Admitting Photons Into The Dirac Equation

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1 Generalizing the Dirac Equation

Within STA, the free Dirac equation is typically written

$$\nabla\psi i = m\psi\gamma_0, \tag{1}$$

where $\psi = \rho^{1/2}e^{I\beta/2}R$.[1] In general, we can define

$$p = m\psi\gamma_0\psi^{-1} = mJ/\rho, \tag{2}$$

where $J = \psi\gamma_0\widetilde{\psi}$ is the usual current density and ρ the usual density, and rewrite Equation 1 as

$$\nabla\psi i = p\psi. \tag{3}$$

At first glance, it may seem that nothing has been done and we've just redefined terms. However, notice that Equation 3 has a larger solution set than Equation 1, since Definition 2 is only valid for massive particles.

Conceptually, the Dirac equation describes particles in their rest frame, and ψ contains a rotor R which boosts γ_0 to the velocity p/m in some lab frame. Except, there is no rest frame for massless particles. Equation 3 minimally extends Equation 1 to admit a new class of null solutions.

2 Null Plane Wave Solutions

Equation 3 admits null plane wave solutions of the form

$$\psi = \psi_0 e^{Ip\cdot x}, \tag{4}$$

where $p^2 = 0$ and $\psi_0 = \rho e^{I\beta}$ is constant.

A circularly polarized electromagnetic plane relates to this equation in a direct manner. Right circularly polarized electromagnetic waves can be written in the form[2]

$$F = Ipn\psi, \tag{5}$$

where $n^2 = 1$ and $n \cdot p = 0$. Then F has a potential

[1] See Geometric Algebra for Physicists (GAP) 8.86 and 8.87.

[2] See GAP 7.142

$$A = \psi n, \tag{6}$$

satisfying $F = \nabla A$,[3] that is a solution to Equation 3 in the following way:

$$\begin{aligned} \nabla\psi &= \nabla A n & (7)\\ &= Fn & (8)\\ &= Ip\psi & (9)\\ &= -pi\psi, & (10) \end{aligned}$$

where $i = I\hat{p}$ is a spatial bivector, using the fact that null vectors are idempotent.[4] Hence,

$$\nabla\psi i = p\psi. \tag{11}$$

3 Comments

What other solutions satisfy Equation 3 but not Equation 1?

Do these solutions correspond to null solutions to the standard Dirac equation?

How does the potential A used above relate to the normal electromagnetic potential and the potential that plays the role of a connection in Dirac theory:

$$\nabla\psi i - eA\psi = m\psi\gamma_0? \tag{12}$$

Does the fact that these solutions arise out of purely a scalar+pseudoscalar spinor, which form a representation of U(1), have anything to do with the fact that electromagnetism is a U(1) gauge theory?

ψ defines F up to specification of n, which determines the direction of electric and magnetic fields. ψ is insufficient to fully characterize classical electromagnetic waves on its own. Is this a bug or a feature? Are the directions of E and B well defined quantum mechanically?

In Dirac theory, the $e^{I\beta}$ factor in ψ is a bit of a mystery and corresponds to particle/anti-particle states for plane-wave solutions for $\beta = 0$ and $\beta = \pi$. Here $e^{I(p\cdot x+\beta)}$ is not a mystery; it plays the role of phase and duality transformations.

Is there a connection to Vaz and Rodriguez's approach, where F is defined as a spinor current?[5]

[3] This is different from the usual potential which defines $F = \nabla \wedge A$.

[4] $p = \hbar\omega(1+\hat{p})\gamma_0 = \hbar\omega(1+\hat{p})\hat{p}\gamma_0 = -\hbar\omega(1+\hat{p})\gamma_0\hat{p} = -p\hat{p}$

[5] https://arxiv.org/pdf/hep-th/9511181v1.pdf