An Extension of the Dirac Equation

Luke Burns

December 3, 2016

Abstract

A minimal extension of the Dirac equation is shown to admit a class of massless, electrically charged fermions that violate charge conjugation (TP) and time reversal (CP) symmetries in the presence of an electromagnetic potential. These solutions have twice the degrees of freedom as solutions to the Dirac equation and so are distinct from massless Dirac fermions. Additionally, solutions exhibit rotation in the particle's spin plane, analogous to the zitterbewegung of massive Dirac theory. The rotation is shown to be identical to electric and magnetic fields of circularly polarized electromagnetic waves.

A bilinear covariant of the spinor field is constructed that satisfies Maxwell's equations to formalize the similarity with electromagnetic waves. The gauge field is shown to play the same role as the duality gauge field of the gauged Maxwell equations.

1 Introduction

The free Dirac equation as expressed in the Space Time Algebra (STA) is[1]

$$\nabla\psi I\sigma_3 = \psi p_0, \tag{1}$$

where

$$p_0 = m\gamma_0. \tag{2}$$

This equation in part describes the dynamics of a spinor $\psi = \rho^{1/2}e^{I\beta/2}R$ that rotates, boosts, and dilates the momentum of a particle in its rest frame $p_0 = m\gamma_0$ onto a probability current $mJ = \psi p_0 \widetilde{\psi}$ in some other frame. However, this description fails for massless particles, since they have no rest frame.

A minimal extension to this equation that accomodates a similar description for massless particles is to simply require that p_0 be constant, so that ψ describes the dynamics of a spinor that rotates, boosts, and dilates the momentum p_0 of a particle in some arbitrary "initial" frame onto a probability current $\psi p_0 \widetilde{\psi}$ in some other frame (it will be shown that this works in general in Section 4).

This extension including the electromagnetic gauge field can be written

$$\nabla\psi I\sigma_3 - eA\psi = \psi p_0. \tag{3}$$

In Section 2, we'll work out the physical constraints on p_0 and show that Equation 3 admits the usual solutions to the Dirac equation and a new class of null solutions, where $p_0^2 = 0$ but $\psi p_0 \neq 0$. Then we'll work out the symmetries of Equation 3 in Section 3. Finally, we will show that a probablistic interpretation extends to Equation 3 in Section 4 and present plane wave solutions in Section 5.

2 Constraints

Equation 3 is only invariant under the usual replacements $\psi \mapsto \psi e^{\alpha I\sigma_3}$ and $A \mapsto A - \nabla\alpha$ if

$$p_0 I\sigma_3 = I\sigma_3 p_0; \tag{4}$$

that is, if p_0 is perpendicular to the plane $I\sigma_3$. This is an essential feature of Dirac theory, so we'll restrict p_0 to preserve it. We require that

$$p_0 = E\gamma_0 \pm |\vec{p}|\gamma_3. \tag{5}$$

Additionally, if solutions are to satisfy the Klein-Gordon equation, we must have

$$p_0^2 \geq 0. \tag{6}$$

If $p_0^2 > 0$, then

$$p_0 = Rm\gamma_0\widetilde{R} \tag{7}$$

for $R = e^{\gamma_3\gamma_0\alpha/2}$ with α given by $\tanh(\alpha) = \pm|\vec{p}|/E$.

If ψ is a spinor valued solution to Equation 3 and $p_0^2 > 0$, then ψR is a solution to the Dirac equation.[1] On the other hand, if ψ is instead a solution to the Dirac equation, then $\psi\widetilde{R}$ is a solution to Equation 3. That is, solutions to Equation 3 are in one-to-one correspondence to solutions of the Dirac equation.

On the other hand, if $p_0^2 = 0$, then

$$p_0 = \omega_0(1 \pm \sigma_3)\gamma_0, \tag{8}$$

for $\omega_0 = E = |\vec{p}|$. Note that ψ can be decomposed into

$$\psi = \psi\frac{1+\sigma_3}{2} + \psi\frac{1-\sigma_3}{2} = \psi_+ + \psi_-. \tag{9}$$

If ψ is a solution to Equation 3 and $p_0^2 = 0$, then the projection $\psi_\pm$ gives a solution to the (massless) Dirac equation, since $p_0\frac{1\pm\sigma_3}{2} = 0$. However, p_0 is not invertible, so the only way to recover ψ would be from $\psi_\mp$. The problem is that $\psi_\mp$ is another solution to Equation 3 and *not* a solution to the Dirac equation, since $p_0\frac{1\mp\sigma_3}{2} = 2p_0$. So there is no way to recover general solutions to Equation 3 from solutions to the massless Dirac equation.

