Spherical Fabricius-Bjerre Theorem

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Introduction

Fabricius-Bjerre gives a theorem relating the number of double points; interior and exterior double tangents; and inflectional points of generic planar curves. Weiner extends Fabricius-Bjerre's theorem to the sphere, providing a similar formula for generic spherical curves which relates these same properties also with antipodal pairs. From this formula, Weiner concludes that a closed space curve whose tangent indicatrix is generic must have a pair of parallel tangents or a pair of parallel osculating planes.

Definitions

Let $\gamma : I \mapsto S^2$ or $\gamma : I \mapsto R^2$ be a closed spherical or planar curve. A point $P \in \gamma$ is a *double point* if $\gamma^{-1}(P)$ contains more than one preimage. An *inflectional point* P is one where the geodesic curvature of γ, denoted k_g, is such that $k_g = 0$. On S^2, this denotes a great circle and in $\mathbb{R}^2$ this denotes a line in the plane. A *vertex* is a point with torsion equal to zero.

A *double tangent* of γ is a great circle that is tangent to γ at exactly two distinct points. We call a double tangent an *exterior double tangent* if the curve around each point of tangency lies on the same side of γ. If a double tangent is not an exterior double tangent, it is an *interior double tangent*.

A *generic* closed space curve γ is defined as a curve where no point is more than one of the following: an inflection point, a double point, a double tangent, or an antipodal point (a point such that the direct opposite point on the sphere is also a point of γ). Furthermore, γ must have that all double points have exactly two preimages, at each inflectional point the arclength derivative of curvature is nonzero, and antipodal points must posses linearly independent derivatives. We can also call a curve generic if it has a generic tangent indicatrix.

We define:
$d =$ the number of double points of γ
$a =$ the number of antipodal points of γ
$2s =$ the number of inflectional points of γ
$i =$ the number of γ's interior double tangents
$e =$ the number of γ's exterior double tangents

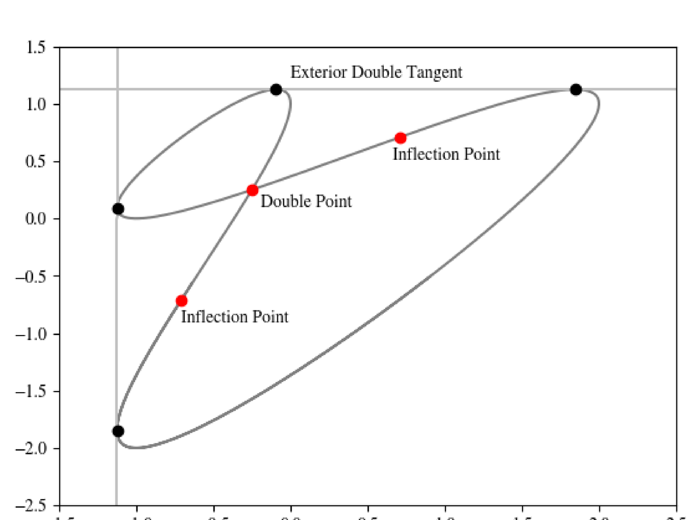


Figure 1: $\gamma(t) = (\sin t - \cos 2t, \cos t - \cos 2t)$ In the above example, $2s = 2$, $e = 2$, and $d = 1$, so the theorem holds as substitution yields $2 - 0 = 1 + 1$.

Theorem In The Plane

$$e - i = d + s \quad [1]$$

Fabricius-Bjerre's proof is by counting the points gained and lost by the positive and negative tangent's to γ. These quantities can only change at a double point, an inflectional point, or a point of double tangency in the plane.

