An Extension of the Dirac Equation

Luke Burns

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Abstract

A minimal extension of the Dirac equation is shown to describe a pair of massless, electrically charged fermions. Solutions exhibit rotation in the particle's spin plane, analogous to the zitterbewegung of massive Dirac theory and identical to electric and magnetic fields of circularly polarized electromagnetic waves.

1 Introduction

The free Dirac equation as expressed in the Space Time Algebra (STA) is[1]

$$\nabla\psi I\sigma_3 = \psi p_0, \tag{1}$$

where

$$p_0 = m\gamma_0. \tag{2}$$

This equation in part describes the dynamics of a spinor $\psi = \rho^{1/2}e^{I\beta/2}R$ that rotates, boosts, and dilates the momentum of a particle in its rest frame $p_0 = m\gamma_0$ onto a probability current $mJ = \psi p_0\widetilde{\psi}$ in some other frame. However, this description fails for massless particles, since they have no rest frame.

A minimal extension to this equation that accomodates a similar description for massless particles is to simply require that p_0 be constant, so that ψ describes the dynamics of a spinor that rotates, boosts, and dilates the momentum p_0 of a particle in some arbitrary "initial" frame onto a probability current $\psi p_0\widetilde{\psi}$ in some other frame (it will be shown that this works in general in Section 4).

The extension including the electromagnetic gauge field can be written

$$\nabla\psi I\sigma_3 - eA\psi = \psi p_0. \tag{3}$$

In Section 2, we'll work out the physical constraints on p_0 and show that Equation 3 admits the usual solutions to the Dirac equation and a new class of null solutions, where $p_0^2 = 0$ but $\psi p_0 \neq 0$. Then we'll work out the symmetries of Equation 3 in Section 3. We will show that a probablistic interpretation extends to Equation 3 in Section 4 and present plane wave solutions in Section 6. Lastly, an apparent similarity to circularly polarized electromagnetic waves is formalized in Section 7. We will set $\hbar = c = 1$ throughout.

2 Constraints

There are two constraints that we must impose on p_0. Firstly, Equation 3 must be invariant under gauge transformations if it is to be compatible with the gauge field A, and secondly, solutions must satisfy the Klein-Gordon equation.

Equation 3 is only gauge invariant under the replacements $\psi \mapsto \psi e^{\alpha I\sigma_3}$ and $A \mapsto A - \nabla\alpha$ if

$$p_0 I\sigma_3 = I\sigma_3 p_0. \tag{4}$$

That is, if p_0 is perpendicular to the plane $I\sigma_3$, so p_0 is of the form

$$p_0 = E_0\gamma_0 \pm |\vec{p}_0|\gamma_3, \tag{5}$$

where E_0 and $\vec{p}_0 = \pm|\vec{p}_0|\gamma_3\gamma_0$ are constant.

Additionally, if solutions are to satisfy the Klein-Gordon equation, we must have

$$p_0^2 \geq 0. \tag{6}$$

If $p_0^2 > 0$, then

$$p_0 = Rm\gamma_0\widetilde{R} \tag{7}$$

for $R = e^{\pm\gamma_3\gamma_0\alpha/2}$ with α given by $\tanh(\alpha) = |\vec{p}_0|/E_0$. This implies that if ψ is a spinor valued solution to Equation 3, then ψR is a solution to the Dirac equation. On the other hand, if ψ is instead a solution to the Dirac equation, then ψR and $\psi\widetilde{R}$ are solutions to Equation 3. That is, spinor valued solutions to Equation 3 are in two-to-one correspondence to solutions of the Dirac equation.

We will see in Section 3 that a general solution to Equation 3 is a full multivector. However, the full multivector solutions are equivalent to a pair of decoupled spinor valued solutions to Equations 17 and 19, corresponding precisely to Equation 3 for each sign of $\vec{p}_0 = \pm|\vec{p}_0|\gamma_3\gamma_0$.

This two-to-one correspondence is reflective of the fact that one can always find two frames in which a massive particle is propagating with the same momentum but in opposite directions. This is the reason helicity (the projection of the momentum onto spin) is not a Lorentz invariant quantity for massive particles but is for massless particles. On the other hand, given a massless particle propagating in one direction, there does not exist a frame in which that particle travels in the opposite direction. Helicity *is* Lorentz invariant for massless particles for this reason.

