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Circle Packing and its Applications

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Contents

Acknowledgments

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1 Introduction and preliminaries

Constructing arrangements of circles according to specified tangency patterns has become a fruitful area of research in recent years. Circle packing goes back to Koebe [5], and more recently, Thurston brought the theory to prominence in 1985 [8], conjecturing a connection, later proved by Rodin and Sullivan [6], between circle packing and the Riemann mapping theorem. Since then, a comprehensive theory has been developed, and deeper connections to analytic function theory, conformal geometry, and an ample array of applications have been discovered.

Most of this thesis will be devoted to providing an exposition of the main ideas and theorems for circle packing. This includes a detailed examination of circle packings with finitely many circles on a topological disc, and a survey of deeper results and extensions of this basic result, to generalize to circle packings having infinitely many circles, and circle packings on general surfaces.

Secondly, as part of this thesis I have developed a computer program utilizing circle packing in a surprising fashion to draw knotted graphs projections in the plane. This application of circle packing is discussed in detail later, and has now been used to draw many thousands of knotted graphs in a canonical way.

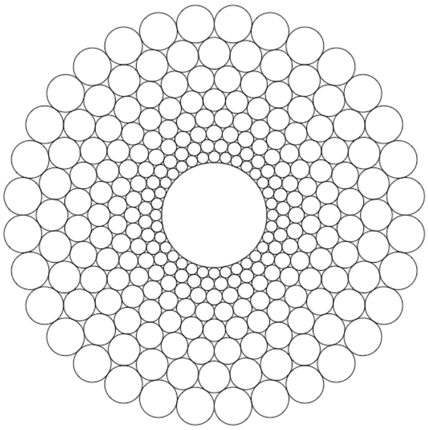


Figure 1: Example circle packing

Basics

The central idea of circle packing is: given a triangulated orientable topological surface K, to produce a corresponding set of circles on some surface S, in bijection with the vertices of K, such that for each edge in K, the corresponding circles meet tangentially and are disjoint. We will see that the geometry required for S, whether Euclidean, hyperbolic, or spherical, arises naturally from the topological and combinatorial nature of K.

Bookkeeping

Before discussing the main results, we will need some terminology to be used from here on. We denote the Euclidean plane by $\mathbb{C}$, the hyperbolic plane by $\mathbb{D}$, and the sphere by $\mathbb{S}$. We will use the Poincaré disc model as our model for hyperbolic geometry.

Definition 1.1. *A* ***complex*** *K is a connected set of 2-simplices (faces), together with the edges (1-simplices) and vertices (0-simplices), with edges and vertices* *of K identified in such a way that following conditions are satisfied:*

(1) Each edge of K is contained in exactly one face of K (called external edges), or two faces of K (called internal edges).

(2) Each vertex $v \in K$ is contained in some integer number $n \geq 2$ of faces of K, and v is contained in exactly two external edges, or no external edges.

(3) Each distinct pair of faces in K either intersect in exactly one edge, or intersect in exactly one vertex, or are disjoint.

(4) The vertices of each face of K can be assigned an ordering such that any pair of distinct faces $f_1, f_2 \in K$ intersecting in an edge in K induce opposite ordering on the common vertices.

A consequence of this definition is that K can be thought of as a triangulation of a topological surface, and it will be convenient to sometimes refer to a complex K as a triangulation of a surface S, meaning K is homeomorphic to S.

Definition 1.2. *A **packing** P for a complex K is a set of (metric) circles C on a surface S, homeomorphic to K such that there is a bijection between the vertices of K and circles of C. For each pair of vertices of K sharing an edge in K, the corresponding circles in P meet tangentially and are disjoint.*

Definition 1.3. *Given a complex K and a packing P for K, a **packing map** $p : K \to S$ is a homeomorphism endowing K with a geometric structure, induced by the geometry of S, such that p maps vertices of K to centers of circles in S, and edges of K to geodesic segments of S.*

Definition 1.4. *Given a complex K, a **label** R is a map $R : V \to (\mathbb{R}, \infty)$, assigning to each vertex $v \in K$ a radius for the corresponding circle of K. We also allow R to take the value ∞ if we are attempting to pack K on a hyperbolic surface.*

In this definition, R does not necessarily qualify as radii for an actual packing. That is, R can be chosen arbitrarily, and there is no guarantee that there exists a packing P for K with radii given by R. Regardless, any such R still gives well defined angles and areas using appropriate formulae for calculating angles in Euclidean, hyperbolic, and spherical geometry.

Lemma 1.5. *For $r_1, r_2, r_3 \in (0, \infty)$, there exists a configuration of circles c_1, c_2, c_3 in hyperbolic or Euclidean geometry (r_i may take the value ∞ in hyperbolic geometry), with c_i having radius r_i. This configuration is unique up to hyperbolic or Euclidean isometry.*

Proof. For a configuration of circles satisfying the above conditions, by connecting the centers of c_i with hyperbolic geodesics, we have the system of equations

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \tag{1}$$

where l_i is the length of the edge opposite to c_i. Solving (1) for r_i gives

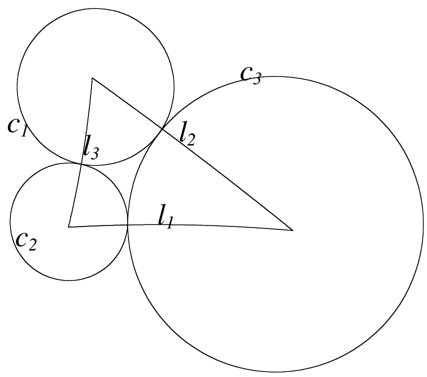


Figure 2: Hyperbolic triangle

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -l_1 + l_2 + l_3 \\ l_1 - l_2 + l_3 \\ l_1 + l_2 - l_3 \end{bmatrix} \tag{2}$$

Moreover, since $l_i + l_j = r_i + r_j + 2r_k > r_i + r_j = l_k$, where $k \neq i, j$, the l_i satisfy the triangle inequality, so can be realized as the edges of a geodesic triangle in hyperbolic or Euclidean geometry, unique up to isometry. $\square$

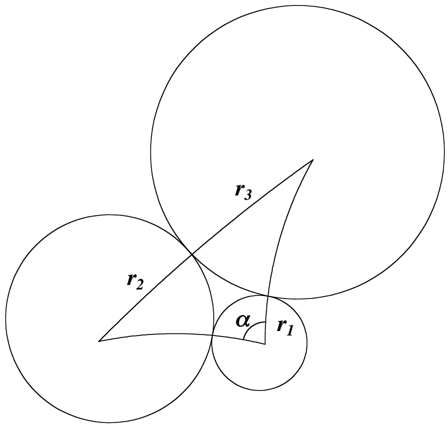


Figure 3: Calculating angles

We now introduce the idea of angles corresponding to a given label R. This will then be used to establish the fundamental condition needed to decide whether R corresponds to a circle packing or not.

Definition 1.6. *Let v be a vertex in K, and let R be a label for K. The* ***angle sum*** *is a map $\theta_R : V \to [0, \infty)$, given by*

$$\theta_R(v) = \sum_{i=1}^{n} \alpha_i$$

where α_i is the angle at $p(v)$ in the ith face containing $p(v)$, and the sum is taken over the n faces of K containing v as a vertex (see fig (3)).

The α_i are given by

Case 1: *hyperbolic*:

$$\alpha_i = \arccos \frac{\cosh(r_1 + r_2)\cosh(r_1 + r_2) - \cosh(r_2 + r_3)}{\sinh(r_1 + r_2)\sinh(r_1 + r_3)} \tag{3}$$

Case 2: *Euclidean*:

$$\alpha_i = \arccos \frac{(r_1 + r_2)^2 + (r_1 + r_3)^2 - (r_2 + r_3)^2}{2(r_1 + r_3)(r_1 + r_3)} \tag{4}$$

Case 3: *spherical*:

$$\alpha_i = \arccos \frac{\cos(r_2 + r_3) - \cos(r_1 + r_2)\cos(r_1 + r_3)}{\sinh(r_1 + r_2)\sinh(r_1 + r_3)} \tag{5}$$

Definition 1.7. *Let K be a complex, P a packing for K, and p a corresponding packing map. The* ***carrier*** *for a complex K, and packing P for K, denoted $Carr(K)$ is the set $p(K)$, that is, the image of K under p, where p is a packing map for P.*

2 Finite packings in $\mathbb{D}$

The goal of this section will be to prove the fundamental result, that given any triangulation K of a closed disc, there exists a packing for K, and subject to conditions on the radii of the boundary circles, this packing is unique up to conformal automorphisms. Surprisingly, it turns out that if we work in the hyperbolic disc $\mathbb{D}$, the calculations become quite manageable, and since under the Poincaré metric, hyperbolic circles and Euclidean circles are the same thing, nothing is lost by working in this setting.