When γ approaches a double point, it's positive half tangent intersects γ before crossing, but after crossing this point moves to the negative half tangent as shown below:

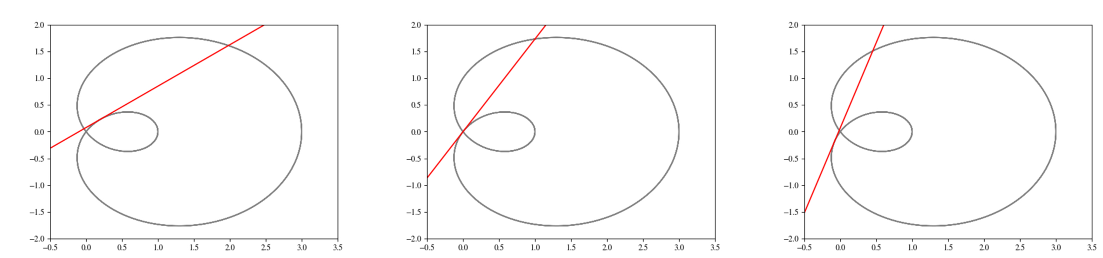


Figure 2: The tangent line to γ as it passes a double point.

When the curve is traversed once, it follows that each double tangent will be passed twice: once for each point of tangency. There are three possible types of both interior and exterior double tangents: those with positive half tangents facing the same direction, away from one another, or towards one another.

When γ has an exterior double tangent with positive half tangents in the same direction, case one of six (e_1), two points are gained on the positive half tangent on one passing, and two points on the negative half tangent are lost at the other. Loss of two points on the negative half tangent are shown below:

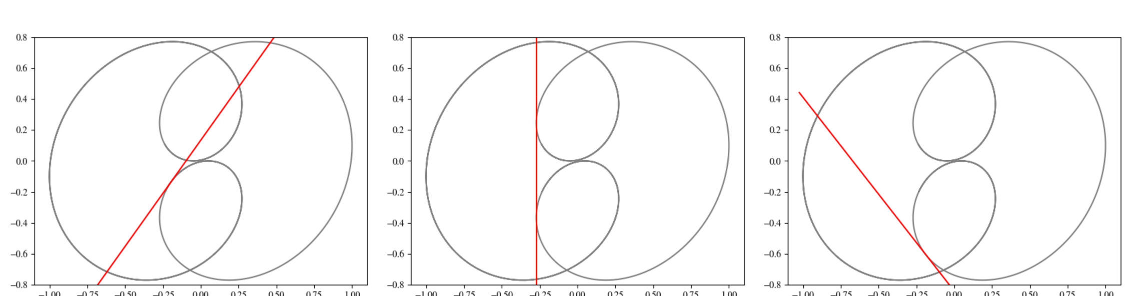


Figure : The tangent line to γ as it passes a double tangent.
Note: This is an exterior double tangent as the double tangent lies on the same side of γ

Similarly, the opposite effect is had for an interior double tangent of the same type (i_1). When double tangents are pointed at one another, two points are gained or lost by exterior (e_2) and interior (i_2) double tangents respectively; for opposite double tangents, two points are lost or gained on the negative half tangent for exterior (e_3) and interior (e_3) double tangents respectively.

From this perspective, this means one traversal of the curve results in $2e_1 + 4e_2$ points gained by the positive half tangent and $2i_1 + 4i_2 + 2d + 2s$ lost by the negative half tangent, resulting in the equality $2e_1 + 4e_2 = 2i_1 + 4i_2 + 2d + 2s$. Traversing the other way results in $2e_1 + 4e_3 = 2i_1 + 4i_3 + 2d + 2s$.

This makes sense as changing direction does not effect the type of same direction facing half tangents since both are reversed, yet this action does make away pointing double tangents point towards one another and vice versa.

Noting $(e_1 + e_2 + e_3)$ is the total number of exterior double tangents and $(i_1 + i_2 + i_3)$ is the total number of interior double tangents, summing the equalities provides [1, Theorem 1]:

$$(2e_1+4e_2)+(2e_1+4e_3) = (2i_1+4i_2+2d+2s)+(2i_1+4i_3+2d+2s)$$
$$(4(e_1 + e_2 + e_3) = 4(i_1 + i_2 + i_3) + 4d + 4s$$
$$e - i = d + s$$

On the Sphere

$$e - i = d - a + s$$

[2, Theorem 1]

On the sphere, our same arguments hold now considering the positive and negative half tangents to be halves of great circles. We still need to see what happens at antipodal points. When a spherical curve γ approaches an antipodal point, it will have this antipodal point intersect the positive half tangent as any geodesic at either point intersects the other. This is the opposite effect that is had from passing a double point as we've seen earlier, thus the theorem can be modified simply by subtracting a from the right hand side.

