



Efficient Succinct Data Structures on Directed Acyclic Graphs

Tesi Triennale in Matematica

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The Subset Membership Problem

Querying Collections Efficiently and Compactly

Consider a large universe of items $U = \{1, \dots, n\}$, and a specific subset $S \subseteq U$ of m items.

Core Task & Desired Properties

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- Store S using minimal space (**Compact Representation**)



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Definition (Shannon Entropy $H(X)$)

The average uncertainty, or information content, per symbol of source X :

$$H(X) = E_{P_X}[-\log_2 P_X(x)] = - \sum_{x \in \mathcal{X}} P_X(x) \log_2 P_X(x) \quad [\text{bits/symbol}]$$



Information-Theoretic Limits for Subsets

From General Entropy to Specific Subsets

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Definition (Zero-Order Empirical Entropy $\mathcal{H}_0(S)$)

Information content of sequence S based on its symbol counts (n_s):

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Uses observed frequencies $\frac{n_s}{n}$ instead of unknown $P_X(x)$.



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For our subset S of m items from n :

- There are $\binom{n}{m}$ such distinct subsets.
- To uniquely identify one, we need at least $\lceil \log_2 \binom{n}{m} \rceil$ bits. This is the **information content** of specifying the subset.



Bitvectors: Querying the Implicit Representation

From Compact Storage to Element Access

We can represent our subset S as a **bitvector** $B[1..n]$ ($B[i] = 1 \iff i \in S$). We are encoding the choice of m positions for the '1's, allowing us to store B using $\approx \lceil \log_2 \binom{n}{m} \rceil$ bits (i.e., $\approx n\mathcal{H}_0(B)$ bits). This means B is not stored as an explicit array of n bits.

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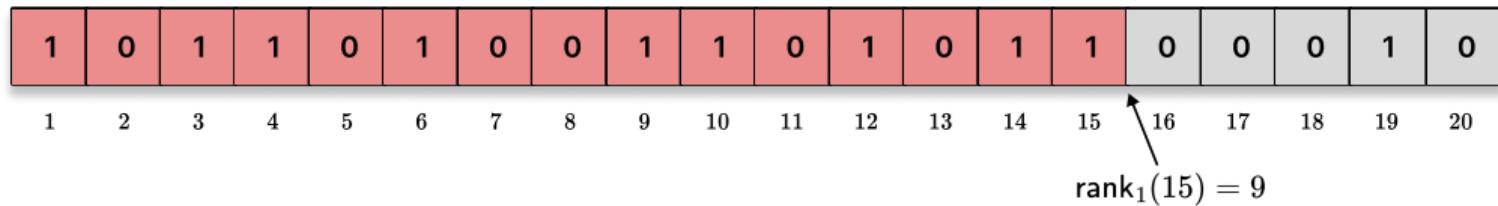
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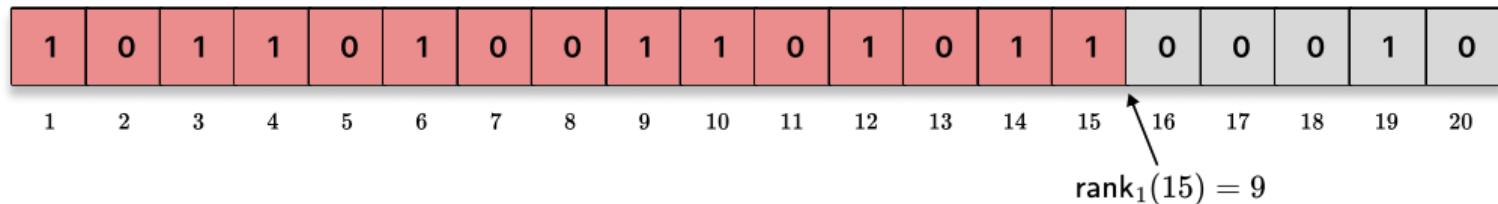
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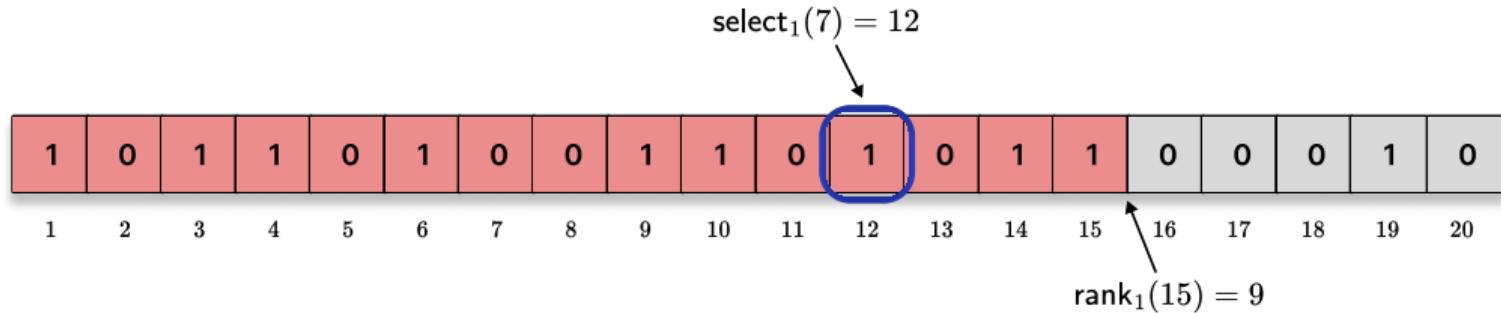
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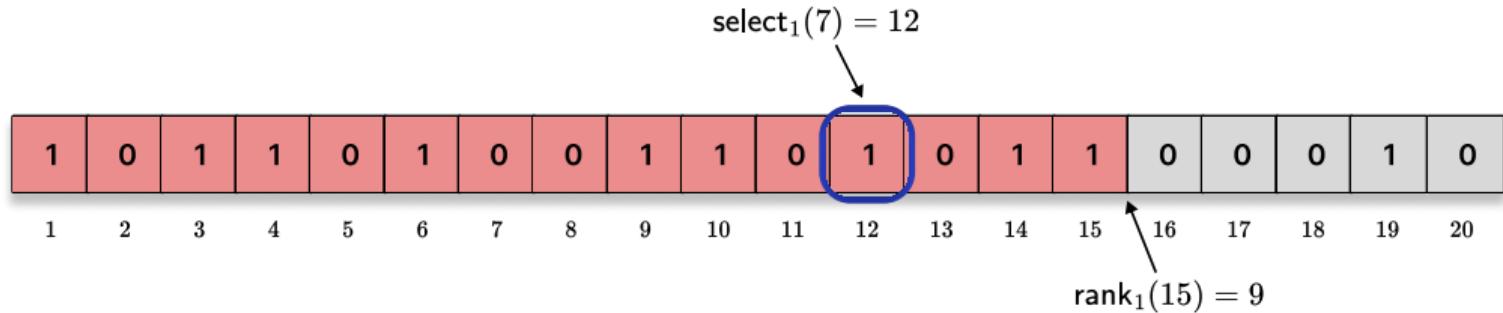
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Access Queries

$$B[i] = 1 \iff \text{rank}_1(B, i) > \text{rank}_1(B, i - 1)$$

- **rank_b(B, i)**: How many bits b are in the prefix $B[1..i]$?
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RRR Structure: The Bitvector Solution

$n\mathcal{H}_0(B)$ Space & $O(1)$ Queries

- **Goal:** Support rank and select in $O(1)$ time.
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Theorem (RRR Structure)

A bitvector $B[1..n]$ with m set bits can be represented using

$$B(n, m) + o(n) + O(\log \log n) \text{ bits},$$

where $B(n, m) = \lceil \log_2 \binom{n}{m} \rceil \approx n \cdot \mathcal{H}_0(B)$, while supporting rank and select in $O(1)$ time.



