Efficient Succinct Data Structures on Directed Acyclic Graphs

Tesi Triennale in Matematica

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The Subset Membership Problem

Querying Collections Efficiently and Compactly

Consider a large universe of items $U = \{1, \ldots, n\}$, and a specific subset $S \subseteq U$ of m items.

Core Task & Desired Properties

- Quickly answer: "Is item x in S?" (**Membership Query**)
- Store S using minimal space (**Compact Representation**)

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To understand ”minimal space”, we turn to Information Theory. *What is the least number of bits needed to uniquely identify S?*

Definition (Shannon Entropy $H(X)$)

The average uncertainty, or information content, per symbol of source X:

$$H(X) = E_{P_X}[-\log_2 P_X(x)] = -\sum_{x \in \mathcal{X}} P_X(x) \log_2 P_X(x) \quad [\text{bits/symbol}]$$

Information-Theoretic Limits for Subsets

From General Entropy to Specific Subsets

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Definition (Zero-Order Empirical Entropy $\mathcal{H}_0(S)$)

Information content of sequence S based on its symbol counts (n_s):

$$\mathcal{H}_0(S) = \sum_{s \in \Sigma} \frac{n_s}{n} \log_2 \frac{n}{n_s} \quad \text{[bits/symbol]}$$

Uses observed frequencies $\frac{n_s}{n}$ instead of unknown $P_X(x)$.

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For our subset S of m items from n:

- There are $\binom{n}{m}$ such distinct subsets.
- To uniquely identify one, we need at least $\lceil \log_2 \binom{n}{m} \rceil$ bits. This is the information content of specifying the subset.

Bitvectors: Querying the Implicit Representation

From Compact Storage to Element Access

We can represent our subset S as a **bitvector** $B[1..n]$ ($B[i] = 1 \iff i \in S$). We are **encoding the choice of m positions for the '1's**, allowing us to store B using $\approx \lceil \log_2 \binom{n}{m} \rceil$ bits. *This means B is not stored as an explicit array of n bits.*

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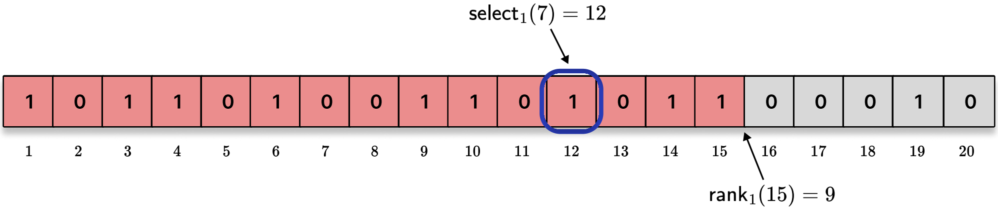

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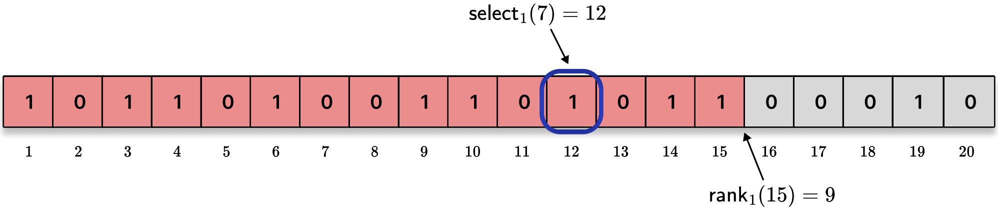

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Access Queries

$B[i] = 1 \iff \text{rank}_1(B, i) > \text{rank}_1(B, i-1)$

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RRR Structure: The Bitvector Solution

$n\mathcal{H}_0(B)$ Space & $O(1)$ Queries

Succinct Data Structure for Bitvectors

- **Goal:** Support rank and select in $O(1)$ time.
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A bitvector $B[1..n]$ with m set bits can be represented using

$$B(n, m) + o(n) + O(\log \log n) \quad \textit{bits},$$

where $B(n, m) = \lceil \log_2 \binom{n}{m} \rceil$, while supporting rank and select queries in $O(1)$ time.

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A Cornerstone Result

RRR shows that **optimal space** *and* **efficient queries** are possible for subsets.

Why Succinct Data Structures?

