

Construction and Properties of Cospectral Non-Backtracking Matrices of Graphs

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Non-backtracking Walks

Non-backtracking Random Walk: (NBRW) A sequence of vertices (v_0, v_1, \dots) such that v_i is chosen uniformly at random from among the neighbors of v_i other than v_{i-1} .

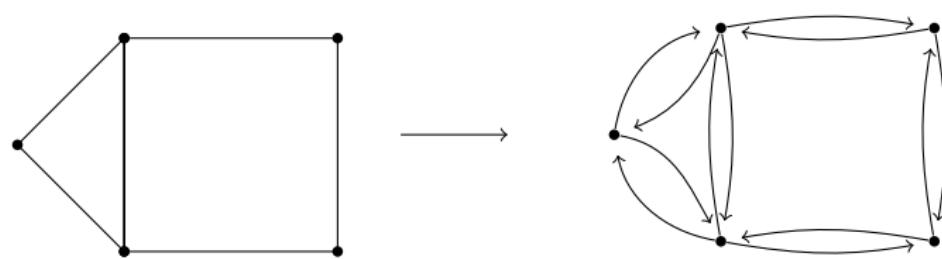
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Notice a NBRW as described is not a Markov Chain.

Living on the Edge

....However, a NBRW can be turned into a Markov chain by changing the state space to directed edges.



Edge Walk Matrix

Non-backtracking matrix: $B((u, v), (x, y)) = \begin{cases} 1 & \text{if } v = x \text{ and } u \neq y \\ 0 & \text{otherwise.} \end{cases}$

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Theorem (Ihara's Theorem, 1966)

$$\det(I - uB) = (1 - u^2)^{m-n} \det(I - uA + u^2(D - I))$$

Properties of the Ihara Matrix

We define the Ihara matrix K as

$$K = \begin{bmatrix} A & D - I \\ -I & 0 \end{bmatrix}.$$

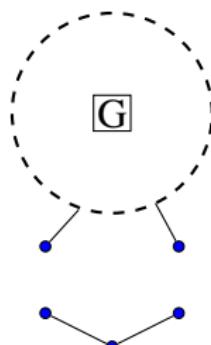
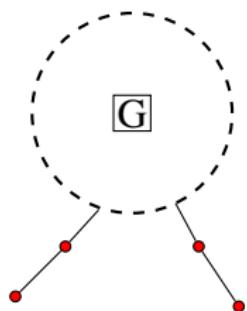
The characteristic polynomial of K is $\det(\mu^2 I - \mu A + (D - I))$.

Properties of the Ihara matrix include

- ① The geometric multiplicity of 1 as an eigenvalue of K counts the number of components in G .
- ② The geometric multiplicity of -1 as an eigenvalue of K counts the number of bipartite components in G .

Comparing the K Matrix to the NB Matrix

Cospectrality with respect to the K matrix is stronger than cospectrality with respect to the NB matrix.



Special Case of Godsil-McKay Switching

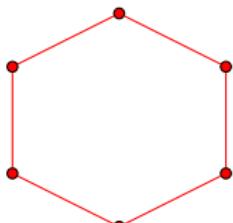
Theorem: Let G_1 be a d -regular graph with adjacency matrix A_1 , and G_2 be an arbitrary graph with adjacency matrix A_2 . Suppose the order m (resp. k) of G_1 (resp. G_2) is even. Let C be an $m \times k(0, 1)$ -matrix such that $Ce_m = \frac{k}{2}e_m$ and $Ce_k = \frac{m}{2}e_k$, where e_m (resp. e_k) is all-one vector of order m (resp. k). Let the adjacency matrices of graphs G and H be given as follows:

$$A_G = \begin{bmatrix} A_1 & C \\ C^T & A_2 \end{bmatrix}, \quad A_H = \begin{bmatrix} A_1 & J - C \\ J^T - C^T & A_2 \end{bmatrix},$$

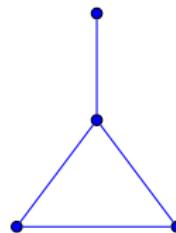
where J is an m by k all-one matrix. Then A_G and A_H are cospectral in regards to their Ihara matrices and Non-Backtracking matrices.

