Riemann's Zeta Function and Applications to Number Theory

Luke Askew

April 2020

1 Introduction and Historical Background

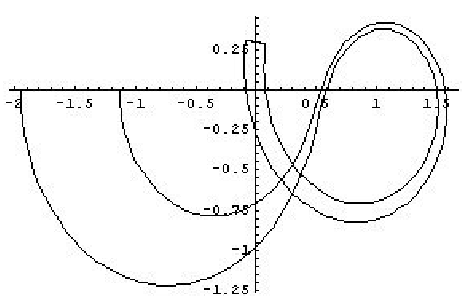
Euler was the first mathematician to consider what would later become known as Riemann's zeta function. Euler was interested in the infinite sum

$$\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + ...$$

where n is an integer. This led to some of the first evaluations of Riemann's function; after Leibniz failed to provide the value of the sum when $n = 2$, Euler proved it converges to $\frac{\pi^2}{6}$ by finding a clever power series for sine. [1, p.445] Riemann became interested in the sum when n is complex valued, defining Riemann's zeta function $\zeta : \mathbb{C} \to \mathbb{C}$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Riemann is responsible for identifying and proving an analytic continuation of $\zeta(s)$ which was used by Hadamard and de la Vallee Poussin to prove the prime number theorem.
[1, p.554]
When evaluating the function along the rectangle in the complex plane with vertices $0.4, 0.6, 0.6 + 14.5i, 0.4 + 14.5i$ the following image is produced:



[2, Course Webpage]

An open problem in mathematics in the Riemann Hypothesis: that all non-trivial zeros of $\zeta(s)$ lie on the line $Re(s) = \frac{1}{2}$ where trivial zeroes are those occurring at negative even numbers in the analytic continuation of $\zeta(s)$. Notice that although the analytic continuation of $\zeta(2) = 0$, the definition given so far of ζ evaluates to $\zeta(2) = 1^2 + 2^2 + 3^2 + ...$ which diverges to infinity.
We begin by proving a number of properties of $\zeta(s)$ before using knowledge of the function to prove the infinitude of prime numbers and to derive a number of integral formulas relating the function with the distribution of prime numbers.

2 Properties of Riemann's Zeta Function

$\zeta(s)$ converges for $Re(s) > 1$. To show this, we will show it converges absolutely which implies convergence.

Proof. Write $s = x + iy$ for $x, y \in \mathbb{R}$, as $|z^{i\theta}| = 1$ we have

$$\sum_{n=1}^{\infty} |\frac{1}{n^s}| = \sum_{n=1}^{\infty} |\frac{1}{n^{x+iy}}| = \sum_{n=1}^{\infty} |(\frac{1}{n^x})(\frac{1}{n^{iy}})| = \sum_{n=1}^{\infty} |\frac{1}{n^x}| = \sum_{n=1}^{\infty} \frac{1}{n^{Re(s)}}$$

By the p-Series Test, we have $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, so $\zeta(s)$ converges for $Re(s) > 1$.

$\square$

The next theorem demonstrates a fascinating link between $\zeta(s)$ and prime numbers through Euler's product formula which expresses the function as a product indexed by prime numbers denoted $\prod_p$.

Theorem 1.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, Re(s) > 1$$

[2]

Proof. Consider $\prod_p \frac{1}{1-p^{-s}}$. $\prod_p$ denotes a product over primes p. Since $p^{-s} < 1$ for $Re(s) > 1$, we can use geometric series properties to deduce

$$\prod_p \frac{1}{1 - p^{-s}} = \prod_p \sum_{n=0}^{\infty} (p^{-s})^n = \prod_p \sum_{n=0}^{\infty} p^{-sn}$$

By the fundamental theorem of arithmetic, we can write any natural number n as the product of primes uniquely. Write $n = 2^{c_{2,n}} \cdot 3^{c_{3,n}} \cdot ... \cdot p_k^{c_{k,n}} = \prod_p p^{c_p}$, a finite product. Consider $\zeta(s)$ in $Re(s) > 1$ so convergence will allow changing of term order.

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} (\prod_p p^{c_p})^{-s} = \sum_{n=1}^{\infty} (\prod_p p^{-sc_p})$$

Notice rearrangement of terms allows

$$\sum_{n=1}^{\infty}(\prod_p p^{-sc_p}) = \sum_{n=1}^{\infty} 2^{-sc_{2,n}} \cdot 3^{-sc_{3,n}} \cdot ... \cdot p_k^{-sc_{k,n}} = \prod_p \sum_{n=0}^{\infty} p^{-sn}$$

because the factorization of each n is unique, so we don't double count when we take the product over primes.
This gives the Euler product formula for $\zeta(s)$ in $Re(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

□

We can use this formula to prove that there are infinitely many primes.

