

Likelihoods

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This material loosely follows Chapter 2 of Roback and Legler.

Topics

- ▶ What is a likelihood
- ▶ Principle of a maximum likelihood estimator
- ▶ How to obtain a maximum likelihood estimator

Notation

Consider observed data $x_{1:n}$ and a fixed, but unknown parameter θ .

Our data can come from any type of distribution. Let's consider a situation where the data is not normally distributed.

Example

Assume we observe one coin flip (observed data) and a success is the coin landing on heads.

What is the distribution of our data?

Distribution of our data

Each trial can be summarized as a Bernoulli coin flip with unknown parameter θ .

The probability mass function is given by

$$P(X = x \mid \theta) = \theta^x (1 - \theta)^{1-x}.$$

Likelihood

A likelihood is function that tells us how likely we are to observe our data for a given parameter value θ .

Likelihood Function

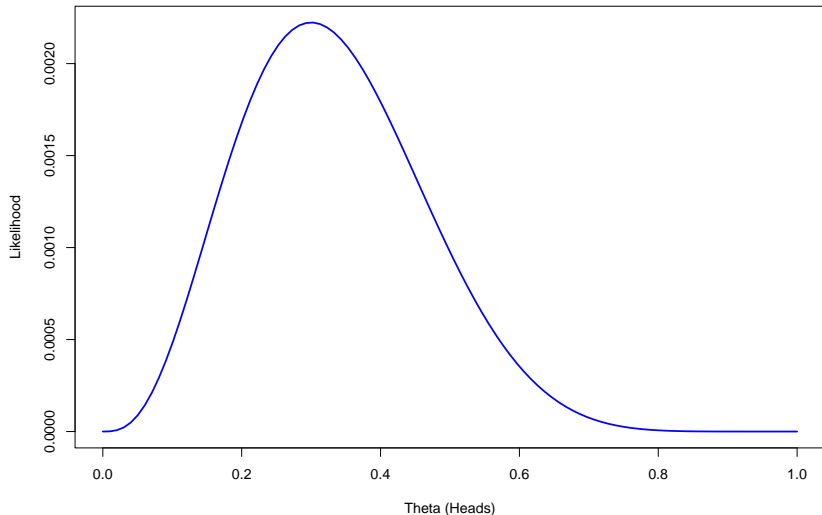
The likelihood function for the Bernoulli distribution becomes

$$L(\theta) = \prod_{i=1}^n \theta^{x_i} (1 - \theta)^{1-x_i}$$

Example

Suppose we observe 3 heads and 7 tails.

Likelihood Function for Bernoulli Distribution



Graphical Maximum Likelihood Estimate

To graphically approximate the Maximum Likelihood Estimator (MLE) of θ for the Bernoulli distribution, we can visually identify the value of θ where the likelihood function reaches its maximum.

Since the Bernoulli distribution's likelihood function is unimodal (it has a single peak), the MLE corresponds to the value of θ that maximizes the likelihood function.

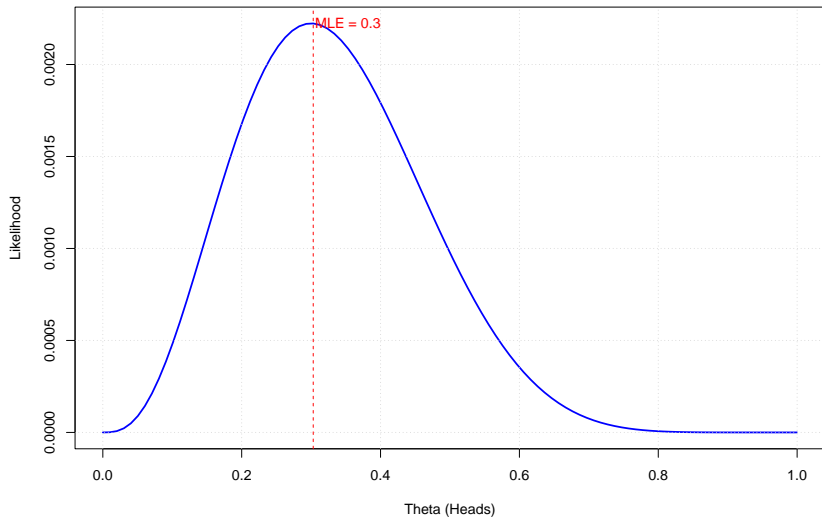
Steps

1. Set up a grid of θ values. We will create a sequence of values between 0 and 1 to evaluate the likelihood function at each point.
2. Evaluate the likelihood function at each point: For each θ value in the grid, we will compute the likelihood for the observed data.
3. Find the value of θ that maximizes the likelihood. The value of θ that gives the highest likelihood is the MLE.

Grid Search

The MLE for theta is: 0.303

The maximum likelihood value is: 0.0022



Finding the MLE using Calculus

A more general way to find the MLE is using calculus, which provides us with a generalized solution.

Perhaps think about why we would not want to perform a grid-search in practice? Think about potentially computational issues!

Finding the MLE

Recall that

$$L(\theta) = \prod_{i=1}^n \theta^{x_i} (1 - \theta)^{1-x_i} = \theta^{\sum_i x_i} (1 - \theta)^{n - \sum_i x_i}$$

Consider the log-likelihood

$$\ell(\theta) = \log \left[\theta^{\sum_i x_i} (1 - \theta)^{n - \sum_i x_i} \right] \quad (1)$$

$$= \sum_i x_i \log \theta + (n - \sum_i x_i) \log(1 - \theta) \quad (2)$$

Finding the MLE

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{\sum_i x_i}{\theta} - \frac{(n - \sum_i x_i)}{(1 - \theta)} =: 0$$

Now, we solve for θ .

Finding the MLE

$$\frac{\sum_i x_i}{\theta} = \frac{(n - \sum_i x_i)}{(1 - \theta)} \quad (3)$$

$$\sum_i x_i(1 - \theta) = (n - \sum_i x_i)\theta \quad (4)$$

$$\sum_i x_i - \theta \sum_i x_i = n\theta - \theta \sum_i x_i \quad (5)$$

$$\sum_i x_i = n\theta - \theta \sum_i x_i + \theta \sum_i x_i \quad (6)$$

$$\sum_i x_i = n\theta \quad (7)$$

$$\theta = \frac{1}{n} \sum_i x_i \quad (8)$$

Second derivative check

The second derivative of $\ell(\theta)$ is

$$-\frac{\sum_i x_i}{\theta^2} - \frac{n - \sum_i x_i}{(1 - \theta)^2}$$

Because θ is between 0 and 1, both terms are negative, so the second derivative is negative, confirming that the critical point corresponds to a maximum.

Circuling back to our example

Thus, for our particular example, we can calculate the MLE directly (and can do so for any problem moving forward instead of performing a grid search).

Practice with the Poisson Distribution

The probability mass function (PMF) of a Poisson-distributed random variable X with parameter λ is given by:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots$$

where $\lambda > 0$ is the rate parameter, which represents both the mean and variance of the distribution.

Questions

1. Write out the likelihood for the Poisson distribution for $x_{1:n}$.
2. Derive using calculus based methods the MLE of λ is $\sum_i x_i / n$ (sample mean) and show that it is in fact a maximum.
3. Verify using a grid-search that your solution matches to the calculus based one.