The Continuity Equation Implies Maxwell's Equations

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Abstract

It is shown that the antiderivative of a coclosed (closed) multivector field fails to be closed (coclosed) by at most a harmonic term, from which it follows that *any* vector valued current density J in four dimensions that is conserved (i.e. is coclosed) possesses a bivector valued antiderivative F that satisfies Maxwell's equations $\partial F = J$ under physically reasonable boundary conditions.

1 Introduction

After establishing a mapping between n-vector fields and differential forms of degree n, I present two key results of geometric calculus: a generalized Integral Formula and Helmholtz decomposition for fields.

Using these results, I show that all closed fields are inexact by at most a monogenic term. Monogenic fields are characterized by the property that they are fully determined by boundary conditions, analogous to complex analytic functions. I present some conditions under which these fields are exact. A field whose antiderivative is closed is dubbed *faithful*, by which it follows that *"the derivative of a closed field is faithful, and the antiderivative of a faithful field is closed."*

This establishes an equivalence between the statements *"an electromagnetic field F is a closed bivector"* and *"an electromagnetic current J is a faithful vector,"* both of which fully determine the structure of Maxwell's equations. I then show that any conserved vector valued field J is faithful under physically reasonable boundary conditions.

2 Fields and forms

If $F_n \equiv \langle F \rangle_n$ is the grade n part of the multivector field (hereafter, just field) F, then its corresponding *differential form f_n of degree n* is a scalar field given by [1]

$$f_n \equiv d^n x^\dagger \cdot F_n, \tag{1}$$

which is the projection of the n-vector field F_n onto the directed measure $d^n x^\dagger = dx_n \wedge \cdots \wedge dx_1$, where dx_i are vector valued differentials.

The hodge star operation $*$ acts on fields as

$$*F \equiv F^\dagger I, \tag{2}$$

where I is the pseudoscalar of some oriented vector manifold.

The exterior derivative d behaves identically to the curl

$$df_n \equiv d^{n+1} x^\dagger \cdot (\partial \wedge F_n), \tag{3}$$

and the "adjoint operator" δ behaves identically to (minus) the divergence

$$\delta f_n \equiv d^{n-1} x^\dagger \cdot (-\partial \cdot F_n). \tag{4}$$

The word *form* will be reserved for scalar fields corresponding to some n-vector field via Equation 1. Lowercase letters will be used for forms and uppercase letters for fields. Subscripts denote grade (degree).

3 Derivatives

A field F is called *closed or curl free* when

$$\partial \wedge F = 0 \tag{5}$$

and *coclosed or divergence free* when

$$\partial \cdot F = 0. \tag{6}$$

A field with no divergence or curl (closed and coclosed)

$$\partial F = \partial \cdot F + \partial \wedge F = 0 \tag{7}$$

is called *monogenic*. It possesses the property of complex analytic functions that, in any region, it is fully determined by its values on the boundary of that region. Hence, the form ω is closed if $d\omega = 0$, coclosed if $\delta\omega = 0$, and monogenic if $d\omega = \delta\omega = 0$.

A field H that satisfies

$$\partial^2 H = 0 \tag{8}$$

might be called *harmonic*, although the term is inappropriate in mixed signature spaces. For instance, in Minkowski space, $\partial^2 H = (\partial_t^2 - \vec{\nabla}^2)H = 0$ is the wave equation and its properties are differ dramatically from the usual harmonic functions in Euclidean spaces. Nonetheless, we will abuse the term here for lack of a better one. A form γ is then harmonic if $d\delta\gamma + \delta d\gamma = 0$.

4 Potentials

If a field J is written as

$$J = \partial \cdot G + \partial \wedge H, \tag{9}$$

then G and H are called *potentials* for J. By extension, if a form ω is given by

$$\omega = d\alpha + \delta\beta \tag{10}$$

then α and β will be called potentials for ω.

