5.1.22

Exercise 5.1.22 Consider a cart attached to a wall by a spring, as shown in Figure 5.5

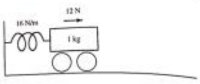


Figure 5.5 Solve for the motion of the cart.

time zero the cart is at rest at its equilibrium position $x = 0$. At that moment a steady force of 12 newtons is applied, pushing the cart to the right. Assume that the rolling friction is $-k\,\dot{x}(t)$ newtons, where $k \geq 0$. Do parts (a) through (d) by hand.

(a) Set up a system of two first-order differential equations of the form $\dot{x} = Ax - b$ for $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$.

(b) Find the steady-state solution of the differential equation.

(c) Find the characteristic equation of A and solve it by the quadratic formula to obtain an expression (involving k) for the eigenvalues of A.

(d) There is a *critical value* of k at which the eigenvalues of A change from real to complex. Find this critical value.

(e) Using MATLAB, solve the initial value problem for the cases (i) $k = 1$, (ii) $k = 6$, (iii) $k = 10$, and (iv) $k = 14$. Rather than reporting your solutions, simply plot $x_1(t)$ for $0 \leq t \leq 3$ for each of your four solutions on a single set of axes. (Do not overlook the help given in Exercises 5.1.19 and 5.1.20.) Comment on your plots (e.g. rate of decay to steady state, presence or absence of oscillations).

(f) What happens when $k = 0$?

$\dot{x} = Ax - b$

$F = ma$ $\quad x(0) = 0$ $\quad \dot{x}(0) = 0$

$x_1 = x \Rightarrow \dot{x}_1 = \dot{x}$

$x_2 = \dot{x} \quad \dot{x}_2 = \ddot{x}$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$

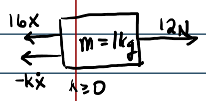
$\dot{x}_1 - x_2 = 0$

$\dot{x}_1 = x_2$

$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16 & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -16x_1 - kx_2$



$\dot{x}_2 + kx_2 + 16x_1 = 12$

$\dot{x}_2 = 12 - kx_2 - 16x_1$

$\ddot{x} + k\dot{x} + 16x = 12$

$\dot{x}_2 + kx_2 + 16x_1 = 12$

$\dot{x}_1 = x_2$

$\dot{x}_2 = 12 - kx_2 - 16x_1$

a) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16 & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -12 \end{pmatrix}$

b) $0 = \begin{pmatrix} 0 & 1 \\ -16 & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -12 \end{pmatrix}$

$\begin{pmatrix} 0 \\ -12 \end{pmatrix} = \begin{pmatrix} x_2 \\ -16x_1 - kx_2 \end{pmatrix} \Rightarrow x_2 = 0, \quad x_1 = 12/16 = \frac{3}{4}$

Steady State Solution is $\begin{pmatrix} 3/4 \\ 0 \end{pmatrix} = \begin{pmatrix} \text{position} \\ \text{velocity} \end{pmatrix}$

c) $\det \begin{pmatrix} -\lambda & 1 \\ -16 & -k-\lambda \end{pmatrix} = \lambda(k+\lambda) + 16$

$= \lambda^2 + k\lambda + 16$

$\lambda_{1,2} = \frac{-k \pm \sqrt{k^2 - 4(1)(16)}}{2}$

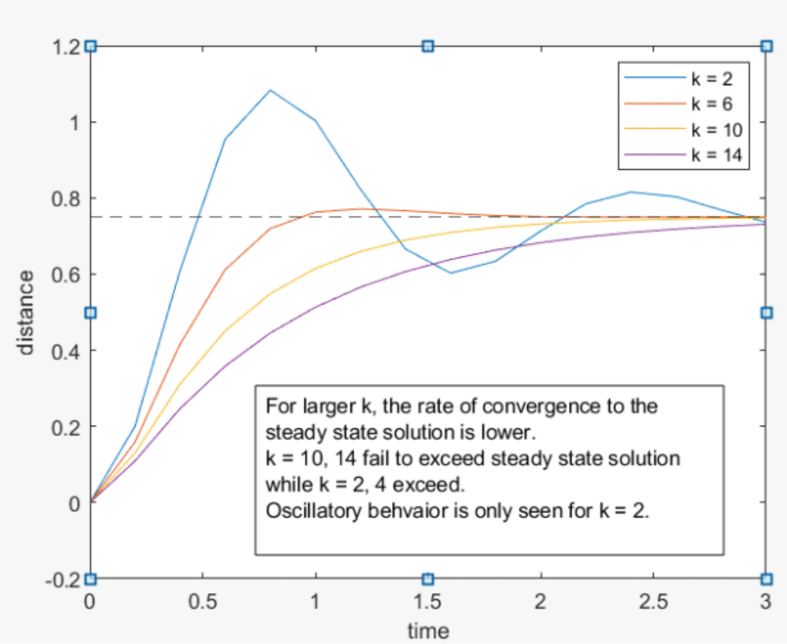
$\lambda_{1,2} = -\frac{k}{2} \pm \frac{\sqrt{k^2 - 64}}{2}$