The General Challenge & Goal

RRR is a specific solution to a general problem:

Massive Data & Auxiliary Structures Overhead

Modern datasets (Science, Web, AI...) are enormous. Complex analysis demands data in RAM, but auxiliary structures (indexes, trees) needed for queries often **occupy more space than the data itself**. $\Longrightarrow$ Fitting everything in RAM is a major bottleneck.

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- *Compression:* Minimal space, but slow/no direct queries.
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The Succinct Goal: Best of Both Worlds

Can we achieve **both**?

- Space near information-theoretic minimum.
- Efficient queries directly on compact data.

Beyond Bitvectors: General Alphabets

Wavelet Trees

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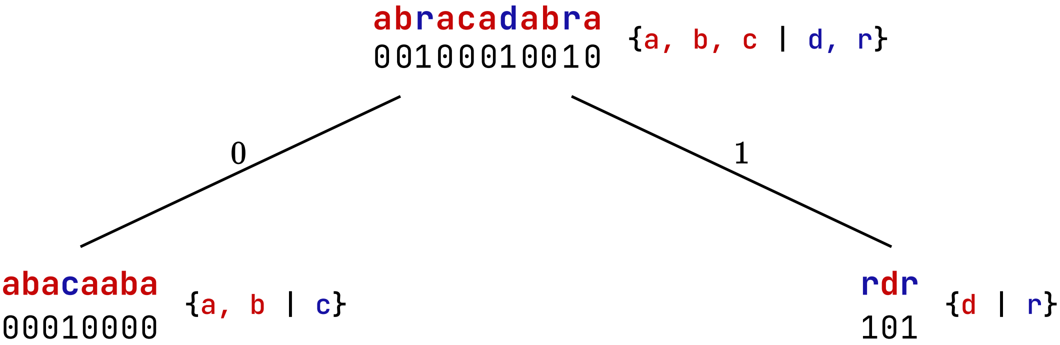

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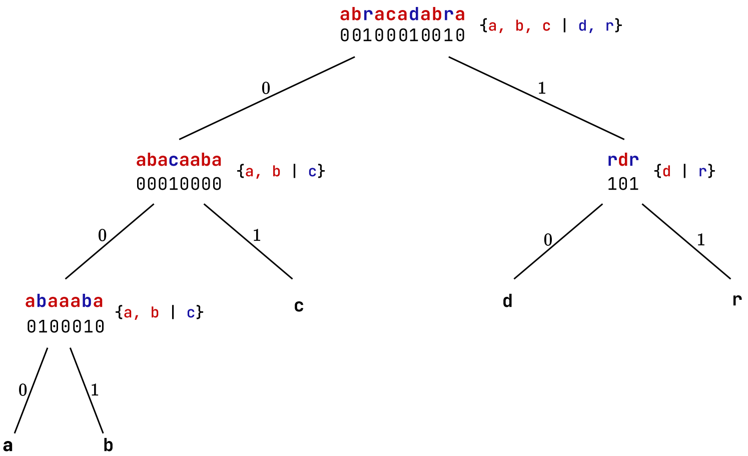

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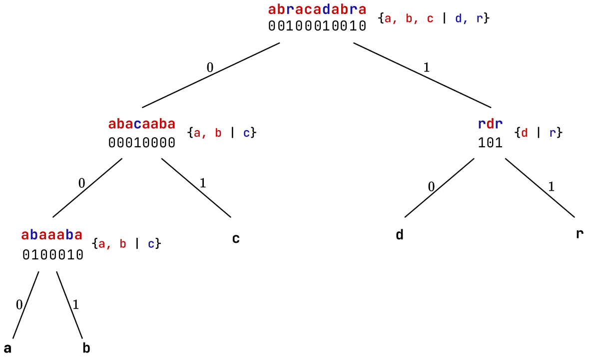

$\mathcal{H}_0$-Compressed Wavelet Tree

Using RRR for bitvectors:

- **Space**: $n\mathcal{H}_0(S) + o(n \log \sigma)$ bits.
- **Query Time**: $O(\log \sigma)$ for access, rank$_c$, select$_c$.

Adapts space to the sequence's zero-order entropy.

Representing Sequence Variation: Degenerate Strings

Definitions and Core Operations

Definition and Rank & Select Adaptation

A **degenerate string** is a sequence $X = X_1X_2 \ldots X_n$, where each X_i is a *subset* of the alphabet Σ with cardinality σ. We can define the following operations:

- subset-rank$_X(i, c)$: Counts sets X_k $(k \leq i)$ where $c \in X_k$.
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To compute subset-rank$_X(i, c)$: Let p be the end position in S for prefix $X_1..X_i$, found using select$_1(R, i+1)$. The result is rank$_c(S, p)$.

