An Extension of the Dirac Equation

Luke Burns

November 11, 2016

1 Introduction

The free Dirac equation in Space Time Algebra is[1]

$$\nabla\psi I\sigma_3 = \psi p_0, \tag{1}$$

where

$$p_0 = m\gamma_0. \tag{2}$$

A minimal extension to this is to simply require that p_0 be constant. With this lightened constraint, Equation 1 admits a new class of null solutions, where $p_0^2 = 0$ but $\psi p_0 \neq 0$.

These null solutions split into a pair of independent right-handed and left-handed solutions, much like solutions to the massless Dirac equation, with the key difference that each solution is a Dirac spinor, not a Weyl spinor, with twice the number of degrees of freedom and distinct symmetries.

2 Constraints

The extension of the full Dirac equation, including the electromagnetic gauge field

$$\nabla\psi I\sigma_3 - eA\psi = \psi p_0 \tag{3}$$

is only invariant under the usual replacements $\psi \mapsto \psi e^{\alpha I\sigma_3}$ and $A \mapsto A - \nabla\alpha$ if $p_0 I\sigma_3 = I\sigma_3 p_0$ (p_0 is perpendicular to the plane $I\sigma_3$). This is an essential feature of Dirac theory, so we'll restrict p_0 to preserve it. We require that

$$p_0 = E\gamma_0 \pm |\vec{p}|\gamma_3. \tag{4}$$

Additionally, if solutions are to satisfy the Klein-Gordon equation, we must have $p_0^2 \geq 0$. If $p_0^2 > 0$, then

$$p_0 = Rm\gamma_0\widetilde{R} \tag{5}$$

for $R = e^{\gamma_3\gamma_0\alpha/2}$ with α given by $\tanh(\alpha) = |\vec{p}|/E$. Then $p_0 \mapsto \widetilde{R}p_0R$ reduces to Equation 2, which yields nothing new. On the other hand, if $p_0^2 = 0$, then

$$p_0 = \omega_0(1 \pm \sigma_3)\gamma_0, \tag{6}$$

for $\omega_0 = E = |\vec{p}|$, which gives the two equations

$$\nabla\psi I\sigma_3 - eA\psi = \omega_0\psi(1 \pm \sigma_3)\gamma_0. \tag{7}$$

[1]GAP 8.87

In the next section, we will find that we can make the choice of $p_0 = \omega_0(1+\sigma_3)\gamma_0$ without loss of generality. Anticipating this result, the final form is

$$\nabla\psi I\sigma_3 - eA\psi = \omega_0\psi(1+\sigma_3)\gamma_0. \tag{8}$$

3 Symmetries

Unlike the Dirac equation, Equation 8 is not invariant under

$$\psi \mapsto \psi\gamma_0. \tag{9}$$

If ψ is a solution to

$$\nabla\psi I\sigma_3 - eA\psi = \omega_0\psi(1+\sigma_3)\gamma_0, \tag{10}$$

then $\psi' = \psi\gamma_0$ is a solution to

$$\nabla\psi' I\sigma_3 - eA\psi' = \omega_0\psi'(1-\sigma_3)\gamma_0. \tag{11}$$

That is, Equation 8 distinguishes between even and odd fields, so ψ is in general a full multivector and can be decomposed into

$$\psi = \langle\psi\rangle_+ + \langle\psi\rangle_-, \tag{12}$$

where $\langle\psi\rangle_+$ and $\langle\psi\rangle_-$ are even and odd multivectors respectively that are independent solutions to

$$\nabla\langle\psi\rangle_\pm I\sigma_3 - eA\langle\psi\rangle_\pm = \omega_0\langle\psi\rangle_\pm(1+\sigma_3)\gamma_0. \tag{13}$$

The choice of $p_0 = \omega_0(1+\sigma_3)\gamma_0$ just as good as the other, because if $\psi_+ = \langle\psi\rangle_+$ is a solution to

$$\nabla\psi_+ I\sigma_3 - eA\psi_+ = \omega_0\psi_+(1+\sigma_3)\gamma_0, \tag{14}$$

then the even multivector

$$\psi_- = \langle\psi\rangle_-\gamma_0 \tag{15}$$

is a solution to

$$\nabla\psi_- I\sigma_3 - eA\psi_- = \omega_0\psi_-(1-\sigma_3)\gamma_0. \tag{16}$$

ψ_+ and ψ_- have opposite helicity, analogous to right-handed and left-handed Weyl spinor solutions to the massless Dirac equation. However, they are wholly distinct, because the spinors ψ_+ and ψ_- are full Dirac spinors, each with 8 degrees of freedom, whereas Weyl spinors only have 4 degrees of freedom.

