# Spinor gauge fields

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Let $\psi \in G_{1,3}$, and consider equations that can be written in the form

$$\nabla\psi = f(\psi; x). \tag{1}$$

A general local transformation will have the form

$$\psi \mapsto \psi e^{\phi(x)}, \tag{2}$$

where $\phi \in G_{1,3}$. This transformation contributes an extra term to Equation 1 in the following way.

$$\nabla(\psi e^{\phi}) = \nabla\psi e^{\phi} + \dot{\nabla}\psi\dot{\phi}e^{\phi}. \tag{3}$$

To make this equation invariant under these transformations requires the introduction of a covariant derivative with an added gauge field that transforms in such a way to absorb these contributions, the grade of which is determined by $\nabla$ and $\phi$ in the second term on the RHS of Equation 3.

Call the covariant derivative $D$. Its action on $\psi$ will be of the form

$$D\psi = \nabla\psi + g(\psi; x), \tag{4}$$

where $g$ is a function involving the gauge field.

As an example, if $\psi$ is a spinor and $\phi(x) = i\alpha(x)$ for a constant bivector $i$ and a scalar field $\alpha$, as in Dirac theory, then Equation 3 becomes

$$\nabla(\psi e^{i\alpha}) = \nabla\psi e^{i\alpha} + \nabla\alpha\psi e^{i\alpha}i. \tag{5}$$

In this case, $g$ might be given by

$$g(\psi; x) = eA\psi i, \tag{6}$$

where $A$ transforms as

$$A \mapsto A - \nabla\alpha, \tag{7}$$

so that Equation 4 is

$$D\psi = \nabla\psi + eA\psi i. \tag{8}$$

Since $A$ transforms as in Equation 7 and $\nabla\alpha$ is a vector, $A$ must also be a vector (this argument is made by Doran and Lasenby in Section 13.3.3 in GAP).

We could equally well talk about the related field

$$M = eA\psi i\psi^{-1} \tag{9}$$

instead of $A$, that transforms as

$$M \mapsto M - \nabla\alpha\psi i\psi^{-1}, \tag{10}$$

so that the covariant derivative can instead be written as

$$D\psi = (\nabla + M)\psi. \tag{11}$$

$M$ can be rightfully called a gauge field, because it performs the same function as $A$. In either case, the grade of the gauge field $A$, or the related gauge field $M$, is odd. If it were even, then it would be a spinor field (i.e. a fermionic field).[1]

Consider a less specific example, where $\psi$ is simply taken to be invertible. Then Equation 3 can be generally written in the form

$$\nabla(\psi e^{\phi}) = \nabla\psi e^{\phi} + M_0\psi e^{\phi}, \tag{12}$$

where

$$M_0 = \dot{\nabla}(\psi\dot{\phi}\psi^{-1}). \tag{13}$$

Then the gauge field $M$ (analogous to $M$ in Equation 9) must transform as

$$M \mapsto M - M_0 \tag{14}$$

and have the same grade as $M_0$.

The key point: *if $\phi$ has even grade, then $M$ (the gauge field) has odd grade, and if $\phi$ has odd grade, then $M$ has even grade.* Usually, gauge transformations $e^{\phi}$ are even (i.e. spinors), and hence the corresponding gauge fields $M$ are odd valued. On the other hand, if $\phi$ is odd, then $e^{\phi}$ is odd, and the corresponding gauge field $M$ is even valued (i.e. a spinor). In this case, the gauge field could be used to describe fermions. The most immediate question that arises is, what do odd transformations mean?[2]

[1] If bosonic gauge fields are always odd valued, I'm curious to know if one could exploit the isomorphism between even and odd fields to identify a relationship between bosonic and fermionic gauge fields.

[2] Could one use the isomorphism between even and odd fields to make sense of odd valued transformations?