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A Cornerstone Result

RRR shows that **optimal space and efficient queries** are possible for subsets.



Why Succinct Data Structures?

The General Challenge & Goal

RRR is a specific solution to a general problem:

Massive Data & Auxiliary Structures Overhead

Modern datasets (Science, Web, AI...) are enormous. Complex analysis demands data in RAM, but auxiliary structures (indexes, trees) needed for queries often **occupy more space than the data itself.** \implies Fitting everything in RAM is a major bottleneck.



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- *Compression:* Minimal space, but slow/no direct queries.
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The Succinct Goal: Best of Both Worlds

Can we achieve **both**?

- Space **near information-theoretic minimum**.
- Efficient queries **directly** on compact data.



Beyond Bitvectors: General Alphabets

Wavelet Trees

Consider a sequence $S[1..n]$ over a **general alphabet** $\Sigma = \{1, \dots, \sigma\}$. How can we answer *succinctly* queries like $\text{rank}_c(i)$, $\text{select}_c(j)$, and $\text{access}(i)$?



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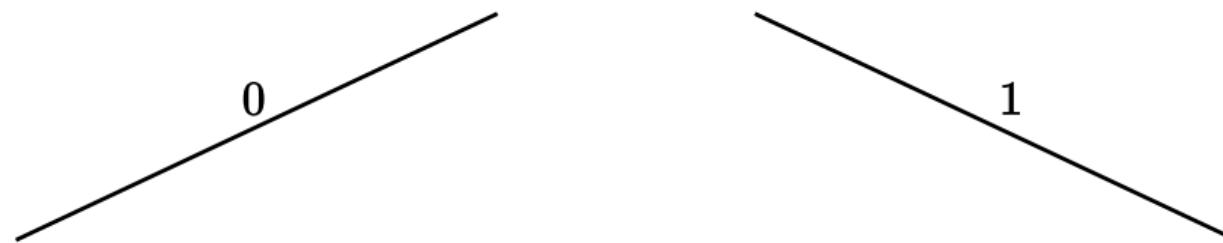


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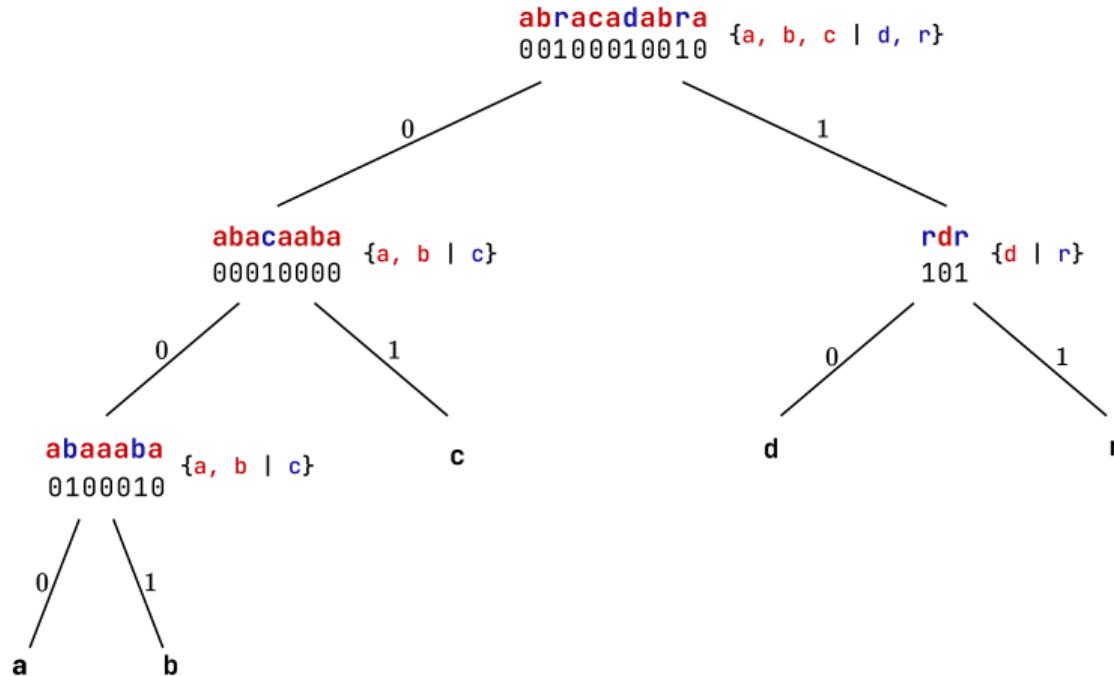
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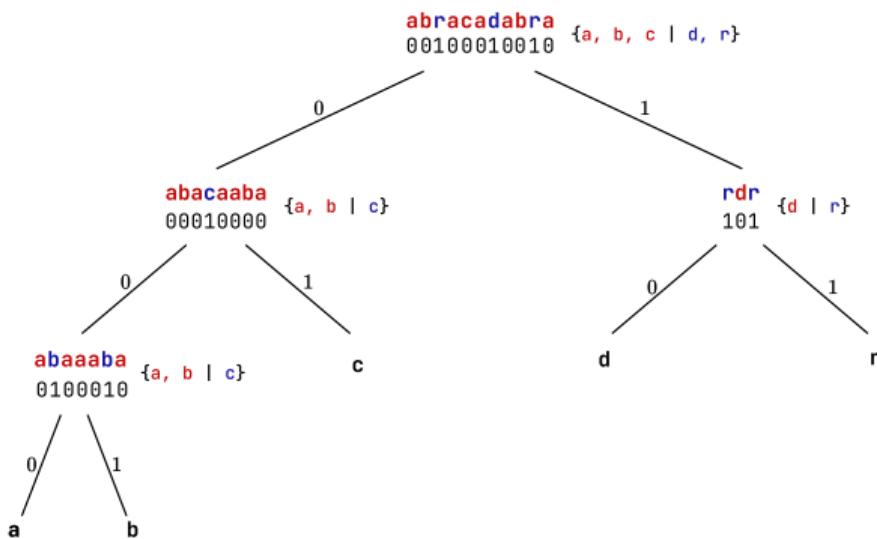




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\mathcal{H}_0 -Compressed Wavelet Tree

Using RRR for bitvectors:

- **Space:** $n\mathcal{H}_0(S) + o(n \log \sigma)$ bits.
- **Query Time:** $O(\log \sigma)$ for access, rank_c , select_c .