From Degenerate Strings to Weighted DAGs

A New Perspective

Recall our degenerate string X:

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- **Edges** E_c: Connect s to $k = 1$. Connect all $v_{k,a}$ to all $v_{k+1,b}$. (Represents sequence adjacency).

Path Weight Aggregation: The $\mathcal{O}$-Set

Capturing All Path Weights

Given a path $P = (v_0 = s, \ldots, v_k = v)$ we define $W(P) = \sum_{j=1}^{k} w(v_j)$

Goal

Characterize the set of all possible distinct cumulative path weights arriving at each node.

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$\mathcal{O}$-Set Definition (Recursive)

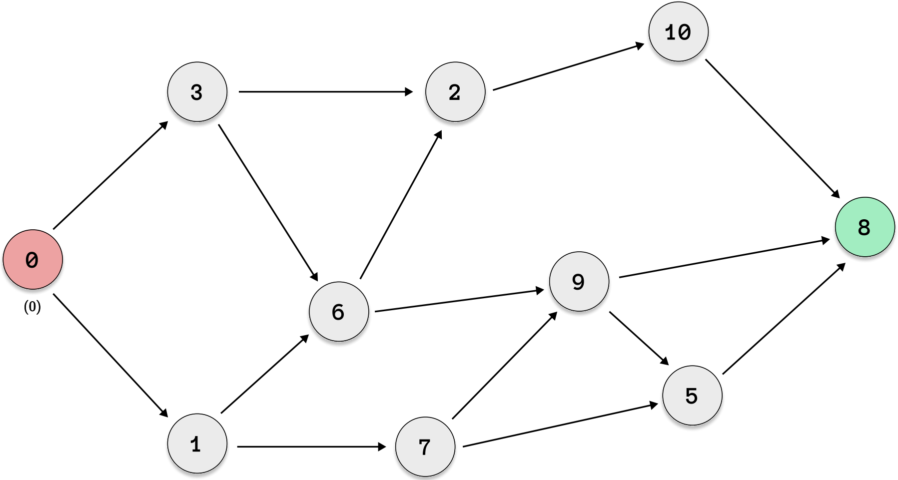
- **Base Case (Source):** $\mathcal{O}_s = \{0\}$
- **Recursive Step ($v \neq s$):**

$$\mathcal{O}_v = \bigcup_{u \in Pred(v)} \{\gamma + w(v) \mid \gamma \in \mathcal{O}_u\} = \{W(P) \mid P \in Path(s, v)\}$$

Keep only distinct values. Store as a sorted sequence.

$\mathcal{O}$-Set Construction Example

Visualizing the Recursive Definition



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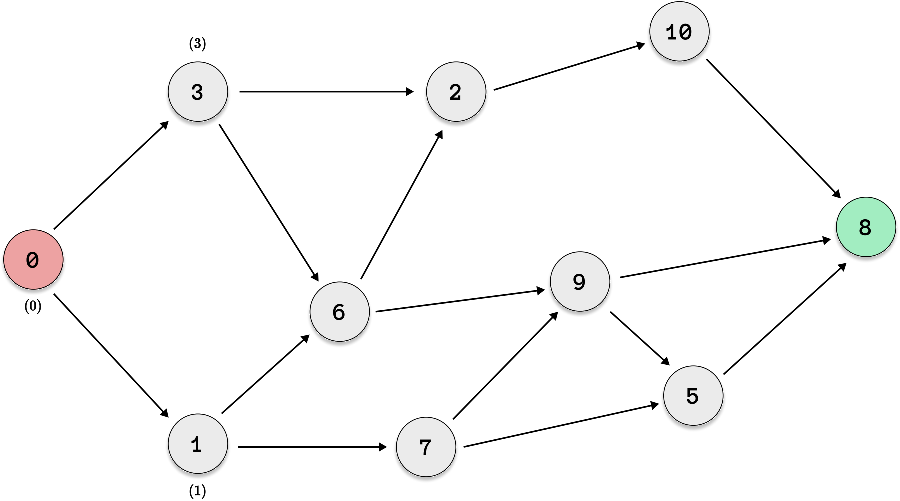

