Simplicial Sets Notes

1 The Idea of Simplicial Sets

Simplicial sets are a generalization of simplicial complexes. Let's start by looking at an example of a simplicial set coming from topology.

Example 1.1. Let Y be a topological space. The functor $SY := \hom_{\mathsf{Top}}(|-|, Y)$ is a simplicial set. Intuitively, it is assigning to each n the set of continuous maps of the n-simplex into Y. Applying the free abelian group functor to these sets yields the singular chain groups.

Our first goal will be to understand singular homology in terms of simplicial sets before branching out. The definition of simplicial sets requires thinking of functors to Set coming from the category Δ.

Definition 1.1. $\overline{\Delta}$ is the category of totally ordered sets with order preserving functions.

We will usually think of $\overline{\Delta}$ as its skeleton, Δ, which has objects sets of the form

$$[n] := \{0, 1, \cdots, n\}$$

for all $n \in \mathbb{Z}_{\geq 0}$. In this case, order preserving just means $x \geq y$ implies $f(x) \geq f(y)$.

Recall that a total order must have $x \geq x$.

Example 1.2. The following are order preserving maps:

$$\begin{tikzcd}
3 \arrow[dr] & \\
2 \arrow[dr] & 2 \\
1 \arrow[r] & 1 \\
0 \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
 & 4 \\
3 \arrow[ur] & 3 \\
2 \arrow[ur] & 2 \\
1 \arrow[ur] & 1 \\
0 \arrow[ur] & 0
\end{tikzcd}
$$

while the following is not:

$$
\begin{tikzcd}
3 \arrow[dr] & \\
2 \arrow[dr] & 2 \\
1 \arrow[ur] & 1 \\
0 \arrow[ur] & 0
\end{tikzcd}
$$

Remark 1.1. A category is *skeletal* if objects that are isomorphic if and only if they are equal. For example, we can take the category of finite dimensional vector spaces which will have a skeleton given by taking one vector space of each dimension since any two vector spaces of the same dimension are isomorphic. Any two skeletons of a category are unique, and assuming choice allows for finding the skeleton of any accessable category. Here is a relevant stackexchange post.

Definition 1.2. A *simplicial set* is a presheaf on Δ valued in Set.

Recall that a presheaf on $\mathcal{E}$ valued in $\mathcal{C}$ is a contravariat functor from $\mathcal{E}$ to $\mathcal{C}$. It is common to write F_n for $F[n]$. Elements of F_n are called n-simplicies.

Example 1.3 (Standard Simplicies)**.** The Yoneda embedding gives us a family of simplicial sets.
$y[n] = \hom_\Delta(-, [n])$ is a simplicial set for all $[n]$, and is called the simplicial set representing the standard n-simplex.

Post composition of order preserving maps gives natural transformations between these functors.

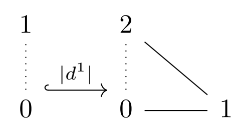
Example 1.4 (Total Singular Complex)**.** Define the (covariant) functor $|-|: \Delta \to \mathsf{Top}$ on objects via

$$[n] \mapsto \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} | \sum x_i = 1, x_i \geq 0\}$$

We will define the effect on general morphisms shortly, but for intuition we consider the order preserving map $d^i : [n-1] \to [n]$ which is the inclusion skipping i. Then $|d^i|$ is found by setting $x_i = 0$, this is geometrically the inclusion of the face not touching the ith vertex.

Consider $d^1 : [1] \to [2]$

$$\begin{tikzcd} & 2 \\ 1 \arrow[ur, "d^1"] & 1 \\ 0 \arrow[r, "d^1"] & 0 \end{tikzcd}$$



Let Y be a topological space. We claim

$$SY := \hom_{\mathsf{Top}}(|-|, Y)$$

is a simplicial set.

On objects, $SY_n = \hom_{\mathsf{Top}}(\Delta_n, Y)$ realizes the nth singular chain group.