Theorem 2. *There exist infinitely many prime numbers.*

Proof. Suppose for contradiction that there are finitely many primes, then since we can write $\zeta(s)$ as a product indexed by primes for any s with $Re(s) > 1$, $\zeta(1)$ is equal to a finite product.
However by the original definition of the function, $\zeta(1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ...$, the divergent harmonic series. Since $\zeta(1)$ cannot be both finite and infinite, we have a contradiction and there are infinitely many primes. □

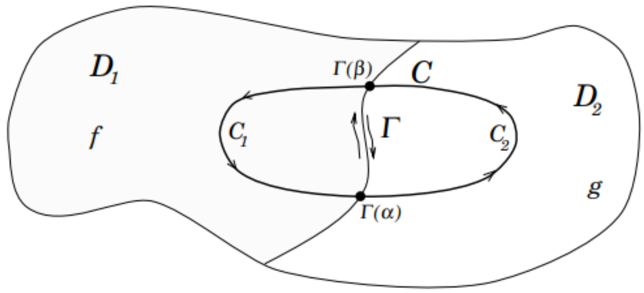
3 Analytic Continuation

We state the following theorem without proof with our goal being to evaluate ζ outside of $Re(s) > 1$.

Theorem 3. *Analytic continuation utilises uniqueness properties of analytic functions to find functions that behave consistently with a given function defined on a different domain. We will use the following theorem to see what we can say about $\zeta(s)$ where $Re(s) \leq 1$.*
Let D_1 and D_2 be two disjoint domains with boundaries sharing a common contour Γ, if $f(z)$ is analytic in D_1 and continuous in $D_1 \cup \Gamma$, if $g(z)$ is is analytic in D_2 and continuous in $D_2 \cup \Gamma$, and if $g(z) = f(z)$ for all $z \in \Gamma$, then

$$H(z) = \begin{cases} f(z);\ z \in D_1 \\ g(z);\ z \in D_2 \\ f(z) = g(z);\ z \in \Gamma \end{cases}$$

is analytic in $D_1 \cup D_2 \cup \Gamma$. $g(z)$ is called the analytic continuation of $f(z)$.



[3, p.122]
Importantly, the values taken by an analytic continuation of a function are equal to the values taken by the function in any shared domain between the two.
We can extend our original definition of Riemann's zeta function to $Re(z) > 0$ by examining $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$.

Theorem 4. *The analytic continuation of $\zeta(s)$ to $Re(z) > 0$ is given by*

$$\frac{1}{2^{1-s}-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}.$$

[4]

Proof. First, checking the absolute convergence of this new series using $s = x+iy$ gives

$$\sum_{n=1}^{\infty}|\frac{(-1)^n}{n^s}| = \sum_{n=1}^{\infty}|\frac{(-1)^n}{n^x}| = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}.$$

Which by the alternating series test converges for $x > 0$.
Taking its sum with $\zeta(s)$ provides

$$\sum_{n=1}^{\infty}\frac{1}{n^s}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s} = (1^{-s}+2^{-s}+3^{-s}+4^{-s}+...)+(-1^{-s}+2^{-s}-3^{-s}+4^{-s}-...)$$

and

$$2(2^{-s}+4^{-s}+^{-s}) = 2(\sum_{n=2,4,6,...}\frac{1}{n^s}) = 2\sum_{k=1}^{\infty}\frac{1}{(2k)^s} = 2\sum_{k=1}^{\infty}2^{-s}\frac{1}{k^s}.$$

So we have

$$\zeta(s)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s} = 2^{1-s}\zeta(s).$$

Solving for $\zeta(s)$, we find

$$\zeta(s) = \frac{1}{2^{1-s}-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}.$$

Since both manipulated series converge for $Re(z) > 1$, define the contour $\Gamma = \{z|Re(z) = 2\}$. Then on Γ,

$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{2^{1-s}-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}.$$

Notice that this is an analytic function times the series we started with, so their product is convergent for $\zeta(s)$ to $Re(z) > 0$ defining the analytic continuation of ζ to this larger region. $\square$