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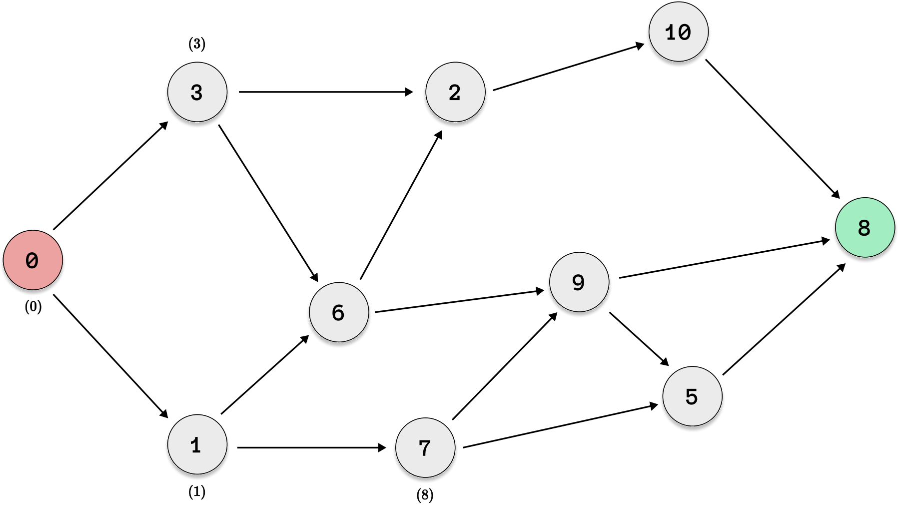

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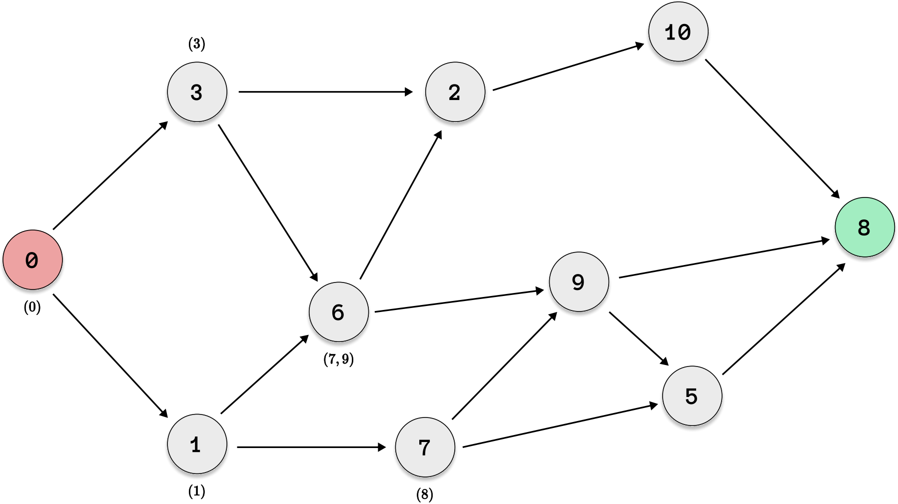

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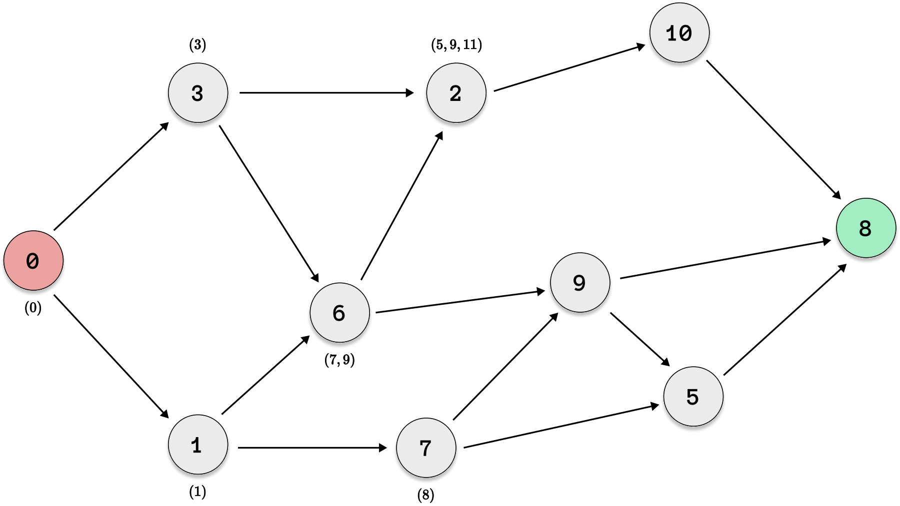