Unlike the massive case, there is no correspondence between massless solutions to Equation 3 and the Dirac equation. If $p_0^2 = 0$, then p_0 can be written

$$p_0 = \omega_0(1 \pm \sigma_3)\gamma_0, \tag{8}$$

where $\omega_0 \equiv E_0 = |\vec{p}_0|$. The decomposition

$$\psi = \psi\frac{1+\sigma_3}{2} + \psi\frac{1-\sigma_3}{2} = \psi_+ + \psi_-, \tag{9}$$

implies that if ψ is a solution to Equation 3, then the Weyl spinor $\psi_\pm$ is a solution to the massless Dirac equation. We can see this by multiplying Equation 3 by $(1 \pm \sigma_3)/2$ on the right, which gives

$$\nabla\psi_\pm I\sigma_3 - eA\psi_\pm = \frac{1}{2}\omega_0\psi(1 \pm \sigma_3)(1 \mp \sigma_3)\gamma_0 = 0, \tag{10}$$

using the facts that $I\sigma_3(1\pm\sigma_3) = (1\pm\sigma_3)I\sigma_3$, $\gamma_0(1\pm\sigma_3) = (1\mp\sigma_3)\gamma_0$, and $(1\pm\sigma_3)(1\mp\sigma_3) = 0$.

However, $\psi_\mp$ is simply another solution to Equation 3 and *not* a solution to the Dirac equation. This can be seen by multiplying Equation 3 on the right by by $(1 \mp \sigma_3)/2$, which gives

$$\nabla\psi_\mp I\sigma_3 - eA\psi_\mp = \frac{1}{2}\omega_0\psi(1 \pm \sigma_3)(1 \pm \sigma_3)\gamma_0 = \psi p_0, \tag{11}$$

since $\frac{1}{2}(1 \pm \sigma_3)(1 \pm \sigma_3) = (1 \pm \sigma_3)$.

Since the projection operator $(1 \pm \sigma_3)/2$ is not invertible, there is no way to recover solutions to Equation 3 from solutions to the massless Dirac equation. Hence, Equation 3 contains null solutions that are distinct from solutions to the massless Dirac equation and, furthermore, we have shown that these are the *only* new solutions admitted by the extension. In this sense, the extension is minimal.

3 Symmetries

In this section, we'll show that solutions to Equation 3 are, in general, multivectors but that a multivector solution is just a pair of independent spinor valued solutions to Equation 3. Then we'll construct charge, parity, and time reversal conjugations.

Unlike the Dirac equation, Equation 3 is not invariant under

$$\psi \mapsto \psi\gamma_0. \tag{12}$$

If ψ is a solution to

$$\nabla\psi I\sigma_3 - eA\psi = \psi p_0, \tag{13}$$

then $\psi' = \psi\gamma_0$ is a solution to

$$\nabla\psi' I\sigma_3 - eA\psi' = \psi'\overline{p}_0, \tag{14}$$

where $\overline{M} \equiv \gamma_0 M \gamma_0$ is minus the reflection of any multivector M across the γ_0 axis.

That is, Equation 3 distinguishes between even and odd fields. In general, ψ is a full multivector that can be decomposed into

$$\psi = \langle\psi\rangle_+ + \langle\psi\rangle_-, \tag{15}$$

where $\langle\psi\rangle_+$ and $\langle\psi\rangle_-$ are even and odd multivectors respectively that are independent solutions to

$$\nabla\langle\psi\rangle_\pm I\sigma_3 - eA\langle\psi\rangle_\pm = \langle\psi\rangle_\pm p_0, \tag{16}$$

because ∇, A, and p_0 are all odd valued. If, say, the gauge field A were even valued, then $\langle\psi\rangle_+$ and $\langle\psi\rangle_-$ would be coupled.[1]

Equations 16 can be re-expressed in terms of two equations involving only spinors. Doing this will allow for an straightforward comparison to Dirac theory. If $\psi_+ = \langle\psi\rangle_+$ is a solution to

$$\nabla\psi_+ I\sigma_3 - eA\psi_+ = \omega_0\psi_+(1+\sigma_3)\gamma_0, \tag{17}$$

then the even multivector

$$\psi_- \equiv \langle\psi\rangle_-\gamma_0 \tag{18}$$

is a solution to

$$\nabla\psi_- I\sigma_3 - eA\psi_- = \omega_0\psi_-(1-\sigma_3)\gamma_0, \tag{19}$$

but these are precisely Equation 3 for each sign of p_0.