Adapts space to the sequence's zero-order entropy.



Representing Sequence Variation: Degenerate Strings

Definitions and Core Operations

Definition and Rank & Select Adaptation

A **degenerate string** is a sequence $X = X_1 X_2 \dots X_n$, where each X_i is a subset of the alphabet Σ with cardinality σ . We can define the following operations:

- $\text{subset-rank}_X(i, c)$: Counts sets X_k ($k \leq i$) where $c \in X_k$.
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To compute $\text{subset-rank}_X(i, c)$: Let p be the end position in S for prefix $X_1..X_i$, found using $\text{select}_1(R, i + 1)$. The result is $\text{rank}_c(S, p)$.



From Degenerate Strings to Weighted DAGs

A New Perspective

Recall our degenerate string X :

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- **Edges E_c :** Connect s to $k = 1$. Connect all $v_{k,a}$ to all $v_{k+1,b}$. (Represents sequence adjacency).



Path Weight Aggregation: The \mathcal{O} -Set

Capturing All Path Weights

Given a path $P = (v_0 = s, \dots, v_k = v)$ we define $W(P) = \sum_{j=1}^k w(v_j)$

Goal

Characterize the set of **all possible distinct** cumulative path weights arriving at each node.



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\mathcal{O} -Set Definition (Recursive)

- **Base Case (Source):** $\mathcal{O}_s = \{0\}$
- **Recursive Step ($v \neq s$):**

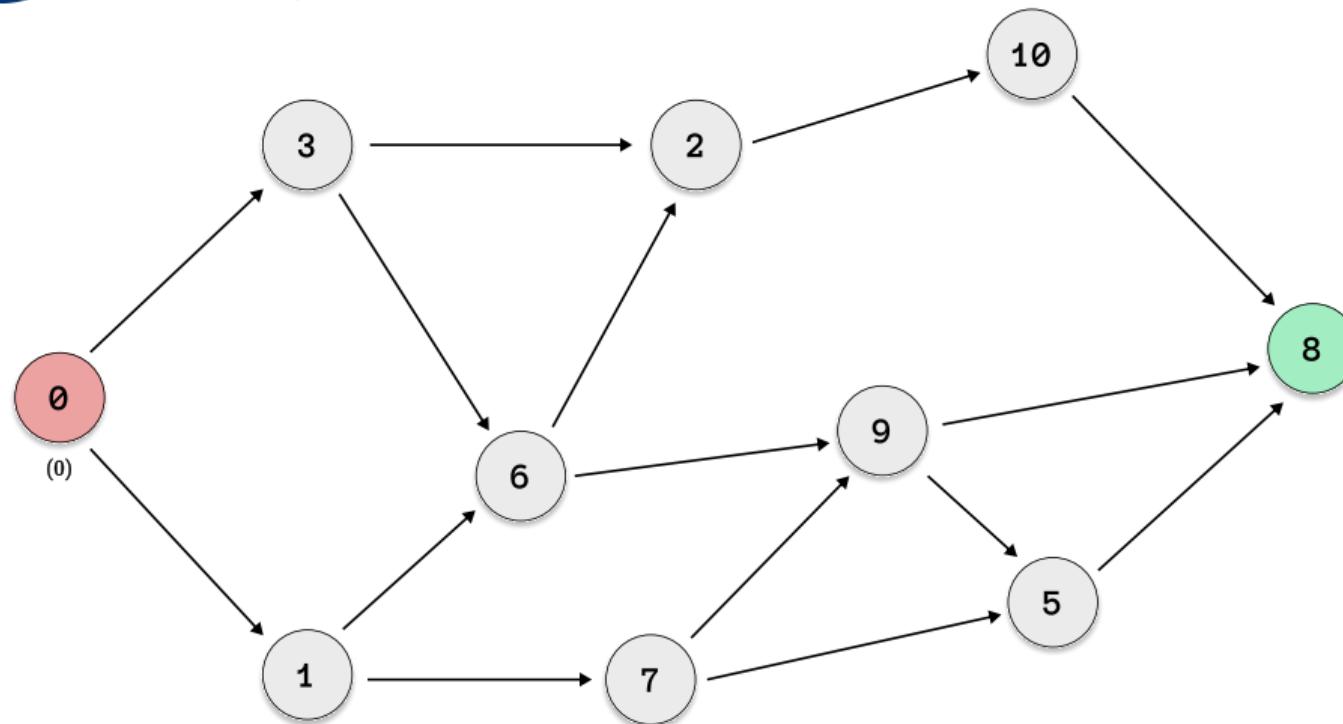
$$\mathcal{O}_v = \bigcup_{u \in \text{Pred}(v)} \{\gamma + w(v) \mid \gamma \in \mathcal{O}_u\} = \{W(P) \mid P \in \text{Path}(s, v)\}$$

Keep only **distinct** values. Store as a sorted sequence.



