g"] & X
\end{tikzcd}$$

What is a pullback here in $\mathcal{C}$?

$$\begin{tikzcd}
? \arrow[r, dashed] \arrow[d, dashed] & 5 \arrow[d] \\
4 \arrow[r] & 3
\end{tikzcd}$$

Example 1.2.3. $\mathbb{Z}_p = a_0 + a_1 p + ... \mid 0 \leq a_i < p$ with the obvious ring structure is the limit over the inclusions $\mathbb{Z}/p^1\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^1\mathbb{Z} \to ...$ in ring.

Finally, if a category has a limit for all finite index categories $\mathcal{I}$, we say it has **finite limits**.

1.3 Topoi

1.3.1 Exponential Objects

Exponential objects generalize our idea of a power set. Since we met them last year, we'll be breif.

Definition 1.3.1. Let X and Y be objects in a category $\mathcal{C}$ with finite products. Then an **exponential object** is an object X^Y with a universal evaluation arrow $v : X^Y \times Y \to X$ in that for all objects Z and maps $t : Z \times Y \to X$, there exists a unique map $u : Z \to X^Y$ such that

$$\begin{tikzcd} Z \times Y \arrow[r, dashed, "u \times id"] \arrow[rr, bend right, "t"'] & X^Y \times Y \arrow[r, "v"] & X \end{tikzcd}$$

commutes.

Example 1.3.1. In Set, $X^Y = \text{Hom}(Y, X)$ where the evaluation map sends $(f, x) \in X^Y \times X$ to $f(x)$. Let's be bad category theorists and find the u using elements, for the sake of time.

We have $t(z, y)$ should commute with $v(u(z), y) = u(z)(y)$, so let $u(z) = t(z, -)$.

Definition 1.3.2. A category with all finite products and exponential objects is called a **cartesian closed category**

Recall that last year we showed an equivelance of categories between the category of small cartesian closed categories and the category of simply typed λ-theories.

1.3.2 Subobject Classifiers

Taking us one step closer to the definition of a topos are subobject classifiers, which will also clear up what Ω is.

In the category Set, suppose we have $S \subseteq X$. There are two ways that this can be represented. First, we can look at the inclusion arrow of S into X.

$$S \hookrightarrow X$$

Alternatively, we can look into S and pick out it's elements with a function that evaluates to 0 on S and 1 outisde of S. Define $2 := \{1, 0\}$ and $\phi_S : X \to 2$

$$\phi_S(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{if } x \notin S \end{cases}$$

How do we make these characterizations compatible? We'll first need to define the map $True : 1 \to 2$ where 1 is the final object in Set. Now what is the pullback (limit) of the following diagram?

$$\begin{tikzcd} & 1 \arrow[d, "True"] \\ X \arrow[r, "\phi_S"] & 2 \end{tikzcd}$$

We call 2 a subobject classifier in Set.

Definition 1.3.3. A **subobject classifier** Ω in a category with finite limits is an arrow

$$1 \xrightarrow{True} \Omega$$

such that for all diagrams

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & 1 \\ X & \end{tikzcd}$$

there exists a unique map ϕ making the following diagram a pullback:

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & 1 \arrow[d] \\ X \arrow[r, dashed, "\phi"] & \Omega \end{tikzcd}$$

Is there a subbject classifier in our example category $\mathcal{C}$?

1.3.3 Topoi at last!

Definition 1.3.4. A topos is a cartesian closed category with finite limits having a subbobject classifier.

Example 1.3.2. *Set* is a topos. It has cartesian products, exponential objects, and internal Hom sets are exponential objects.

To conclude our discussion of the motivating theorem, a topos $\mathcal{E}$ is Boolean roughly if Ω behaves like 2 does in *Set*. The intuition is that every subobject has a complement.

One boolean topos for which the axiom of choice holds and the continuum hypothesis fails is called the Coen topos which is built as a sheaf on a poset.

We give one more example of a topos.

Example 1.3.3. G-set, the category of sets equipped with a group action by G a finite group with functions compatible with the action of G is a boolean topos.

Its subobject classifier is 2, the power object corresponds to Hom sets, is given by the subsets of X that have an open stabilizer.

Finite limits pull back along the forgetful functor to *Set*.

This extends to Lie groups nontrivially using an adjuntion to the associated discrete group.

Chapter 2

Day 2: Introduction to Grothendieck Topoi

Today we present a point of view on topoi that starts with geometry, and by the end of the day, we should bump into the content we covered last week.

2.1 Presheaves

Definition 2.1.1. For a topological space X define $Top(X)$ to be the category whose objects are open sets and arrows are inclusions.