This means that Equation 3 contains null solutions that are distinct from solutions to the massless Dirac equation and, furthermore, that these are the *only* new solutions admitted by the extension. In this sense, the extension is minimal.

3 Symmetries

Unlike the Dirac equation, Equation 3 is not invariant under

$$\psi \mapsto \psi\gamma_0. \tag{10}$$

If ψ is a solution to

$$\nabla\psi I\sigma_3 - eA\psi = \psi p_0, \tag{11}$$

then $\psi' = \psi\gamma_0$ is a solution to

$$\nabla\psi' I\sigma_3 - eA\psi' = \psi'\overline{p}_0, \tag{12}$$

where $\overline{M} \equiv \gamma_0 M\gamma_0$ is minus the reflection of any multivector M across the γ_0 axis.

That is, Equation 3 distinguishes between even and odd fields, another reason we are not able to find a one-to-one correspondence between massless solutions to Equation 3 and the Dirac equation. In general, ψ is a full multivector and can be decomposed into

$$\psi = \langle\psi\rangle_+ + \langle\psi\rangle_-, \tag{13}$$

[1] We will see in Section 3 that ψ is in general a multivector, so if ψ is a solution to Equation 3, then its even and odd parts satisfy Equation 3 independently. The argument can be made for each of these solutions independently using Equations 15 and 17.

where $\langle\psi\rangle_+$ and $\langle\psi\rangle_-$ are even and odd multivectors respectively that are independent solutions to

$$\nabla\langle\psi\rangle_\pm I\sigma_3 - eA\langle\psi\rangle_\pm = \langle\psi\rangle_\pm p_0. \tag{14}$$

For multivector fields, the choice of $p_0 = \omega_0(1\pm\sigma_3)\gamma_0$ is arbitrary, because each choice gives the same two equations. If $\psi_+ = \langle\psi\rangle_+$ is a solution to

$$\nabla\psi_+ I\sigma_3 - eA\psi_+ = \omega_0\psi_+(1+\sigma_3)\gamma_0, \tag{15}$$

then the even multivector

$$\psi_- \equiv \langle\psi\rangle_-\gamma_0 \tag{16}$$

is a solution to

$$\nabla\psi_- I\sigma_3 - eA\psi_- = \omega_0\psi_-(1-\sigma_3)\gamma_0. \tag{17}$$

This justifies our treatment of solutions to Equation 3 as spinors in the last section. We just need to keep in mind that solutions where $p_0 = \omega(1+\sigma_3)\gamma_0$ are distinct from those where $p_0 = \omega(1-\sigma_3)\gamma_0$. We will confirm shortly that there is no grade-preserving mapping between solutions to Equation 15 and Equation 17.

At first glance, it appears that Equations 15 and 17 describe particles of opposite helicity. However, this is not exactly the case, because Equations 15 and 17 admit solutions of both positive and negative helicity. See the plane wave solutions in Section 5 for an example. The characteristic quantity that distinguishes between ψ_+ and ψ_- is the "projection" of charge onto helicity, which one can see by inspecting Equations 15 and 17.

We can make better sense of this by looking at charge, parity, and time reversal conjugations. It turns out while there are maps between solutions to Equation 15 and Equation 17, they are not grade preserving in the presence of a gauge field. These conjugations are given by

$$\hat{C}\psi : \psi \mapsto \psi\gamma_1 \iff eA \mapsto -eA \tag{18}$$

$$\hat{P}\psi : \psi(x) \mapsto \overline{\psi}(\overline{x}) \iff \nabla \mapsto \overline{\nabla}, eA(x) \mapsto e\overline{A}(\overline{x}), \text{ and } p_0 \mapsto \overline{p}_0 \tag{19}$$

$$\hat{T}\psi : \psi(x) \mapsto -I\overline{\psi}(-\overline{x})\gamma_1 \iff \nabla \mapsto -\overline{\nabla}, I\sigma_3 \mapsto -I\sigma_3, eA(x) \mapsto -e\overline{A}(-\overline{x}), \text{ and } p_0 \mapsto -\overline{p}_0. \tag{20}$$

Parity is grade preserving, and of course, so is the combined CPT conjugation, given by

$$\psi(x) \mapsto -I\psi(-x) \iff A(x) \mapsto A(-x) \text{ and } p_0 \mapsto -p_0. \tag{21}$$

The mapping $p_0 \mapsto -p_0$ is equivalent to the mappings $I\sigma_3 \mapsto -I\sigma_3$ and $eA \mapsto -eA$, which change spin (helicity) and charge. However, charge and time reversal conjugations are not grade preserving. This means that charge and time reversal conjugations are not well defined for solutions to Equations 15 and 17.