d) $k^2 - 64 = 0$

$k^2 = 64 \Rightarrow k = 8$

5.1.22 Continued 1

e) Plots k = 2 : 4 : 14



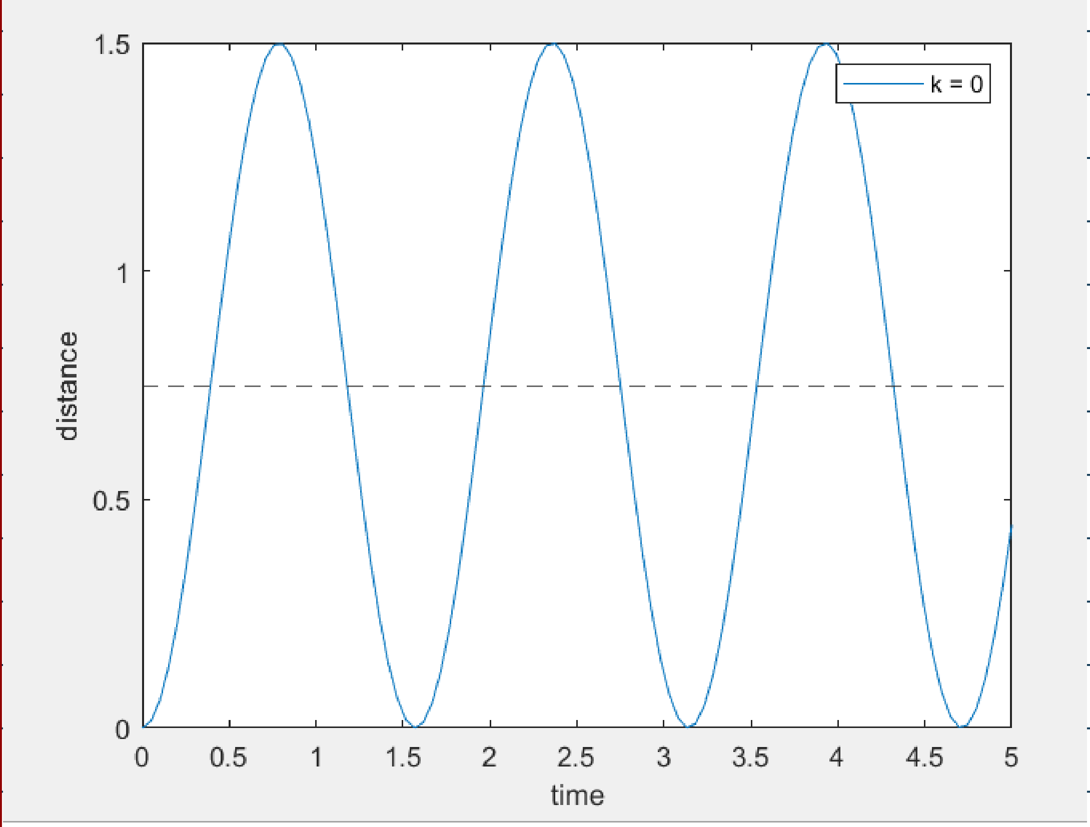
Physically reasonable.

```
x_plot = zeros(8, 16); %initialize solution
for k = 2:4:14
        A = [0, 1; -16, -k]; %matrix
        x_o = [0; 0]; %initial condition
        [V, D] = eig(A); %mat4rix of e-vectors and diagonalization
        b = [0; -12]; %forcing function
        z = (A)^(- 1) * b; %calculate z
        c = V^(-1) * (x_o -z); %calculate c
        t = 0:0.2:3; %time
        x = z * ones(size(t)); %initialize x
        for j = 1:2
            x = x + V(:, j) * exp(t * D(j, j)) * c(j); %the money
        end
    x_plot((k / 2), :) = x(1, :); %store solution
    x_plot((k / 2) + 1, :) = x(2, :); %store solution
end
p = plot(t, x_plot(1, :), t, x_plot(3, :), t, x_plot(5, :), t, x_plot(7, :)) %plot
yline(3/4, '--')
legend('k = 2', 'k = 6', 'k = 10', 'k = 14')
xlabel('time')
ylabel('distance')
```