Definition 2.1.2. A **presheaf on a topological space** X values with values in $\mathcal{C}$ is a contravariant functor $F : Top(X) \to \mathcal{C}$

Recall that sheaves are presheaves satisfying a gluing axiom making them compatible with the topology through an exactness property.

Example 2.1.1 (Skyscraper Sheaf)**.** For a set $S \in Set$, and a point $x \in X$, the skyscraper sheaf s_x is a set valued presheaf. For any U open in X,

$$s_x(U) = \begin{cases} S & \text{if } x \in U \\ \{*\} & \text{if } x \notin U \end{cases}$$

What does s_x do to morphisms?

Example 2.1.2. Let M be a real manifold, then the functor $F : Top(M) \to$ *Ring* taking $U \to C(U)$ is a sheaf.

Example 2.1.3. In algebraic geometry we can associate the structure sheaf $\mathcal{O}(U)$ to Zariski open sets.

2.1.1 Categories of Presheaves

We begin by slightly generalizing the notion of a presheaf, removing the need for a topological space.

Definition 2.1.3. A set valued **presheaf** is a contravariant functor $F : \mathcal{C} \to Set$.

Definition 2.1.4. The category of presheaves $PSh(\mathcal{C})$ for a small category $\mathcal{C}$ has objects presheaves in Set and morphisms natural transformations.

Example 2.1.4. Let $\mathcal{C}$ be a category with one object and the identity morphism, then the objects of $PSh(\mathcal{C})$ correspond to a functor F picking out an object in Set.

$$\begin{tikzcd} \mathcal{C} & & Set \\ * \arrow[loop below, "id"] & & F(*) \arrow[loop below, "F(id)=id"] \end{tikzcd}$$

Let's consider a natural transformation α between two presheaves F and G

$$\begin{tikzcd} F(*) \arrow[r, "Ff"] \arrow[d, dashed, "\alpha"] & F(*) \arrow[d, dashed, "\alpha"] \\ G(*) \arrow[r, "Gf"] & G(*) \end{tikzcd}$$

Why are the α maps the same?

Notice that the only such f is the identity morphism, so we have that

$$\begin{tikzcd} F(*) \arrow[r, "id"] \arrow[d, dashed, "\alpha"] & F(*) \arrow[d, dashed, "\alpha"] \\ G(*) \arrow[r, "id"] & G(*) \end{tikzcd}$$

will commute for any α a map in Set. We see that $PSh(\mathcal{C})$ is just Set (isomorphism of categories)!

Example 2.1.5. Let $\mathcal{D}$ be a category with n objects and only the identity morphisms. What is $PSh(\mathcal{D})$?

What will be the exponential objects?

What will be the terminal object?

What will be the subobject classifier?

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & 1 \arrow[d] \\
X \arrow[r, dashed, "\phi"] & \Omega
\end{tikzcd}$$

This is another example of a Boolean topos.

Example 2.1.6 (Presheaf of Evolving Sets)**.** Let $\mathcal{M}$ be a monoid freely generated by one morphism.

$$\begin{tikzcd}
\ast \arrow[loop above, "id"] \arrow[loop below, "g"]
\end{tikzcd}$$

Let's begin by fleshing out what the morphisms in $PRe(\mathcal{M})$.

$$\begin{tikzcd}
\mathcal{M} \arrow[d, bend right, "G"'] \arrow[d, bend left, "F"] \\
Set
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\ast) \arrow[r, "Fg"] \arrow[d, dashed, "\alpha"] & F(\ast) \arrow[d, dashed, "\alpha"] \\
F(\ast) \arrow[r, "Gg"] & F(\ast)
\end{tikzcd}$$

Now what are the endomorphisms of an object (functor) F?

$End_{\mathcal{M}}(A)$ are morphisms such that $Ff \circ \alpha = \alpha \circ Ff$, i.e. $End_{\mathcal{E}}(A) \cong C_{\mathcal{M}}(Ff)$

Let's do a limit, specifically, what is the product of G and H?

$$\begin{tikzcd}
 & L\ast \arrow[dl, "a"] \arrow[dr, "c"] & L\ast \arrow[dl, "b"'] \arrow[dr, "d"] & \\
G\ast \arrow[r, "Gg"] & G\ast & H\ast \arrow[r, "Gg"] & H\ast
\end{tikzcd}$$

Example 2.1.7. Let $\mathcal{F}$ be a monoid generated by a morphism σ such that σ^2 is the identity. Then $PSh(\mathcal{F})$ is the presheaf of involution sets.