O-Set Construction Example

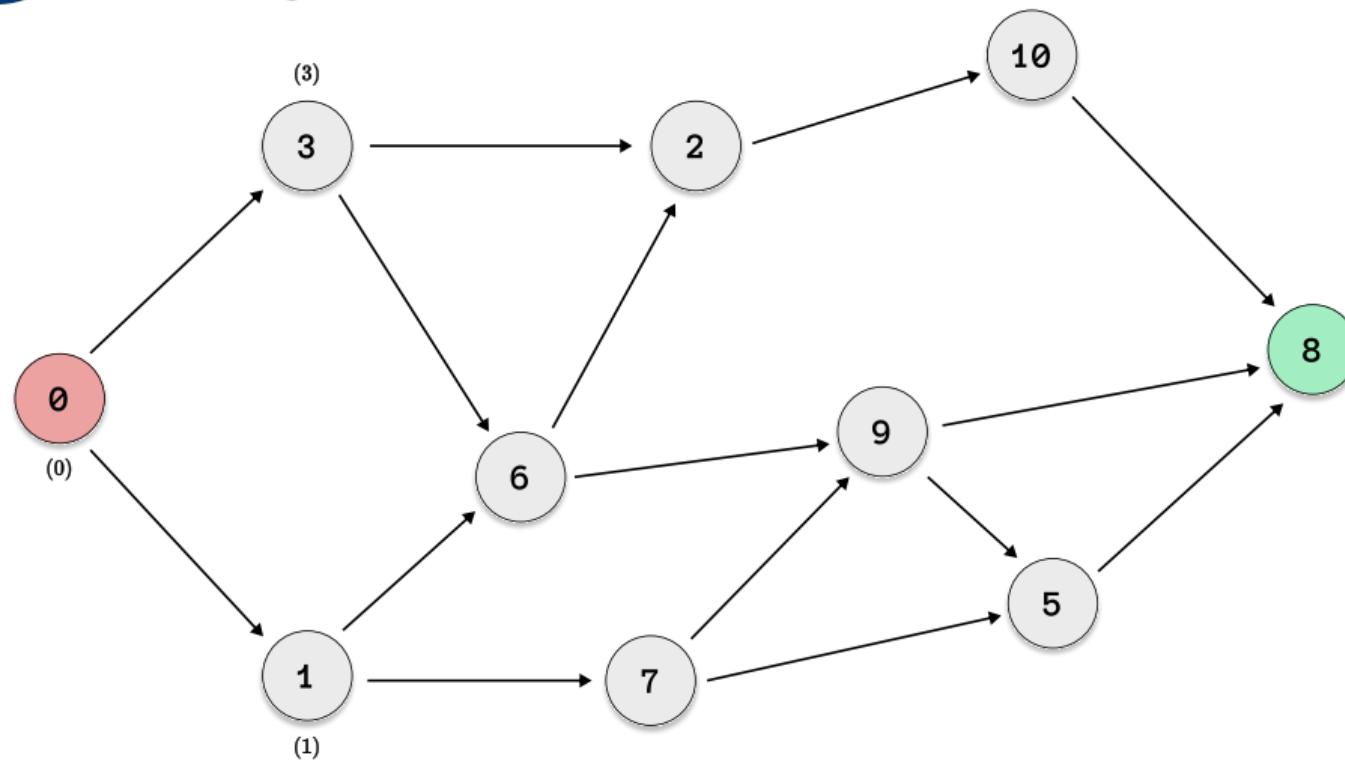
Visualizing the Recursive Definition





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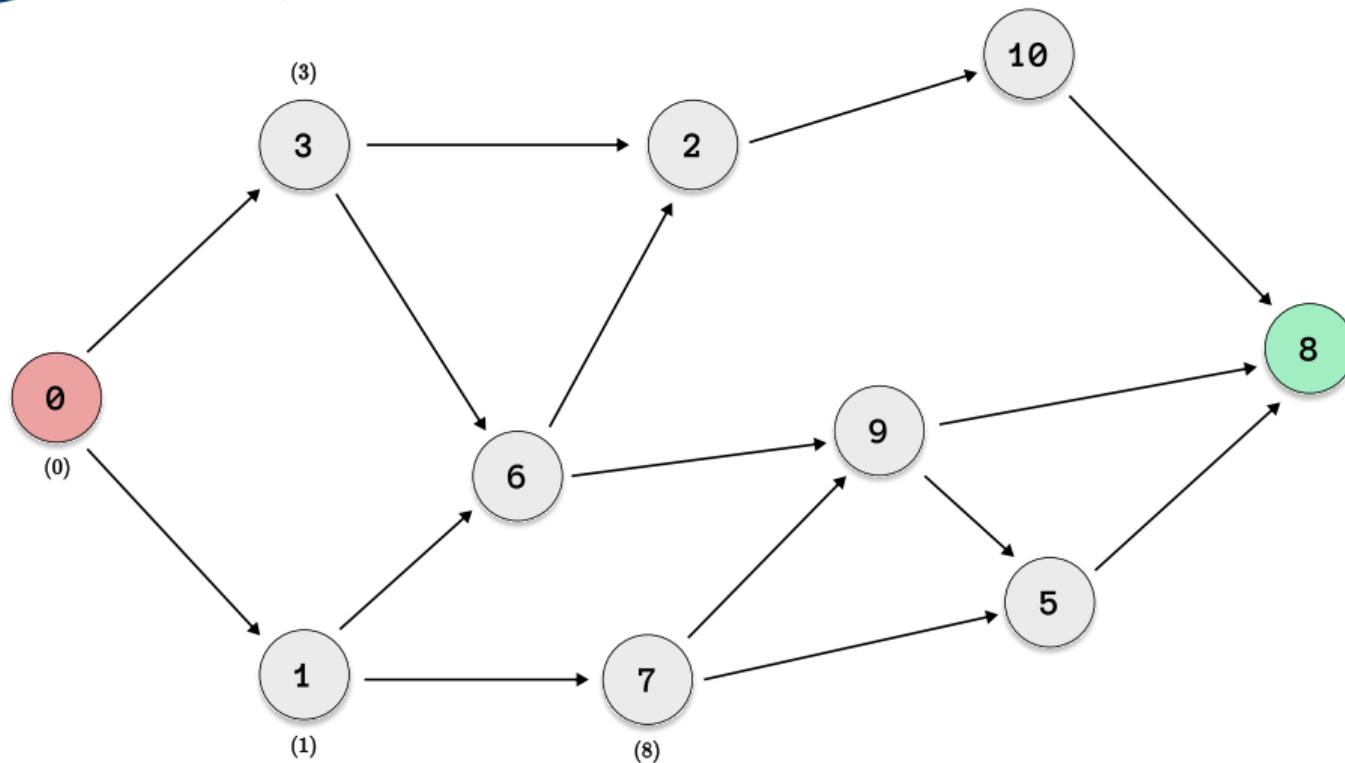
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