To begin, we will need some monotonicity results that hold in both the Euclidean and hyperbolic case.

2.1 Monotonicity results

We now develop the fundamental monotonicity results that are crucial to proving existence of finite packings.

Lemma 2.1. *Let T be a hyperbolic triangle constructed by connecting the centres of pairwise tangent circles having radii r_1, r_2, r_3 (which exists by the lemma (1.5). If r_2, r_3 are held fixed while r_1 is increased continuously, β, γ will increase continuously (see fig (3)), and the area of T is continuously increasing in r_1. If r_2 or r_3 is continuously increased while r_1 is held constant, α will continuously decrease.*

Proof. Define $x_1 = e^{2r_1}$, $x_2 = e^{2r_2}$, $x_3 = e^{2r_3}$. From the hyperbolic law of cosines (3), we have

$$\alpha = \arccos\left(\frac{(x_1x_2)(x_1x_3) - 2x_1(x_2x_3+1)}{(x_1x_2-1)(x_1x_3-1)}\right)$$

So that

$$\frac{\partial\alpha}{\partial x_2} = \frac{1}{x_1x_2-1}\sqrt{\frac{x_1(x_1-1)(x_3-1)}{(x_2-1)(x_1x_2x_3-1)}}$$

$$\frac{\partial\alpha}{\partial x_1} = \frac{1-x_1^2x_2x_3}{(x_1x_2-1)(x_1x_3-1)}\sqrt{\frac{(x_2-1)(x_3-1)}{x_1(x_1-1)(x_1x_2x_3-1)}}$$

Then since $x_1, x_2, x_3 > 1$, we have $\frac{\partial\alpha}{\partial x_1} < 0$, and $\frac{\partial\alpha}{\partial x_2} > 0$, so α is continuously decreasing in x_1, and α is continuously increasing with β. A symmetric argument interchanging the roles of x_2, x_3 implies α is also continuously increasing with x_3. Since $r_1 = \frac{1}{2}\log x_1$ is strictly continuously increasing with x_1, α is also continuously decreasing with x_1, and continuously increasing with x_2, x_3. Since both β, γ are increasing with r_1, the triangle obtained increasing r_1 will contain T, hence the area of T increases with r_1. □

The Euclidean case is similar, and the details for the computation are omitted.

Definition 2.2. *A* packing label *R for a complex K is a label such that there exists a circle packing for K with the property that for each vertex $v \in K$, $r(v) = R(v)$, where $r(v)$ is the radius of the circle corresponding to v in the appropriate geometry.*

Lemma 2.3. ***(Main condition on labels)***

Let K be a closed combinatorial disc, and let R be a label for K. Then R is a packing label for K if and only if $\theta_v = 2\pi$ for all internal vertices $v \in K$.

Proof. If R is a packing label for K, it follows immediately that $\theta_v = 2\pi$ since the faces K containing v wrap exactly once around v in any packing of K.

Now assume R is a label for K such that $\theta_R(v) = 2\pi$ for all internal vertices $v \in K$. Select an interval vertex in K, say v_0, and place a circle of radius $R(v_0)$ at the origin in $\mathbb{D}$. Then select a neighbour of v_0, say v_1, and place an

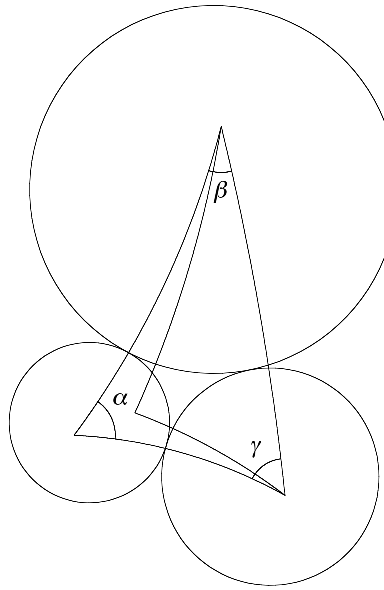


Figure 4: Hyperbolic angle monotonicity

externally tangent circle to v_0 along the positive real axis with radius $R(v_1)$. Then to position the remaining circles, we proceed iteratively:

For each neighbouring $v_i, v_j \in K$ with corresponding tangent circles $c_i, c_j \in \mathbb{D}$, choose a neighbour v_k of both v_i, and v_j that has not had a circle placed for it, if such a v_k exists. If v_k does exist, there is a unique circle $c_k \in \mathbb{D}$ tangent to c_i, c_j, with radius $R(v_k)$ such that the ordering of the vertices $v_i, v_j, v_k \in \mathbb{D}$ agrees with the ordering of $v_i, v_j, v_k \in K$.

Since K is finite, applying this process must terminate after some finite number of steps. It remains to prove that the positions for the c_i are well defined, that is, that the position of c_i is independent of the ordering of the v_i during the above construction. To prove this, we need to introduce the notion of local modifications.

Definition 2.4. *A **chain** of faces of a complex K is a sequence $\mathcal{C} = \{f_1, f_2, ..., f_n\}$ of faces of K with the property that each face is adjacent to the previous, that is, $f_i \cap f_{i+1} = e_i$, where e_i is an edge of K, or $f_i \cap f_{i+1} = f_i$, if $f_i = f_{i+1}$.*

Definition 2.5. *Let K be a complex, and let $\mathcal{C} = \{f_1, f_2, ..., f_n\}$ be a chain of faces in K such that there exists some vertex $v \in K$ with $v \in f_j$, $1 \leq j \leq n$. A* ***local modification*** *of $\mathcal{C}$ at v is any chain $\mathcal{C}' = \{g_1, g_2, ..., g_m\}$ of faces $\mathcal{C}$ are the same of those of $\mathcal{C}'$, except possibly for faces of $\mathcal{C}$ and $\mathcal{C}'$ containing the vertex v (see fig (5)).*

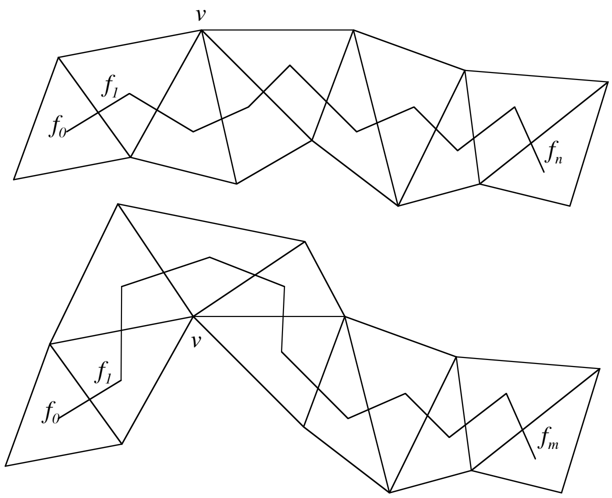


Figure 5: Local modification about vertex v

If any two chains $\mathcal{C}_1, \mathcal{C}_2$ are related by a finite sequence of local modifications, we say $\mathcal{C}_1, \mathcal{C}_2$ are ***homotopic****.*

Continuing with the proof of 2.3, let $\mathcal{C}, \mathcal{C}'$ be chains beginning at a fixed face $f_1 \subset K$. Each of these chains will give a unique list of vertices $\{v_i\}, (v_i')$ along which we use the above procedure to position $\{v_i\}, \{v_i'\} \in \mathbb{D}$. Fix two vertices in f_1, say $v_0, v_1 \in \mathbb{D}$, then placing the $\{v_i\}$ along $\mathcal{C}$ will place f_n at the same position as placing the $\{v_i'\}$ along $\mathcal{C}'$ if and only if positioning backwards along $\{v_{n-i}\}$, and then forwards along $\{v_i'\}$ places the initial face and the final face in the same position. That is, it will suffice to consider only closed chains f_i with $f_1 = f_n$.

Since the angle sum at all interior vertices $v \in K$ is 2π, given a chain $\mathcal{C} = \{f_1, f_2, ..., f_n\}$, any local modification of $\mathcal{C}$ at $v \in f_i$ will preserve the final face f_n of $\mathcal{C}$, since the local modification must wrap around v some integer number of times, preserving the final face of the modification, and hence f_n.

Combining this with that fact that since K is a closed disc, for all closed chains $\mathcal{C} = \{f_1, f_2, ..., f_{n-1}, f_1\}$ there is a sequence local modifications of $\mathcal{C}$ taking $\mathcal{C}$ to the single face $f_1 \in K$ implies all closed chains will place the first and last face in the same position, completing the proof. $\square$

Definition 2.6. *A **maximal packing** P for K is a packing with the further property that for each boundary vertex $v \in K$, $R(v) = \infty$. That is, boundary circles in P are circles tangent to the unit disk (horocycles).*

Definition 2.7.

