COMPUTING A BASIS FOR POLYNOMIAL SOLUTIONS TO SYSTEMS OF HOMOGENEOUS LINEAR PDES WITH CONSTANT COEFFICIENTS ALGEBRAICALLY

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Background

For the following, let R be a ring and I, J ideals in R.

We will require some definitions relating to ideals. A *colon ideal* $(I : J)$ is a set defined from two other ideals equal to $\{r \in R | rJ \subseteq I\}$
A sequence of colon ideals

$$(I : J) \subseteq (I : J^2) \subseteq (I : J^3) \subseteq ...$$

stabilizes to the *saturation* of I in J denoted $(I : J^\infty)$. It is the set $\{r \in R | \exists n \in \mathbb{N} : rJ^n \subseteq I\}$
The *radical of an ideal* I, denoted $\sqrt{I}$ or $rad(I)$ is the ideal defined by the set $\{r \in R | r^n \in I \text{ for some } n \in \mathbb{N}\}$. An ideal I is radical if $\sqrt{I} = I$.
A *Gröbner basis* and the reduced Gröbner basis of an ideal is a special set of generators for the ideal. If G is the reduced Gröbner basis for I, then $\langle G \rangle = I$, furthermore the reduced Gröbner basis of an ideal is unique. For an algorithm that calculates Gröbner basis and further properties, see [1, Chapter 2].

We will also require definitions relating to primary ideal decomposition. An ideal P is *prime* in R if the only way for a product to be in P is if at least one factor is an element of P. P is a minimal prime is it does not contain within it any smaller prime ideal. P is *associated* to I if there is an element $r \in R$ such that $(I : \langle r \rangle) = P$. Importantly, every ideal I in R is an intersection of primary ideals. To see why we care about primary ideal decomposition, consider the polynomial ring $R = \mathbb{Q}[x]$ and $I = \langle x^2 - 1 \rangle \trianglelefteq R$. The primary decomposition of $I = \langle x - 1 \rangle \cap \langle x + 1 \rangle$. We see that the primary decomposition provides the solutions to the polynomial.

The polynomial ring we work with to solve differential equations is $\mathbb{Q}[\partial_1, ..., \partial_n]$ denoted $\mathbb{Q}[\partial]$. This can be regarded as an equivalent ring to rational polynomials in x variables since differential operators commute. The elements of the ring are differential operators. We now consider ideals $K \trianglelefteq \mathbb{Q}[\partial]$ in this polynomial ring. We also require a working understanding of the module of analytic functions over this polynomial ring which is like a vector space with elements in the ring instead of a field. A polynomial $D \in \mathbb{Q}[\partial]$ acts on a function f in this module by differentiation, so when we want to solve a system of PDEs, we're looking for the functions in the module which are annihilated by the operators in our ideal.

The initial ideal and all Gröbner basis in this project were chosen using lexicographic ordering with $\partial_n > \partial_{n+1}$ and $x_n > x_{n+1}$. Using this ordering, we define the *initial ideal* of K $in(K)$ to be the ideal generated by the leading monomials of K's reduced Gröbner basis. The monomials annihilated by all operators in $in(K)$ are called the *standard monimials* of K. When an ideal K is generated by a system of PDEs, we'll denote the solutions in the module by $sol(K)$ and the polynomial solutions to this system by $Polysol(K)$.

