Exposition with proof

lgeel

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1 Introduction

In this paper I will be proving the fundamental theorem of algebra using complex analysis by incorporating Roche's theorem.

I going to show that for a large enough circle with it's center at the origin, the image of this circle will wrap around the origin n times, but, under the assumption that the image of the polynomial lies in $\mathcal{C}\backslash 0$, it does not wrap around the origin at all.

Consider a circle $\iota = Re^2 ix\pi$, with x $\subset$ I and R which is large enough that $|R^n e^{2ixn\pi}| > |a_{n-1}R^{n-1}e^{2\pi ix(n-1)} + ... + a_0|$. Then, using Rouche's Theorem,
$\int_{f(\Gamma)} \frac{dz}{z} = \int_\Gamma \frac{f'(z)}{f(z)}dz$
$= \int_\Gamma \frac{z^{n\prime}}{z^n}dz$
$= \int_\Gamma \frac{n}{z}dz$
$= \int_0^1 \frac{n}{Re^{2\pi i}}Re^2\pi xdz$
$= \int_0^1 2indx\pi dz$
$= 2\pi in$

Now I'll prove that under the assumption that f is never 0, the integral must be equal to 0. Polynomials are holomorphic functions, which means they're differentiable in a neighborhood of each point in a domain in a complex coordinate space C^n. The inverses of holomorphic functions are holomorphic wherever the function is nonzero. If the image of f is contained in $\mathcal{C}$ 0, then $\frac{1}{f}$ is holomorphic everywhere. Also, f' is a polynomial of degree $n-1$, so it's holomorphic everywhere. Then, $\frac{f'}{f}$ is holomorphic everywhere. By Cauchy's integral theorem, the integral of $\frac{f'}{f}$ over a closed path is 0. Then,
$\int_\Gamma \frac{f'(z)}{f(z)}dz = \int_\Gamma \frac{d(z)}{z}dz = 0$

This is a contradiction, therefore the image of f must contain 0. Thus, there's a minimum of one zero of f. Hence, the proof is complete.