On the d_i maps, we get a function $\hom_{\mathsf{Top}}(|n|, Y) \to \hom_{\mathsf{Top}}(|n-1|, Y)$ under SY by restricting a map in $\hom_{\mathsf{Top}}(|n|, Y)$ to the ith face of $|n|$.

2 Order Preserving Maps

In the category Δ, we define the following maps:

Definition 2.1. We define the ith coface map $d^i : [n-1] \to [n]$ to be the map given by

$$d^i(k) = \begin{cases} k, k < i \\ k+1, k \geq i \end{cases}$$

We define the ith codegeneracy map $s^i : [n+1] \to [n]$ as

$$s^i(k) = \begin{cases} k, k \leq i \\ k-1, k > i \end{cases}$$

Example 2.1. s^i asks for the ith image element to be mapped to twice.

$$\begin{tikzcd}
3 \arrow[dr, "s^1"] & \\
2 \arrow[dr, "s^1"] & 2 \\
1 \arrow[r, "s^1"] & 1 \\
0 \arrow[r, "s^1"] & 0
\end{tikzcd}$$

Theorem 2.1. *Compositions of the coface and codegeneracy maps generate all morphisms in Δ.*

Definition 2.2. If X is a simplicial set, we define the face maps

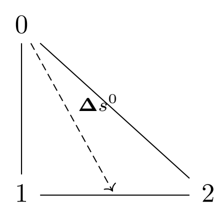
$$d_i := Xd^i$$

and degeneracy maps

$$s_i := Xs^i.$$

Example 2.2. Applying the total singular complex functor SY for a topological space Y to the codegeneracy map s^i yields $s_i = SYs^i$. Geometrically, $s_i : \hom_{\mathsf{Top}}(\Delta_{n-1}, Y) \to \hom_{\mathsf{Top}}(\Delta_n, Y)$ is composition of the map in the domain with collapse of the i and $i+1$th verticies to a single point.

$$\begin{tikzcd}
2 \arrow[dr, "s^0"] & \\
1 \arrow[dr, "s^0"] & 1 \\
0 \arrow[r, "s^0"] & 0
\end{tikzcd}$$



Considering the relations on the face and degeneracy maps imposed by functoriality, we get the following original definition of simplicial sets.

Definition 2.3. A simplicial set X is a collection of sets X_n for each integer $n \geq 0$ together with functions $d_i : X_n \to X_{n-1}$ and $S_i : X_n \to X_{n+1}$ satisfying the following relations:

$$d_i d_j = d_{j-1} d_i, i < j$$

$$s_i s_j = d_{j+1} d_i, i \leq j$$

$$d_i s_j = \begin{cases} 1, i = j, j+1 \\ s_{j-1} d_i, i < j \\ s_j d_{i-1}, i > j+1 \end{cases}$$

3 The category sSet

Definition 3.1. The category of simplicial sets, $\mathsf{sSet} := \mathsf{Set}^{\Delta^{op}}$.
The objects are simplicial sets and the morphisms are natural transformations.

Yoneda's lemma helps us understand the standard n simplicies, and we can use the density theorem to understand all simplicial sets as a colimit.

Recall that we have $y : \Delta \hookrightarrow \mathsf{sSet}$, providing the standard n simplicies. Applying Yoneda's lemma, we see that for any simplicial set X we have that

$$X_n \cong \hom_{\mathsf{sSet}}(|y[n]|, X)$$

is a natural isomorphism of sets.
Furthermore, density theorem states that

$$X \cong \mathrm{colim}_{x \in X_n} |y[n]|.$$

where x are understood via the Yoneda lemma to be maps into X from $y|n|$.

So we've established that understanding Δ and using the Yoneda lemma allows us to understand simplicial sets. Before I'm done I want to introduce the fun functor which formalizes our former photographs: $F : \Delta \to Cat$ which embeds a simplicial set $[n]$ as the free category with $n+1$ objects and n generating arrows. Functors between these images can be given by their action on the generating arrows.

4 Useful Links

A Leisurely Introduction to Simplicial Sets by Emily Riehl

Simplicial Sets Wikipedia

nLab