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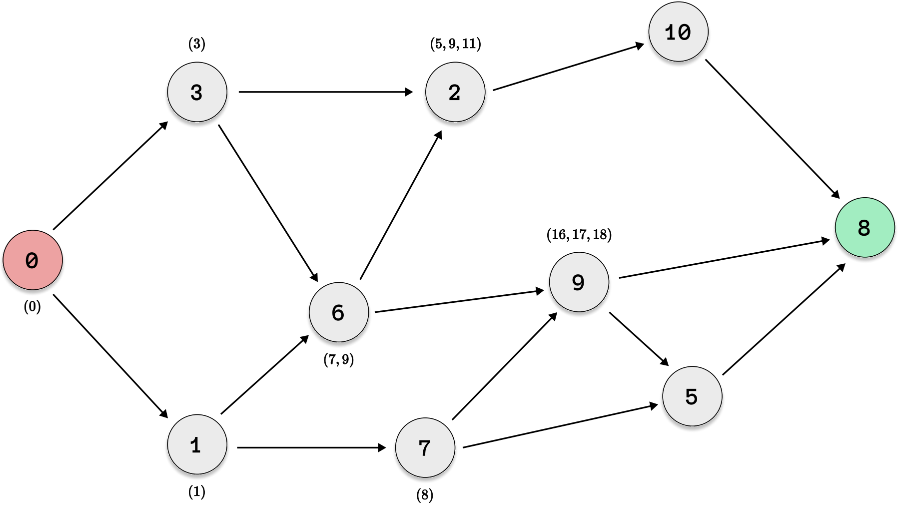

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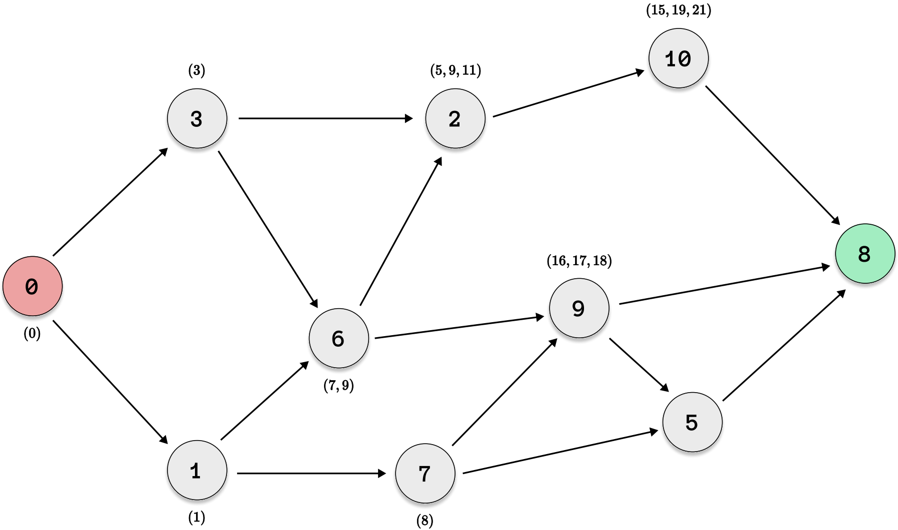

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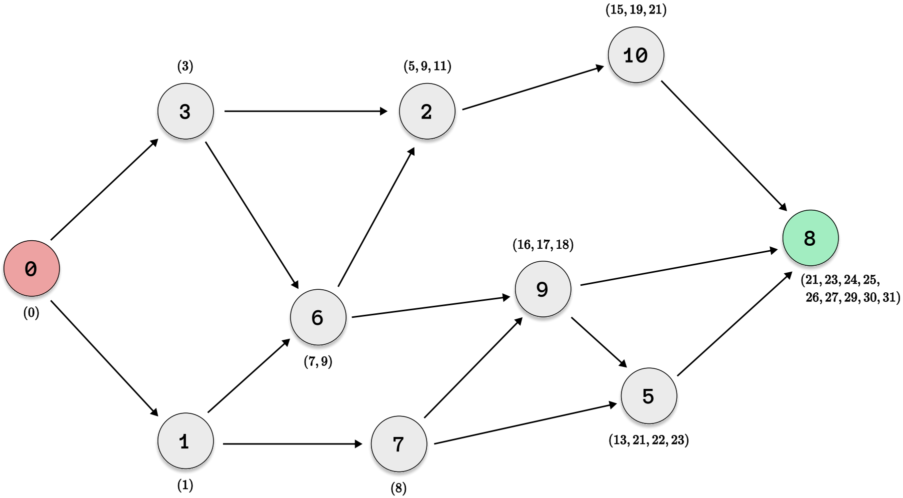