5.1.22 Continued 2

"Undamped"

f) k=0 case Pure Oscillation



Theortical System - Not physically possible

```
k = 0; %k is zero
A = [0, 1; -16, -k]; %matrix
    x_o = [0; 0]; %intial condition
    [V, D] = eig(A); %matrix of e-vectors and diagonalization
    b = [0; -12]; %forcing function
    z = (A)^(- 1) * b; %calc z
    c = V^(-1) * (x_o -z); %calc c
    t = linspace(0, 5); %initialize time
    x = z * ones(size(t)); %intial solution
    for j = 1:2
      x = x + V(:, j) * exp(t * D(j, j)) * c(j); % the money
    end

p = plot(t, x(1, :)) %plot
yline(3/4, '--')
legend('k = 0')
xlabel('time')
ylabel('distance')
```

5.1.23

$F = ma$

SYSTEMS OF DIFFERENTIAL EQUATIONS 303

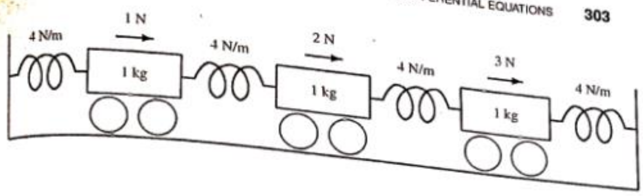


Figure 5.6 Solve for the motion of the carts.

5.1.23 Consider a system of three carts attached by springs, as shown in Figure 5.6. The carts are initially at rest. At time zero the indicated forces are applied, causing the carts to move toward a new equilibrium. Let x_1, x_2, and x_3 denote the displacements of the carts, and let x_4, x_5, and x_6 denote their respective velocities. Suppose the coefficients of rolling friction of the three carts are k_1, k_2, and k_3, respectively.

(a) Apply Newton's second law to each cart to obtain a system of three second-order differential equations for the displacements x_1, x_2, and x_3. You may find it useful to review Example 1.2.10.

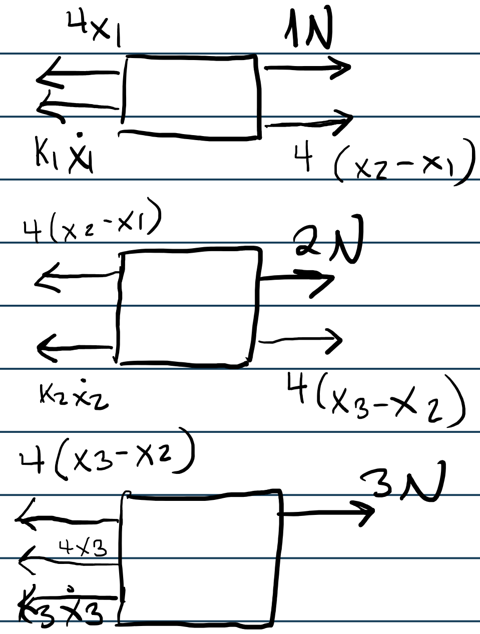
(b) Introducing the velocity variables x_4, x_5, and x_6, rewrite your system as a system of six first-order differential equations. Write your system in the form $\dot{x} = Ax - b$.

(c) Find the steady-state solution of the system.

(d) Solve the initial value problem under each of the conditions listed below. In each case plot x_1, x_2, and x_3 on a single set of axes for $0 \le t \le 20$, and comment on the plot.