Incorporating the Tangent Indicatrix

Consider an arclength perametrized curve β taken to be the tangent to a space curve α which is generic in the sense that α is such that $\alpha'(s) = \beta(s)$ is generic. A double point on β corresponds to a pair of parallel tangents on α. Similarly, a pair of antipodal points on β correspond to a pair of oppositely parallel tangents on α. An important result in relating these curves is that the geodesic curvature of β, κ_g is equal to α's torsion divided by its curvature, so as we assume torsion of β is non-vanishing, we have that geodesic inflectional points of β correspond to vertices on α.

We will also consider β's unit binormal from the Frenet–Serret frame. Notice that two antipodal points will have unit binormals with equivalent span which will also occur at points of double tangency. γ near interior double tangents will pass the osculating planes of the points of double tangency in opposite general directions, so their osculating planes are labeled *discordant*. For exterior double tangents, the corresponding osculating planes are considered *concordant*, a label reflecting that β near exterior double tangents passes in the same general direction. We have now established a connection between all variables in the Spherical Fabricius-Bjerre Theorem with properties of a generic space curve allowing for statement of the following theorem:

Applications to Closed Space Curves

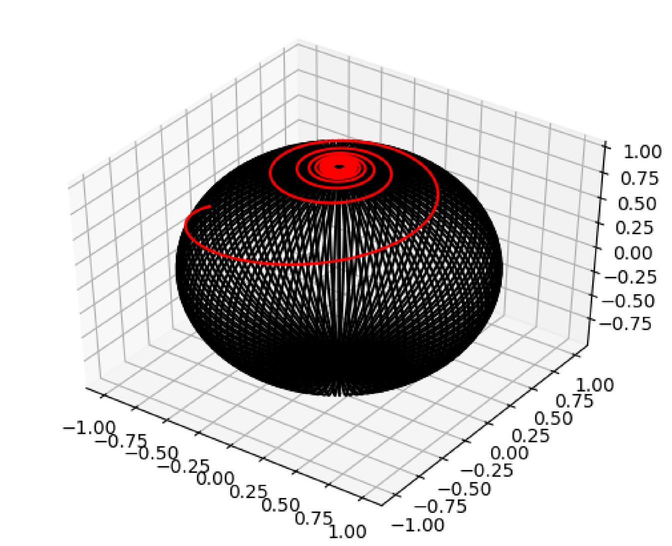
Let $\gamma : I \mapsto S^2$ be a generic non-degenerate closed space curve, then $i = t - s - d + a$ where:

2i = the number of vertices points of γ
d = the number of pairs of directly parallel tangents of γ
a = the number of pairs of oppositely parallel tangents of γ
s = the number of pairs of discordant osculating planes of γ
t = the number of pairs of concordant parallel osculating planes of γ
[2, Theorem 2]

As a corollary, we can show that for any closed generic space curve γ, that γ possesses at least one pair of parallel tangents or a pair of parallel osculating planes.

[2, Theorem 3]

We can prove this by contradiction. If there are no pairs of parallel tangents or pairs of parallel osculating planes, $d = a = t = s = 0 = 2i$ as any of $d, a, t, s \neq 0$ will invoke a pair. $2i = 0$ implies torsion of γ has constant sign. We know that γ must never have zero curvature by the definition of regular curves, so the curvature of γ also has constant sign. κ_g of $\gamma'(s)$ is a ratio of these values so additionally has constant sign. Consider, without loss of generality, that $\gamma'(s)$ has positive geodesic curvature $k_g > 0$. $\gamma'(s)$ is clearly constantly turning in the same direction relative to the great circle defined by the intersection of the $\gamma'(s)$, $\gamma''(s)$ plane with the sphere, so if $\gamma'(s)$ is to not intersect itself, it must then follow a spiraling path which cannot ever be closed. We therefore have shown γ possesses at least one pair of parallel tangents or a pair of parallel osculating planes.



References

[1] Fr. Fabricius-Bjerre, *On double tangents of plane closed curves,* Math. Scand. 11 (1962), 113-116.
[2] J Weiner, *A Spherical Fabricius-Bjerre Formula with Applications to Closed Space Curves.* Math. Scand. 61 (1987), 286-291