# An Extension of the Dirac Equation

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## 1 Introduction

The free Dirac equation in STA is

$$\nabla\psi I\sigma_3 = \hat{p}\psi, \tag{1}$$

where $\psi \in G^+_{1,3}$ such that[1]

$$\hat{p}\psi = m\psi\gamma_0. \tag{2}$$

A less restrictive constraint is just that

$$\hat{p}^2 = m^2 \tag{3}$$

be constant. With this lightened restriction, Equation 2 can be replaced with

$$\hat{p}\psi = \psi p_0 \tag{4}$$

where $p_0$ is a constant vector. This admits a new class of null solutions, where $\hat{p}^2 = 0$ but $\hat{p}\psi \neq 0$.[2]

## 2 Symmetries

The extension of the full Dirac equation, including the electromagnetic gauge field

$$\nabla\psi I\sigma_3 - eA\psi = \psi p_0 \tag{5}$$

is only invariant under the usual replacements $\psi \mapsto \psi e^{\alpha I\sigma_3}$ and $A \mapsto A - \nabla\alpha$ if

$$p_0 \cdot I\sigma_3 = 0. \tag{6}$$

This is an essential feature of Dirac theory, so we'll restrict $p_0$ preserve it. We require that

$$p_0 = E\gamma_0 + |p|\gamma_3. \tag{7}$$

If $p_0^2 = m^2 \neq 0$, then

$$p_0 = Rm\gamma_0\widetilde{R} \tag{8}$$

for $R = e^{\gamma_3\gamma_0\alpha/2}$ with $\alpha$ given by $\tanh(\alpha) = |p|/E$. Then $p_0 \mapsto \widetilde{R}p_0R$ gives Equation 2.

On the other hand, if $p_0^2 = 0$, then

[1]See GAP 8.86

[2]$\hat{p}$ is used in order to reserve $p$ for $p = Rp_0\widetilde{R}$.

$$p_0 = \omega(\gamma_0 + \gamma_3), \tag{9}$$

for $\omega = E = |p|$. Spinors satisfying Equation 4 with $p_0 = \omega(\gamma_0 + \gamma_3)$ are distinct from those satisfying Equation 2. This is a minimal extension that preserves the usual gauge field of Dirac theory, while admitting a new class of null solutions.[3]

Note that the conservation of the current density $J$ also depends on $p_0$.[4]

$$\begin{aligned}
\nabla \cdot J &= \langle \nabla\psi\gamma_0\widetilde{\psi}\rangle + \langle \psi\gamma_0\dot{\widetilde{\psi}}\dot{\nabla}\rangle & (10)\\
&= 2\langle Is(eA + pe^{-I\beta}))\rangle & (11)\\
&= 2\langle Ie^{-I\beta}sp\rangle & (12)\\
&= 2\sin(\beta)s \cdot p & (13)
\end{aligned}$$

where $s = \psi\gamma_3\widetilde{\psi}$ and $pe^{-I\beta} = \psi p_0 \psi^{-1}$. See the *Scratch Work* section for more details. This means that

$$\nabla \cdot J = 0 \iff \gamma_3 \cdot p_0 = 0, \tag{14}$$

which holds for $p_0 = m\gamma_0$ and does not hold for $p_0 = \omega(\gamma_0 + \gamma_3)$.

In the context of the normal Dirac equation, the conservation of $J$ is usually taken to mean that single fermions cannot be created or destroyed.[5] However, this interpretation may not make sense for null $p_0$. We'll try to make sense of these solutions next.

# 3 Scratch work

## 3.1 $\nabla \cdot J$

$$\begin{aligned}
\nabla\psi &= -(eA\psi + \psi p_0)I\sigma_3 & (15)\\
\nabla\psi\gamma_0\widetilde{\psi} &= -(eA\psi + \psi p_0)I\sigma_3\gamma_0\widetilde{\psi} & (16)\\
&= -I(eA\psi + \psi p_0)\gamma_3\widetilde{\psi} & (17)\\
&= -I(eA + pe^{-I\beta})s & (18)\\
\langle\nabla\psi\gamma_0\widetilde{\psi}\rangle &= -\langle I(eA + pe^{-I\beta})s\rangle & (19)\\
&= \langle IseA\rangle + \langle Ispe^{-I\beta})\rangle & (20)
\end{aligned}$$

$$\begin{aligned}
\dot{\widetilde{\psi}}\dot{\nabla} &= -(\widetilde{eA\psi I\sigma_3} + \widetilde{\psi p_0 I\sigma_3}) & (21)\\
&= eI\sigma_3\widetilde{\psi}A + I\sigma_3 p_0\widetilde{\psi} & (22)\\
\psi\gamma_0\dot{\widetilde{\psi}}\dot{\nabla} &= \psi\gamma_0 I\sigma_3\widetilde{\psi}eA + \psi\gamma_0 I\sigma_3 p_0\widetilde{\psi} & (23)\\
&= I(\psi\gamma_3\widetilde{\psi}eA + \psi\gamma_3 p_0\widetilde{\psi}) & (24)\\
&= I(seA + s\widetilde{\psi p_0 \psi^{-1}}) & (25)\\
&= Is(eA + pe^{-I\beta}) & (26)\\
\langle\psi\gamma_0\dot{\widetilde{\psi}}\dot{\nabla}\rangle &= \langle IseA\rangle + \langle Ispe^{-I\beta}\rangle & (27)
\end{aligned}$$

[3] Do these relate to the null solutions of the Dirac equation?
[4] GAP 8.93
[5] GAP p283

where $pe^{-I\beta} = \psi p_0 \psi^{-1}$.

### 3.1.1 $\widetilde{\psi^{-1}} = \widetilde{\psi}^{-1}$

$$\widetilde{\psi^{-1}} = \rho^{-1/2}\widetilde{e^{-I\beta/2}}\widetilde{R} = \rho^{-1/2}e^{-I\beta/2}R$$
$$\widetilde{\psi}^{-1} = (\rho^{1/2}e^{I\beta/2}\widetilde{R})^{-1/2} = \rho^{-1/2}e^{-I\beta/2}R$$