Example of Ihara Cospectral Graphs Using GM-Switch

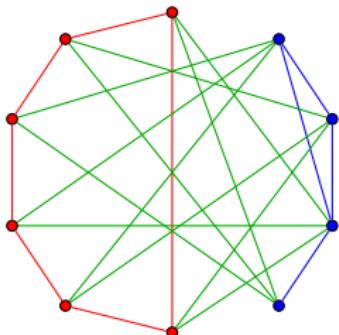
A_1



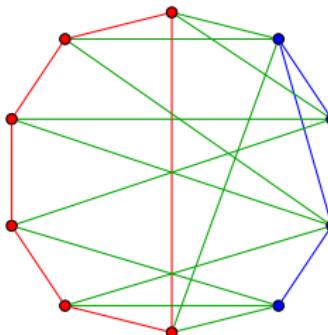
A_2



A_G



A_H



Summary of Proof

We show that the Ihara matrices of A_G and A_H are similar matrices. We start by defining the matrix Q :

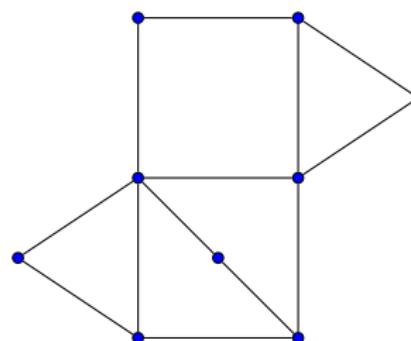
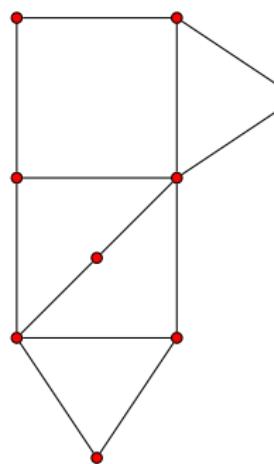
$$Q = \begin{bmatrix} \frac{2}{m}J - I_m & 0 \\ 0 & I_k \end{bmatrix}. \quad (1)$$

Q has the useful property that $Q = Q^{-1}$ and $QA_GQ^{-1} = A_H$. We know that $D_G = D_H$ based on the way we constructed A_G and A_H . Since A_1 is a regular graph, we get the following similarity:

$$\begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A_G & D_G - I_m \\ -I_k & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} A_H & D_H - I_m \\ -I_k & 0 \end{bmatrix}. \quad (2)$$

Other Ihara Cospectral Graphs

Our special case of GM switching does not generate all Ihara cospectral graphs. The following graphs are Ihara cospectral, but are not adjacency cospectral. We hope to characterize more families of Ihara cospectral graphs.



Degree Sequence

Conjecture

If two graphs are Ihara cospectral, then they have the same degree sequence.

Approach: Determine what information about vertices and edges we can gather from the Ihara matrix. We can deduce the number of edges from the spectrum of K^2 . Higher powers of K may reveal more information about the degree sequence.

Stricter Conditions

If we assume the Ihara matrices are similar, we obtain

$$\begin{bmatrix} A_1 & D_1 - I \\ -I & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A_2 & D_2 - I \\ -I & 0 \end{bmatrix}.$$

This leads to the system of equations

- ① $A_1 S_{11} + (D_1 - I) S_{21} = S_{11} A_2 - S_{12},$
- ② $-S_{11} = S_{21} A_2 - S_{22},$
- ③ $-S_{12} = S_{21} (D_2 - I),$
- ④ $A_1 S_{12} + (D_1 - I) S_{22} = S_{11} (D_2 - I).$

Stricter Conditions

We can use the system of equations to derive

$$(D_1 - I)(S_{11} + S_{21}A_2) = (S_{11} + A_1S_{21})(D_2 - I).$$

If $S_{21}A_2 = A_1S_{21}$, then we would obtain a similarity between $D_1 - I$ and $D_2 - I$. Since $D_1 - I$ is diagonal, this would imply the degree sequences are the same. Assumptions:

- The Ihara matrices are similar.
- S_{21} is invertible and is the similarity matrix for A_1 and A_2 .
- The matrix $S_{11} + S_{21}A_2$ is invertible.

This is a lot of restriction, so this may only represent very special cases of non-backtracking cospectral pairs (if any). However, it does show that there seems to be a connection between the degree sequences and similarity with respect to the adjacency matrices.

Further Research

Conjecture

If two graphs are non-backtracking cospectral and adjacency cospectral, then they have the same degree sequence.

Questions:

- Can any two of the adjacency matrix, Laplacian, signless Laplacian, or non-backtracking matrix guarantee the same degree sequence?
- Can we determine the degrees of the vertices from the Ihara matrix?
- Is there an easy way of creating non-backtracking cospectral pairs with minimum degree at least 2?
- Can we characterize more families of Ihara cospectral graphs?