An Extension of the Dirac Equation

Luke Burns

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1 Introduction

The free Dirac equation in STA is

$$\nabla \psi I \sigma_3 = \psi p_0, \tag{1}$$

where

$$p_0 = m\gamma_0. \tag{2}$$

A less restrictive constraint is just that

$$p_0^2 = m^2 \tag{3}$$

be constant. With this lightened restriction, Equation 1 admits a new class of null solutions, where $p_0^2 = 0$ but $\psi p_0 \neq 0$.

2 General Results

2.1 Constraints on p_0

The extension of the full Dirac equation, including the electromagnetic gauge field

$$\nabla \psi I \sigma_3 - eA\psi = \psi p_0 \tag{4}$$

is only invariant under the usual replacements $\psi \mapsto \psi e^{\alpha I \sigma_3}$ and $A \mapsto A - \nabla \alpha$ if $p_0 \cdot I\sigma_3 = 0$:

$$\begin{aligned}
\nabla(\psi e^{\alpha I \sigma_3}) I \sigma_3 - e(A - \nabla\alpha)\psi e^{\alpha I \sigma_3} &= (\nabla \psi I \sigma_3 - eA\psi) e^{\alpha I \sigma_3} && (5)\\
&= \psi p_0 e^{\alpha I \sigma_3} && (6)\\
&= \psi e^{\alpha I \sigma_3} p_0. && (7)
\end{aligned}$$

This is an essential feature of Dirac theory, so we'll restrict p_0 preserve it. We require that

$$p_0 = E\gamma_0 \pm |\vec{p}|\gamma_3 = (E + \vec{p})\gamma_0. \tag{8}$$

If $p_0^2 = m^2 \neq 0$, then

$$p_0 = Rm\gamma_0\widetilde{R} \tag{9}$$

for $R = e^{\gamma_3\gamma_0\alpha/2}$ with α given by $\tanh(\alpha) = |\vec{p}|/E$. Then $p_0 \mapsto \widetilde{R} p_0 R$ reduces to Equation 2. On the other hand, if $p_0^2 = 0$, then

$$p_0 = \omega(1+\hat{p})\gamma_0, \tag{10}$$

for $\omega = E = |\vec{p}|$, which gives an equation of the form

$$\nabla\psi I\sigma_3 = \omega\psi(1 \pm \sigma_3)\gamma_0. \tag{11}$$

Solutions to Equation 1 satisfying Equation 10 are distinct from those satisfying Equation 2. In particular, these solutions are distinct from the massless solutions to Equation 2, because they yield different observables. This will be demonstrated in the next section.

2.2 The Conserved Vector Current

In general, if v_0 is a constant vector, then

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = \langle v_0 \wedge p_0(I\sigma_3\psi\widetilde{\psi})\rangle + \langle v_0 \cdot I\sigma_3(\widetilde{\psi}eA\psi)\rangle, \tag{12}$$

and

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = 0 \iff v_0 \wedge p_0 = 0. \tag{13}$$

That is, energy-momentum density $\rho p = \psi p_0 \widetilde{\psi}$ is the only vector-valued bilinear covariant conserved generally (up to a constant multiple).

In particular, notice that the conservation of the usual probability current $J = \psi\gamma_0\widetilde{\psi} = \rho v$ does *not* vanish for $p_0 = \omega(1+\hat{p})\gamma_0$, since $\gamma_0 \wedge p_0 = -\vec{p} = \mp\omega\sigma_3$:

$$\begin{aligned} \nabla \cdot J &= \langle \gamma_0 \wedge p_0(I\sigma_3\psi\widetilde{\psi})\rangle & (14)\\ &= \mp\omega\rho\langle Ie^{I\beta}\rangle & (15)\\ &= \pm\omega\rho\sin\beta. & (16) \end{aligned}$$

A consequence of this result is that the usual interpretation of J as a probability current does not extend to Equation 10. If solutions to this equation prove useful, then producing an interpretation that applies to both Equation 2 and Equation 10 may be worthwhile.

Is there an obvious route to a probabilistic interpretation that works for general p_0?