(1) $k_1 = 1$, $k_2 = 0$, and $k_3 = 0$.
(2) $k_1 = 1$, $k_2 = 8$, and $k_3 = 8$.
(3) $k_1 = 8$, $k_2 = 8$, and $k_3 = 8$.

a)



$\ddot{x}_1 + k_1\dot{x}_1 - 4(x_2-x_1) + 4x_1 = 1$

$\ddot{x}_2 + k_2\dot{x}_2 + 4(x_2-x_1) - 4(x_3-x_2) = 2$

$\ddot{x}_3 + k_3\dot{x}_3 + 4(x_3-x_2) + 4x_3 = 3$

b) $x_4 = \dot{x}_1$, $x_5 = \dot{x}_2$, $x_6 = \dot{x}_3$

$\dot{x}_4 = \ddot{x}_1$, $\dot{x}_5 = \ddot{x}_2$, $\dot{x}_6 = \ddot{x}_3$

$\dot{x}_4 + k_1 x_4 - 4(x_2 - x_1) + 4x_1 = 1 \rightarrow \dot{x}_4 = -k_1x_4 + 4(x_2-x_1) - 4x_1 + 1$
$= -8x_1 + 4x_2 - k_1x_4 + 1$

$\dot{x}_5 + k_2 x_5 + 4(x_2 - x_1) - 4(x_3 - x_2) = 2 \rightarrow \dot{x}_5 = -k_2x_5 - 4(x_2-x_1) + 4(x_3-x_2) + 2$
$4x_1 - 8x_2 + 4x_3 - k_2x_5 + 2$

$\dot{x}_6 + k_3x_6 + 4(x_3 - x_2) + 4x_3 = 3$ $\quad \dot{x}_6 = 4x_2 - 8x_3 - k_3x_6$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -8 & 4 & 0 & -k_1 & 0 & 0 \\ 4 & -8 & 4 & 0 & -k_2 & 0 \\ 0 & 4 & -8 & 0 & 0 & -k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -2 \\ -3 \end{pmatrix}$$

c) Steady State Sol'n

$\vec{0} = A\vec{x} - \vec{b} \Rightarrow A\vec{x} = \vec{b}$

$x_4 = x_5 = x_6 = 0$ &

$$\left.\begin{aligned} -8x_1 + 4x_2 &= -1 \\ 4x_1 - 8x_2 + 4x_3 &= -2 \\ 4x_2 - 8x_3 &= -3 \end{aligned}\right\}$$

