Maxwell's Equations from a Bounded Current

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Abstract

It is shown that Maxwell's equations follow directly from the ansatz "an electric current density is a conserved, integrable, and bounded timelike vector field in spacetime."

1 Derivatives and Antiderivatives

The vector derivative of a multivector field M is given by

$$\nabla M = \nabla \cdot M + \nabla \wedge M. \tag{1}$$

Hence, denoting $M_n \equiv \langle M \rangle_n$,

$$\langle \nabla M \rangle_n = \nabla \cdot M_{n+1} + \nabla \wedge M_{n-1}. \tag{2}$$

[1] shows that any integrable multivector function is the derivative of some other multivector function. This is a generalization of the Helmholtz theorem. The antiderivative of a multivector function is given explicitly as an integral in [1] on page 261, although it is not needed for this result.

An antiderivative G of some multivector field J satisfies

$$\nabla G = J. \tag{3}$$

Constraints on G can be determined using Equation 2. J_n is given by

$$J_n = \nabla \cdot G_{n+1} + \nabla \wedge G_{n-1}. \tag{4}$$

Hence, if $J_k = 0$, then

$$\nabla \cdot G_{k+1} = 0 \text{ and } \nabla \wedge G_{k-1} = 0 \tag{5}$$

are constraints on G.

2 Maxwell's Equations

Suppose J_1 is an conserved (i.e. divergenceless), integrable vector field. That J_1 is integrable implies that it possesses an antiderivative G satisfying

$$\nabla G = J_1. \tag{6}$$

Using Equation 5, the potential can be placed in the form

$$G = G_0 + G_2 + C, \tag{7}$$

with the constraint

$$\nabla \wedge G_2 = 0 \tag{8}$$

and where C is monogenic, i.e.

$$\nabla C = 0. \tag{9}$$

This means that

$$J_1 = \nabla G = \nabla G_0 + \nabla \cdot G_2. \tag{10}$$

Thus, since J_1 is divergenceless, i.e.

$$\nabla \cdot J_1 = \nabla^2 G_0 + \nabla \cdot (\nabla \cdot G_2) = \nabla^2 G_0 = 0, \tag{11}$$

the only constraint on G is

$$\nabla^2 G_0 = 0. \tag{12}$$

More generally, J_1 is of the form

$$J_1 = \nabla \cdot G_2 + B, \tag{13}$$

where B is a monogenic vector field.

The fundamental property of a monogenic field is that, in any region it is uniquely determined by its values on the boundary of that region. Because of this, it shares many similar properties to complex analytic functions. In particular, if B is timelike (i.e. positive-definite $B^2 > 0$) and B^2 is bounded over all space, then it must be constant.[2] This is a generalization of Liouville's theorem. Hence, if J_1 is timelike and bounded over all space, then B must be a constant.

Remarkably, this means that the antiderivative of any conserved, integrable, and bounded timelike vector field $J \in G_{1,3}$ in spacetime is a bivector field F satisfying Maxwell's equations

$$\nabla F = J. \tag{14}$$

References

[1] D. Hestenes. *Clifford algebra to geometric calculus.* D. Reidel Publishing Company (1984).

[2] C. Doran and A. Lasenby. *Geometric Algebra for Physicists.* Cambridge University Press (2003).