The Rank Query on Weighted DAGs

What Values are ”Active” at Node N?

Rank Query on a Node N: $\mathrm{rank}_G(N)$

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2. These intervals are then maximally merged. The query $\mathrm{rank}_G(N)$ returns a minimal collection of disjoint closed integer intervals

$$\mathcal{R}_N = \{[l_1, r_1], [l_2, r_2], \ldots, [l_p, r_p]\}$$

such that their union exactly covers S_N.

$\mathcal{R}_N$ captures the range of possible cumulative sums during the *activity* at node N

The Challenge: Storing Path Information

$\mathcal{O}$-Sets Can Be Huge!

- **Problem**: The size $|\mathcal{O}_v|$ can grow very large!
- **Question:** Can we represent the necessary information more compactly?

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- **Question:** Can we represent the necessary information more compactly?

Core Idea: Partitioning + Indirection

Partition vertices V into two types:

1. Explicit Vertices (V_E)

Store $\mathcal{O}_v$ directly.
(*Simple, but potentially large*)

2. Implicit Vertices (V_I)

Do *not* store $\mathcal{O}_v$ explicitly
(*Reconstruct on-the-fly.*)

The Challenge: Storing Path Information

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Reconstruction for $v \in V_I$ using:

- Designated Successor $\sigma(v)$
- Offset Sequence $\mathcal{J}_v$ (at v)

Implicit Reconstruction: Successor & Offset

How V_I Nodes Refer to Others

1. Designated Successor $\sigma(v)$ (for $v \in V_I$)

Which successor should v point to? **Heuristic**: Choose $u = \sigma(v)$ that minimizes $|\mathcal{O}_u|$.

$$\sigma(v) \in \operatorname*{argmin}_{u \in Succ(v)} \{|\mathcal{O}_u|\}.$$

2. Offset Sequence $\mathcal{J}_v$ (for $v \in V_I$)

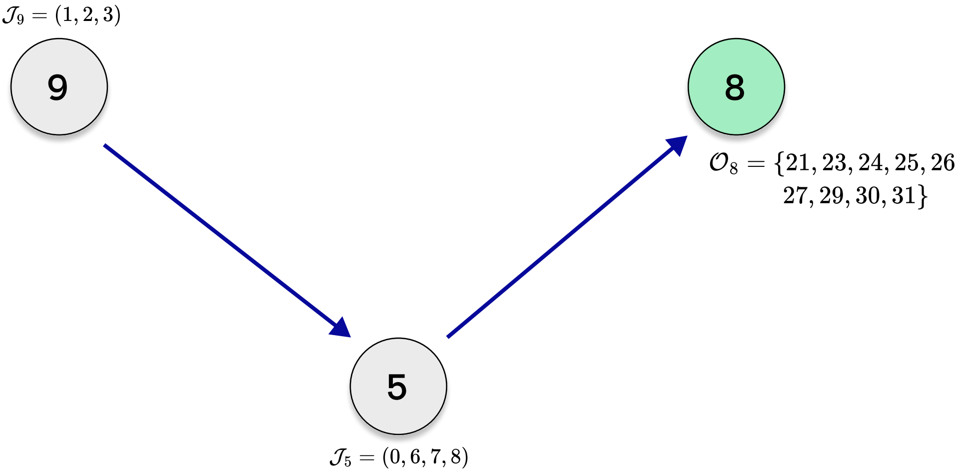
How to get $\mathcal{O}_v$ from $\mathcal{O}_{\sigma(v)}$? Let $u = \sigma(v)$.