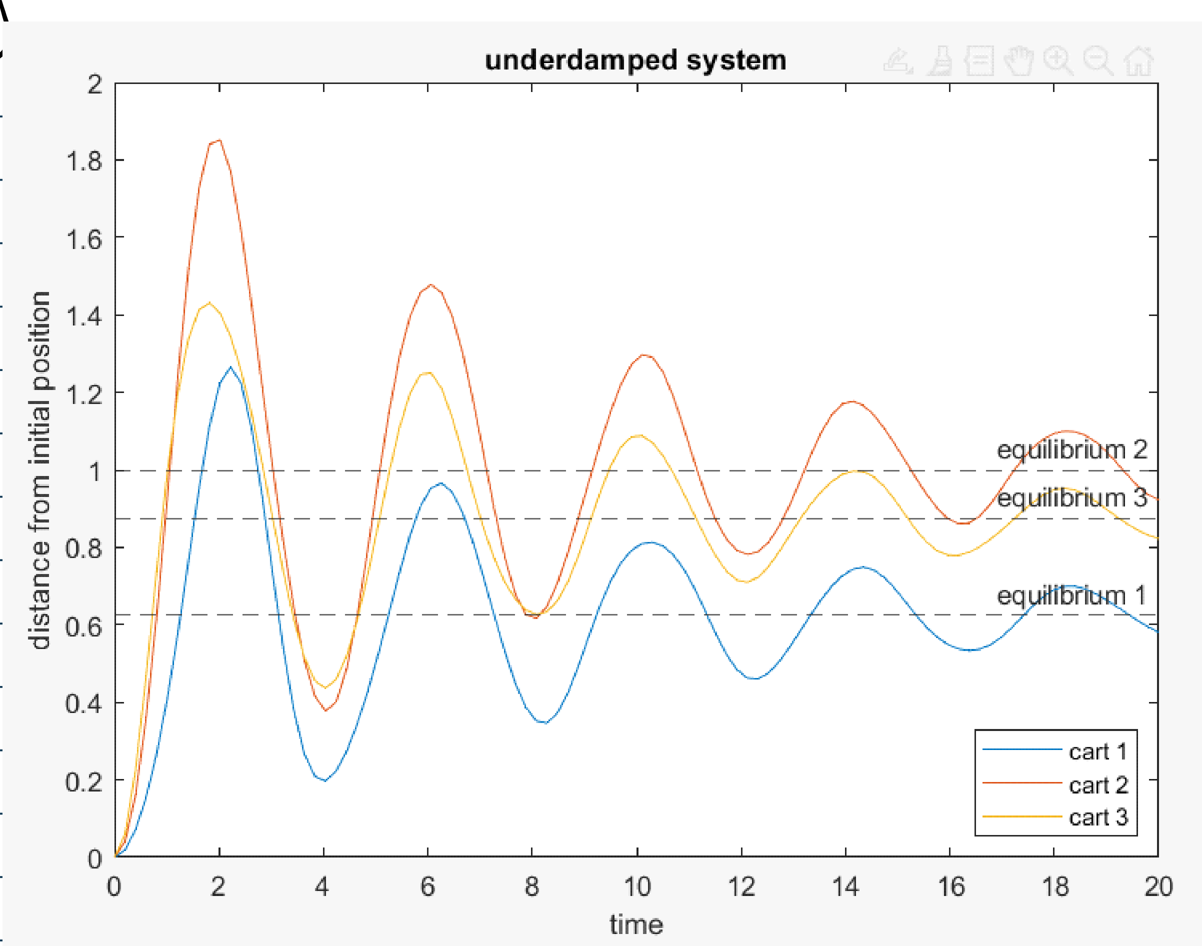
Same system as Ex 1.2.10

So, $\boxed{\vec{x} = \begin{pmatrix} 0.625 \\ 1 \\ 0.875 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$

$$4x_2 - 8x_3 = -3 \quad \left\| \begin{pmatrix} 0.875 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|$$