A **maximal packing label** R is a label for K such that there exists a maximal packing P with corresponding label L and $L(v) = R(v)$ for all vertices $v \in K$.

Lemma 2.8. *A label R is a maximal label for K if and only if there exists a maximal packing for K with label R.*

The proof of this lemma follows from the section on Perron labels.

Theorem 2.9. ***Existence of hyperbolic packing labels***

Let K be a closed combinatorial disc. There exists a unique hyperbolic maximal label R for K, and and a corresponding maximal packing P. Moreover, if P' is a second packing for R, then P and P' are related by a Möbius transformation.

The theorem will be established in two stages: first we assume there is some packing for K, not necessarily a maximal packing, and show that this implies existence of a maximal packing for K. Then we argue by induction, that existence of maximal packings for all complexes having $n-1$ vertices implies existence of packings (not necessarily maximal) for all complexes having n vertices.

The proof that we can go from arbitrary packings to maximal packing is based on the Perron method, and the inductive part of the argument is based on [7], and avoids many the difficulties found in the more technical earlier proofs of the theorem [9].

We need a method to obtain maximal packings for K from arbitrary packings for K. For this we use the Perron method.

2.2 Perron method

Let K be a combinatorial closed disc, and define $\mathcal{R}(K)$ to be the set of all labels R for K such that if v is an interior vertex of K,

$$\theta_R(v) \geq 2\pi \tag{6}$$

call elements of $\mathcal{R}$ Perron labels. Define $\max\{R_1, R_2\}$ to be the label obtained by taking the maximum of $R_1(v), R_2(v)$ for each vertex v. Define $\sup\{R_i\}$, where $\{R_i\}$ is a family of labels to be the label obtained by taking the supremum of $R_i(v)$ over all labels in the family.

Lemma 2.10. *Let K be a closed combinatorial disc, and let $R_1, R_2 \in \mathcal{R}$, then:*

(1) $\max\{R_1, R_2\} \in \mathcal{R}$

(2) For each interior vertex $v \in K$, $R_1(v) < \infty$.

(3) Let $\mathcal{M} := \sup\{R : R \in \mathcal{R}\}$. Then for all interior vertices $v \in K$, $\theta_{\mathcal{M}}(v) = 2\pi$, and for all external vertices $w \in K$, $\mathcal{M}(w) = \infty$.

Proof. To prove the first statement, let $R_1, R_2 \in \mathcal{R}$. Let v_0 be a vertex in K, and assume $R_1(v_0) \geq R_2(v_0)$. Then the label $\hat{R} = \max\{R_1, R_2\}$ has the properties that $\hat{R}(v_0) = R_1(v_0)$, and for all neighbouring vertices v of v_0, $\hat{R}(v) \geq R_1(v)$. By the monotonicity result (2.1), in the label $\hat{R}$, the angle at v_0 in each face of K containing v_0, is greater than or equal to the corresponding angle in the label R_1. Since this applies to each face in the flower of v_0, summing of the faces containing v_0 gives $\theta_{\hat{R}}(v_0) \geq \theta_{R_1}(v_0) \geq 2\pi$. That is, $\hat{R} \in \mathcal{R}$.

The second statement follows by constructing an upper bound on the radii of an interior vertex in terms of the number of vertices in its flower. Given a label $R \in \mathcal{R}$, choose an interior vertex v_0. Let $r = R(v_0)$, and let $\{r_1, r_2, ..., r_k\}$, be the radii of the neighbours of v_0. By applying a conformal automorphism of $\mathbb{D}$, we can send v_0 to the origin without affecting the angles at v_0, and hence leaving

$\theta_R(v_0)$ unchanged. By lemma (2.1), $\theta_R(v_0)$ is increasing in r_i, $i \in 1, 2, ..., k$, so $\theta_R(v_0)$ is maximised by setting $r_i = \infty$, $i \in 1, 2, ..., k$. In this case, since v_0 is centered at the origin, the angles in each face containing v_0 are all equal to $\frac{2\pi}{k}$, so from the hyperbolic law of cosines

$$\cos\left(\frac{2\pi}{k}\right) = 1 - 2e^{-2r_0}$$

or

$$r_0 = -\log\left(\sin\left(\frac{\pi}{k}\right)\right) < \infty$$

Since this is the case in which r_0 is maximised for a given number k of neighbours, we conclude for all interior vertices $v \in K$, $R(v) < \infty$.

To show the final statement of the lemma, assume there exists an interior vertex $v \in K$ with $\theta_{\mathcal{M}}(v) > 2\pi$ or an exterior vertex v with $\theta_{\mathcal{M}}(v) < \infty$. If there exists an interior vertex $v' \in K$ with $\theta_{\mathcal{M}}(v') > 2\pi$, we know part (2) of the lemma, that $\theta_{\mathcal{M}}(v') < \infty$, and by the continuous monotonicity for angle sums, we could replace $\mathcal{M}(v')$ with $\mathcal{M}(v') + \epsilon$ for some $\epsilon > 0$ without $\theta_{\mathcal{M}}(v')$ dropping below 2π, contrary to the definition of the label $\mathcal{M}$. Alternatively, if $\theta_{\mathcal{M}}(v') < \infty$ for an exterior v', replacing $\theta_{\mathcal{M}}(v')$ by ∞ will increase the angle sums of any interior tangent vertices to v, again contradicting the definition of $\mathcal{M}$. Therefore, $\mathcal{M}$ is a maximal packing label for K. $\square$

Having proved that arbitrary packing labels give rise to maximal packing labels, it remains to show that for K a closed combinatorial disc, there exists *some* packing $P(K)$, and here we need to consider two cases:

Case 1: there are no interior edges of K having both vertices as boundary vertices.

Case 2: there is at least one interior edge of K having both vertices as boundary vertices.

Proof of case 1: Proceeding by induction, we check the statement holds when K has three vertices, since there is only one possible triangulation with three

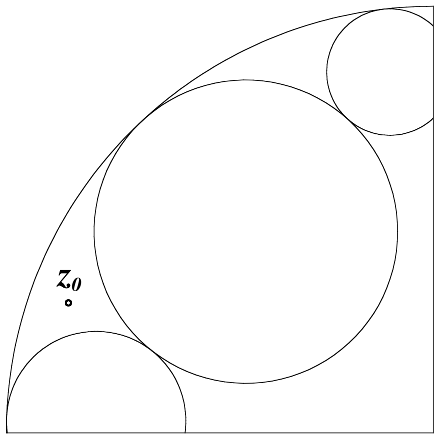


Figure 6: Choosing z_0

vertices, namely, a single triangle, which can be maximally packed by taking P to be the three hyperbolic circles centred at the cube roots of unity, having infinite radius.

Now assume that for all closed combinatorial discs L with less than N vertices, there exists a packing for L. Consider the complex obtained by deleting a boundary vertex v from K, and any edges and faces of K containing v, call this complex K'. Since no interior edges of K have both ends as boundary vertices, the end of any edges deleted will necessarily be interior vertices, and this implies K has not been split into two subcomplexes, that is, K' is be connected, and since v is not an interior vertex of K, removing it leaves K' a combinatorial disc. By the inductive hypothesis, there exists a packing P for K', and by the Perron method, this can be made into a maximal packing $\mathcal{P}$ for K. Because the maximal packing P for the reduced complex K' are tangent to $\mathbb{D}$, it follows that the neighbours $\{v_i\}$ of the deleted vertex v from K are tangent to the boundary. Fix a point $z_0 \in \mathbb{D}$ in an area of $\mathbb{D}$ between the unit disc and two external circles of K (see figure (6)).

By stereographic projection, $\mathcal{P}$ can be mapped to the sphere $\mathbb{S}$, and this maps circles in $\mathbb{D}$ to circles in $\mathbb{S}$ (see the appendix for details of stereographic projection).

We can now apply a conformal automorphism ϕ of $\mathbb{S}$, taking circles to circles, and mapping the image of z_0 under stereographic projection to the north pole of $\mathbb{S}$.

Note that the boundary of $\mathbb{D}$ is mapped to the equator of $\mathbb{S}$ under stereographic projection. After applying ϕ, we stereographically project back to the plane. Finally, normalize so the projection lies in $\mathbb{D}$. Let this sequence of mappings be called $\psi : \mathbb{D} \to \mathbb{D}$, that is,

$$\psi(z) = s^{-1}(\phi(s(z)))$$

where $s : \mathbb{D} \to \mathbb{S}$ is stereographic projection. We claim the resulting image is a packing for K, that is, $\psi(P)$ is a circle packing for K.