A field J is called *exact* when

$$J = \partial \wedge F \tag{11}$$

and *coexact* when

$$J = \partial \cdot F, \tag{12}$$

whereby a form ω is exact if $\omega = d\alpha$ and coexact if $\omega = \delta\beta$.

5 Antiderivatives

A field F is called an *antiderivative* of J if

$$J = \partial F = \partial \cdot F + \partial \wedge F, \tag{13}$$

which is unique up to a monogenic term. That is, $F + C$ such that $\partial C = 0$ is also an antiderivative. Furthermore, given an antiderivative, one has possession of constraints on F. For every $J_k = 0$,

$$J_k = \partial \cdot F_{k+1} + \partial \wedge F_{k-1} = 0. \tag{14}$$

As an example, if $J = J_n$ is an n-vector field, then

$$J_n = \partial F = \partial \cdot F_{n+1} + \partial \wedge F_{n-1}, \tag{15}$$

and the constraints due to $J_{n-1} = J_{n+1} = 0$ are

$$\partial \cdot F_{n-1} = \partial \wedge F_{n+1} = 0. \tag{16}$$

Of course, F *could* contain terms of higher and lower grades, but they make no contribution to J_n. In this case, it will generally be of the form $F = F_{n-1} + F_{n+1} + C$.

If j_n and f_n are the forms given by J_n and F_n, then Equation 15 is equivalent to

$$j_n = \delta f_{n+1} + df_{n-1}, \tag{17}$$

and Equation 16 is equivalent to

$$\delta f_{n-1} = df_{n+1} = 0. \tag{18}$$

Given potentials f_{n-1} and f_{n+1} under these constraints, one is in possession of an antiderivative of j_n.

6 The Fundamental Theorem

Let $\mathcal{M}$ be an m-dimensional smooth oriented vector manifold with a piecewise smooth boundary $\partial\mathcal{M}$ and L be a linear function, differentiable on $\mathcal{M}$ and $\partial\mathcal{M}$. Then, [1]

$$\int L(\dot{x}, d^m x \cdot \dot{\partial}) = \oint L(x, d^{m-1}x), \tag{19}$$

where $L(\dot{x}, d^m x \cdot \dot{\partial})$ denotes right and left differentiation all x dependent terms by ∂.

Stokes' theorem of differential forms is

$$\int \langle L(\dot{x}, d^m x \cdot \dot{\partial})\rangle = \oint \langle L(x, d^{m-1}x)\rangle, \tag{20}$$

for scalar valued integrands.

7 Integral Formula

Let J be a field on a simple (not self-intersecting) manifold $\mathcal{M}$ subject to the same criteria in the fundamental theorem. Suppose J satisfies the equation

$$\partial F = J. \tag{21}$$

Then F is given by [1]

$$F(x) = (-1)^m I^{-1}(x) \left(\int g(x,x') d^m x' J(x') - \oint g(x,x') d^{m-1} x' F(x') \right), \tag{22}$$

where g is a Green's function of ∂ satisfying $\partial g(x,x') = -g(x,x')\partial' = \delta(x - x')$. This result says that *any integrable field has an antiderivative, and it's given by Equation 22.*

8 Helmholtz decomposition

The integral forumla tells us that J has an antiderivative F such that

$$J = \partial F = \partial \cdot F + \partial \wedge F = (-1)^m I^{-1} \left(\int g d^m x \cdot \partial^2 F - \oint g d^{m-1} x \partial F \right). \tag{23}$$

In addition, we can say

$$\partial \cdot F = (-1)^m I^{-1} \left(\int g d^m x \cdot \partial(\partial \cdot F) - \oint g d^{m-1} x \partial \cdot F \right) \tag{24}$$

and

$$\partial \wedge F = (-1)^m I^{-1} \left(\int g d^m x \cdot \partial(\partial \wedge F) - \oint g d^{m-1} x \partial \wedge F \right), \tag{25}$$

which gives a generalized Helmholtz decomposition into divergence free (or coclosed) and curl free (or closed) fields, $\partial \cdot F$ and $\partial \wedge F$ respectively. This is because $\partial \wedge (\partial \wedge M) = \partial \cdot (\partial \cdot M) = 0$ for any field M.