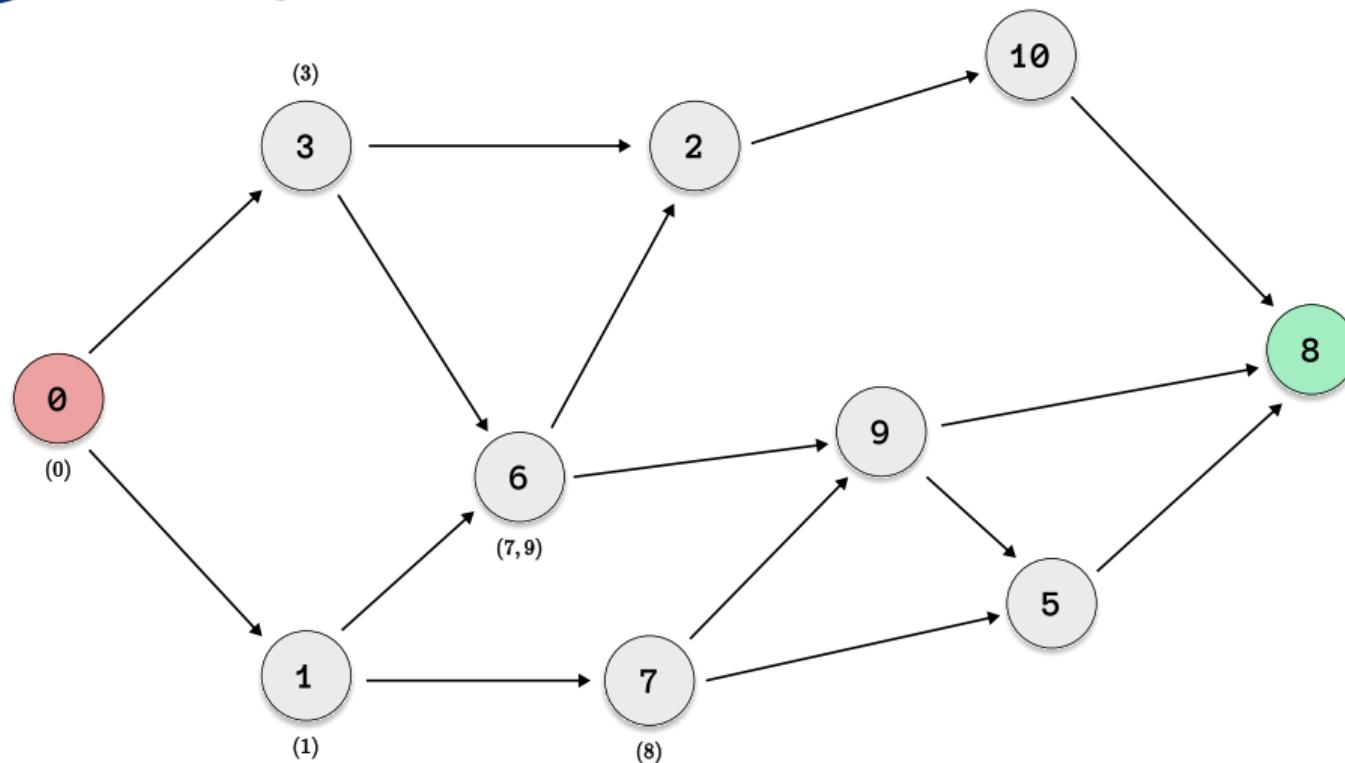
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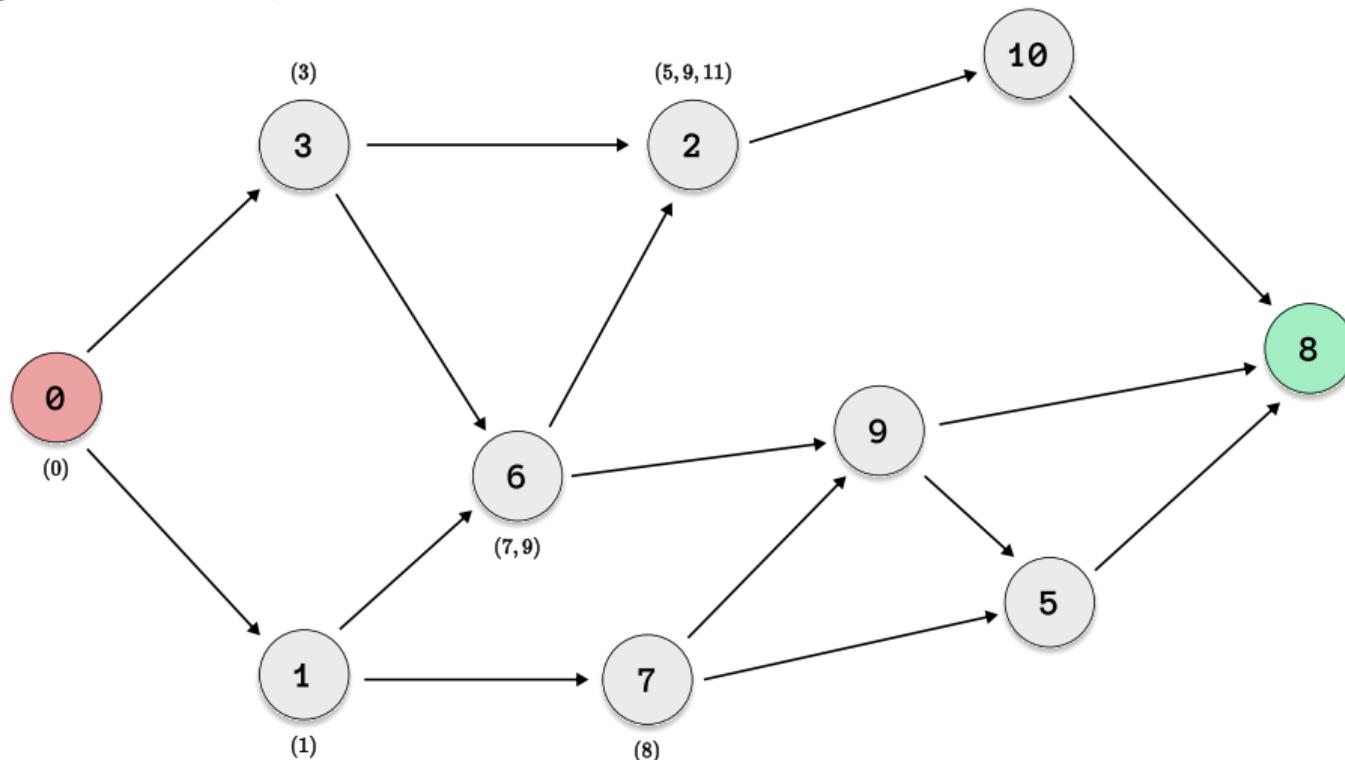
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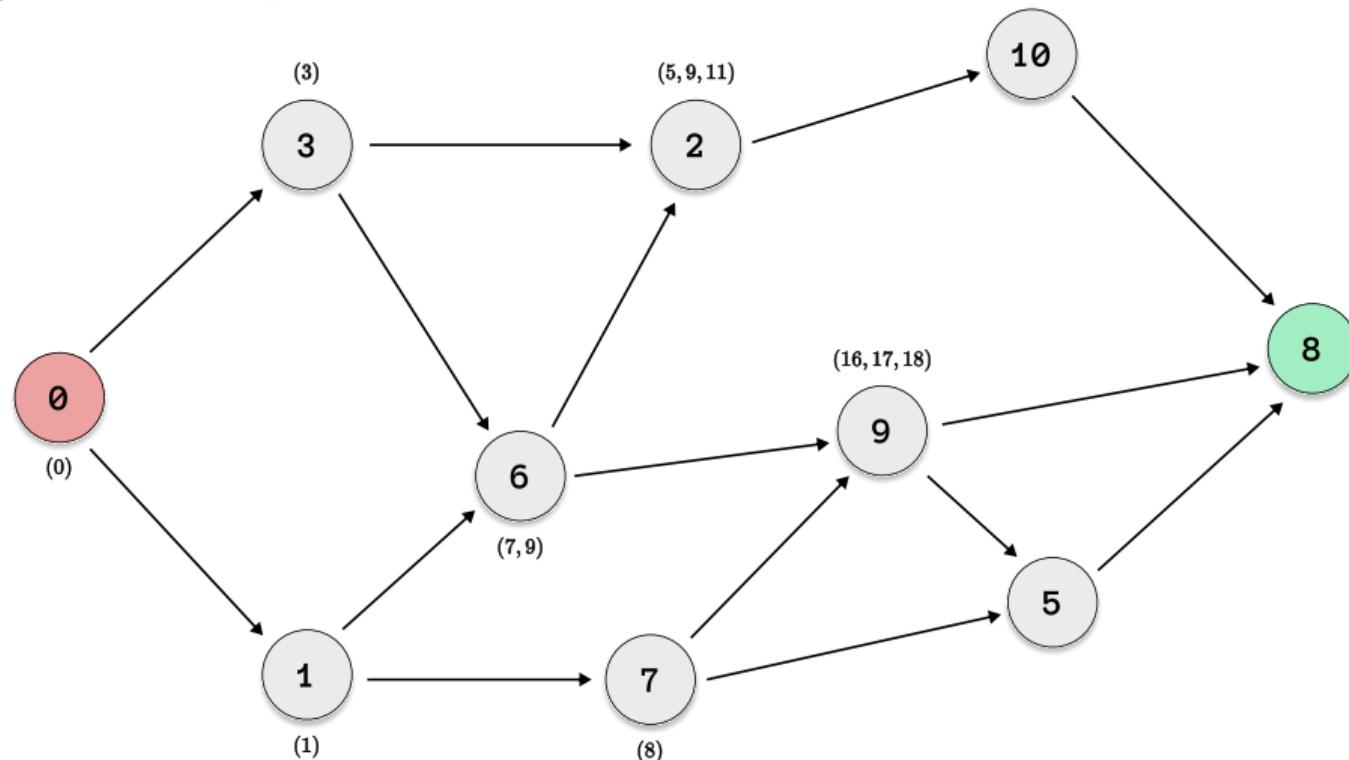
