

Exposition with proof

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Proof of Fundamental Theorem of Algebra

In this paper I will be proving the fundamental theorem of algebra using complex analysis by incorporating Roche's theorem.

The fundamental theorem of algebra states that states that every polynomial equation of degree n with complex number coefficients has n roots, or solutions, in the complex numbers.

Also, let me explain Rouché's theorem as I will be using it soon in this proof. Rouché's theorem, named after mathematician Eugene Rouché, states that for any two complex-valued functions f and g holomorphic inside some region \mathcal{K} with closed contour ∂K , if $\|g(z)\| \leq \|f(z)\|$ on ∂K , then f and $f + g$ have the same number of zeros inside K , where each zero is counted as many times as its multiplicity. This theorem assumes that the contour ∂K is simple, that is, without self-intersections.

I'm going to show that for a large enough circle with its center at the origin, the image of this circle will wrap around the origin n times, but, under the assumption that the image of the polynomial lies in $C \setminus \{0\}$, it does not wrap around the origin at all.

Consider a circle $\Gamma = Re^{2ix\pi}$, with $x \in I$ and R which is large enough that $|R^n e^{2ixn\pi}| > |a_{n-1}R^{(n-1)}e^{2\pi ix(n-1)} + \dots + a_0|$. Then, using Rouché's Theorem,

$$\begin{aligned}\int_{f(\Gamma)} \frac{dz}{z} &= \int_{\Gamma} \frac{f'(z)}{f(z)} dz \\ &= \int_{\Gamma} \frac{z^{n'}}{z^n} dz \\ &= \int_{\Gamma} \frac{n}{z} dz \\ &= \int_0^1 \frac{n}{Re^{2\pi i}} 2\pi i Re^{2\pi ix} dx \\ &= \int_0^1 2\pi i n dx \\ &= 2\pi i n\end{aligned}$$

Now I'll show that under the assumption that f is never 0, the integral must be equal to 0. Polynomials are holomorphic functions, which means they're dif-

ferentiable in a neighborhood of each point in a domain in a complex coordinate space C^n . The inverses of holomorphic functions are holomorphic wherever the function is nonzero. If the image of f is contained in $C \setminus 0$, then $\frac{1}{f}$ is holomorphic everywhere. Also, f' is a polynomial of degree $n - 1$, so it's holomorphic everywhere. Then, $\frac{f'}{f}$ is holomorphic everywhere. By Cauchy's integral theorem, the integral of $\frac{f'}{f}$ over a closed path is 0. Then,

$$\int_{\Gamma} \frac{f'(z)}{f(z)} dz = \int_{\Gamma} \frac{d(z)}{z} dz = 0$$

This is a contradiction, therefore the image of f must contain 0. Thus, there's a minimum of one zero of f . Hence, the proof is complete.

References

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