At first glance, it appears that Equations 17 and 19 describe particles of opposite helicity. However, this is not exactly the case, because Equations 17 and 19 each admit solutions with both positive and negative helicity. For instance, if ψ is a solution to Equation 17, then $I\psi$ is a solution to the same equation with opposite charge and helicity. The characteristic quantity that distinguishes between Equations 17 and 19 is the "projection" of charge onto helicity, which one can see by inspecting Equations 17 and 19.

This is an example of a conjugation: a mapping between solutions to an equation. The most important conjugations are charge (C), parity (P), and time reversal (T) conjugations. Combined CPT symmetry is a fundamental symmetry, the violation of which would indicate a violation of Lorentz invariance.

For general multivector solutions, these conjugations are given by

$$\hat{C}\psi = \psi\gamma_1 \iff eA \mapsto -eA \tag{20}$$

$$\hat{P}\psi = \overline{\psi}(\overline{x}) \iff \nabla \mapsto \overline{\nabla}, eA(x) \mapsto e\overline{A}(\overline{x}),\ \text{and}\ p_0 \mapsto \overline{p}_0 \tag{21}$$

$$\hat{T}\psi = -I\overline{\psi}(-\overline{x})\gamma_1 \iff \nabla \mapsto \overline{\nabla}, eA(x) \mapsto -e\overline{A}(-\overline{x}),\ \text{and}\ p_0 \mapsto -\overline{p}_0. \tag{22}$$

[1] This raises the interesting possibility of having a spinor valued (i.e. fermionic) gauge field.

Parity and the combined CPT conjugation

$$\hat{C}\hat{P}\hat{T}\psi(x) = -I\psi(-x) \iff A(x) \mapsto A(-x) \text{ and } p_0 \mapsto -p_0 \tag{23}$$

are grade preserving conjugations and so apply directly to ψ_+ and ψ_- in Equation 17 and 19. However, charge and time reversal conjugations swap even and odd parts of ψ and so transform $\psi_+ = \langle\psi\rangle_+$ and $\psi_- = \langle\psi\rangle_-\gamma_0$ differently.

Under charge conjugation, $\langle\psi\rangle_\pm$ transform as

$$\psi \mapsto \psi\gamma_1 \implies \langle\psi\rangle_+ \mapsto \langle\psi\rangle_+\gamma_1 = \langle\psi\gamma_1\rangle_- \tag{24}$$

$$\text{and } \langle\psi\rangle_- \mapsto \langle\psi\rangle_-\gamma_1 = \langle\psi\gamma_1\rangle_+. \tag{25}$$

So ψ_+ and ψ_- transform as

$$\psi_+ \mapsto \psi'_- = \psi_+\sigma_1 \text{ and } \psi_- \mapsto \psi'_+ = -\psi_-\sigma_1, \tag{26}$$

where $\psi'_- \equiv \langle\psi'\rangle_-\gamma_0$ and $\psi'_+ \equiv \langle\psi'\rangle_+$, which induces a sign change in both charge and helicity. Therefore, C, P, and T conjugations for ψ_+ and ψ_- are given by

$$\hat{C}\psi_\pm = \pm\psi_\pm\sigma_1 \iff eA \mapsto -eA \text{ and } p_0 \mapsto \overline{p}_0 \tag{27}$$

$$\hat{P}\psi_\pm = \overline{\psi}_\pm(\overline{x}) \iff \nabla \mapsto \overline{\nabla}, eA(x) \mapsto e\overline{A}(\overline{x}), \text{ and } p_0 \mapsto \overline{p}_0 \tag{28}$$

$$\hat{T}\psi_\pm = \mp I\overline{\psi}_\pm(-\overline{x})\sigma_1 \iff \nabla \mapsto \overline{\nabla}, eA(x) \mapsto -e\overline{A}(-\overline{x}), \text{ and } p_0 \mapsto -p_0. \tag{29}$$

Note that if $A = 0$, then charge conjugation provides a map between solutions to Equations 17 and 19. Otherwise, they are distinct, and describe two distinct particles: one with correlated charge and helicity, one with anti-correlated charge and helicity.