Additionally, this decomposition comes with constraints given by Equation 14.

9 Antiderivatives of divergence free (or coclosed) fields

The above result implies that antiderivatives of divergence free fields fail to be curl free, and antiderivatives of curl free fields fail to be divergence free, by at most a harmonic term H satisfying $\partial^2 H = 0$.

Suppose $J = \partial F$ is divergence free (the dual result for curl free fields follows identically). Then,

$$\partial \cdot J = \partial \cdot (\partial F) = \partial \cdot (\partial \wedge F) = \partial(\partial \wedge F) = 0, \tag{26}$$

which means that $C \equiv \partial \wedge F$ is monogenic and J is *cohomologous* with C

$$J - C = \partial F - C = \partial \cdot F, \tag{27}$$

because their difference is coexact.

Employing the integral theorem, C has an antiderivative H such that

$$C = \partial H. \tag{28}$$

With $G \equiv F - H$, this implies that F can then be written

$$F = G + H \tag{29}$$

where $\partial G = \partial \cdot F$ and $\partial^2 H = 0$. Hence, F fails to be closed by at most a harmonic term H.

As an example, if C is an r-vector field, then C can be written $C = \partial \cdot (x \wedge C)/r = \partial \wedge (x \cdot C)/(n-r)$, and C is both exact and coexact, in which case[1]

$$J = \partial F = \partial \cdot (F + x \wedge C/r) \tag{30}$$

is coexact — although, F is not closed.

If $C = 0$ on the boundary, then $C = 0$ everywhere, and its antiderivative is closed

$$J = \partial F = \partial \cdot F. \tag{31}$$

Let us call a field J *faithful* if its antiderivative is closed. Faithful fields are coexact, and all coclosed fields differ from faithful fields by at most a monogenic field, which depends solely on the manifold and boundary conditions. Note, however, that coexact fields are not necessarily faithful.

10 Maxwell's equations

Maxwell's equations follow from directly from the statement that *"an electromagnetic field F is a closed bivector field and its derivative is its current J,"* which means that

$$\partial \wedge F = 0. \tag{32}$$

This implies that its derivative J is

$$J = \partial F = \partial \cdot F, \tag{33}$$

which are Maxwell's equations with no magnetic monopoles or currents. As is well known, Maxwell's equations imply the continuity equation

$$\partial \cdot J = \partial \cdot (\partial \cdot F) = 0, \tag{34}$$

which means charge is conserved. F is the antiderivative of J, so to say that F is closed is the same as saying J is faithful. Hence, the above is equivalent to a dual statement *"an electromagnetic current J is a faithful vector field and its antiderivative is its electromagnetic field F."*

Suppose J is a conserved current. Then by Equation 16 and Equation 29, its antiderivative F can be decomposed into

$$F = F_0 + F_2, \tag{35}$$

where F_0 is harmonic.

Equation 23 tells us that, in four dimensions, J can be written

[1] Under what conditions are monogenic fields (co)exact?

$$J = \partial F = I^{-1} \left(\int g d^4 x \cdot \partial^2 F - \oint g d^3 x J \right). \tag{36}$$

If we integrate over all of spacetime, the boundary term vanishes for reasonable charge distributions, and J is given by

$$J = \partial F = I^{-1} \int g d^4 x \cdot \partial^2 F, \tag{37}$$

which is only dependent on $\partial^2 F = \partial^2 (F_0 + F_2) = \partial^2 F_2$, since F_0 is harmonic. Hence, J is independent of F_0, and J can at last be written as

$$J = \nabla F = \nabla \cdot F, \tag{38}$$

and J is faithful.

References

[1] D. Hestenes. *Clifford algebra to geometric calculus*. D. Reidel Publishing Company (1984).