The Algorithm

Input: A System of Linear PDEs
Output: A basis for the space of polynomial solutions to the system

1. Convert Equations to a Ideal $J \trianglelefteq \mathbb{Q}[\partial_1, ..., \partial_n]$
2. Compute $I = (J : (J : \langle \partial_1, ..., \partial_n \rangle^\infty))$
3. Check Solution Space is Zero Dimensional
4. Compute the Reduced Gröbner Basis of I
5. Compute β, the Standard Monomials for I
6. Output $f_\beta(x_1, ..., x_n)$ for all $x^\beta \in \beta$

[2, Chapter 10.3]

Converting a System of PDEs to an Ideal

To convert from a system of PDEs to their ideal representation, we first write the system set equal to zero and we will rewrite $\frac{\partial^i y}{\partial x_i^i}$ as ∂_i, so

$$\frac{\partial^2 y}{\partial x_1 \partial x_3} = 0, \frac{\partial^2 y}{\partial x_1 \partial x_4} = 0, ..., \frac{\partial y}{\partial x_2} + \frac{\partial y}{\partial x_3} = \frac{\partial y}{\partial x_5}$$

[2, Chapter 10.3]

becomes

$$\partial_1\partial_3, \partial_1\partial_4, \partial_2\partial_4, \partial_2\partial_5, \partial_3\partial_5, \partial_1 + \partial_2 - \partial_4, \partial_2 + \partial_3 - \partial_5$$

This is a system of polynomial equations in ∂ variables with coefficients in $\mathbb{Q}$, so we next consider the ideal J generated by the polynomials

$$J = \langle \partial_1\partial_3, \partial_1\partial_4, \partial_2\partial_4, \partial_2\partial_5, \partial_3\partial_5, \partial_1 + \partial_2 - \partial_4, \partial_2 + \partial_3 - \partial_5 \rangle$$

Computing Colon Ideals

We next compute $J' = (J : \langle \partial_1, ..., \partial_n \rangle^\infty)$,
The ideal J' is the intersection of all primary components of J whose associated minimal prime is not the ideal $\langle \partial_1, ..., \partial_n \rangle$.
In the above example, $J' = \langle 1 \rangle$, so $I = (J : \langle 1 \rangle) = J$

Importantly, $Polysol(I) = Polysol(J)$. We also have that if one of the generators of J' has a nonzero constant term that $Polysol(J)$ is zero dimensional (i.e. has a finite basis); we need that the solution set is zero dimensional for step 6. If $J = J'$ that there are no nonzero polynomial solutions to the system.

We can check the last two conditions for our example. As $J' = \langle 1 \rangle$ has one element with constant term $1 \neq 0$ and $J \neq J'$ we continue.

Computing the Reduced Gröbner Basis of I and Finding Standard Monomials

Computing a Reduced Gröbner Basis of a polynomial ring is a task Sage can handle for us. For the previously defined ideal I, Sage returns the Reduced Gröbner Basis

$$\{\partial_1 - \partial_3 - \partial_4 + \partial_5, \partial_2 + \partial_3 - \partial_5, \partial_3^2 + \partial_4\partial_5, \partial_3\partial_4 - \partial_4\partial_5, \partial_3\partial_5, \partial_4^2, \partial_5^2\}$$

We call this set of generators G and will also label the set β of polynomials annihilated by $in(I)$. Standard monomials of I are the monomials which are annihilated by in(I), so we must first compute $in(I)$ by taking the leading monomials of G. As the polynomials in G are already written in lexographic order, we see this is the set $\{\partial_1, \partial_2, \partial_3^2, \partial_3\partial_4, \partial_3\partial_5, \partial_4^2, \partial_5^2\}$. We simply generate an ideal with these elements to find

$$in(I) = \langle \partial_1, \partial_2, \partial_3^2, \partial_3\partial_4, \partial_3\partial_5, \partial_4^2, \partial_5^2 \rangle$$

Remembering that we are looking for polynomials annihilated by $in(I)$, consider a generator of I written in x instead of ∂ variables. For example, consider x_3^2 from the generator ∂_3^2. The corresponding operator ∂_3^2 does not annihilate x_3^2 and clearly won't annihilate $x_3^2 * p(x)$ for any polynomial $p(x) \neq 0$. Knowing this, we conclude that we can find β by finding all operator monomials not in $in(I)$ and writing them in x variables.

$$\beta = \{x_4x_5, x_5, x_4, x_3, 1\}$$

For an ideal K it is convenient to call the monomials not in $in(K)$ the *standard monomials* of K and those in $in(K)$ *non-standard* . For any two elements $b_1, b_2 \in \beta$, we see that for any $c_1, c_2 \in \mathbb{Q}$, $c_1b_1 + c_2b_2$ will provide a monomial eliminated by $in(I)$ as differential operators are linear operators, so β is a rational basis for the solutions to $in(I)$.