Example 2.1.8 ($G - Set$)**.** Consider G to be a category as $\mathcal{M}$ above, then $PRe(G)$ is the category of G-Sets.

For any groupoid, we have that a morphism between any two objects is an isomorphism, so applying the equivalence of categories to its skeleton provides us with a set of groups with no morphisms between them - call it $\mathcal{G}$. which we've seen how to handle with the presheaf on n objects.

Theorem 2.1.1. *$PSh(\mathcal{C})$ is a topos for any small category $\mathcal{C}$.*

2.2 Categorical Adjunctions

Adjunctions will be important in the definition of Grothendieck topoi, so let's review them here! I will go a bit fast, since this information is a review from last year. I will advise the ACTors to speak up with any questions.
Let $\mathcal{C}$, $\mathcal{D}$ be categories with functors $R : \mathcal{D} \to \mathcal{C}$ and $L : \mathcal{C} \to \mathcal{D}$. We say R, L are adjoint functors is the following functors are naturally isomorphic:

$$\mathrm{Hom}_{\mathcal{D}}\left(L(-), -\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(-, R(-)\right)$$

In this context, we write $L \dashv R$ and say that L is left adjoint to R/R is right adjoint to L (will review naturally isomorphic, and it's context here, if prompted).
TODO: Definition in terms of unit and counit? Or is this spending too much time here?
EXAMPLES (will go over some subset, depending on time):

- A pair of adjoint functors between posets is a Galois correspondence.
- Forgetful functor and free functor. Forgetful is R, free is L.

FUN FACT: If $\mathcal{C}$ and $\mathcal{D}$ have finite limits, and if L preserves these limits, we call L and R geometric morphisms. If R is full and faithful, we call the adjunction a geometric embedding. (This fact will be helpful later. I will define full and faithful.)

2.3 Grothendieck Topoi

A Grothendieck topoi is a category $\mathcal{T}$ that admits a full and faithful functor $F : \mathcal{T} \to \mathrm{Pre}(\mathcal{C})$ that has a left exact left adjoint $L : \mathrm{Pre}(\mathcal{C})$.
Aside: a left exact functor preserves finite limits.
Examples of Grothendieck topoi include: the catgeory of continuous G-sets
What is the difference between Grothendieck topoi and elementary topoi?

- For one, Grothendieck topoi are used primarily in *algebraic topology*, while elementary topoi are useful in *logic*.
- Every Grothendieck topoi is an elementary topoi, but the converse is not true.

Chapter 3

Topoi: A Crossover Episode!

3.1 Title & Abstract

Topoi: A Crossover Episode!

Topoi, a friendly categorification of topology, are ubiquitous in modern mathematics, being utilized everywhere from geometry, to topology, to logic. In this talk, we will introduce topoi through two themes that historically motivated their development: limits and presheaves. With a few definitions from category theory, we define elementary and Grothendieck topoi, drawing on familiar examples from geometry throughout.

3.2 L: Introduction

Thank you for the opportunity to speak today! We are going to be talking about topoi, which have a reputation for falling into the abstract nonsence bin of mathematics. Having read some of the past abstracts from this seminar, we ask why should the gometer or number theorist care about Topoi?

Whenever we're working in a geometric category, the underlying strucure of this category will be a topos, as we will see later. Whenever you hear topoi, you should think of the sheaves coming up in your area of geometry. My understanding is that the theory of topoi has led to a wealth of existence results for universal objects that may come up in your research when you are trying to show the existence of some object of geometric or number theoretic interest.

Without an existence result, we can be in a perilous position. My favorite example is assuming the existence of a universal natural number that is the largest.

Then $n^2 \leq n$, so $n^2 - n \leq 0$ and we see n is 0 or 1, so the number satisfying the universal property of being the largest is either 0 or 1. We must prove the largest natural number exists before we start working with it.

This is a toy example, but the same thing can come up in research. I'm at the end of a grapevine where I've been told a hole in the supposed proof of the ABC conjecture is the assumption that a category has all limits. What's great about Topoi is that when we work in them, we can guarantee that we will always have finite limits.

The two biggest names usually associated with the notion of a topos are Grothendieck and Lawvere, where Grothendieck is usually associated with the topoi arising geometrically and Lawvere with the first workable set of axioms. It will likely ring a bell to the folks in this seminar that Topoi play a starring role in EGA 4. Soon after these geometric origins, topoi became popular among logicians as their internal logic allows for a categorificiation of much of 19th century logic.