Symmetries of Equation 3 include the following mappings, which induce the substitutions listed next to them.

$$\begin{aligned}
\psi \mapsto \psi\gamma_0 &\iff p_0 \mapsto \overline{p}_0 && (17)\\
\psi \mapsto \psi\gamma_1 &\iff A \mapsto -A && (18)\\
\psi \mapsto \psi\gamma_2 &\iff A \mapsto -A && (19)\\
\psi \mapsto \psi\gamma_3 &\iff p_0 \mapsto -\overline{p}_0 && (20)\\
\psi \mapsto \gamma_0\psi &\iff \nabla \mapsto \overline{\nabla} \text{ and } A \mapsto \overline{A} && (21)\\
\psi \mapsto I\psi &\iff p_0 \mapsto -p_0 && (22)
\end{aligned}$$

where $\overline{M} = \gamma_0 M \gamma_0$ for any multivector M. Note that these mappings are not grade preserving and therefore are not symmetries of Equations 14 and 16 individually, analogous to how CPT symmetries are symmetries of the Dirac equation but not the Weyl equations.

The decomposition given by Equation 15 is one of many possible decompositions of $\langle\psi\rangle_-$, each of these symmetries yielding an additional decomposition of $\langle\psi\rangle_-$. For instance, the decompositions $\langle\psi\rangle_- = \psi_-^1\gamma_1$ and $\langle\psi\rangle_- = \psi_-^3\gamma_3$ give

$$\nabla\psi_-^1 I\sigma_3 + eA\psi_-^1 = \omega_0\psi_-^1(1+\sigma_3)\gamma_0, \tag{23}$$

which has opposite the charge of ψ_+ and

$$\nabla\psi_-^3 I\sigma_3 - eA\psi_-^3 = \omega_0\psi_-^3(-1+\sigma_3)\gamma_0, \tag{24}$$

which is pointing in the opposite timelike direction (i.e. an anti-particle of ψ_+).

In other words, given a pair of solutions with opposite helicity *or* opposite charge *or* a pair of solutions pointing in opposite timelike directions, one has a solution to Equation 10. So while these mappings are not proper conjugations, because they are mappings between entirely independent solutions, they do determine an equivalence between these properties.

Furthermore, just as one cannot find a Lorentz transformation that maps between massless particles of opposite helicity, the above indicates that it is not possible to find a Lorentz transformation between solutions with opposite charge or solutions pointing in opposite timelike directions.

This means that Equation 10 does not have well defined charge or time-reversal symmetries. The fact that $\psi \mapsto \psi\gamma_0$ leaves the Dirac equation invariant allows for well defined charge and time-reversal conjugations.

3.1 CPT

The following symmetries are the same as the CPT symmetries for the Dirac equation as presented by GAP (8.90) applied to Equation 3. The problem here is that the usual charge and time reversal conjugations induce additional changes to p_0 that invalidate their usage as charge and time reversal conjugations.

"Charge conjugation" is given by

$$\psi \mapsto \psi\gamma_1\gamma_0 \iff A \mapsto -A \text{ and } p_0 \mapsto \overline{p}_0. \tag{25}$$

It induces $p_0 \mapsto \overline{p}_0$, whereas it should induce $p_0 \mapsto p_0$.

"Time reversal" is given by

$$\psi(x) \mapsto I\gamma_0\psi(-\overline{x})\gamma_1 \iff \nabla \mapsto -\overline{\nabla}, I\sigma_3 \mapsto -I\sigma_3, A \mapsto -\overline{A}, \text{ and } p_0 \mapsto -p_0. \tag{26}$$

However, p_0 does not transform as it should under time reversal. Proper time reversal induces $p_0 \mapsto -\overline{p}_0$.

On the other hand, parity reversal is given by

$$\psi(x) \mapsto \overline{\psi}(\overline{x}) \iff \nabla \mapsto \overline{\nabla}, A \mapsto \overline{A}, \text{ and } p_0 \mapsto \overline{p}_0, \tag{27}$$

which can be fairly called parity reversal.

Additionally, the combined CPT symmetry

$$\psi(x) \mapsto -I\psi(-x) \iff p_0 \mapsto -p_0, \tag{28}$$

does induce the appropriate transformations.

4 Observables

Here we'll work out the observables of individual solutions ψ to Equation 14 and 16. Unlike solutions to the massless solutions to the Dirac equation, ψ can be written in the canonical form[2]

$$\psi = \rho^{1/2} e^{I\beta/2} R, \tag{29}$$

and determines a moving frame of vectors

$$e_\mu = R\gamma_\mu \widetilde{R} \tag{30}$$

and currents

$$\rho e_\mu = \psi\gamma_\mu\widetilde{\psi}. \tag{31}$$

4.1 The Probability Current

Essential to Dirac theory is its probabilistic intepretation, which depends on a conserved probability current J satisfying the continuity equation

$$\nabla \cdot J = 0. \tag{32}$$

The usual probability current $\psi\gamma_0\widetilde{\psi}$ of Dirac theory is not conserved here. To see this, consider the following, for a constant vector v_0.