4 Probability Current

Essential to Dirac theory is its probabilistic intepretation, which depends on a conserved probability current J satisfying the continuity equation

$$\nabla \cdot J = 0. \tag{30}$$

The usual probability current $\psi\gamma_0\widetilde{\psi}$ of Dirac theory is not conserved for spinor valued solutions ψ to Equation 3. To see this, consider the following, for a constant vector v_0.

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = \langle v_0 \wedge p_0(I\sigma_3\psi\widetilde{\psi})\rangle + \langle v_0 \cdot I\sigma_3(\widetilde{\psi}eA\psi)\rangle, \tag{31}$$

which gives a condition for conservation

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = 0 \iff v_0 \wedge p_0 = 0. \tag{32}$$

The second term in Equation 31 vanishes, because $v_0 \wedge p_0 = 0$ implies $v_0 \cdot I\sigma_3 = 0$, due to Equation 4.

This means that $\psi p_0\widetilde{\psi}$ is the only vector valued bilinear covariant conserved in general (up to a constant multiple). Furthermore, the fact that $\nabla\cdot(\psi p_0\widetilde{\psi}) = 0$ implies the existence of streamlines with tangents given by $p = Rp_0\widetilde{R}$, which are timelike if $p_0^2 > 0$ and lightlike if $p_0^2 = 0$.[2] The usual probability current $\psi\gamma_0\widetilde{\psi}$ is not conserved because $\gamma_0 \wedge p_0 \neq 0$.

The normalization procedure

$$\int d^3x\gamma_0 \cdot J = 1 \tag{33}$$

can be extended straightforwardly. In Dirac theory, Equation 33 is equivalent to

$$\int d^3x\gamma_0 \cdot (\psi p_0 \widetilde{\psi}) = m, \tag{34}$$

which simply ensures that integrating energy density (in the γ_0 frame) over all of space is just the rest energy of the particle.

Since massless particles do not have rest energy, a reasonable generalization of this is

$$\int d^3x \gamma_0 \cdot (\psi p_0 \widetilde{\psi}) = c. \tag{35}$$

for a constant c. Any choice of $c \neq 0$ determines a probability current $J = \psi p_0 \widetilde{\psi}/c$ with a normalized probability density $J_0 = \gamma_0 \cdot J$. Selecting $c = \gamma_0 \cdot p_0$ may be a convenient choice, because it coincides with ω_0 in the massless theory and aligns with the usual choice m in the massive theory. Alternatively, there's nothing preventing us from choosing $c = 1$ and simply referring to $\rho p = \psi p_0 \widetilde{\psi}$ as the probability current.

5 Observables

What is the appropriate definition of spin? The spin vector of hestenes and spin bivector of doran/lasenby are incompatible, due to duality factor.

Why does Hestenes use a lightlike spin bivector? Is there a relation between this bivector and F below?

If I were to reverse engineer the process, following the strategy of Vaz, seeking a spinorial description of circularly polarized electromagnetic fields, would I end up with the same equation?
- This would contribute to the Vaz paper, which depended on invertibility.

That is, are all circularly polarized electromagnetic waves described via this equation?

Note other forms of the Equation 1:

$$\begin{aligned}
\nabla \psi I \sigma_3 &= \rho^{1/2} R' p_0 = p \psi && (36)\\
\nabla \psi &= p s^{-1} \psi = I \sigma_3' p \psi = I p \psi && (37)\\
-I \nabla \psi &= \psi p_0 && (38)\\
&= \rho^{1/2}(x) e^{I\beta'(x)} R_0 && (39)
\end{aligned}$$

which have different symmetries. In particular, the second form requires that the only kinematical part is due to $\rho^{1/2} e^{I\beta/2}$ (otherwise, there would be a bivector term in the derivative). This has fewer degrees of freedom than Equation 3, doesn't it? R_0 is constant and can pick out any frequency for p.

Gauging:

$$\begin{aligned}
-I \nabla \psi &= p \psi. && (40)\\
\psi &\mapsto \psi e^{I\phi(x)} && (41)\\
&\implies && (42)\\
-I \nabla \psi' &= -I \nabla \psi e^{I\phi} - \nabla \phi \psi e^{I\phi} && (43)\\
&\implies D\psi = -I \nabla \psi - eA\psi = p\psi, && (44)
\end{aligned}$$

where $A \mapsto A - \nabla \phi$.