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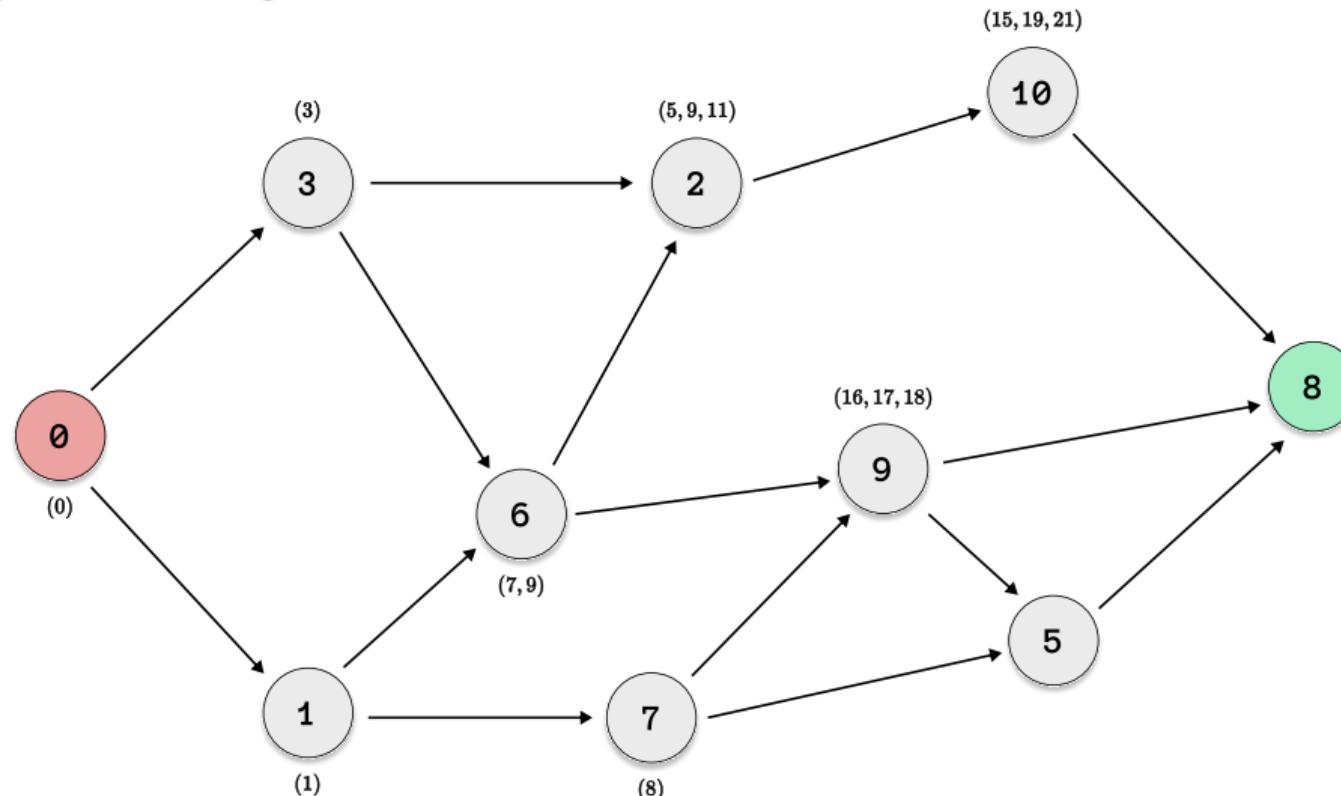
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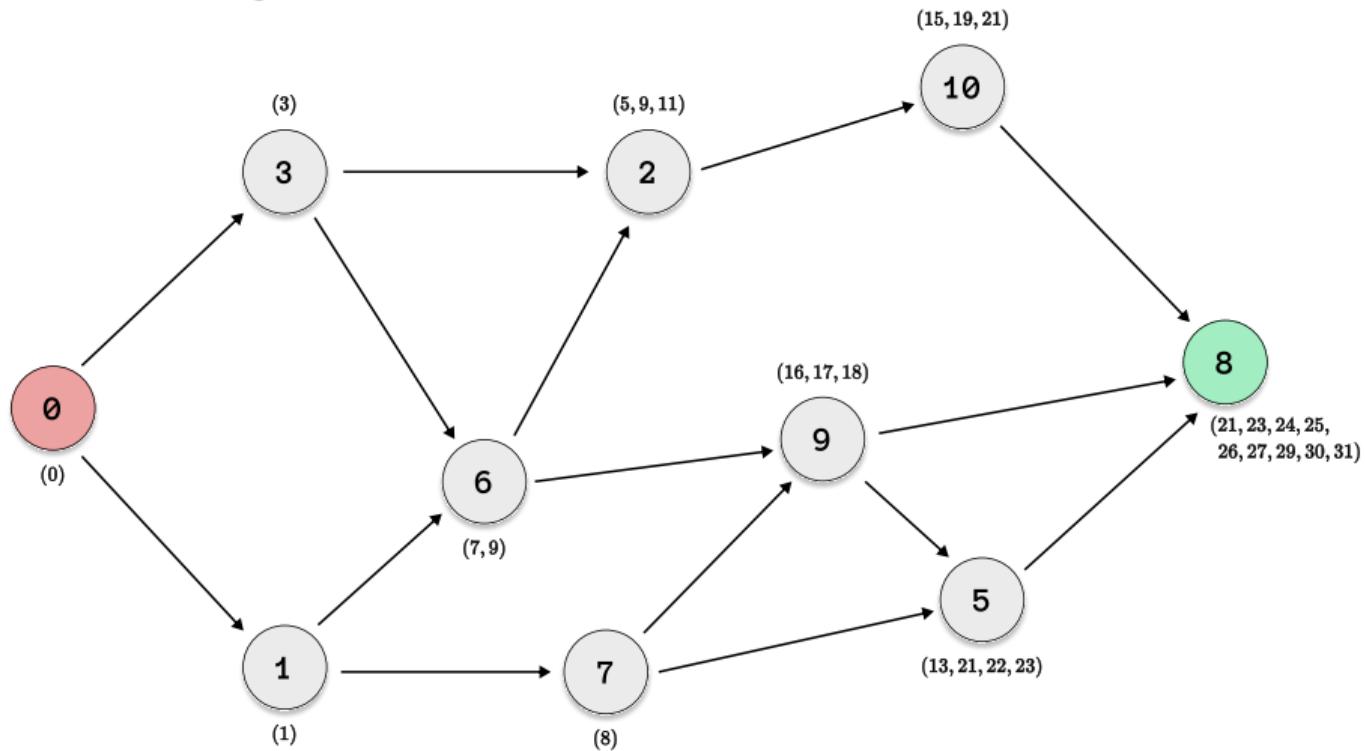
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The Rank Query on Weighted DAGs