This extension can allow us to answer new questions about $\zeta(s)$, such as where $\zeta(s) = 0$ in this new domain. Take for example the line $\{z|z = 1 + it\}$ for t real. Evaluating this line provides

$$|\zeta(1+it)| = |\frac{1}{2^{1-(1+it)}-1}||\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1+it}}| \geq |\frac{1}{2^{-it}-1}|\sum_{n=1}^{\infty}|\frac{(-1)^n}{n^{1+it}}| = |\frac{1}{2^{-it}-1}|\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

So as

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - ...$$

converges we can rearrange

$$(-1 + \frac{1}{2}) + (-\frac{1}{3} + \frac{1}{4}) + ... = -\frac{1}{2} - \frac{1}{12} - \frac{1}{30} - ...$$

whose series of partial sums is negative and decreasing implying

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} < 0, |\zeta(1+it)| > 0.$$

Furthermore $|\frac{1}{1-p^{-s}}| > 1$ whenever $Re(s) > 1$ and $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ so we can strengthen this statement to say there are no zeroes of $\zeta(s)$ with $Re(s) \geq 1$.

4 Interesting Integral Formulas

$\zeta(s)$ is related to prime numbers through a number of integral formulas, some of which will be explored here. Peron's formula a general result for comparing a finite sum with an integral and alongside Riemann's functional equation can be used to prove the prime number theorem. To prove Peron's formula, we'll start by proving another integral formula serves as an indicator function. We'll also prove Abel's summation formula and state the Wiener-Ikehara Theorem which when combined will give a proof of a result equivalent to the prime number theorem.

Theorem 5. *Suppose $x \in \mathbb{R}, x \notin \mathbb{Z}$, $c > 0$, then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\frac{x}{n})^w \frac{dw}{w} = \{1; n < x, 0; n > x\}$$

[5]

Proof. Consider the contour $\gamma = \{z | z = c + it\}$ that we are evaluating along to be the diameter of a circle C_R of radius R centered at c. Define the points of C_R to the left of γ C_R^- and to the right C_R^+. This provides two simple closed curves L and R where $L = \gamma + C_R^-$ and $R = \gamma + C_R^+$. Define $a = \frac{x}{n}$, then $n < x$ implies $a > 1$ and $n > x$ implies $a < 1$. We take the branch cut of $log(z)$ with $0 \leq arg(z) < 2\pi$.
Case 1: Suppose $a > 1$ Since the pole $w = 0$ is contained in the region bounded by L we can apply the Cauchy Residue Theorem to see,

$$\frac{1}{2\pi i}\int_L \frac{a^w}{w} = 1$$

Dissecting this integral,

$$\frac{1}{2\pi i}\int_L \frac{a^w}{w} = \lim_{R\to\infty}\int_{C_R^-} \frac{a^w}{w} dw + \int_\gamma \frac{a^w}{w} dw$$

By Jordan's lemma from complex analysis, since $\frac{1}{w} \to 0$ uniformly as $R \to \infty$, $\lim_{R\to\infty}\int_{C_R^-} \frac{a^w}{w} dw = 0$

$$\lim_{R\to\infty}\int_{C_R^-} \frac{a^w}{w} dw = 0$$

So if $n < x$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\frac{x}{n})^w \frac{dw}{w} = 1$$

Case 2: Suppose $a < 1$, then we can proceed similarly showing the integral over C_R^+ goes to zero by Jordan's lemma, yet since there is no pole enclosed by R, we see if $n > x$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\frac{x}{n})^w \frac{dw}{w} = 0$$

So $\int_{c-i\infty}^{c+i\infty} (\frac{x}{n})^w \frac{dw}{w}$ is an indicator function for $n < x$. □

Theorem 6 (Peron's Formula)**.** *Suppose $x \in \mathbb{R}, x \notin \mathbb{Z}$, $c > 0$, and $\{a_n\}$ is a complex sequence, then*

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s+w)\frac{x^w}{w} dw$$

where

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Proof.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s+w)\frac{x^w}{w}dw = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^{\infty}\frac{a_n}{n^{s+w}}\frac{x^w}{w}dw$$

$$= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{c-i\infty}^{c+i\infty}\frac{a_n}{n^{s+w}}\frac{x^w}{w}dw = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c-i\infty}^{c+i\infty}(\frac{x}{n})^w\frac{dw}{w}.$$

Since $\int_{c-i\infty}^{c+i\infty}(\frac{x}{n})^w\frac{dw}{w}$ is an indicator function for $n < x$,