If $\phi = \phi_e + I\phi_b$, then $A = A_e + IA_b$ gives

$$F = \langle \nabla A \rangle_2 = F_e + F_b = \nabla \wedge A_e + \nabla \cdot IA_b \tag{45}$$

such that

$$\nabla F = J \equiv J_e + IJ_b. \tag{46}$$

6 Plane Waves

Plane wave solutions for Equation 17 are given by

$$\nabla\psi_+ I\sigma_3 = \pm p\psi_+, \tag{47}$$

where $p = Rp_0\widetilde{R}$ is constant and $\omega = p\cdot\gamma_0 > 0$.

This implies that ρ is constant, $\beta = 0, \pi$, and $R = R_0 e^{\mp I\sigma_3(p\cdot x + c)}$, where R_0 is constant and c is a monogenic phase shift satisfying $\nabla c(x) = 0$. For every ρ and R_0, which completely determine the constant probability current $\rho p = \psi p_0\widetilde{\psi} = \rho R_0 p_0 \widetilde{R}_0$, following the conventions of [1], we have two solutions (taking $c = 0$)

$$\psi_+^{(+)}(x) = \rho^{1/2} R_0 e^{-I\sigma_3 p\cdot x} \text{ and } \psi_+^{(-)}(x) = \rho^{1/2} I R_0 e^{I\sigma_3 p\cdot x}, \tag{48}$$

which are CPT conjugates of one another. $\psi_+^{(+)}$ and $\psi_+^{(-)}$ describe particles propagating in the $\vec{p}_0 = p\wedge\gamma_0$ direction (in the γ_0 frame) with opposite spin, or helicity.

There are two corresponding solutions for Equation 19 of the form

$$\psi_-^{(+)} = \psi_+^{(+)}\sigma_1 \text{ and } \psi_-^{(-)} = \psi_+^{(-)}\sigma_1, \tag{49}$$

which propagate in $-\vec{p}_0$ direction.

These solutions are similar to the plane wave solutions to the Dirac equation in that there are four of them, each of which is described by a Dirac (four component) spinor, and are unlike the right and left handed plane wave solutions to the massless Dirac equation, for which there are only two distinct solutions, each of which is given by a Weyl (two component) spinor. On the other hand, they are similar to Weyl plane waves, and unlike Dirac plane waves, in that their momentum is lightlike and their helicity is Lorentz invariant.

The main qualitative feature that distinguish $\psi_\pm^{(\pm)}$ from Weyl plane waves is that their (constant) spin plane

$$\rho S = \psi I\sigma_3\widetilde{\psi} = (\pm)\rho R I\sigma_3\widetilde{R} = (\pm)\rho R_0 I\sigma_3\widetilde{R}_0 \tag{50}$$

does *not* vanish, and the vectors ($i = 1, 2$)

$$e_i = R\gamma_i\widetilde{R} = R_0\gamma_i\widetilde{R}_0 e^{\mp S2p\cdot x} \tag{51}$$

rotate at a frequency 2ω in this plane, analogous to the rotation of electric and magnetic fields of circularly polarized electromagnetic waves.

In fact, this analogy can be made precise. Left and right circularly polarized electromagnetic waves have the form[1]

$$F_\pm = kn\alpha e^{\mp Ik\cdot x} = Akne^{\pm I\hat{k}(k\cdot x + c)}, \tag{52}$$

where n is a constant vector perpendicular to k (i.e. $k\cdot n = 0$), $\alpha = Ae^{\mp Ic}$ is a constant amplitude and duality transformation, and $\hat{k} = k\wedge\gamma_0/|k\wedge\gamma_0|$. As noted by [1], positive and negative helicities correspond to left and right handedness respectively. The plus and minus in $F_\pm$ refer to positive and negative helicity, rather than handedness.