The reason that charge and time reversal conjugations are grade preserving in Dirac theory is that Equation 10 leaves the Dirac equation invariant. The situation here is different. There is no odd-valued conjugation that leaves Equations 15 and 17 invariant in general that would enable us to construct grade preserving charge and time reversal conjugations out of Equations 18 and 20, which one can confirm with a glance at the possible vector-valued conjugations:

$$\psi \mapsto \psi\gamma_0 \iff p_0 \mapsto \overline{p}_0 \tag{22}$$

$$\psi \mapsto \psi\gamma_{1,2} \iff eA \mapsto -eA \tag{23}$$

$$\psi \mapsto \psi\gamma_3 \iff p_0 \mapsto -\overline{p}_0 \tag{24}$$

$$\psi \mapsto \gamma_0\psi \iff \nabla \mapsto \overline{\nabla} \text{ and } eA \mapsto -e\overline{A}. \tag{25}$$

In the special case where the electromagnetic gauge field vanishes, Equation 18 is an odd-valued conjugation that leaves Equation 3 unchanged, and so

$$\psi \mapsto \psi\sigma_1, \tag{26}$$

provides a grade preserving map between solutions to Equations 15 and 17. So when $A = 0$, time reversal conjugation is simply

$$\hat{T}\psi : \psi(x) \mapsto -I\overline{\psi}(-\overline{x}) \iff \nabla \mapsto -\overline{\nabla}, I\sigma_3 \mapsto -I\sigma_3, \text{ and } p_0 \mapsto -\overline{p}_0. \tag{27}$$

4 Probability Current

Essential to Dirac theory is its probabilistic intepretation, which depends on a conserved probability current J satisfying the continuity equation

$$\nabla \cdot J = 0. \tag{28}$$

The usual probability current $\psi\gamma_0\widetilde{\psi}$ of Dirac theory is not conserved for even valued solutions ψ to Equation 3 (which includes solutions to Equations 15 and 17). To see this, consider the following, for a constant vector v_0.

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = \langle v_0 \wedge p_0(I\sigma_3\psi\widetilde{\psi})\rangle + \langle v_0 \cdot I\sigma_3(\widetilde{\psi}eA\psi)\rangle, \tag{29}$$

which gives a condition for conservation

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = 0 \iff v_0 \wedge p_0 = 0. \tag{30}$$

The second term in Equation 29 vanishes, because $v_0 \wedge p_0 = 0$ implies $v_0 \cdot I\sigma_3 = 0$, due to Equation 4.

This means that $\psi p_0\widetilde{\psi}$ is the only vector-valued bilinear covariant conserved in general (up to a constant multiple). Furthermore, the fact that $\nabla \cdot (\psi p_0 \widetilde{\psi}) = 0$ implies the existence of streamlines with tangents given by $p = Rp_0\widetilde{R}$, which are timelike if $p_0^2 > 0$ and lightlike if $p_0^2 = 0$.[2] The usual probability current $\psi\gamma_0\widetilde{\psi}$ is not conserved because $\gamma_0 \wedge p_0 \neq 0$.

The normalization procedure

$$\int d^3x\gamma_0 \cdot J = 1 \tag{31}$$

can be extended straightforwardly. In Dirac theory, Equation 31 is equivalent to

$$\int d^3x\gamma_0 \cdot (\psi p_0\widetilde{\psi}) = m, \tag{32}$$

which simply ensures that integrating energy density (in the γ_0 frame) over all of space is just the rest energy of the particle.

Since massless particles do not have rest energy, a reasonable generalization of this is

$$\int d^3x\gamma_0 \cdot (\psi p_0\widetilde{\psi}) = c. \tag{33}$$

for a constant c. Any choice of c determines a probability current $J = \psi p_0\widetilde{\psi}/c$ with a normalized probability density $J_0 = \gamma_0 \cdot J$. Selecting $c = \gamma_0 \cdot p_0$ may be a convenient choice, because it coincides with ω_0 in the massless theory and aligns with the usual choice m in the massive theory. Alternatively, there's nothing preventing us from choosing $c = 1$ and simply referring to $\rho p = \psi p_0\widetilde{\psi}$.