What Values are "Active" at Node N?

Rank Query on a Node N: $\text{rank}_{\mathcal{G}}(N)$

1. Returns a representation of a set of integers derived from the \mathcal{O} -set \mathcal{O}_N .

$$S_N = \bigcup_{x \in \mathcal{O}_N} \{z \in \mathbb{N}_0 \mid \max(0, x - w(N) + 1) \leq z \leq x\}.$$



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2. These intervals are then maximally merged. The query $\text{rank}_{\mathcal{G}}(N)$ returns a **minimal collection of disjoint closed integer intervals**

$$\mathcal{R}_N = \{[l_1, r_1], [l_2, r_2], \dots, [l_p, r_p]\}$$

such that their union exactly covers S_N .

\mathcal{R}_N captures the range of possible cumulative sums during the *activity* at node N



The Challenge: Storing Path Information

\mathcal{O} -Sets Can Be Huge!

- **Problem:** The size $|\mathcal{O}_v|$ can grow very large!
- **Question:** Can we represent the necessary information more compactly?



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- **Problem:** The size $|\mathcal{O}_v|$ can grow very large!
- **Question:** Can we represent the necessary information more compactly?

Core Idea: Partitioning + Indirection

Partition vertices V into two types:

1. Explicit Vertices (V_E)

Store \mathcal{O}_v directly

(Simple, but potentially large)



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Partition vertices V into two types:

1. Explicit Vertices (V_E)

Store \mathcal{O}_v directly
(Simple, but potentially large)

2. Implicit Vertices (V_I)

Do not store \mathcal{O}_v explicitly
(Reconstruct on-the-fly)

Reconstruction for $v \in V_I$ using:

- Designated Successor $\sigma(v)$
- Offset Sequence \mathcal{J}_v (at v)



Implicit Reconstruction: Successor & Offset

How V_I Nodes Refer to Others

1. Designated Successor $\sigma(v)$ (for $v \in V_I$)

Which successor should v point to? **Heuristic:** Choose $u = \sigma(v)$ that minimizes $|\mathcal{O}_u|$.

$$\sigma(v) \in \operatorname{argmin}_{u \in \text{Succ}(v)} \{|\mathcal{O}_u|\}.$$

2. Offset Sequence \mathcal{J}_v (for $v \in V_I$)

How to get \mathcal{O}_v from $\mathcal{O}_{\sigma(v)}$? Let $u = \sigma(v)$.

- **Relationship:** Each element $x_k \in \mathcal{O}_v$ comes from some $y_{j_k} \in \mathcal{O}_u$ via $x_k = y_{j_k} - w(u)$.
- **Offset Sequence \mathcal{J}_v :** Stores the index j_k corresponding to each x_k .

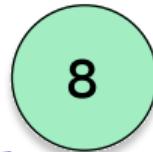
$$\mathcal{J}_v = (j_0, j_1, \dots, j_{m-1}), \quad \text{where } m = |\mathcal{O}_v|$$



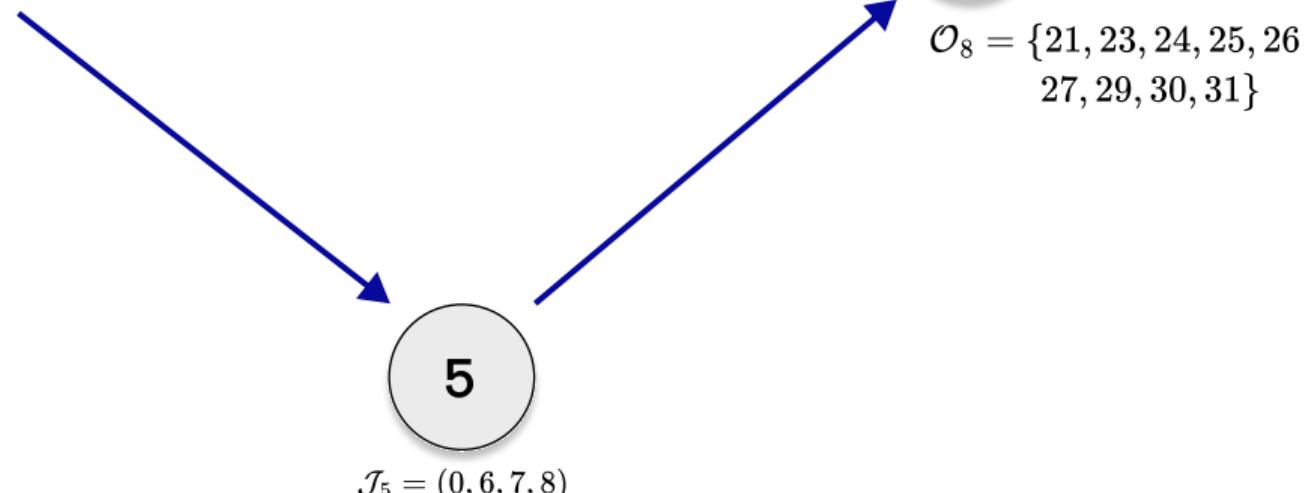
Example: Computing $\mathcal{O}_9[1]$

Following the Successor Path: $9 \rightarrow 5 \rightarrow 8$

$$\mathcal{J}_9 = (1, 2, 3)$$



$$\mathcal{O}_8 = \{21, 23, 24, 25, 26, 27, 29, 30, 31\}$$





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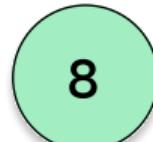
$$k = 0$$
$$sum = 0$$

$$\mathcal{J}_9[1] = 2$$
$$+\omega(\sigma(9)) = 5$$

$$k = 2$$
$$sum = 5$$



$$\mathcal{J}_5 = (0, 6, 7, 8)$$



$$\mathcal{O}_8 = \{21, 23, 24, 25, 26, 27, 29, 30, 31\}$$



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$$\mathcal{J}_5 = (0, 6, 7, 8)$$



$$\begin{aligned} k &= 7 \\ sum &= 13 \end{aligned}$$

$$\begin{aligned} \mathcal{J}_5[2] &= 1 \\ +\omega(\sigma(5)) &= 8 \end{aligned}$$



$$\begin{aligned} \mathcal{O}_8 &= \{21, 23, 24, 25, 26 \\ &\quad 27, 29, 30, 31\} \end{aligned}$$