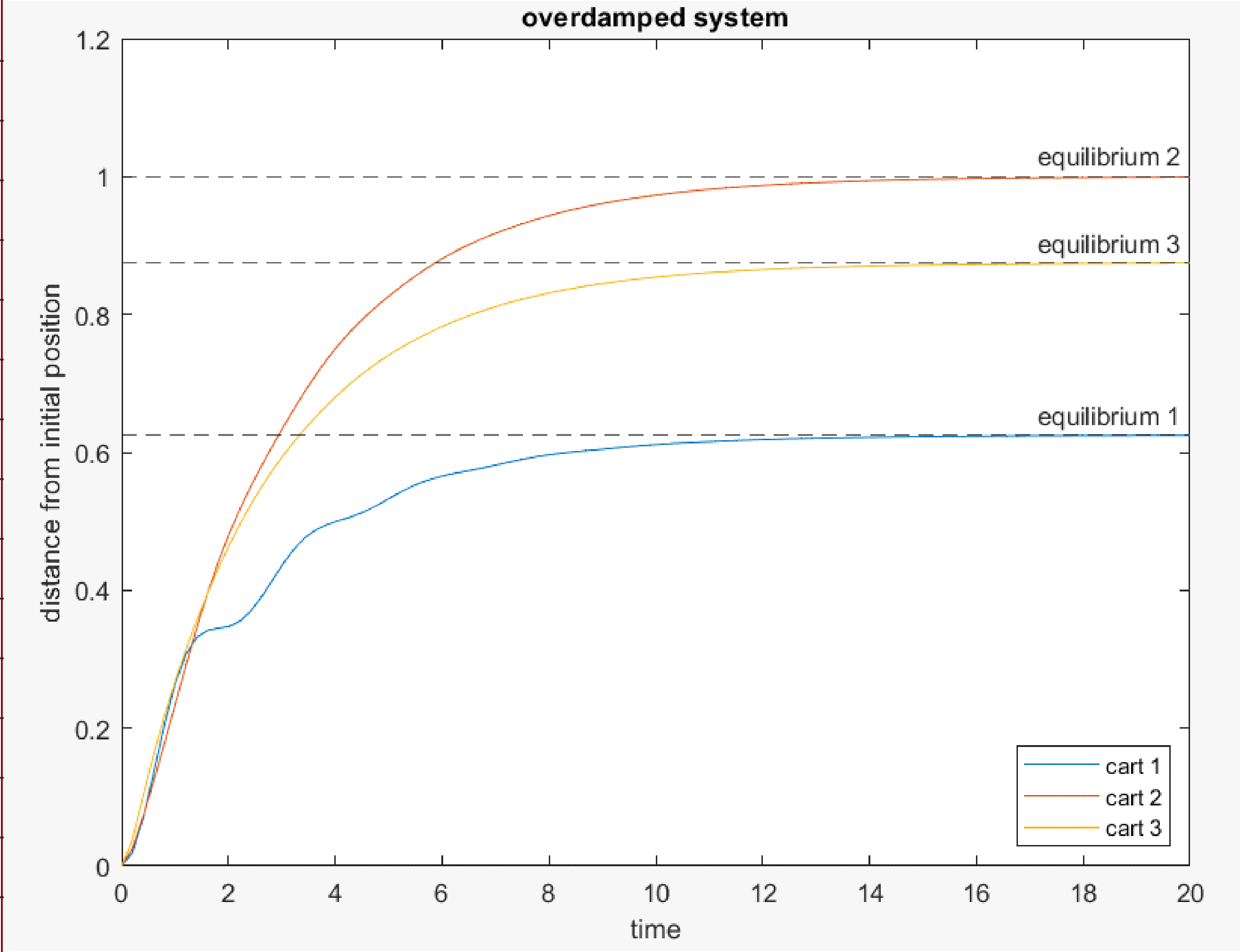
5.1.23 Continued 1

d)



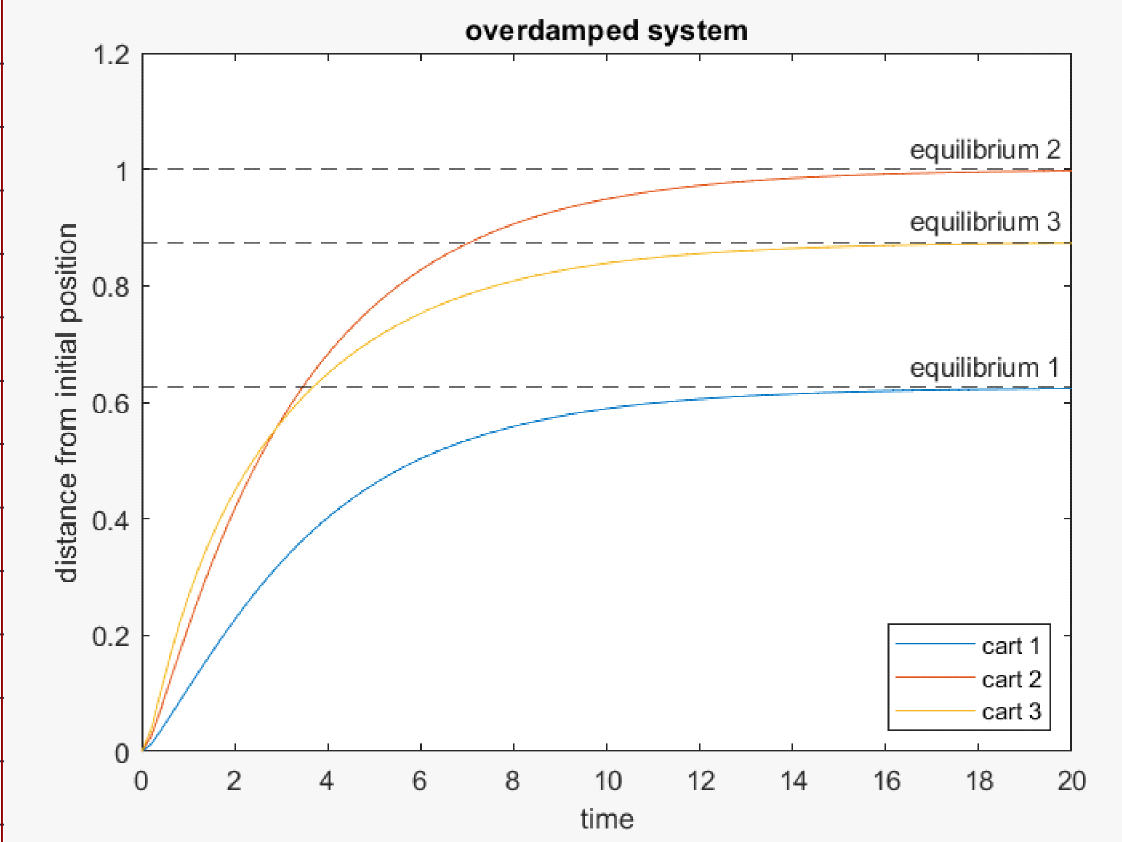

```
k1 = 1; %intialize k's
k2 = 0;
k3 = 0;
A = [zeros(3, 3) eye(3); -8 4 0 -k1 0 0; 4 -8 4 0 -k2 0; 0 4 -8 0 0 -k3]; %intialize A
        x_o = transpose([0 0 0 0 0 0]); %intial condition
        [V, D] = eig(A); %matrix of e-vectors and diagonalization
        b = transpose([0 0 0 -1 -2 -3]); %forcing function
        z = (A)^(- 1) * b; %calc z
        c = V^(-1) * (x_o - z); %calc c
        t = linspace(0, 20); %intialize time
        x = z * ones(size(t)); %intialize solution
        for j = 1:6
            x = x + V(:, j) * exp(t * D(j, j)) * c(j); %the money
        end
x = real(x);
figure
p = plot(t, x(1, :), t, x(2, :), t, x(3, :)) %plot
xlabel('time')
ylabel('distance from initial position')
yline(0.625, '--', 'equilibrium 1')
yline(1, '--', 'equilibrium 2')
yline(0.875, '--', 'equilibrium 3')
legend('cart 1', 'cart 2', 'cart 3')
legend('Location','southeast')
ylim([0 2])
title('underdamped system')
```