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\int_{c-i\infty}^{c+i\infty}(\frac{x}{n})^w\frac{dw}{w} = \sum_{n<x}\frac{a_n}{n^s}.$$

□

Using Peron's Formula with sequences of prime related numbers will give us an integral formula involving ζ.
First, define Λ such that if $n = p^k$ for p prime and $k \geq 1$, $\Lambda(n) = log(n)$ otherwise $\Lambda(n) = 0$.
Applying Peron's Formula to a sum of $\Lambda(n)$ for $n < x$ provides

$$\sum_{n<x}\frac{\Lambda(n)}{n^s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+w}}\frac{x^w}{w}dw.$$

Simplifying,

$$\sum_{n<x}\Lambda(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^w}\frac{x^w}{w}dw = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{p^k}\frac{log(p)}{p^{kw}}\frac{x^w}{w}dw.$$

Now we consider $-\frac{\zeta'(s)}{\zeta(s)}$

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds} - log(\zeta(s)) = \frac{d}{ds} - log(\prod_p \frac{1}{1-p^{-s}}) = \sum_p \frac{d}{ds}log(1-p^{-s})$$

$$= -\sum_p \frac{log(p)p^{-s}}{1-p^{-s}} = \sum_p \frac{log(p)}{1-p^s}.$$

Remembering the following identity for $|z| > 1$,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$$

we arrive at an expression for $-\frac{\zeta'(s)}{\zeta(s)}$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{log(p)}{1-p^s} = \sum_p log(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{(-p^s)^{n+1}} = \sum_{p^k} \frac{log(p)}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Substituting, we arrive at a nice formula

$$\sum_{n<x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (-\frac{\zeta'(s)}{\zeta(s)}) \frac{x^s}{s} ds.$$

We now explore theorems that will help us prove the prime number theorem.

Theorem 7 (Abel's partial summation formula)**.** *Suppose a_n is a complex sequence and $f(x)$ a complex function defined for all positive real numbers with continuous first derivative. Defining $A(x) = \sum_{1 \le n \le x} a_n$, we have*

$$\sum_{1 \le n \le x} a_n f(n) = A(x)f(x) - \int_1^x A(x) f'(x) dx.$$

[6]

Proof. Let k be an integer such that $k \le x < k+1$, then noting that $a_i = A(i) - A(i-1)$,

$$\begin{aligned}
\sum_{1 \le n \le x} a_n f(n) = \sum_{1 \le n \le k} a_n f(n) &= a_1 f(1) + a_2 f(2) + ... + a_k f(k) \\
&= A(1)f(1) + (A(2) - A(1))f(2) + ... + (A(k) - A(k-1))f(k) \\
&= -A(1)(f(2) - f(1)) - ... - A(k-1)(f(k+1) - f(k)) \\
&= A(k)f(k) - \sum_{1 \le n \le k-1} A(n)(f(n+1) - f(n)) \\
&= A(k)f(k) - \sum_{1 \le n \le k-1} A(n) \int_n^{n+1} f'(t) dt
\end{aligned}$$

For all t where $n < t < n+1$, $A(t) = A(n)$ so we can incorporate $A(n)$ as $A(t)$ in the integral.

$$\begin{aligned}
A(k)f(k) - \sum_{1 \le n \le k-1} A(n) \int_n^{n+1} f'(t) dt &= A(k)f(k) - \sum_{1 \le n \le k-1} \int_n^{n+1} f'(t) A(t) dt \\
&= A(k)f(k) - \int_1^k f'(t) A(t) dt
\end{aligned}$$

□

5 Prime Distribution

To demonstrate how powerful the integral formulas previously derived are we incorporate some facts proven elsewhere to describe how what we've seen so far relates to the prime number theorem.

Theorem 8 (Wiener-Ikehara Theorem)**.** *Let g be a non-negative, non-decreasing real function defined for all positive real numbers. Suppose the integral*

$$\int_0^\infty g(x)e^{-xs}dx = f(s)$$

converges for $Re(s) > 1$ and is analytic for $Re(s) \geq 1$ except for a simple pole at $s = 1$ with residue 1. Then,

$$\lim_{x\to\infty} e^{-x}g(x) = 1.$$

[7, p.16]
It is a fact that $-\frac{\zeta'(s)}{\zeta(s)}$ has a simple pole at $s = 1$ with residue 1 and is analytic for $Re(s) \geq 1$. Finally we need two new functions: $\psi(x) = \sum_{n\leq x} \Lambda(n)$ and $\pi(x) = \#$ primes $\leq$ x. [7, p.17]
First recall our derivation for $-\frac{\zeta'(s)}{\zeta(s)}$ in $Re(z) > 1$ and Abel's partial summation formula:

$$-\frac{\zeta'(s)}{s\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \sum_{1\leq n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(x)f'(x)dx.$$

