Exposition with proof

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April 2022

Proof of Fundamental Theorem of Algebra

In this paper I will be proving the fundamental theorem of algebra using complex analysis by incorporating Roche's theorem.

The fundamental theorem of algebra states that states that every polynomial equation of degree n with complex number coefficients has n roots, or solutions, in the complex numbers.

Also, let me explain Rouche's theorem as I will be using it soon in this proof. Rouche's theorem, named after mathematician Eugene Rouche, states that for any two complex-valued functions f and g holomorphic inside some region $\mathcal{K}$ with closed contour ∂K, if $\|g(z)\| \leq \|f(z)\|$ on ∂ K, then f and $f + g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity. This theorem assumes that the contour ∂K is simple, that is, without self-intersections.

I'm going to show that for a large enough circle with it's center at the origin, the image of this circle will wrap around the origin n times, but, under the assumption that the image of the polynomial lies in $C\backslash\{0\}$, it does not wrap around the origin at all.

Consider a circle $\Gamma = Re^{2ix\pi}$, with x $\subset$ I and R which is large enough that $|R^n e^{2ixn\pi}| > |a_{n-1}R^{(n-1)}e^{2\pi ix(n-1)} + ... + a_0|$. Then, using Rouche's Theorem,

$\int_{f(\Gamma)} \frac{dz}{z} = \int_\Gamma \frac{f'(z)}{f(z)} dz$

$= \int_\Gamma \frac{z^{n\prime}}{z^n} dz$

$= \int_\Gamma \frac{n}{z} dz$

$= \int_0^1 \frac{n}{Re^{2\pi i}} 2\pi i Re^{2\pi ix} dx$

$= \int_0^1 2indx\pi$

$= 2\pi in$

Now I'll show that under the assumption that f is never 0, the integral must be equal to 0. Polynomials are holomorphic functions, which means they're dif-

ferentiable in a neighborhood of each point in a domain in a complex coordinate space C^n. The inverses of holomorphic functions are holomorphic wherever the function is nonzero. If the image of f is contained in $\mathcal{C}\ \ 0$, then $\frac{1}{f}$ is holomorphic everywhere. Also, f' is a polynomial of degree $n-1$, so it's holomorphic everywhere. Then, $\frac{f'}{f}$ is holomorphic everywhere. By Cauchy's integral theorem, the integral of $\frac{f'}{f}$ over a closed path is 0. Then,
$\int_\Gamma \frac{f'(z)}{f(z)} dz = \int_\Gamma \frac{d(z)}{z} dz = 0$

This is a contradiction, therefore the image of f must contain 0. Thus, there's a minimum of one zero of f. Hence, the proof is complete.

References

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