To check this claim, we need to show:

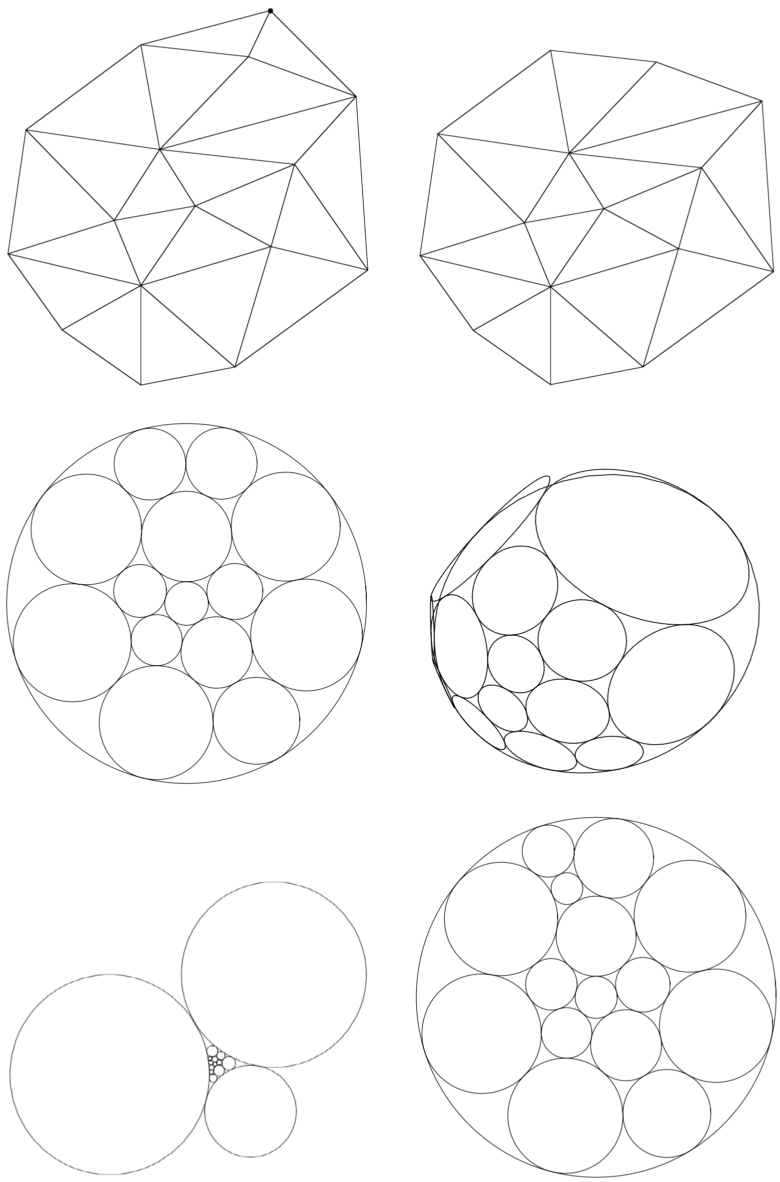
(1) circles remain circles under ψ.

(2) ψ maps interior vertices of K to interior vertices

(3) $\theta_R(v) = 2\pi$ for all interior vertices $v \in K$

The first statement follows immediately since s, s^{-1}, and ϕ all take circles to circles. The second statement follows since the exterior of $\psi(K)$ is a neighbourhood of z_0, and z_0 was defined so that it lies between the boundary of $\mathbb{D}$ and two exterior circles of K. This means no interior circles are mapped to exterior circles, hence ψ maps interior vertices of K to interior vertices. For the final statement, we need only consider neighbouring interior vertices of the deleted vertex v, since by part (2), $\theta_R(w)$ already holds for all other interior vertices. But it also holds for the interior neighbours u of v since the ψ maps u to an interior vertex by the choice of z_0.

We summarise pictorially below: The first image is the complex K we seek to pack, then the reduced complex K'. The third image shows the maximal packing for P' for K', followed by P' under stereographic projection. The fourth image is the Möbius transformation, and finally the maximal packing P for K.



Proof of the second case: There exists an interior edge e of K with both ends as boundary vertices.

In this case, we cut the complex along e, we obtain two subcomplexes L, L' (including copies of the vertices of e in both L and L') each having strictly fewer vertices than K. If either L or L' still have interior edges ending in boundary vertices, continue this process, and since there are finitely many vertices of K, this process must eventually terminate. Therefore, assume that by cutting

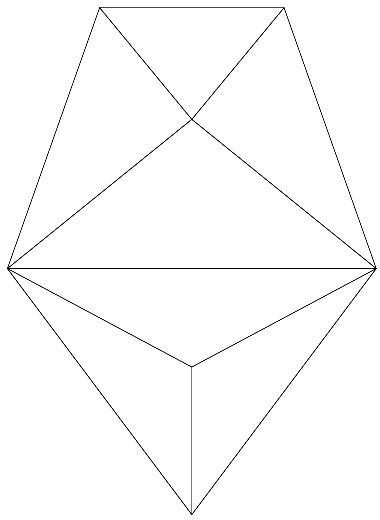


Figure 7: Example complex for case two

K along e, the resultant subcomplexes L, L' have no interior edges ending at boundary vertices. Let the vertices at the ends of e be denoted by u, v.

Then since the reduced subcomplexes have fewer vertices than the original complex, by the first case, there exists maximal univalent packings P, P' for L, L' in $\mathbb{D}$. Moreover, since a Möbius transformation is uniquely defined by specifying the images of three points, we may normalize P, P' so that the the centers of the circles corresponding to u, v are mapped to $(-1/2, 0)$, and $(1/2, 0)$, and the point of tangency of the circles corresponding to u, v is mapped to $(0, 0)$.

After normalization, P, P' lie in opposite halves of $\mathbb{D}$, with the circles corresponding to u, v in the same positions, so by overlaying P and P', we obtain a packing for K, completing the proof.

2.3 Univalence

Definition 2.11. *A circle packing P for K is* ***univalent*** *if the circles in P are pairwise disjoint. P is* locally univalent *if the neighbouring circles to each fixed circle $c \in P$ are pairwise disjoint.*

Lemma 2.12. *Let K be a combinatorial closed disc, and P a maximal hyper-*

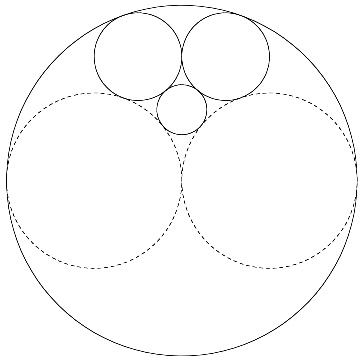


Figure 8: Packing for L

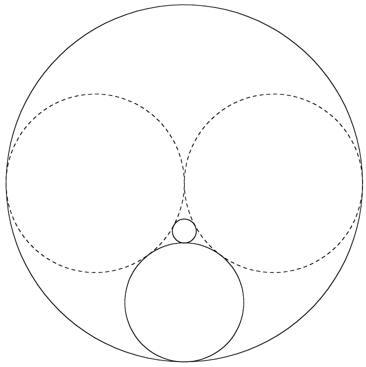


Figure 9: Packing for L'

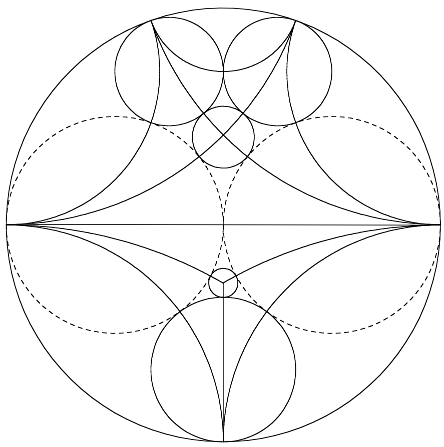


Figure 10: Packing for K

bolic packing. Then P is univalent.

Proof. The idea behind the proof comes from the fact that the there are no branch points of the packing map p. Since any label R for K endows each triangular face of K with a unique (up to isometry) hyperbolic metric, and maps K injectively onto this hyperbolic triangle, at all points except possibly the vertices of K, p is locally injective. At a vertex v of K, since $\theta_R(v) = 2\pi$, the faces in K containing v wrap around v exactly once, so there is no branching of p. Therefore, small neighbourhoods of vertices are also mapped injectively, so p is locally injective for all points in K.