5 Plane Waves

Plane wave solutions for Equation 15 are given by

$$\nabla\psi_+ I\sigma_3 = \pm p\psi_+, \tag{34}$$

where $p = Rp_0\widetilde{R}$ is constant and $\omega = p \cdot \gamma_0 > 0$.

This implies that ρ is constant, $\beta = 0, \pi$, and $R = R_0 e^{\mp I\sigma_3(p\cdot x + c)}$, where R_0 is constant and c is a monogenic phase shift satisfying $\nabla c(x) = 0$. For every ρ and R_0, which completely determine the constant probability current $\rho p = \psi p_0 \widetilde{\psi} = \rho R_0 p_0 \widetilde{R}_0$, following the conventions of [1], we have two solutions (taking $c = 0$)

$$\psi_+^{(+)}(x) = \rho^{1/2} R_0 e^{-I\sigma_3 p\cdot x} \text{ and } \psi_+^{(-)}(x) = \rho^{1/2} I R_0 e^{I\sigma_3 p\cdot x}, \tag{35}$$

which are CPT conjugates of one another. $\psi_+^{(+)}$ and $\psi_+^{(-)}$ describe particles propagating in the $\vec{p} = p \wedge \gamma_0$ direction (in the γ_0 frame) with opposite spin, or helicity.

There are two corresponding solutions for Equation 17 of the form

$$\psi_-^{(+)} = \psi_+^{(+)}\sigma_1 \text{ and } \psi_-^{(-)} = \psi_+^{(-)}\sigma_1, \tag{36}$$

using the fact that $A = 0$ and employing Equation 26, which propagate in $-\vec{p}$ direction.

These solutions are similar to the plane wave solutions to the Dirac equation in that there are four of them, each of which is described by a Dirac (four component) spinor, and are unlike the right and left handed plane wave solutions to the massless Dirac equation, for which there are only two distinct solutions, each of which is given by a Weyl (two component) spinor. On the other hand, they are similar to Weyl plane waves, and unlike Dirac plane waves, in that their momentum is lightlike and their helicity is Lorentz invariant.

The main qualitative feature that distinguish $\psi_\pm^{(\pm)}$ from Weyl plane waves is that their (constant) spin plane

$$\rho S = \psi I\sigma_3 \widetilde{\psi} = (\pm)\rho R I\sigma_3 \widetilde{R} = (\pm)\rho R_0 I\sigma_3 \widetilde{R}_0 \tag{37}$$

does *not* vanish, and the vectors ($i = 1, 2$)

$$e_i = R\gamma_i \widetilde{R} = R_0 \gamma_i \widetilde{R}_0 e^{\mp S 2 p\cdot x} \tag{38}$$

rotate at a frequency 2ω in this plane, analogous to the rotation of electric and magnetic fields of circularly polarized electromagnetic waves.

In fact, this analogy can be made precise. Left and right circularly polarized electromagnetic waves have the form[1]

$$F_\pm = kn\alpha e^{\mp I k\cdot x} = Akne^{\pm I\hat{k}(k\cdot x + c)}, \tag{39}$$

where n is a constant vector perpendicular to k (i.e. $k \cdot n = 0$), $\alpha = Ae^{\mp Ic}$ is a constant amplitude and duality transformation, and $\hat{k} = k \wedge \gamma_0 / |k \wedge \gamma_0|$. As noted by [1], positive and negative helicities correspond to left and right handedness respectively. The plus and minus in $F_\pm$ refer to positive and negative helicity, rather than handedness.

Making the identifications $p = \hbar k$, $R_0\gamma_1\widetilde{R}_0 = n$, and $\rho = \pm A$, then e_1 can be seen to rotate identically to the factor $ne^{\pm I\hat{k}(k\cdot x + c)}$ with twice the frequency and the corresponding electromagnetic field can be placed in the form

$$F_\psi \equiv \psi p_0 \gamma_1 \widetilde{\psi} = \rho p e_1 e^{I\beta}, \tag{40}$$

which in this case is

$$F_\psi = \pm \rho p e_1. \tag{41}$$

In this setting, the $e^{I\beta}$ factor is less mysterious than it is in Dirac theory, as it is simply a duality rotation of the elecromagnetic field. In fact, since p is null here, it simplifies to an ordinary spatial rotation of e_1:

$$F_\psi = \rho p e_1 e^{-S\beta}. \tag{42}$$

F_ψ is a proper electromagnetic field satisfying

$$\nabla F_\psi = 0 \text{ and } F_\psi^2 = 0 \tag{43}$$

which are the usual specifications of circularly polarized electromagnetic fields. In fact, it will be shown in the next section that Equation 40 satisfies Maxwell's equations for all massless *and* massive solutions ψ (although, with a disclaimer) to Equation 1. Furthermore, when $A \neq 0$, A plays the same role as the gauge field that arises in a gauging duality symmetry of Maxwell's equations.