Retrive $\mathcal{O}_8[7] = 30$

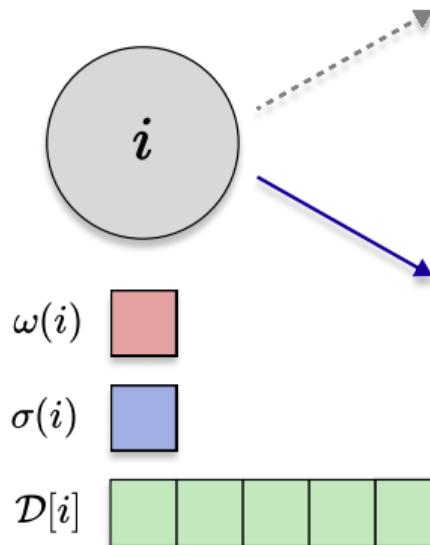
Result = $\mathcal{O}_9[1] = 30 - 13 = 17$



Succinct Data Structure: Components

Arrays Indexed by Vertex ID

Each node stores 3 components

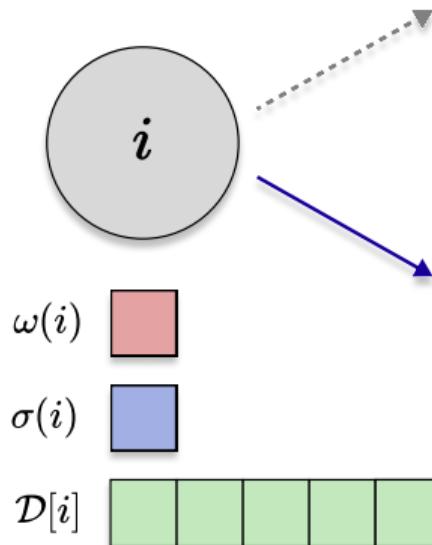




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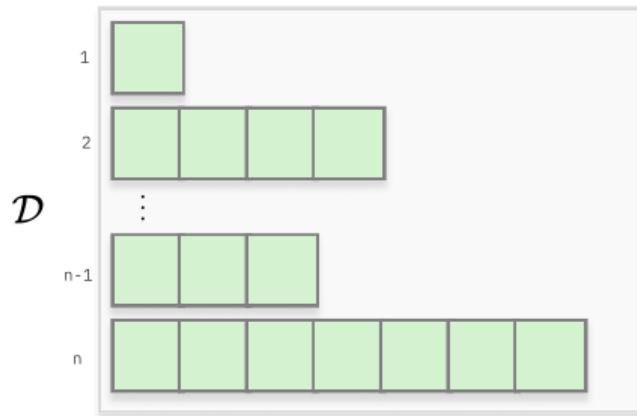
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Succinct DAG as a Struct of Arrays

ω	0	$\omega(1)$	$\omega(2)$	$\omega(n)$
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σ	0	$\sigma(1)$	$\sigma(2)$	$\sigma(n)$
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Compression Strategies

Reducing Memory Footprint

Component	Description
\mathcal{W} (Node weights)	Array of positive integers.
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Compression Options

- Variable-Length Integer Coding —→ we published a Rust library^a for this!
- Wavelet Trees (*for nodes with small weight range*)

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- Wavelet Trees (*for nodes with small weight range*)
- Elias-Fano Encoding (*for monotonic sequences*)
- Run-Length Encoding (RLE) (*for clustered monotonic sequences*)

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Space Efficiency: Baseline Comparison

How Much Information is in the Graph?

To evaluate our structure's space, we need a baseline.

0th-Order Graph Entropy $H_0(G)$

A theoretical lower bound for storing the *entire* weighted DAG (V, E, w) losslessly.

$$H_0(G) = \underbrace{H_W(G)}_{\text{Cost for Weights}} + \underbrace{H_E(G)}_{\text{Cost for Topology (Edges)}}$$



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Any method saving the *full* graph structure needs at least $H_0(G)$ bits!



Space Comparison: Succinct Structure vs. Baselines

Bitcoin DAG Example ($n \approx 22k, m \approx 50k$)

Method	Estimated Bits
Theoretical Lower Bound	1,525,730
Weights $H_W(G)$	60,824
Topology $H_E(G)$	1,464,906

Precomputed Rank Queries:

- Explicit Binary Storage
- Elias-Fano Compressed

Our Succinct DAG

- Weights \mathcal{W}
- Successors Σ
- Assoc. Data \mathcal{D} (RLE)



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Achieving Sub-Entropy Space: How?

Our structure is **lossy** regarding the full graph topology:

- It **does not store** the complete edge set.
- It only stores the chosen successor $\sigma(v)$ for each implicit node (in Σ).

However, it is **lossless** for computing the specific **Rank Query**.



Future Direction: Bounded Query Time

Guaranteeing Predictable Performance

Performance Consideration

Query time for implicit node v depends on the length of the successor path

$$v \rightarrow \sigma(v) \rightarrow \sigma(\sigma(v)) \rightarrow \dots \rightarrow e \in V_E$$

Problem: Can be large/variable in deep DAGs \implies slow worst-case query time.



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Solution, Challenges & Trade-offs

- **Solution:** Ensure every implicit node can reach an explicit node within k steps.
- **Challenges:** Finding the smallest possible V'_E that satisfies this condition is NP-hard (*minimum distance- k dominating set*).
- **Trade-off:** More explicit nodes \implies faster queries, but larger space.



Efficient Succinct Data Structures on Directed Acyclic Graphs

Thank you for listening!



Worst-Case \mathcal{O} -Set Size: Is Exponential Growth Possible?

Understanding the \mathcal{O} -set Size

Exponential Growth Can Occur

The cardinality of an \mathcal{O} -set, $|\mathcal{O}_v|$, is not generally bounded by a polynomial in the number of vertices $|V|$. It can grow exponentially.

Underlying Reason: Path Count

The number of distinct paths from a source s to a vertex v , denoted $|Path(s, v)|$, can itself be exponential in certain DAG structures. Since $|\mathcal{O}_v| \leq |Path(s, v)|$, the potential for exponential size exists.



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Key Condition for Exponential Growth

The exponential potential is realized if the vertex weights $w(v)$ are assigned such that distinct paths $P_1 \neq P_2$ almost always lead to distinct cumulative weights $W(P_1) \neq W(P_2)$.



Achieving Exponential \mathcal{O} -Set Size

A Strategy for Path Weight Uniqueness

Start with a DAG structure that naturally admits an exponential number of paths between two nodes. An example is a layered graph with multiple choices at each layer transition.

Strategic Weight Assignment

Assign vertex weights $w(v)$ carefully to ensure path weight uniqueness.

$$w(v) = 2^k \quad (\text{using a unique exponent } k \text{ for each node})$$



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Mechanism: Unique Binary Representation

With power-of-2 weights, the cumulative path weight $W(P) = \sum_{v \in P \setminus \{s\}} w(v)$ becomes a sum of distinct powers of 2. Due to the uniqueness of binary representation, different sets of nodes (i.e., different paths) produce different sums. Therefore, $|Path(s, v)|$ distinct paths yield $|\mathcal{O}_v| = |Path(s, v)|$ distinct weights.