Gauging Duality Transformations

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1 Maxwell's Equations

An electromagnetic field F is a bivector valued field with vanishing curl. The four Maxwell equations are a consequence of this statement and are given by the single equation[1]

$$\nabla F = J, \tag{1}$$

where

$$F = E + IB \tag{2}$$

is a frame dependent decomposition of F into a timelike bivector E and spacelike bivector IB. In general,

$$J = J_e + IJ_b, \tag{3}$$

where $J_e = \nabla \cdot F$ is a vector and $IJ_b = \nabla \wedge F$ is a trivector. As suggested by notation, J_e is the electrical current density, and J_b is a hypothetical magnetic current density.

The restriction that F has vanishing curl

$$\nabla \wedge F = 0 \tag{4}$$

is the same as requiring that $J_b = 0$. The "generalized" Maxwell's equations, without the constraint that $J_b = 0$, are solely determined by the fact that F is a bivector. In other words, *any* bivector field contains a decomposition into electric and magnetic parts that satisfy the generalized Maxwell's equations (Equation 1).

2 Generalized Potentials

By allowing F to be an arbitrary bivector field, it can be defined in terms of a potential M as

$$F = \langle \nabla M \rangle_2, \tag{5}$$

where the brackets are an indication to only take the bivector part of ∇M.

∇M only contains even grade terms if M contains odd grade terms. Assuming M contains no even terms (which would have no contribution to F), it is of the form

$$M = M_e + IM_b, \tag{6}$$

where M_e is a vector and IM_b is a trivector such that

[1] See Equation 7.14 in Doran and Lasenby's Geometric Algebra for Physicists

$$F = \nabla \wedge M_e + \nabla \cdot IM_b, \tag{7}$$

which ensures that the electric and magnetic source terms

$$J_e = \nabla \cdot (\nabla \wedge M_e) \text{ and } J_b = \nabla \wedge (\nabla \cdot IM_b) \tag{8}$$

are sourced by the corresponding vector potentials M_e and M_b.

3 Duality Transformations

A duality rotation of F

$$F \mapsto Fe^{I\phi} = \cos(\theta)E - \sin(\theta)B + I(B\cos(\theta) + E\sin(\theta)), \tag{9}$$

mixes up electric and magnetic fields. If this is a global, constant transformation, then J transforms identically

$$J \mapsto Je^{I\phi}, \tag{10}$$

which mixes up electric and magnetic sources. Because of this, the duality transformation does not preserve the physical content of the theory, and so cannot be considered a true symmetry of the equation (unless $J = 0$). On the other hand, duality transformations are a *formal* symmetry of Equation 1. And seeing that the presence of magnetic sources already challenges the physical integrity of J, it is arguably less wrong to call it a symmetry in the general case where J includes magnetic sources.

For the case $J = 0$, duality is a true global symmetry of Equation 1, so one might ask whether this symmetry can be promoted to a local symmetry. Interestingly, while Equation 1 with $J = 0$ is invariant under duality transformations, the standard Maxwell action is not. This has caused some confusion. [5] reviewed attempts of [3], [6], and [7] to gauge this symmetry. [6] and [7] concluded that duality symmetry could not be gauged, in contradiction to the result of [3]. [5] concluded that the discrepancy between these results was due to the fact that [6] and [7] required preservation of gauge invariance, whereas [3] failed to take this into account (i.e. that the gauging process breaks gauge invariance of the original field).

The confusion is cleared up by introduction of a full multivector (complex) valued potential — one that sources both electric *and* magnetic sources. By this method, [1], [8], and [9] constructed dual symmetric Lagrangians. [8] and [9] went on to construct the same equations as [3] at the level of the action and generalized the result for the case where $J \neq 0$.