Over the course of our talk, we hope to convey an intuition for topoi as being the natural base category to do mathematics involving limits and to think about sheaves. We've written some big checks when it comes to universal objects, before we get ahead of ourselves, we should talk about what general categorical limits (universal cones) are, and you've definitely seen them before, but maybe nor in their categorical formality.

3.3 Limits

Example 3.3.1 (T: Kernels)**.**

Example 3.3.2 (T: Products)**.** Consider the index category $I = 2$ where 2 is the category with two objects and where the only morphisms are the identity arrows. Then a functor $F : \mathcal{I} \to Set$ picks out two sets, and the limit L of this $\mathcal{I}$ shaped diagram is a diagram

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ S_1 & & S_2 \end{tikzcd}$$

such that for any other such L', we have a universal function f making the following commute:

$$\begin{tikzcd}
& L' \arrow[d, dashed, "f"] \arrow[ddl] \arrow[ddr] & \\
& L \arrow[dl] \arrow[dr] & \\
S_1 & & S_2
\end{tikzcd}$$

What set is L?

What is L in the category of $R - mod$?

Example 3.3.3 (T: p-adic ring)**.** $\mathbb{Z}_p = a_0 + a_1 p + ... \mid 0 \leq a_i < p$ with the obvious ring structure is the limit over the inclusions $\mathbb{Z}/p^1\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^1\mathbb{Z} \to$ ... in ring.

3.4 Presheaves

A topos is a category with additional structure. A category is a collection of objects with morphisms between them, satisfying associativity and identity.

Definition 3.4.1 (T:)**.** A set valued **presheaf** is a contravariant functor $F : \mathcal{C} \to Set$.

Definition 3.4.2 (T:)**.** For a topological space X define $Top(X)$ to be the category whose objects are open sets and arrows are inclusions.

Definition 3.4.3 (L:)**.** A **presheaf on a topological space** X values with values in $\mathcal{C}$ is a contravariant functor $F : Top(X) \to \mathcal{C}$

Example 3.4.1 (L: Stalk of a Sheaf)**.** Limit among the opens

Recall that sheaves are presheaves satisfying a gluing axiom making them compatible with the topology through an exactness property.

Example 3.4.2 (L: Skyscraper Sheaf)**.** For a set $S \in Set$, and a point $x \in X$, the skyscraper sheaf s_x is a set valued presheaf. For any U open in X,

$$s_x(U) = \begin{cases} S & \text{if } x \in U \\ \{*\} & \text{if } x \notin U \end{cases}$$

Example 3.4.3 (L:)**.** Let M be a real manifold, then the functor $F : Top(M) \to Ring$ taking $U \to C(U)$ is a sheaf.

Example 3.4.4 (L:)**.** In algebraic geometry we can associate the structure sheaf $\mathcal{O}(U)$ to Zariski open sets.

We won't need sheaves for the rest of this talk outside of mentioning examples from geometry, but presheaves will play an important role.

Definition 3.4.4 (T?:)**.** The category of presheaves $PSh(\mathcal{C})$ for a small category $\mathcal{C}$ has objects presheaves in Set and morphisms natural transformations.

Example 3.4.5 (T?:)**.** Let $\mathcal{C}$ be a category with one object and the identity morphism, then the objects of $PSh(\mathcal{C})$ correspond to a functor F picking out an object in Set.

$$\begin{tikzcd}
\mathcal{C} & & Set \\
* \arrow[loop below, "id"] & & F(*) \arrow[loop below, "F(id)=id"]
\end{tikzcd}$$

Let's consider a natural transformation α between two presheaves F and G

$$\begin{tikzcd}
F(*) \arrow[r, "Ff"] \arrow[d, dashed, "\alpha"] & F(*) \arrow[d, dashed, "\alpha"] \\
G(*) \arrow[r, "Gf"] & G(*)
\end{tikzcd}$$

Why are the α maps the same?

Notice that the only such f is the identity morphism, so we have that

$$\begin{tikzcd}
F(*) \arrow[r, "id"] \arrow[d, dashed, "\alpha"] & F(*) \arrow[d, dashed, "\alpha"] \\
G(*) \arrow[r, "id"] & G(*)
\end{tikzcd}$$

will commute for any α a map in Set. We see that $PSh(\mathcal{C})$ is just Set (isomorphism of categories)!

Example 3.4.6 (L:)**.** Let $\mathcal{D}$ be a category with n objects and only the identity morphisms. What is $PSh(\mathcal{D})$?

This is just the beginning of sheaves as universes of sets.

Definition 3.4.5 (T:)**.** A Grothendieck topos is a category $\mathcal{T}$ that admits a full and faithful functor $F : \mathcal{T} \to \mathrm{Pre}(\mathcal{C})$ that has a left exact left adjoint $L : \mathrm{Pre}(\mathcal{C})$.