Combining these results with $a_n = \Lambda(n)$ and $f(n) = \frac{1}{n^s}$ achieves

$$\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\Lambda(n)}{n^s} \\
&= \lim_{N\to\infty} (\sum_{1\leq n\leq x} \Lambda(N))\frac{1}{N} + s\int_1^N \frac{\Lambda(x)}{x^{s+1}}dx \\
&= s\int_1^\infty \frac{\psi(x)}{x^{s+1}}dx \\
&= s\int_0^\infty \psi(e^x)e^{-sx}dx.
\end{aligned}$$

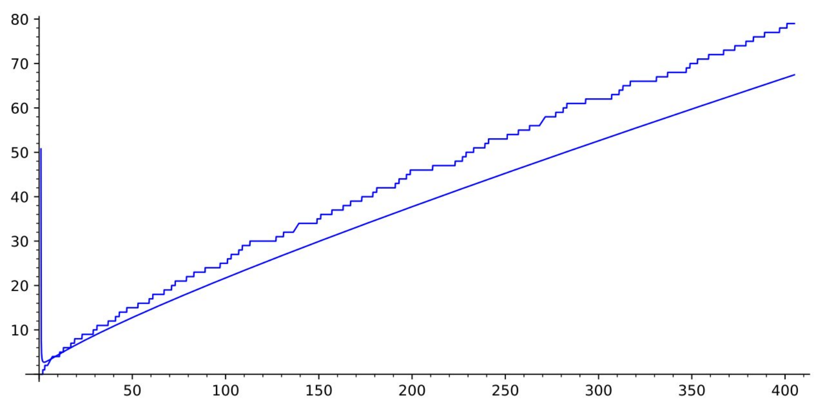
Where for the last step we let $x = e^u, dx = e^u du$. So now

$$-\frac{\zeta'(s)}{s\zeta(s)} = \int_0^\infty \psi(e^x)e^{-sx}dx.$$

Applying the Wiener-Ikehara Theorem,

$$\lim_{x\to\infty} e^{-x}\psi(e^x) = 1 = \lim_{x\to\infty} \frac{\psi(x)}{x}.$$

This fact is equivalent to the prime number theorem, that $\lim_{x\to\infty} \frac{x}{log(x)} = \lim_{x\to\infty} \pi(x)$. Handwaving aside, this demonstrates that ζ is related to the distribution of prime numbers. [7, p.16]
When comparing the discrete function $\pi(x)$ to the continuous function $\frac{x}{log(x)}$ for $x \leq 405$ we see the following plot:



Sure enough, the functions are quite similar. Proofs involving Peron's Formula utilize more sophisticated machinery, but can provide estimates for $\pi(x)$ which are even closer.

6 Conclusion

From its students to its content, Riemann's zeta function is an example of just interconnected a subject mathematics is. ζ has been studied in some form by Leibniz, Euler, Riemann, Hadamard, de la Vallee Poussin and countless others connecting an infinite series with prime numbers with integral formulas with analytic continuation. Over the course of research for this paper, three proofs of the prime number theorem were identified, all of which utilized properties of this now infamous function. ζ is an interesting mathematical curiosity sure to be of interest to mathematicians and students for centuries to come.

References

[1] Carl B Boyer. *A History of Mathematics*. John Wiley and Sons, 2011.

[2] Noam D. Elkies. Introduction to analytic number theory elementary approaches ii: the euler product.

[3] Athanassios S. Fokas Mark J. Ablowitz. *Complex Variables*. Cambridge University Press, 2003.

[4] Alois Pichler. Strategies to evaluate the riemann zeta function.

[5] Yum-Tong Siu. Perron formula without error estimates.

[6] Craig A. Tracy. Abel partial summation formula.

[7] Michael Filaseta. Math 782: Analytic number theory (instructor's notes).

[8] Jonathan Sondow and Eric W Weisstein. Riemann zeta function.

[9] Maha Mosaad Alghamdi. *Analytic Proof of the Prime Number Theorem*. PhD thesis, 2019.

[10] Andrew Granville. The explicit formula and the proof of the prime number theorem.