Computing Solution Constants

For every non-standard monomial operator ∂^α which does not appear in I, we compute $C_{\alpha.\beta}$ by reducing ∂^α modulo G. Here we're considering ∂^β and x^β for β in β to come from powers of corresponding variables.

$$\partial^\alpha - \Sigma_{x^\beta \in \beta} c_{\alpha.\beta} \partial^\beta \in I$$

α	∂_1	∂_2	$\partial_1\partial_2$	$\partial_1\partial_5$	$\partial_2\partial_3$	$\partial_3\partial_4$	∂_1^2	∂_2^2	∂_3^2
Σ_{x^β}	$\partial_4 - \partial_3 - \partial_5$	$\partial_3 - \partial_5$	$\partial_4\partial_5$	$\partial_4\partial_5$	$\partial_4\partial_5$	$\partial_4\partial_5$	$-\partial_4\partial_5$	$-\partial_4\partial_5$	$-\partial_4\partial_5$

Finding the Solution Basis

In our running example, there are 9 values of α we need to find constants for. Let's consider $\partial^\alpha = \partial_1\partial_5$. We can show $\partial_1\partial_5 - \partial_4\partial_5 \in I$, so the only nonzero value for $C_{x_1x_5,\beta}$ is $C_{x_1x_5,x_4x_5} = 1$. Now that we have our values for $c_{\alpha.\beta}$, we will compute the polynomials

$$f_\beta(x) = \Sigma_\alpha c_{\alpha.\beta} \frac{\beta!}{\alpha!} x^\alpha$$

where $\beta!$ is the product of all powers of beta factorial.For example, if we use $x^\beta = x_4x_5$, we get $\beta! = 0! * 0! * 0! * 1! * 1! = 1$ The process is the same for α, where $\alpha = \partial_1^2$ corresponds to $\alpha! = 2! = 2$. If our solution space were not zero dimensional as we checked earlier, the sum over β would be an series.
Now we can construct our set of functions using our polynomial formula:

$$f_{(1)}(x) = 1$$
$$f_{(x_3)}(x) = x_3 - x_1 - x_2$$
$$f_{(x_4)}(x) = x_4 + x_1$$
$$f_{(x_5)}(x) = x_5 - x_1 - x_2$$
$$f_{(x_4x_5)}(x) = x_4x_5 + x_1x_5 + x_3x_4 + x_2x_3 - \frac{x_3^2}{2} - \frac{x_2^2}{2} - \frac{x_1^2}{2}$$

After verifying these solutions are in fact eliminated by all generators and elements of I that have shown up on this poster, we're done! We now have a vector space basis for the polynomial solution space to our system of PDEs.

$$Polysol(J) = \{1, x_3 - x_1 - x_2, x_4 + x_1, x_5 - x_1 - x_2, f_{(x_4x_5)}(x)\}$$

Partial Sage Implimentation

Code used for all calculations included in this poster is available on GitHub at github.com/lukeask/Differential-Equations. In it's current state, the code is tailored to the specific operator polynomials used in this poster though improvements will come soon.

References

[1] Cox David, Little John, and O'Shea Donal.
Ideals, Varieties, and Algorithms.
Springer, 3 edition, 2008.

[2] Bernd Strumfels.
Solving Systems of Polynomial Equations.
AMS, 2002.

[3] The Sage Developers.
SageMath, the Sage Mathematics Software System (Version 8.0),
2017.
https://www.sagemath.org.