Alternatively, a Grothendieck topos can be thought of as a full subcategory of a category of presheaves.

If $\mathcal{C}$ and $\mathcal{D}$ have finite limits, and if L **preserves limits**, we call L and R **geometric morphisms**. If R is full and faithful, we call the adjunction a geometric embedding. (This fact will be helpful later. I will define full and faithful.)

3.5 Topoi

Now seeing the utility of Topoi as a place to do mathematics, what are they?

Definition 3.5.1 (T:)**.** A **topos** is a cartesian closed category with finite limits having a subbobject classifier.

Example 3.5.1 (T:)**.** *Set* is a topos. It has cartesian products, internal Hom sets are exponential objects, limits are built from equalizers and cartesian products, and 2= {0,1 } is the subobject classifier.

3.5.1 Finite Limits

Definition 3.5.2 (L:)**.** Provided a small index category $\mathcal{I}$, an I shaped diagram in $\mathcal{C}$ is the image of a functor $F : I \to \mathcal{C}$.

$$\begin{tikzcd}
 & \mathcal{I} & \mathcal{C} & \\
A \arrow[dr, "f"] & & FA \arrow[dr, "Ff"] & \\
B & D & FB & FD
\end{tikzcd}$$

Definition 3.5.3. A **limit** of an $\mathcal{I}$ shaped diagram in $\mathcal{C}$ is an object $L \in \mathcal{C}$ with maps to every object in $\{FI\}_{I\in\mathcal{I}}$ that commute with all maps Ff for f a morphism in I that is universal with respect to this property. For any L' with maps to all $\{FI\}_{I\in\mathcal{I}}$, there exists a unique map $L' \to L$.

$$\begin{tikzcd}
 & & L' \arrow[dl, dashed] \arrow[dll, bend right] \arrow[dl, shift left=0] \arrow[d, bend left] \\
 & L \arrow[dl] \arrow[d] \arrow[dr] & \\
FB & FA \arrow[r, "Ff"] & FD
\end{tikzcd}$$

A category with limit objects existing for all finite index categories is said to have finite limits.

3.5.2 Cartesian Closed

[T:] Exponential objects generalize our idea of a function set.

Definition 3.5.4. Let X and Y be objects in a category $\mathcal{C}$ with finite products. Then an **exponential object** is an object X^Y with a universal evaluation arrow $v : X^Y \times Y \to X$.

In the category of sets, these correspond to Hom sets and function evaluation.

Definition 3.5.5 (T:)**.** A category with all finite products and exponential objects is called a **cartesian closed category**

Remark that cartesian closed categories are the same thing as a simply typed λ calculus, so as topoi are cartesian closed categories with extra structure, they are made for computation.

3.5.3 Subobject Classifiers

When we were setting up $Top(X)$, we created a category with inclusion morphisms, but how do we think of inclusion in a categorical way?
In the category Set, suppose we have $S \subseteq X$. There are two ways that this can be represented. First, we can look at the inclusion arrow of S into X.

$$S \hookrightarrow X$$

Alternatively, we can look into S and pick out it's elements with a function that evaluates to 0 on S and 1 outisde of S. Define $2 := \{1, 0\}$ and $\phi_S : X \to 2$

$$\phi_S(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{if } x \notin S \end{cases}$$

How do we make these characterizations compatible? We'll first need to define the map $True : 1 \to 2$ where 1 is the final object in Set. Now what is the pullback (limit) of the following diagram?

$$\begin{tikzcd} & 1 \arrow[d, "True"] \\ X \arrow[r, "\phi_S"] & 2 \end{tikzcd}$$

We call 2 a subobject classifier in Set.

Definition 3.5.6. A **subobject classifier** Ω in a category with finite limits is an arrow

$$1 \xrightarrow{True} \Omega$$

such that for all diagrams

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & 1 \\ X & \end{tikzcd}$$

there exists a unique map ϕ making the following diagram a pullback:

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & 1 \arrow[d] \\
X \arrow[r, dashed, "\phi"] & \Omega
\end{tikzcd}$$

A **pullback** is the limit of a diagram of the following form:

$$\begin{tikzcd}
& Z \arrow[d, "f"] \\
Y \arrow[r, "g"] & X
\end{tikzcd}$$

so is L with the two dashed arrows in the following diagram

$$\begin{tikzcd}
L \arrow[r, dashed] \arrow[d, dashed] & Z \arrow[d, "f"] \\
Y \arrow[r, "g"] & X
\end{tikzcd}$$