Making the identifications $p = \hbar k$, $R_0\gamma_1\widetilde{R}_0 = n$, and $\rho = \pm A$, then e_1 can be seen to rotate identically to the factor $ne^{\pm I\hat{k}(k\cdot x + c)}$ with twice the frequency and the corresponding electromagnetic field can be placed in the form

$$F_\psi \equiv \psi p_0\gamma_1\widetilde{\psi} = \rho p e_1 e^{I\beta}, \tag{53}$$

which in this case is

$$F_\psi = \pm\rho p e_1. \tag{54}$$

In this setting, the $e^{I\beta}$ factor is less mysterious than it is in Dirac theory, as it is simply a duality rotation of the elecromagnetic field. In fact, since p is null here, it simplifies to an ordinary spatial rotation of e_1:

$$F_\psi = \rho p e_1 e^{-S\beta}. \tag{55}$$

F_ψ is a proper electromagnetic field satisfying

$$\nabla F_\psi = 0 \text{ and } F_\psi^2 = 0 \tag{56}$$

which are the usual specifications of circularly polarized electromagnetic fields. In fact, it will be shown in the next section that Equation 53 satisfies Maxwell's equations for all massless *and* massive solutions ψ (although, with a disclaimer) to Equation 1. Furthermore, when $A \neq 0$, A plays the same role as the gauge field that arises in a gauging duality symmetry of Maxwell's equations.

The only discrepancy is that e_1 rotates with twice the frequency, so it may be worth making the replacement $p_0 \mapsto \frac{1}{2}p_0$ and taking the equation

$$\nabla\psi I\sigma_3 = \frac{1}{2}\psi p_0, \tag{57}$$

as primary, so that the frequency of rotation of e_1 is given by ω.

7 An Electromagnetic Bilinear Covariant

Suppose $\psi \in G^+_{1,3}$ satisfies Equation 57, taking p_0 to be arbitrary and F_ψ is given by Equation 53. Then

$$\begin{aligned}
\nabla F_\psi &= J_\psi && (58)\\
&= \nabla(\psi p_0 \gamma_1 \widetilde{\psi}) && (59)\\
&= \nabla\psi p_0 \gamma_1 \widetilde{\psi} + \dot{\nabla}\psi p_0 \gamma_1 \dot{\widetilde{\psi}} && (60)\\
&= \rho p^2 e_2. && (61)
\end{aligned}$$

The current term vanishes for massless solutions precisely because $p^2 = 0$ and does not vanish for massive solutions because $p^2 \neq 0$. Additionally, we see that the contraction of p with itself due to the derivative is precisely what ensures that the current J_ψ is vector valued and so describes an electromagnetic field free from magnetic sources.

In the presence of an electromagnetic gauge field, if ψ satisfies

$$\nabla\psi I\sigma_3 - \frac{1}{2}eA\psi = \frac{1}{2}\psi p_0, \tag{62}$$

then

$$\nabla F_\psi - eAFe_2e_1 = \rho p^2 e_2. \tag{63}$$

In the massless case, $AF_\psi e_2 e_1 = IAF_\psi$ and $p^2 = 0$. This yields the equation

$$\nabla F_\psi - eIAF_\psi = 0, \tag{64}$$

which is identical to the gauged Maxwell equations in vacuum.[3][4][5]

The massive analog of these equations is [5]

$$\nabla F - eIAF = J, \tag{65}$$

but this is distinct from Equation 63.

An additional, more serious concern with Equation 63 in the massive case is that the current $\rho p^2 e_2$ is a *spacelike* vector, because $e_2^2 = -1$. This is arguably non-physical.[2]

[2] Todo: Does $\nabla \cdot (\rho p^2 e_2) = 0$? If not, that would definitively render the massive case useless.

A Matrix Formulation

Following Equation 8.70 in [1], Equations 17 and 19 take the following form in matrix representation.

$$\hat{\gamma}^\mu(i\partial_\mu - eA_\mu)|\psi_+\rangle = \omega_0(1+\hat{\gamma}_5)|\psi_+\rangle \tag{66}$$

and

$$\hat{\gamma}^\mu(i\partial_\mu - eA_\mu)|\psi_-\rangle = \omega_0(1-\hat{\gamma}_5)|\psi_-\rangle, \tag{67}$$

where

$$\hat{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\gamma}^k = \begin{pmatrix} 0 & \hat{\sigma}^k \\ -\hat{\sigma}^k & 0 \end{pmatrix}, \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{68}$$

See [1] for more details on the isomorphism.

References

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