Proof of Fundamental Theorem of Algebra (Exposition)

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In this paper I will be proving the fundamental theorem of algebra using complex analysis by incorporating Roche's theorem.

Background

The fundamental theorem of algebra states that states that every polynomial equation of degree n with complex number coefficients has n roots, or solutions, in the complex numbers. The theorem does not tell us what the solutions are, but rather how many solutions exist for a given polynomial function. For example, the polynomial $x^4 + 6x^3 - 36x^2 - 9x - 8$ has a degree of 4 because its largest exponent is 4. The highest degree of a polynomial tells us the number of solutions of a polynomial. This theorem is the foundation when it comes to solving polynomial equations and is extremely common throughout mathematics. It has practical applications. For example, if you are finding the solutions of a polynomial function of degree 4, you know that you must keep working until you find 4 solutions.

Not all the solutions must be real. It is important to note that the theorem says *complex solutions*, so the solutions can be imaginary. Complex numbers are in the form of a + bi where a and b are real numbers. The term a is the real part, and the term bi is imaginary. If b = 0, then the number is a real number. Therefore, all real numbers are complex numbers, their b's are just 0. Examples: In the complex number 7 + 4i, 7 is the real part and 4i is imaginary. In the complex number 4 + 0i, 4 is the real part and 0i is the imaginary part. Because b = 0, the number simplifies to 4. Lets say you are trying to solve the equation $X^2 + 25 = 0$. This can be factored out to $(x + 5i)(x5i)$ which gives two solutions, both of which are imaginary.

I will explain Rouche's theorem as I will be using it soon in this proof. Rouche's theorem, named after mathematician Eugene Rouche, states that for any two complex-valued functions f and g holomorphic inside some region $\mathcal{K}$ with closed contour ∂K, if $\|g(z)\| \leq \|f(z)\|$ on ∂ K, then f and $f + g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity. This theorem assumes that the contour ∂K is simple, that is, without self-intersections.

Additionally, holomorphic functions are complex-valued functions of at least one complex variables that is complex differentiable in a neighbourhood of each point in a domain in complex coordinate space C^n.

Proof

I'm going to show that for a large enough circle with it's center at the origin, the image of this circle will wrap around the origin n times, but, under the assumption that the image of the polynomial lies in $C\backslash\{0\}$, it does not wrap around the origin at all.

Consider a circle $\Gamma = Re^{2ix\pi}$, with x ⊂ I and R which is large enough that $|R^n e^{2ixn\pi}| > |a_{n-1}R^{(n-1)}e^{2\pi ix(n-1)} + ... + a_0|$.
Then, using Rouche's Theorem,

$\int_{f(\Gamma)} \frac{dz}{z} = \int_\Gamma \frac{f'(z)}{f(z)}dz$

$= \int_\Gamma \frac{z^{n\prime}}{z^n}dz$

$= \int_\Gamma \frac{n}{z}dz$

$= \int_0^1 \frac{n}{Re^{2\pi i}} 2\pi iRe^{2\pi ix}dx$

$= \int_0^1 2indx\pi$

$= 2\pi in$

Now I'll show that under the assumption that f is never 0, the integral must be equal to 0. Polynomials are holomorphic functions, which means they're differentiable in a neighborhood of each point in a domain in a complex coordinate space C^n. The inverses of holomorphic functions are holomorphic wherever the function is nonzero. If the image of f is contained in $\mathcal{C}\ \ 0$, then $\frac{1}{f}$ is holomorphic everywhere. Also, f' is a polynomial of degree $n-1$, so it's holomorphic everywhere. Then, $\frac{f'}{f}$ is holomorphic everywhere. By Cauchy's integral theorem, the integral of $\frac{f'}{f}$ over a closed path is 0. Then,
$\int_\Gamma \frac{f'(z)}{f(z)}dz = \int_\Gamma \frac{d(z)}{z}dz = 0$

This is a contradiction, therefore the image of f must contain 0. Thus, there's a minimum of one zero of f. Hence, the proof is complete.

References

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