Now defining a slightly altered version of the packing map p of K, which we denote by $\tilde{p}$, given by $\tilde{p} = p$ for all points in K not in a boundary face, and for points in a boundary face of K, we deform p homeomorphically so that the geodesic boundary edge of K is pushed to the boundary of $\mathbb{D}$, where we are now including boundary points of the disc. The altered version of $\tilde{p}$ is then that is

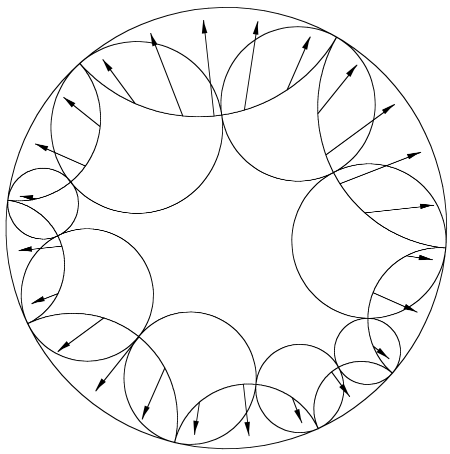


Figure 11: altered packing map $\tilde{p}$

locally injective everywhere, hence a covering map, mapping the boundary of K to the boundary of $\mathbb{D}$, and it is then a standard topological result that since $\mathbb{D}$ is simply connected, $\tilde{p}$ a homeomorphism implies $\mathbb{D}$ is the universal cover for K, and that the degree of the $\tilde{p}$ is 1, hence $\tilde{p}$ is globally injective, and it follows that the circles of P are univalent. $\square$

2.4 Uniqueness of maximal packing labels

Lemma 2.13. *Let K be a closed combinatorial disc, and suppose there exists a univalent maximal hyperbolic packing label $\mathcal{M}$ for K. Then $\mathcal{M}$ is unique.*

Proof. To prove that the maximal packing label $\mathcal{M}$ is unique, we appeal to hyperbolic area considerations. Let F be the number of faces of K. Then in hyperbolic geometry, we have

$$Area\,(T) = \pi - \alpha - \beta - \gamma$$

where T is a hyperbolic triangle, and α, β, γ are the angles at the vertices of T. Defining N to be the number of interior vertices of K, for any packing Q of K, we have

$$Area\,(Carr\,(Q)) \quad = F\pi - \textstyle\sum_{v \in K} \theta\,(v) \quad = (F - 2N)\,\pi \tag{7}$$

since $\theta\,(v) = 2\pi$ for v an interior vertex, and $\theta\,(w) = 0$ for w a boundary vertex.

Therefore, any maximal packing Q' has carrier area dependent only on the combinatorics of K, that is, $Area\,(Carr\,(Q)) = Area\,(Carr\,(Q'))$.

Note that the Perron method implies that if a packing label $R\,(v) \leq \mathcal{M}\,(v)$ for a single vertex v, then $R \leq \mathcal{M}$ for all vertices of K.

Assume we have a second maximal packing label R' with packing P', and $R'\,(w) = \infty$ for boundary vertices w. If $R' < \mathcal{M}$, by the above statement, $R' \leq \mathcal{M}$ for all vertices. By the monotonicity result for areas of hyperbolic triangles, $Area\,(Carr\,(P')) < Area\,(Carr\,(Q))$, contradicting the previous statement that the area is independent of the maximal packing. Therefore we must have $R' = \mathcal{M}$, so $\mathcal{M}$ is unique. $\square$

3 Circle packing general surfaces

So far we have proven existence of packings restricted to closed combinatorial discs. In this section we introduce an extension of the this result for the case when K is an open disc, and when K is a finite triangulation of a multiply connected surface.

3.1 Extension of the main theorem

Theorem 3.1. *Let K be a triangulation of an orientable topological surface S. Then there exists a Riemann surface $\mathcal{S}$, and unique up to conformal equivalence, and homeomorphic to S, and a packing P filling $\mathcal{S}$, unique up to conformal automorphisms of $\mathcal{S}$.*

The proof of the general case is beyond the scope of this thesis, and is due to [2], however, the special case for K a combinatorial sphere follows easily from the case for a finite disc [5][9].

Proof. (Koebe-Andreev-Thurston theorem) Let K be a triangulation of a topological sphere. Choose a vertex v_0, and delete this vertex and all faces and edges of K containing v_0. This will turn K into a new complex K', now a triangulation of a closed topological disc. By the main theorem (2.9), there exists a unique maximal hyperbolic packing L, with boundary circles as horocycles. By projecting P to to the Riemann sphere using stereographic projection, we can take the equator as the removed circle from K, and circles tangent to the equator are precisely those tangent to the removed circle, giving a packing for P for K.

To prove uniqueness of P, assume P' is another packing. By an appropriate Möbius transformation, P' may be transformed so that the circle corresponding to the the deleted vertex v_0 is the equator of the Riemann sphere. This, together with the uniqueness (up to conformal automorphism) of L from Theorem 2.9 imply P and P' are related by a Möbius transformation, which preserves the equator, so P is unique up to Möbius transformations of the Riemann sphere.

$\square$

3.2 Hyperbolic and Euclidean open discs

There is one case when the topological nature of K is not enough to determine the geometry of packings for K, namely, when K is a triangulation of an open disc. In this case, we need to distinguish between two possible situations.

Take an expanding sequence of simply connected finite subcomplexes $\{K_i\}$ of K, that is, the K_i have the property that $K_i \subset K_{i+1}$, and $\bigcup_i K_i = K$, and for each i, Theorem 2.9 means there exists a unique maximal packing label R_i. Then there are two possibilities: either $R_i(v) \to \alpha > 0$, or $R_i(v) \to 0$, and this is independent of the vertex v. The key tool in proving this independence is the Rodin Sullivan ring lemma, see [6].

In the first case, we obtain a hyperbolic packing, and in the second case, we obtain a Euclidean packing after an appropriate rescaling.

A complex is called constant n-degree if each vertex in K has exactly n neighbours. The dichotomy between hyperbolic and Euclidean packings can be readily seen from the constant degree packings. The constant 6-degree complex gives a Euclidean packing, and the constant n-degree complex, for $n \geq 7$ gives a hyperbolic packing (for $n \leq 5$, there can be no constant n-degree complex triangulating an open disc).

If K does not have constant degree, it becomes much harder to decide whether K will be hyperbolic or Euclidean.

3.3 Multiply connected surfaces

We now introduce a method for constructing packings for a special case of the extended version of the main theorem, namely when K triangulates a multiply connected orientable surface without boundary.

Theorem 3.2. *Let K be a triangulation of a multiply connected orientable* *surface S without boundary. There exists a univalent packing P on a Riemann surface $\mathcal{S}$, homeomorphic to K such that $Carr(K)$ fills $\mathcal{S}$. P is unique up to conformal automorphisms of $\mathcal{S}$, and $\mathcal{S}$ is unique up to conformal equivalence.*

We give a sketch of the proof. See [**?**] for a more detailed version.

Proof. Given a complex K as in the statement of the theorem, then the proof is (). Otherwise, it is a standard result that there exists a universal cover $\tilde{K}$, unique up to conformal equivalence. which is either hyperbolic ($\tilde{K} = \mathbb{D}$), or Euclidean ($\tilde{K} = \mathbb{C}$).

We can lift K by a map $\pi : K \to \tilde{K}$, giving a triangulation on $\tilde{K}$. Denote the set of covering transformations of $\tilde{K}$ by Λ. Note that since the elements of Λ have infinite order, $\tilde{K}$ is necessarily an infinite triangulation.

By the extension of the main theorem, there exists a packing $\tilde{P}$ for $\tilde{K}$ in the appropriate geometry. If $\tilde{K}$ is hyperbolic, $\tilde{P}$ is unique up to Möbius transformations, and if $\tilde{K}$ is Euclidean, $\tilde{P}$ is unique up to Euclidean isometries.

Let ϕ be a covering transformation of $\tilde{K}$. Then ϕ acts as simplicial homeomorphisms on the faces of $\tilde{K}$, inducing a map from $\tilde{P}$ to itself. Moreover, ϕ preserves the tangency patterns of circles in $\tilde{P}$. By the uniqueness of $\tilde{P}$, it must be the case that this map from $\tilde{P}$ to itself is given by a conformal automorphism ψ_ϕ such that

$$\psi_\phi\left(c\left(v\right)\right) = c\left(\phi\left(v\right)\right)$$

where $c(v)$ is the circle in $\tilde{P}$ associated with vertex v.

This enables us to take the quotient $\tilde{K}/\Lambda : \tilde{K} \to S$. Since we have shown $\tilde{P}$ is invariant under the elements of Λ, it follows the quotient gives a well defined circle packing on K.

