An Extension of the Dirac Equation

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1 Introduction

The free Dirac equation in STA is

$$\nabla\psi I\sigma_3 = \hat{p}\psi, \tag{1}$$

where $\psi \in G^+_{1,3}$ such that[1]

$$\hat{p}\psi = m\psi\gamma_0. \tag{2}$$

A less restrictive constraint is just that

$$\hat{p}^2 = m^2 \tag{3}$$

be constant. With this lightened restriction, Equation 2 can be replaced with

$$\hat{p}\psi = \psi p_0 \tag{4}$$

where p_0 is a constant vector. This admits a new class of null solutions, where $\hat{p}^2 = 0$ but $\hat{p}\psi \neq 0$.[2]

2 Symmetries

The extension of the full Dirac equation, including the electromagnetic gauge field

$$\nabla\psi I\sigma_3 - eA\psi = \psi p_0 \tag{5}$$

is only invariant under the usual replacements $\psi \mapsto \psi e^{\alpha I\sigma_3}$ and $A \mapsto A - \nabla\alpha$ if

$$p_0 \cdot I\sigma_3 = 0. \tag{6}$$

This is an essential feature of Dirac theory, so we'll restrict p_0 preserve it. We require that

$$p_0 = E\gamma_0 + |p|\gamma_3. \tag{7}$$

If $p_0^2 = m^2 \neq 0$, then

$$p_0 = Rm\gamma_0\widetilde{R} \tag{8}$$

for $R = e^{\gamma_3\gamma_0\alpha/2}$ with α given by $\tanh(\alpha) = |p|/E$. Then $p_0 \mapsto \widetilde{R}p_0R$ gives Equation 2.
On the other hand, if $p_0^2 = 0$, then

[1] See GAP 8.86

[2] $\hat{p}$ is used in order to reserve p for $p = Rp_0\widetilde{R}$.

$$p_0 = \omega(\gamma_0 + \gamma_3), \tag{9}$$

for $\omega = E = |p|$. Spinors satisfying Equation 4 with $p_0 = \omega(\gamma_0 + \gamma_3)$ are distinct from those satisfying Equation 2. [3]

Note that the conservation of the current density J also depends on p_0.[4]

$$\begin{aligned}
\nabla \cdot J &= \langle \nabla\psi\gamma_0\widetilde{\psi}\rangle + \langle \psi\gamma_0\dot{\widetilde{\psi}}\dot{\nabla}\rangle && (10)\\
&= 2\langle Is(eA + pe^{-I\beta}))\rangle && (11)\\
&= 2\langle Ie^{-I\beta} sp\rangle && (12)\\
&= 2\sin(\beta) s\cdot p && (13)
\end{aligned}$$

where $s = \psi\gamma_3\widetilde{\psi}$ and $pe^{-I\beta} = \psi p_0 \psi^{-1}$. See the *Scratch Work* section for more details. This means that

$$\nabla \cdot J = 0 \iff \gamma_3 \cdot p_0 = 0, \tag{14}$$

which holds for $p_0 = m\gamma_0$ and does not hold for $p_0 = \omega(\gamma_0 + \gamma_3)$.

In the context of the normal Dirac equation, the conservation of J is usually taken to mean that single fermions cannot be created or destroyed.[5] However, this interpretation may not make sense for null p_0. We'll try to make sense of these solutions next.

3 Scratch work

3.1 $\nabla \cdot J$

$$\begin{aligned}
\nabla\psi &= -(eA\psi + \psi p_0)I\sigma_3 && (15)\\
\nabla\psi\gamma_0\widetilde{\psi} &= -(eA\psi + \psi p_0)I\sigma_3\gamma_0\widetilde{\psi} && (16)\\
&= -I(eA\psi + \psi p_0)\gamma_3\widetilde{\psi} && (17)\\
&= -I(eA + pe^{-I\beta})s && (18)\\
\langle \nabla\psi\gamma_0\widetilde{\psi}\rangle &= -\langle I(eA + pe^{-I\beta})s\rangle && (19)\\
&= \langle IseA\rangle + \langle Ispe^{-I\beta})\rangle && (20)
\end{aligned}$$

$$\begin{aligned}
\dot{\widetilde{\psi}}\dot{\nabla} &= -(\widetilde{eA\psi I\sigma_3} + \widetilde{\psi p_0 I\sigma_3}) && (21)\\
&= eI\sigma_3\widetilde{\psi}A + I\sigma_3 p_0\widetilde{\psi} && (22)\\
\psi\gamma_0\dot{\widetilde{\psi}}\dot{\nabla} &= \psi\gamma_0 I\sigma_3\widetilde{\psi}eA + \psi\gamma_0 I\sigma_3 p_0\widetilde{\psi} && (23)\\
&= I(\psi\gamma_3\widetilde{\psi}eA + \psi\gamma_3 p_0\widetilde{\psi}) && (24)\\
&= I(seA + s\widetilde{\psi p_0\psi^{-1}}) && (25)\\
&= Is(eA + pe^{-I\beta}) && (26)\\
\langle \psi\gamma_0\dot{\widetilde{\psi}}\dot{\nabla}\rangle &= \langle IseA\rangle + \langle Ispe^{-I\beta}\rangle && (27)
\end{aligned}$$

where $pe^{-I\beta} = \psi p_0\psi^{-1}$.

[3]Do these relate to the null solutions of the Dirac equation?
[4]GAP 8.93
[5]GAP p283

3.1.1 $\widetilde{\psi^{-1}} = \widetilde{\psi}^{-1}$

$$\widetilde{\psi^{-1}} = \rho^{-1/2}\widetilde{e^{-I\beta/2}\widetilde{R}} = \rho^{-1/2}e^{-I\beta/2}R$$
$$\widetilde{\psi}^{-1} = (\rho^{1/2}e^{I\beta/2}\widetilde{R})^{-1/2} = \rho^{-1/2}e^{-I\beta/2}R$$