Luke Beebe Multivariate HW 04 2/20/24

4 (4) a) Find distribution of $3X_1 - 2X_2 + X_3$

$$(3, -2, 1)\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 6 + 6 + 1 = 13 = \mu$$

$$\underbrace{\underset{1\times 3}{(3 \ \ -2 \ \ 1)}\underset{3\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}}}_{1\times 3}\underset{3\times 1}{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}} = \begin{pmatrix} 3-2+1 & 3-6+2 & 3-4+2 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 6 + 2 + 1 = 9 = \sigma^2$$

$$N_3(13, 9)$$

b) Find $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ such that X_2 and $X_2 - a^T\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ are independent

$$Y = X_2 - (a_1, a_2)\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} = X_2 - a_1X_1 - a_2X_3$$
$$= -a_1X_1 + X_2 - a_2X_3$$

$$A = \begin{array}{c} \begin{pmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{pmatrix} \end{array}\begin{array}{l} X_2 \\ Y \end{array}$$

$$A\Sigma A^T = \underset{2\times 3}{\begin{pmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{pmatrix}}\underset{3\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}}\underset{3\times 2}{\begin{pmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_2 \end{pmatrix}} = \begin{pmatrix} 1 & 3 & 2 \\ -a_1+1-a_2 & -a_1+3-2a_2 & -a_1+2-2a_2 \end{pmatrix}\begin{pmatrix} 0 & -a \\ 1 & 1 \\ 0 & -a \end{pmatrix}$$

$$= \begin{array}{c} X_2 \\ Y \end{array}\begin{pmatrix} 3 & -a_1+3-2a_2 \\ -a_1+3-2a_2 & -a(-a+1-a_2) - a_1 + 3 - 2a_2 - a(-a_1+2-2a_2) \end{pmatrix}$$

We want $-a_1 + 3 - 2a_2 = 0$

so $a_1 = 3 - 2a_2$ $\quad a = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + c\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(6) Independent?

a) X_1 and X_2 Yes $\mathrm{cov}(X_1, X_2) = 0$

b) X_1 and X_3 No $\mathrm{cov}(X_1, X_3) = -1$

c) X_2 and X_3 Yes $\mathrm{cov}(X_2, X_3) = 0$

d) (X_1, X_3) and X_2 Yes $\mathrm{cov}(X_1, X_2) = 0$ $\mathrm{cov}(X_3, X_2) = 0$

e) X_1 and $X_1 + 3X_2 - 2X_3 = Y$

No $\mathrm{cov}(X_1, Y) = 6$

$$\underset{2\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix}}\underset{3\times 3}{\begin{pmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{pmatrix}} = \begin{pmatrix} 4 & 0 & -1 \\ 6 & 15 & -5 \end{pmatrix}\underset{3\times 2}{\begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{pmatrix}}$$

$$\begin{array}{c|cc} & X_1 & Y \\ \hline X_1 & 4 & 6 \\ Y & 6 & 61 \end{array} = \begin{pmatrix} 4 & 6 \\ 6 & 61 \end{pmatrix}$$