Finally, the uniqueness of the Riemann surface S is also due to the uniqueness of $\tilde{P}$. Suppose P' is a second packing for K, filling a Riemann surface S'. But since the universal covers of S, S' only depends on the topology of K, both P and P' can be lifted to packings in the same universal cover, and by uniqueness, P and P' must be related by a conformal automorphism. Since $\tilde{P}$, $\tilde{P'}$ fill S,

S' respectively, there is an induced conformal automorphism taking S to S', completing the proof. $\square$

4 Computation of packing radii

So far we have established existence and uniqueness results, but have done little in the way of discussing how to calculate packings in practice. That is the focus of this chapter: given a complex K, to construct a packing label R for K. We follow the algorithm that has been implemented by Collins and Stephenson in [?].

Main case: K a triangulation of a topological closed disc. Most of the progress made into calculating packing labels has been done for the disc, and currently there is no known method to calculate packing labels when K triangulates a topological sphere. To construct a spherical packing, as in the main theorem, we puncture, pack as a disc and project back to the sphere.

4.1 Dirichlet boundary value problems

For the remainder of this section, assume K is a triangulation of a closed combinatorial disc. The approach to finding a packing label for K focuses on solving Dirichlet type boundary value problems, where given radii for the boundary vertices of K, we seek to solve for a packing label R for K on the interior vertices. That is, given a label $B(v)$ defined on the exterior vertices of K, we seek to a label R for K such that

$$\theta_R(v) = \begin{cases} 2\pi \text{ for } v \text{ an interior vertex} \\ B(v) \text{ otherwise} \end{cases}$$

Then to produce a maximal packing label, we simply set the label on the boundary vertices to be as large needed for the desired accuracy. The main result here is that given any B, such a label R exists.

Theorem 4.1. *Let K be a closed combinatorial disc, V the vertices of K, and $B : V\,(0, \infty]$. There exists a unique hyperbolic packing label R that agrees with B on the boundary vertices.*

For a proof, see [4].

4.2 The main algorithm

Given the existence of solutions to boundary value problems, we now seek an algorithm to find such solutions. The procedure described here is an iterative approach, starting with an arbitrary label R_0 with no requirement except that R_0 is fixed with given values for boundary vertices of K.

The main difficulty is that the equations (4.1) for the label R are highly non-linear, and generally impossible to solve analytically. Because of this, we use an algorithm whereby we iteratively adjust the radii of interior vertices, giving a sequence of labels $\{R_i\}$ that converge to a packing label R for K.

We will need a definition before stating the algorithm.

Definition 4.2. *Given an interior vertex $v \in K$ with neighbours $\{v_1, v_2, ..., v_k\}$, and corresponding label $R\,(v) = r$, $R\,(v_i) = r_i$, define $\hat{\theta} : \mathbb{R} \to \mathbb{R}$ by*

$$\hat{\theta}\,(r) = (\theta_L\,(v))\,(r)$$

where L is a label defined on v, v_i, such that $L\,(v_i)$ is a constant for $i = 1, 2, ..., k$, and $\hat{\theta}\,(r_0) = (\theta_L\,(v))\,(r_0)$.

In the definition, we consider r to be the variable, so $\hat{\theta}\,(v)$ is an approximation of the true value of $(\theta\,(v))\,(r)$ for r near r_0.

The algorithm

Let B as above be given as radii of boundary vertices of K.

(1) Choose any starting label R_0 subject to $R_0\,(w) = B\,(w)$ for all boundary vertices w.

(2) For each internal vertex v of K, adjust $R_n(v)$ so that $\hat{\theta}(R_n(v)) = 2\pi$. Call this adjusted value $R_{n+1}(v)$.

(3) Repeat step (2) until the desired level of accuracy of angle sums is achieved.

We claim that using this algorithm to produce a sequence of labels $\{R_i\}$ implies $R_i \to R$ as $i \to \infty$, where R is a packing label for K. For the details of the proof, see [4].

It remains to see how to choose the label L as above, so that $\hat{\theta}(r_0) = (\theta_L(v))(r_0)$. In other words, how to find the fixed radius $\hat{r}$ so that replacing the values of $R(w)$ by $\hat{r}$ for all neighbouring vertices w of v leaves $\theta_R(v)$ unchanged.

We also need to know how to adjust $R_n(v)$ in the second part of the algorithm, so that $\hat{\theta}(R_n(v)) = 2\pi$.

We state the results here, and refer to [4] for the details.

Lemma 4.3. *Let v be an interior vertex of K. Define $\hat{r}$ to be the radius of the neighbours of v such that*

$$\hat{\theta}(r_0) = (\theta_L(v))(r_0)$$

Define u to be the adjusted value of $R(v)$ such that

$$\hat{\theta}_R(v) = 2\pi$$

Then

$$\hat{r} = \begin{cases} \frac{\beta - \sqrt{v}}{\beta v - \sqrt{v}} \text{ for } R \text{ a hyperbolic label} \\ \frac{\beta}{1-\beta} v \text{ for } R \text{ a Euclidean label} \end{cases}$$

and

$$u = \begin{cases} \left(\frac{2\delta}{\sqrt{(1-\hat{r})^2 + 4\delta^2 \hat{r}} + 1 - \hat{r}} \right)^2 \text{ for } R \text{ a hyperbolic label} \\ \frac{1-\delta}{\delta} \hat{r} \text{ for } R \text{ a Euclidean label} \end{cases}$$

where $\beta = \sin\left(\frac{\theta_R(v)}{2k}\right)$, and $\delta = \sin\left(\frac{\pi}{k}\right)$.

This completes the algorithm for constructing packing labels for combinatorial closed discs.

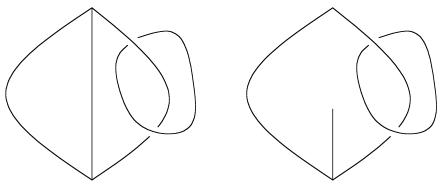


Figure 12: Legal and illegal graph projections

5 Drawing knotted graph projections

5.1 Knotted graph projections

Here we describe an algorithm developed to display projections of knotted graphs given by abstract combinatorial crossing information.

By a **knotted graph** in $\mathbb{R}^3$, we mean a finite set of vertices in R^3, together with a finite set of disjoint edges connecting pairs of vertices. We allow the possibility an edge connecting a vertex to itself, thereby including loops. Furthermore, we require that there are no open vertices (see fig (12)). By a **knotted graph projection** we mean the orthogonal projection to the plane of a knotted graph in such a way that any edge crossings occur transversely.

Let G be a knotted graph. The information required to construct a knotted graph projection for G can be captured by an abstract combinatorial data set. We have developed and implementated an algorithm written in the C programming language and *Mathematica* software package that takes such a data set, and produces a knotted graph projection for G.

Here we use the data format from [3], and is stored as a text file as follows:

The file begins with a header line (used to identify the knotted graph), followed by the total number of vertices and edge crossings on a new line. The bulk of the file is the the combinatorial vertex information, with the information for each vertex being written on a separate line, as follows:

Let $P(G)$ be a knotted graph projection, and let $\{v_0, v_1, ..., v_k\}$ be a list of vertices and edge crossings of $P(G)$. For each v_i, we first list v_i, followed by a

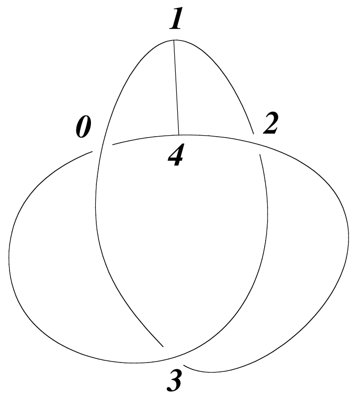


Figure 13: Example projection

symbol, one of $+$, $-$ or $*$, and finally a list $\{v_{i_j}\}$ of neighbours of v_i, written in a clockwise cyclic order. The symbol refers to the type of vertex or crossing v_i is. We write $+$ if the first listed neighbour v_{i_1} of v_i crosses over v_i, $-$ if it crosses under, and $*$ if v_i is a vertex (not a crossing). See fig (13) for an example of the data format. For fig (13), the data file is:

header line

5

0 + 1 4 3 3

1 * 2 4 0

2 + 3 3 4 1

3 + 0 0 2 2

4 * 0 1 2

Note that since the data format stores only the abstract combinatorial information for G, the data format only specifies how to draw G on the sphere. Therefore, to project to the plane, we must fix an external face of G. We usually choose this to be the face of having the greatest number of vertices.