Ultimately, the no-go conclusions reached by [6] and [7] were due to the fact that the resulting gauge field introduces both electric *and* magnetic sources into the Equation 1. This is the reason for both loss of gauge invariance and lack of local symmetry at the level of the action. The usual vector valued gauge field and Lagrangian enforce the absence of magnetic sources, whereas the resulting equations after gauging require the admission of magnetic sources. The potential defined by Equation 6 does not suffer from this deficiency, and a dual symmetric Lagrangian (for the vaccum equations) can then be expressed as[1]

$$\mathcal{L}_{\text{dual}} = \frac{1}{2}\langle(\nabla \wedge M_e)^2 + (\nabla \wedge M_b)^2\rangle_0, \tag{11}$$

whereas the traditional Lagrangian excludes the magnetic contribution

$$\mathcal{L}_{\text{trad}} = \langle(\nabla \wedge M_e)^2\rangle_0. \tag{12}$$

Here, we will present a derivation of the results of [8] and [9] at the field level, directly from Equation 1, without assuming Equation 4. If we allow duality transformations to vary, letting $\phi = \phi(x)$, then we will pick up an extra term in the derivative

$$\nabla(Fe^{I\phi}) = \nabla F e^{I\phi} - I\nabla\phi F e^{I\phi}, \tag{13}$$

where the negative sign is due to the fact that the pseudoscalar I anticommutes with ∇.

We can construct a covariant derviative D with an added gauge field that transforms in such a way to absorb the contributions of this local symmetry to the derivative. Namely, D must transform so that

$$D'F' = D'(Fe^{I\phi}) = DFe^{I\phi}. \tag{14}$$

To accomplish this, we'll tack on a term to the derivative

$$D = \nabla - IA \tag{15}$$

that transforms as

$$A \mapsto A' = A - \nabla\phi \tag{16}$$

in tandem with duality transformations. The quantity A must be the same grade as $\nabla\phi$, so it is a vector whose curvature $F_A = \nabla \wedge A$ satisfies Maxwell's equations without magnetic sources.[2] Equation 16 is equivalent to

$$D \mapsto D' = D + I\nabla\phi, \tag{17}$$

which ensures Equation 14 holds, yielding the equation

$$\nabla F - IAF = J, . \tag{18}$$

Equation 18 is the same as the result of [8], as well as [3] for the case $J = 0$. Note,

IA itself is a pseudo-vector, and was correctly identified to transform as such by [8] , [2], and [3]. Due to its transformation properties, [2], [3], and [4] proposed that it gave rise to long-range spin interactions mediated by an axial vector boson (they called it an "axial photon"). [8] drew connections to axion electrodynamics. Identifying the gauge field

$$IA = \nabla Ia, \tag{19}$$

where Ia is the pseudoscalar axion field, and requiring $J_b = 0$ and $A \cdot F = 0$ results in the equations of axion electrodynamics:

$$\nabla F = J_e + \nabla Ia \cdot F. \tag{20}$$

Interestingly, in this case, $A = -\nabla a$ is a gradient and implies that the curvature $F_A = \nabla \wedge A = 0$, which is normally considered to be absent of physical content.

The fact that IA is a trivector is expected when compared with the U(1) gauge theory in quantum mechanics. Gauge theory usually involves operators $i\partial_\mu$ instead of just ∂_μ. In this case, the pseudoscalar I takes on the role of the unit imaginary i, since $I^2 = -1$, and plays the role as a duality map (the Hodge map) between vectors and pseudo-vectors.

The replacement

[2] Actually, it could technically include a trivector term and preserve the form of the equation. [1] considered electromagnetic potentials of the form $A = A_e - IA_b$ with an additional gauge freedom. It may be interesting to try gauging duality plus "scaling" transformations of the form $e^{\phi_b + I\phi_e}$. These would then allow for magnetic sources.

$$D \mapsto -ID = -I\nabla - A = \gamma^\mu (I\partial_\mu - A_\mu) \tag{21}$$

places the gauging procedure performed here and in quantum mechanics on comparable footing, and we can see that the gauge field here is no different from the U(1) gauge field of electromagnetism in Dirac theory.

This raises the question of whether the field A can be identified directly with an electromagnetic field. From this perspective, the gauge field would not give rise to a new "axial photon," as suggested by [2], [3], and [4]. The gauge boson would simply be a photon. In which case, we'd left with a pair of coupled electromagnetic fields F and $F_A = \nabla \wedge A$ interacting non-linearly.

References

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