- **Relationship**: Each element $x_k \in \mathcal{O}_v$ comes from some $y_{j_k} \in \mathcal{O}_u$ via $x_k = y_{j_k} - w(u)$.
- **Offset Sequence $\mathcal{J}_v$**: Stores the index j_k corresponding to each x_k.

$$\mathcal{J}_v = (j_0, j_1, \ldots, j_{m-1}), \quad \text{where } m = |\mathcal{O}_v|$$

Example: Computing $\mathcal{O}_9[1]$

Following the Successor Path: $9 \to 5 \to 8$



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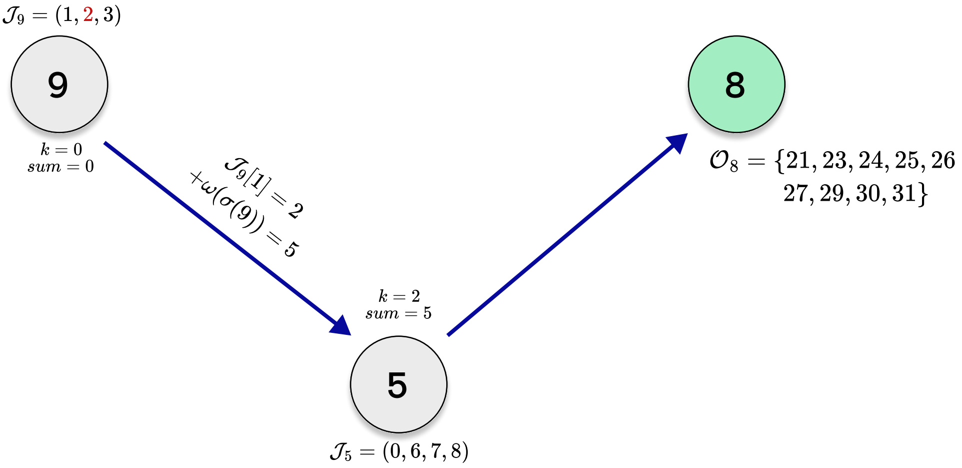

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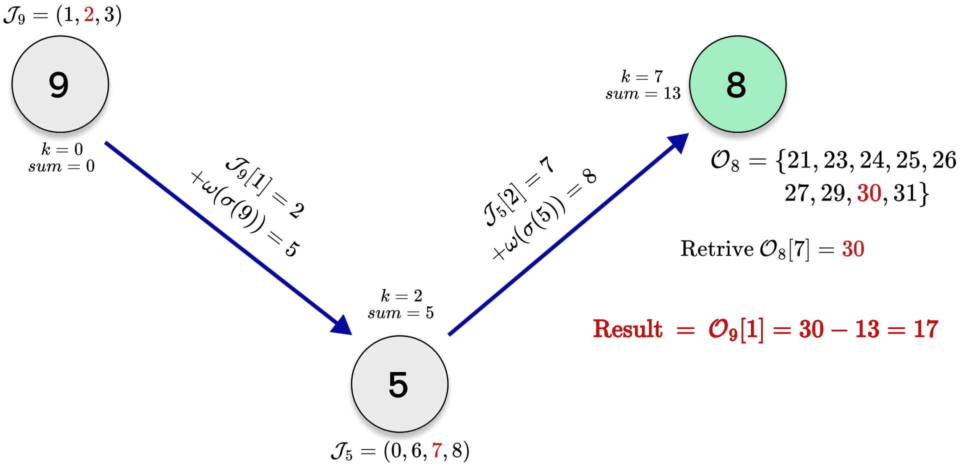
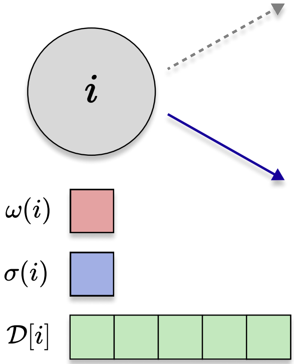


Succinct Data Structure: Components

Arrays Indexed by Vertex ID

Each node stores 3 components



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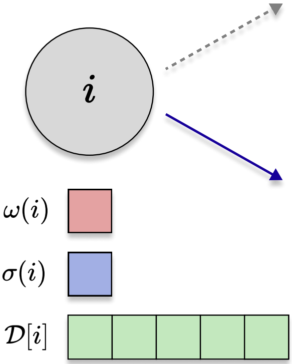
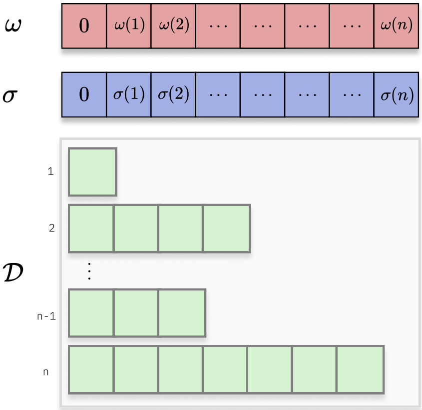


Succinct DAG as a Struct of Arrays



Compression Strategies

Reducing Memory Footprint

Component	Description
$\mathcal{W}$ (Node weights)	Array of positive integers.
Σ (Successor IDs)	Array of positive integers.