5.1.23 Continued 2




```
k1 = 1;
k2 = 8;
k3 = 8;
A = [zeros(3, 3) eye(3); -8 4 0 -k1 0 0; 4 -8 4 0 -k2 0; 0 4 -8 0 0 -k3];
    x_o = transpose([0 0 0 0 0 0]);
    [V, D] = eig(A);
    b = transpose([0 0 0 -1 -2 -3]);
    z = (A)^(-1) * b;
    c = V^(-1) * (x_o - z);
    t = linspace(0, 20);
    x = z * ones(size(t));
    for j = 1:6
        x = x + V(:, j) * exp(t * D(j, j)) * c(j);
    end
x = real(x);
figure
p = plot(t, x(1, :), t, x(2, :), t, x(3, :))
xlabel('time')
ylabel('distance from initial position')
yline(0.625, '--', 'equilibrium 1')
yline(1, '--', 'equilibrium 2')
yline(0.875, '--', 'equilibrium 3')
legend('cart 1', 'cart 2', 'cart 3')
legend('Location','southeast')
ylim([0 1.2])
title('overdamped system')
```

5.1.23 Continued 3




```
k1 = 8;
k2 = 8;
k3 = 8;
A = [zeros(3, 3) eye(3); -8 4 0 -k1 0 0; 4 -8 4 0 -k2 0; 0 4 -8 0 0 -k3];
    x_o = transpose([0 0 0 0 0 0]);
    [V, D] = eig(A);
    b = transpose([0 0 0 -1 -2 -3]);
    z = (A)^(- 1) * b;
    c = V^(-1) * (x_o - z);
    t = linspace(0, 20);
    x = z * ones(size(t));
    for j = 1:6
      x = x + V(:, j) * exp(t * D(j, j)) * c(j);
    end
x = real(x);
figure
p = plot(t, x(1, :), t, x(2, :), t, x(3, :))
xlabel('time')
ylabel('distance from initial position')
yline(0.625, '--', 'equilibrium 1')
yline(1, '--', 'equilibrium 2')
yline(0.875, '--', 'equilibrium 3')
legend('cart 1', 'cart 2', 'cart 3')
legend('Location','southeast')
ylim([0 1.2])
title('overdamped system')
```