## MATH 456 — Mathematical Modeling

**Assignment #3** **Due Date:** February 21 2022, 11:59 PM

**Question 1.** The probability that a child has blue eyes is 1/4. Assume independence between children. Consider a family with 5 children.

(a) If it is known that at least one child has blue eyes, what is the probability that at least 3 children have blue eyes?

(b) If it is known that the youngest child has blue eyes, what is the probability that at least 3 children have blue eyes?

**Question 2.** Let $X = (X_a, X_b) \in \mathbb{R}^n$ be a normal random vector with $X_a \in \mathbb{R}^a$ and $X_b \in \mathbb{R}^b$ with $a + b = n$. Assume that mean and covariance of $X$ are given by

$$\mu = (\mu_a, \mu_b) \text{ and } \Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb}. \end{pmatrix}$$

Prove the following:

(a) The marginal distribution of $X_a$ is $X_a \sim N(\mu_a, \Sigma_{aa})$.

(b) The conditional distribution of $X_b$ given $X_a = x_a$ is

$$X_b | X_a = x_a \sim N(\mu_b + \Sigma_{ba}\Sigma_{aa}^{-1}(x_a - \mu_a), \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab}).$$

(c) For any $a \in \mathbb{R}^n$, then $a^T X \sim N(a^T\mu, a^T\Sigma a)$.

**Question 3.** Suppose we toss a coin once and let $p$ be the probability of heads. Let $X$ denote the number of heads and $Y$ denote the number of tails.

(a) Prove that $X$ and $Y$ are dependent.

(b) Let $N \sim \text{Poisson}(\lambda)$ and suppose that we toss a coin $N$ times. Let $X$ and $Y$ be the number of heads and tails. Show that $X$ and $Y$ are independent.

**Question 4. [Coding]** **(A universal random number generator)**. Let $X$ have a continuous, strictly increasing CDF. Let $Y = F(X)$. Find the density of $Y$. This is called the probability integral transformation. Now let $U \sim \text{Uniform}(0, 1)$ and let $X = F^{-1}(U)$. Show that $X \sim F$. Now write a program that takes $\text{Uniform}(0, 1)$ random variables and generates random variables from a $\text{Exp}(\beta)$ distribution.

**Question 5. [Coding]** Consider tossing a fair die. Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$. Then $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 2/3$ and $\mathbb{P}(AB) = 1/3$. Since $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$, the events $A$ and $B$ are independent. Simulate draws from the sample space and verify that $\hat{P}(AB) = \hat{P}(A)\hat{P}(B)$ where $\hat{P}$ is the proportion of times an event occurred in the simulation. Now find two events $A$ and $B$ that are not independent. Compute $\hat{P}(A)$, $\hat{P}(B)$ and $\hat{P}(AB)$. Compare the calculated values to their theoretical values. Report your results and interpret.

**Note: Document your results for coding questions using word, latex or markdown, and submit the documents along with the codes.**