Graph augmentation

The first step in producing a knotted graph projection $P(G)$ is to augment G to make it a triangulation. We do this by adding extra vertices and edges to G

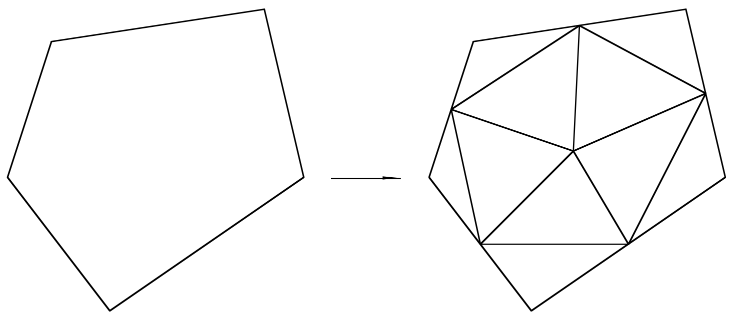


Figure 14: Internal face augmentation

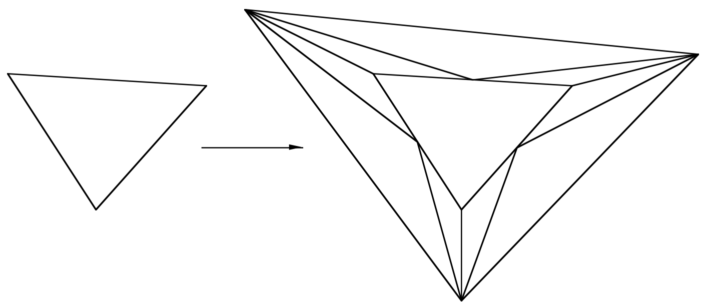


Figure 15: External face augmentation

according to the following pattern:

For each edge of G, add a vertex in the middle of the edge. Then for each internal face, connect each new vertex with the new vertices on the adjacent edges. If the newly created face is not a triangle, add another new vertex to the center of the face, and connect each of the new vertices in the face to this central vertex. This will guarantee all internal faces become triangulated.

For the external face, connect a new vertex to each original vertex on the external face with a new edge. Then connect each newly added vertex to the adjacent newly added vertices, and finally connect newly added vertices to the external middle edge vertices to finish the triangulation. See fig (14) for an example.

Having augmented $P(G)$, we are ready to go ahead and pack the resulting triangulation. The projection can then be reconstructed and displayed by

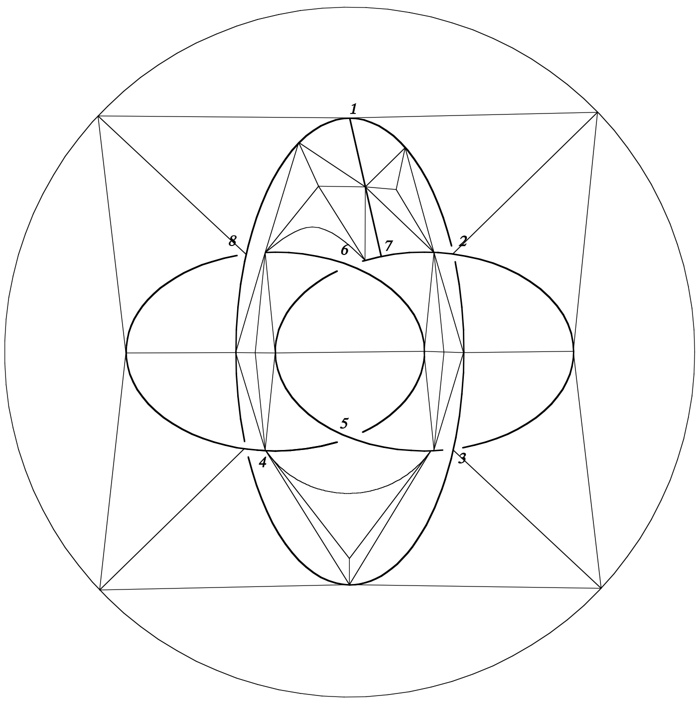


Figure 16: Augmented graph

connecting the centers of the packing $P(G)$ to display the graph. The reconstruction of the original graph is nicely represented by connecting the centers of the circles in the appropriate order using cubic splines.

5.2 Cubic splines

Lemma 5.1. *Given a set points $C = \{a_1, a_2, ..., a_n\}$, $a_i \in \mathbb{R}^2$, there exists a unique closed or open C^2 piecewise cubic curve $\gamma : [0,1] \to \mathbb{R}^2$, intersecting a_i in the given order. If γ is open, we have the boundary conditions $\gamma(0) = \gamma(1) = 0$*

The curve γ is called a cubic spline through $\{x_1, x_2, ..., x_n\}$, and we prove its existence as follows

Proof. We describe the curve γ by the parametric equations $X_i(t) = a_i + b_i t + c_i t^2 + d_i t^3$, where $a_i, b_i, c_i, d_i \in \mathbb{R}^2$.

The strategy is to solve for the first and second derivatives of X_i at the endpoints of each segment to satisfy the boundary conditions (**??**), and to solve for the constants a_i, b_i, c_i, d_i in terms of the values of X_i, and its derivative X_i' evaluated at the endpoints of each segment X_i.

To solve for a_i, b_i, c_i, d_i, evaluating X_i, X_i' at the endpoints gives

$$X_i(0) = a_i$$

$$X_i'(0) = b_i$$

$$X_i(1) = a_i + b_i + c_i + d_i$$

$$X_i'(1) = b_i + 2c_i + 3d_i$$

Solving the system gives

$$a_i = X_i(0)$$

$$b_i = X_i'(0)$$

$$c_i = 3(X_i(1) - X_i(0)) - 2X_i'(0) - X_i'(1)$$

$$d_i = 2\left(X_i\left(0\right) - X_i\left(1\right)\right) + X_i'\left(0\right) + X_i'\left(1\right)$$

Since we insist γ is a C^2 curve, we have the boundary conditions on the endpoints of each segment X_i

$$X_i\left(1\right) = X_{i+1}\left(0\right)$$

$$X_i'\left(1\right) = X_{i+1}'\left(0\right)$$

$$X_i''\left(1\right) = X_{i+1}''\left(0\right)$$

$i = 1, 2, ..., n-1$. We then have two choices for the final boundary conditions on the endpoints of γ, depending on whether γ is to be a smooth closed curve, or an open curve. If we want to curve to be closed, we have, $X_n\left(1\right) = X_0\left(0\right)$, $X_n'\left(1\right) = X_0'\left(0\right)$, $X_n''\left(1\right) = X_0''\left(0\right)$. Otherwise, we set $X_0''\left(0\right) = X_n''\left(1\right) = 0$. The above equations lead to the linear system (after much simplification) (see [1] for details)

$$\begin{bmatrix} 4 & 1 & & & 1 \\ 1 & 4 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ & & 1 & 4 & 1 \\ 1 & & & 4 & 1 \end{bmatrix} \begin{bmatrix} X_1'\left(0\right) \\ X_2'\left(0\right) \\ \vdots \\ X_{n-1}'\left(0\right) \\ X_n'\left(0\right) \end{bmatrix} = \begin{bmatrix} 3\left(X_1\left(0\right) - X_n\left(0\right)\right) \\ 3\left(X_2\left(0\right) - X_0\left(0\right)\right) \\ \vdots \\ 3\left(X_n\left(0\right) - X_{n-2}\left(0\right)\right) \\ 3\left(X_0\left(0\right) - X_{n-1}\left(0\right)\right) \end{bmatrix}$$