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- Run-Length Encoding (RLE) *(for clustered monotonic sequences)*

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Space Efficiency: Baseline Comparison

How Much Information is in the Graph?

To evaluate our structure's space, we need a baseline.

0^{th}-Order Graph Entropy $H_0(G)$

A theoretical lower bound for storing the *entire* weighted DAG (V, E, w) losslessly.

$$H_0(G) = \underbrace{H_W(G)}_{\text{Cost for Weights}} + \underbrace{H_E(G)}_{\text{Cost for Topology (Edges)}}$$

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Any method saving the *full* graph structure needs at least $H_0(G)$ bits!

Space Comparison: Succinct Structure vs. Baselines

Bitcoin DAG Example ($n \approx 22k, m \approx 50k$)

Method	Estimated Bits
Theoretical Lower Bound	**1,525,730**
Weights $H_W(G)$	60,824
Topology $H_E(G)$	1,464,906
Precomputed Rank Queries:	
Explicit Binary Storage	
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Achieving Sub-Entropy Space: How?

Our structure is **lossy** regarding the full graph topology:

- It does not store the complete edge set.
- It only stores the chosen successor $\sigma(v)$ for each implicit node (in Σ).

However, it is **lossless** for computing the specific Rank Query.

Future Direction: Bounded Query Time

Guaranteeing Predictable Performance

Performance Consideration

Query time for implicit node v depends on the length of the successor path

$$v \to \sigma(v) \to \sigma(\sigma(v)) \to \cdots \to e \in V_E$$

Problem: Can be large/variable in deep DAGs $\implies$ slow worst-case query time.

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Solution, Challenges & Trade-offs

- **Solution:** Ensure every implicit node can reach an explicit node within k steps.
- **Challenges:** Finding the smallest possible V_E' that satisfies this condition is NP-hard (*minimum distance-k dominating set*).
- **Trade-off:** More explicit nodes $\implies$ faster queries, but larger space.

Efficient Succinct Data Structures on Directed Acyclic Graphs

Thank you for listening!

Worst-Case $\mathcal{O}$-Set Size: Is Exponential Growth Possible?

Understanding the $\mathcal{O}$-set Size

Exponential Growth Can Occur

The cardinality of an $\mathcal{O}$-set, $|\mathcal{O}_v|$, is not generally bounded by a polynomial in the number of vertices $|V|$. It can grow exponentially.

Underlying Reason: Path Count

The number of distinct paths from a source s to a vertex v, denoted $|Path(s, v)|$, can itself be exponential in certain DAG structures. Since $|\mathcal{O}_v| \leq |Path(s, v)|$, the potential for exponential size exists.

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Key Condition for Exponential Growth

The exponential potential is realized if the vertex weights $w(v)$ are assigned such that distinct paths $P_1 \neq P_2$ almost always lead to distinct cumulative weights $W(P_1) \neq W(P_2)$.

Achieving Exponential $\mathcal{O}$-Set Size

A Strategy for Path Weight Uniqueness

Start with a DAG structure that naturally admits an exponential number of paths between two nodes. An example is a layered graph with multiple choices at each layer transition.

Strategic Weight Assignment

Assign vertex weights $w(v)$ carefully to ensure path weight uniqueness.

$$w(v) = 2^k \quad \text{(using a unique exponent } k \text{ for each node)}$$

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Mechanism: Unique Binary Representation

With power-of-2 weights, the cumulative path weight $W(P) = \sum_{v \in P \setminus \{s\}} w(v)$ becomes a sum of distinct powers of 2. Due to the uniqueness of binary representation, different sets of nodes (i.e., different paths) produce different sums. Therefore, $|Path(s, v)|$ distinct paths yield $|\mathcal{O}_v| = |Path(s, v)|$ distinct weights.