The only discrepancy is that e_1 rotates with twice the frequency, so it may be worth making the replacement $p_0 \mapsto \frac{1}{2}p_0$ and taking the equation

$$\nabla\psi I\sigma_3 = \frac{1}{2}\psi p_0, \tag{44}$$

as primary, so that the frequency of rotation of e_1 is given by ω.

6 An Electromagnetic Bilinear Covariant

Suppose $\psi \in G^+_{1,3}$ satisfies Equation 44, taking p_0 to be arbitrary and F_ψ is given by Equation 40. Then

$$\begin{align}
\nabla F_\psi &= J_\psi \tag{45}\\
&= \nabla(\psi p_0 \gamma_1 \widetilde{\psi}) \tag{46}\\
&= \nabla\psi p_0\gamma_1\widetilde{\psi} + \dot{\nabla}\psi p_0\gamma_1\dot{\widetilde{\psi}} \tag{47}\\
&= \rho p^2 e_2. \tag{48}
\end{align}$$

The current term vanishes for massless solutions precisely because $p^2 = 0$ and does not vanish for massive solutions because $p^2 \neq 0$. Additionally, we see that the contraction of p with itself due to the derivative is precisely what ensures that the current J_ψ is vector valued and so describes an electromagnetic field free from magnetic sources.

In the presence of an electromagnetic gauge field, if ψ satisfies

$$\nabla\psi I\sigma_3 - \frac{1}{2}eA\psi = \frac{1}{2}\psi p_0, \tag{49}$$

then

$$\nabla F_\psi - eAFe_2e_1 = \rho p^2 e_2 \tag{50}$$

which gives

$$\nabla F_\psi - eAF_\psi e_2e_1 = \rho p^2 e_2. \tag{51}$$

In the massless case, $AF_\psi e_2 e_1 = IAF_\psi$. This yields the equation

$$\nabla F_\psi - eIAF_\psi = 0, \tag{52}$$

which is identical to the gauged Maxwell equations in vacuum.[3][4][5]

The massive analog of these equations is [5]

$$\nabla F - eIAF = J, \tag{53}$$

but this is distinct from Equation 51.

An additional, more serious concern with Equation 51 in the massive case is that the current $\rho p^2 e_2$ is a *spacelike* vector, because $e_2^2 = -1$. This is arguably non-physical.[2]

7 Conclusion

WIP. What are reasons to expect or not expect such a pair of particles to exist? Could this be substituted into Hestenes's electron model, that uses a massless charged particle?

[2] Todo: Does $\nabla \cdot (\rho p^2 e_2) = 0$? If not, that would definitively render the massive case useless.

A Matrix Formulation

Following Equation 8.70 in [1], Equations 15 and 17 take the following form in matrix representation.

$$\hat{\gamma}^{\mu}(i\partial_{\mu} - eA_{\mu})|\psi_{+}\rangle = \omega_0(1 + \hat{\gamma}_5)|\psi_{+}\rangle \tag{54}$$

and

$$\hat{\gamma}^{\mu}(i\partial_{\mu} - eA_{\mu})|\psi_{-}\rangle = \omega_0(1 - \hat{\gamma}_5)|\psi_{-}\rangle, \tag{55}$$

where

$$\hat{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\gamma}^k = \begin{pmatrix} 0 & \hat{\sigma}^k \\ -\hat{\sigma}^k & 0 \end{pmatrix}, \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{56}$$

See [1] for more details on the isomorphism.

References

[1] C. Doran and A. Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press (2003).

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[3] L. Burns. *Gauging Duality Symmetry*. `https://github.com/lukeburns/gauge-duality`. (Work in Progress).

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[5] S.C. Tiwari. *Axion electrodynamics in the duality perspective* Modern Physics Letters A. Vol. 30, No. 40 (2015).