If γ is a closed curve, and

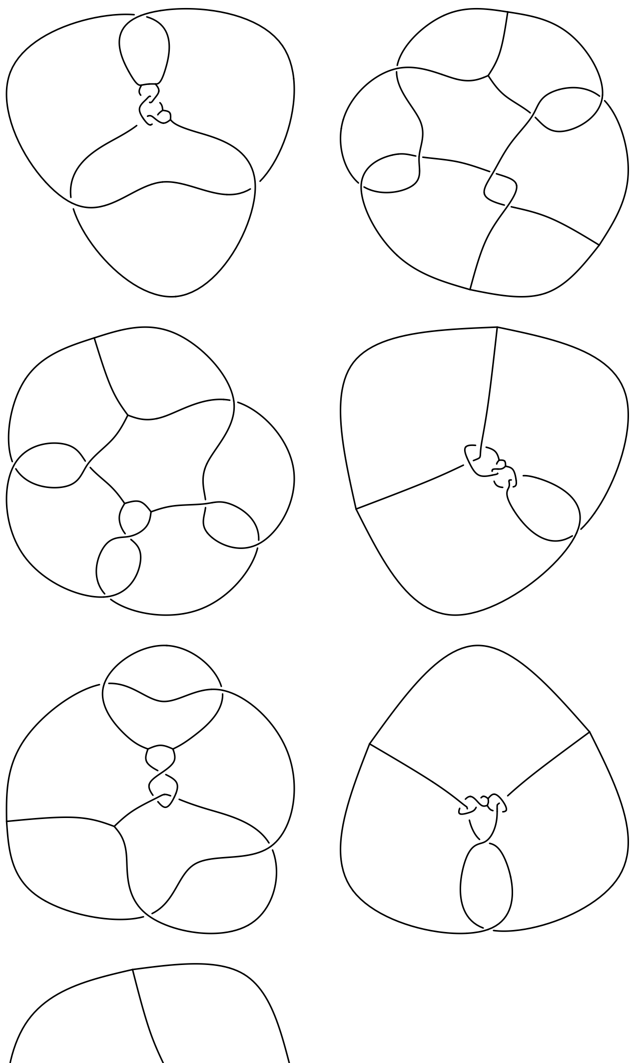
$$\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & 1 & 4 & 1 \\ & & & & 2 & 1 \end{bmatrix} \begin{bmatrix} X_1'\left(0\right) \\ X_2'\left(0\right) \\ X_3'\left(0\right) \\ \vdots \\ X_{n-1}'\left(0\right) \\ X_n'\left(0\right) \end{bmatrix} = \begin{bmatrix} 3\left(X_1\left(0\right) - X_0\left(0\right)\right) \\ 3\left(X_2\left(0\right) - X_0\left(0\right)\right) \\ 3\left(X_3\left(0\right) - X_1\left(0\right)\right) \\ \vdots \\ 3\left(X_n\left(0\right) - X_{n-2}\left(0\right)\right) \\ 3\left(X_n\left(0\right) - X_{n-1}\left(0\right)\right) \end{bmatrix}$$

otherwise. □

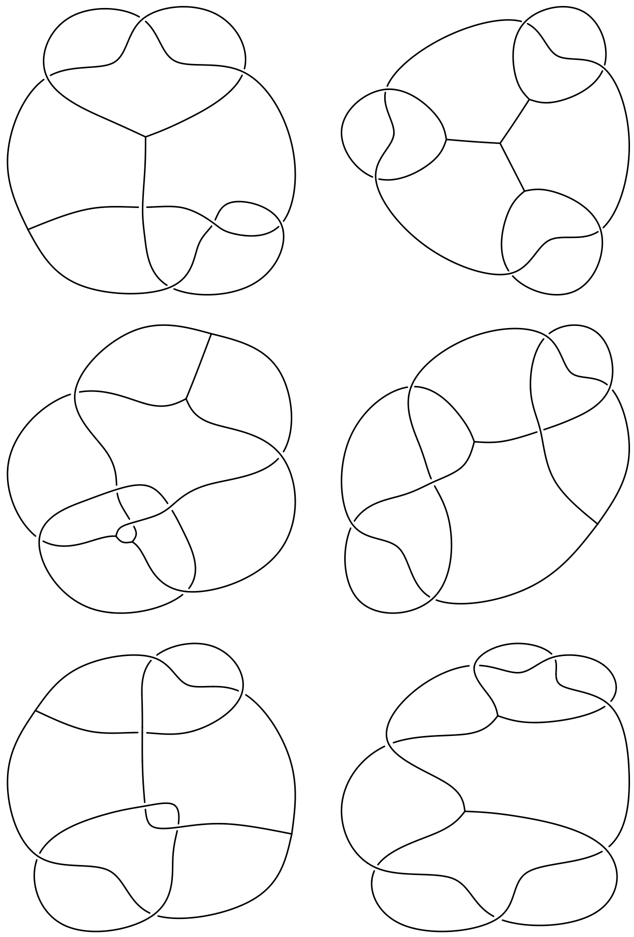
Returning to graph augmentation, we choose the $x_i \in \mathbb{R}^2$ to be the centres of the original circles produced from circlepacking the augmented knotted graph G, listed in the appropriate order. Then plotting a cubic spline through $\{x_i\}$ as above, gives a final knotted graph projection for G. We use closed cubic splines for components of G that are loops, and open cubic splines for components of G ending in vertices (not crossings). In the next section we provide some examples of the output of the program.

Some example knotted graph projections

The following figures are example knotted graph projections, all corresponding to the same knotted graph, but in each figure, the external face has been chosen differently:



Some further examples of knotted graph projections produced by the program:



6 Appendix

6.1 Stereographic Projection

Let $s = (\alpha, \beta, \gamma)$ be a point on the sphere $x_1^2 + x_2^2 + x_3^2 = 1$, $z = (\alpha, \beta, 0)$ be the vertical projection of x to the plane, and let $f : \mathbb{S}^2 \to \mathbb{R}^2$ be the stereographic projection map taking x to the intersection of the plane and the line connecting x to the north pole of the sphere, $(0, 0, 1)$, as in 17. Since both z and $f(x)$ lie along the same ray from the origin, we have

$$\frac{z}{|z|} = \frac{f(x)}{|f(x)|} \tag{8}$$

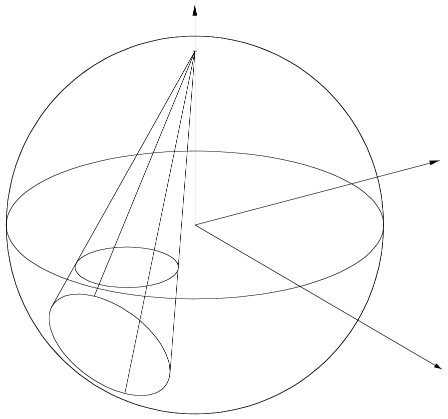


Figure 17: Stereographic projection

By similar triangles,

$$\frac{|z|}{1-\gamma} = |f(x)| \tag{9}$$

Substituting this back into (8) gives

$$f(x) = z\frac{|f(x)|}{|z|} = \left(\frac{\alpha}{1-\gamma}, \frac{\beta}{1-\gamma}, 0\right) \tag{10}$$

Let (u, v) be the coordinates of $f(x)$. Now let $g : \mathbb{R}^2 \to \mathbb{S}^2$ be the inverse map to f. Since x lies on the sphere, $|z|^2 + \gamma^2 = 1$, so

$$\begin{aligned} |f(x)|^2 &= \frac{|z|^2}{(1-\gamma)^2} && (11) \\ &= \frac{1-\gamma^2}{(1-\gamma)^2} && (12) \\ &= \frac{1+\gamma}{1-\gamma} && (13) \end{aligned}$$

Hence

$$\gamma = \frac{f(x)^2 - 1}{f(x)^2 + 1} \tag{14}$$

Substituting into (9) gives the expression for z, and hence x, in terms of $f(x)$:

$$\begin{aligned} z &= (1-\gamma) f(x) && (15)\\ &= \left(1 - \frac{|f(x)|^2 - 1}{|f(x)|^2 + 1}\right) f(x) && (16)\\ &= \frac{2}{|f(x)|^2 + 1} f(x) && (17) \end{aligned}$$

That is,

$$(\alpha, \beta, \gamma) = \left(\frac{2u}{|f(x)|^2 + 1}, \frac{2v}{|f(x)|^2 + 1}, \frac{|f(x)|^2 - 1}{|f(x)|^2 + 1}\right) \tag{18}$$

Lemma 6.1. *Stereographic projection provides a bijective correspondence between circles on the sphere and circles and straight lines in the plane.*

Proof. A circle on the sphere is obtained by intersecting a plane through the sphere. Let (u, v) be a point in the plane. The corresponding point on the sphere under stereographic projection is, by (18),

$$x = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right) \tag{19}$$

If x lies on the plane $Ax_1 + Bx_2 + Cx_3 = D$, then

$$A\frac{2u}{u^2 + v^2 + 1} + B\frac{2v}{u^2 + v^2 + 1} + C\frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} = D \tag{20}$$

So

$$2Au + 2Bv + (C - D)\left(u^2 + v^2\right) = D + C \tag{21}$$

If $C = D$, this is the locus of a straight line in the plane. If $C \neq D$, this is the locus of either a circle or a point in the plane, however, being the image of a circle on the sphere, cannot be a single point, so it must be a circle.

Conversely, if S is the locus of a circle or straight line in the plane, then S has the form $A'(u^2 + v^2) + B'u + C'v = D'$ for some A', B', C', D'. Then setting

$A' = C - D$, $B' = \frac{A}{2}$, $C' = \frac{B}{2}$, $D' = C + D$, and solving for A, B, C, D, so

$$A = 2B' \tag{22}$$

$$B = 2C' \tag{23}$$

$$C = \frac{A'+D'}{2} \tag{24}$$

$$D = \frac{D'-A'}{2} \tag{25}$$

The intersection of the plane $Ax_1+Bx_2+Cx_3 = D$ and the sphere $x_1^2+x_2^2+x_3^2 = 1$ is then the circle corresponding to S. $\square$

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