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = \langle v_0 \wedge p_0 (I\sigma_3 \psi\widetilde{\psi})\rangle + \langle v_0 \cdot I\sigma_3 (\widetilde{\psi} e A \psi)\rangle, \tag{33}$$

which gives a condition for conservation

$$\nabla \cdot (\psi v_0 \widetilde{\psi}) = 0 \iff v_0 \wedge p_0 = 0. \tag{34}$$

This means that $\psi p_0 \widetilde{\psi}$ is the only vector-valued bilinear covariant conserved in general (up to a constant multiple). Furthermore, the fact that $\nabla \cdot (\psi p_0 \widetilde{\psi}) = 0$ implies the existence of lightlike streamlines with tangents given by $p = R p_0 \widetilde{R}$.

The usual probability current $\psi\gamma_0\widetilde{\psi}$ is not conserved because $\gamma_0 \wedge p_0 = -\vec{p} = \mp\omega_0\sigma_3$, which gives

$$\nabla \cdot (\psi\gamma_0\widetilde{\psi}) = \pm\omega_0 \rho \sin\beta. \tag{35}$$

The normalization procedure

$$\int d^3x \gamma_0 \cdot J = 1 \tag{36}$$

can be extended straightforwardly. In Dirac theory, Equation 36 is equivalent to

$$\int d^3x \gamma_0 \cdot (\psi p_0 \widetilde{\psi}) = m, \tag{37}$$

which simply ensures that integrating energy density (in the γ_0 frame) over all of space is just the rest energy of the particle.[3]

Since massless particles do not have a rest frame, a reasonable generalization of this is

$$\int d^3x \gamma_0 \cdot (\psi p_0 \widetilde{\psi}) = c. \tag{38}$$

[2] Hestenes said this once

[3] Note, there is a certain arbitrariness to this, since it is frame dependent.

for a constant c. Any choice of c determines a probability current $J = \psi p_0 \widetilde{\psi}/c$ with a normalized probability density $J_0 = \gamma_0 \cdot J$. Selecting $c = \gamma_0 \cdot p_0$ may be a convenient choice, because it coincides with ω_0 in the massless theory, reducing the number of arbitrary parameters to be specified, and aligns with the usual choice m in the massive theory.[4]

5 Plane Waves

Plane wave solutions are given by

$$\nabla \psi I \sigma_3 = p\psi, \tag{39}$$

where p is a constant, which tells us that

$$p\psi = \psi p_0. \tag{40}$$

Using the canonical decomposition $\psi = \rho^{1/2} e^{I\beta/2} R$,

$$p e^{I\beta} = R p_0 \widetilde{R} \tag{41}$$

implies that $e^{I\beta} = \pm 1$, and hence

$$p = \pm R p_0 \widetilde{R}. \tag{42}$$

p is constant implies that ρ is constant and, given the decomposition $R = R_{\|} R_{\perp}$, where $R_{\perp}$ satisfies $p_0 = R_{\perp} p_0 \widetilde{R}_{\perp}$, that $R_{\|}$ is constant and $R_{\perp} = e^{I\sigma_3 \theta(x)}$.

This implies that

$$\theta = \pm p \cdot x + c(x), \tag{43}$$

where $c(x)$ satisfies $\nabla c(x) = 0$ (is monogenic).

Taking $c(x) = 0$, for every ρ and $R_{\|}$, which determine a constant Lorentz transformation and dilation of p_0, we have two solutions

$$\psi_1 = \rho^{1/2} R_{\|} e^{-I\sigma_3 p\cdot x} \text{ and } \psi_2 = \rho^{1/2} I R_{\|} e^{I\sigma_3 p\cdot x}, \tag{44}$$

which are CPT conjugates of one another.

ψ_1 and ψ_2 describe particles with the opposite spin, propagating in the same direction. In total, there are four solutions: two for Equation 14 which describe two particles of opposite helicity propagating in one direction, and two for Equation 16 which describe two particles of opposite helicity propagating in the other direction.

Important note When $A = 0$, CPT conjugation is the same as helicity conjugation. Need to make corrections throughout to account for this. Solutions to Equation 14 do not necessarily have opposite helicity to solutions to Equation 16 as I've claimed. They describe particles propagating in opposite directions. This inconsistency is traceable to the fact that while Lorentz transformations cannot map between momenta with opposite helicity for massless particles, duality transformations *can*.

[4] I am displeased with this, because of the frame dependency. Is there a frame-independent normalization condition?