Construction and Properties of Cospectral Non-Backtracking Matrices of Graphs

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What is Non-Backtracking?

A non-backtracking random walk is a random walk on a graph with the added requirement that each step cannot travel to the vertex visited on the immediate previous step. The non-backtracking matrix of a graph can be defined by viewing it as a walk on edges rather than vertices. [1]
Regarding the spectrum of the non-backtracking matrix, the following holds for any graph.

Ihara's Theorem: *Given a graph G with n vertexs and m edges, let B be the non-backtracking matrix of G. Let A denote the adjacency matrix of G and D the degree matrix. Then*

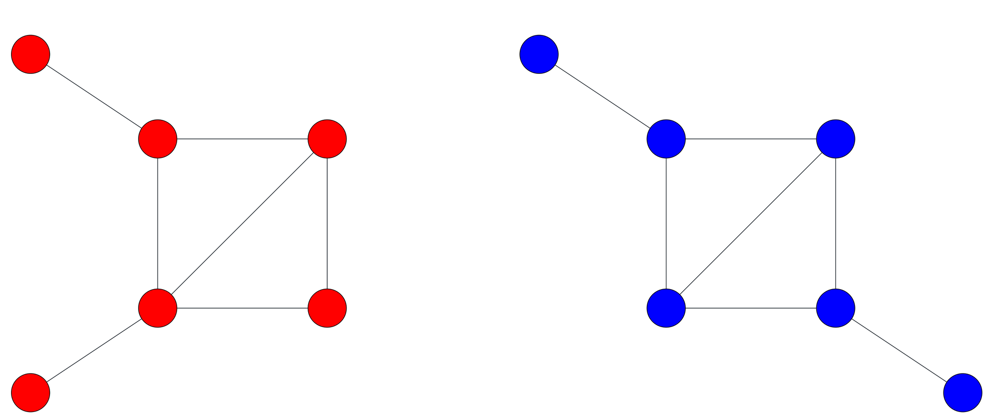
$$det(I - uB) = (1 - u^2)^{m-n} det(u^2(D - I) - uA + I)$$

Graphs with Leaf Nodes

Glover and Kempton in [1] show that **leaf nodes** contribute an eigenvalue of **0** to the spectrum of the non-backtracking matrix.
Likewise, **connected components** contribute an eigenvalue of **1**.
Hence, we have chosen to narrow our research to **connected graphs** with **minimum degree 2.**

Example of Cospectral Graphs with Leaf Nodes

Note that the following pair of graphs below are non-isomorphic extensions of the same graph by appending leaf nodes to it (but at different nodes). From the discussion above, we can determine they are Non-Backtracking cospectral.



Non-backtracking cospectral graphs are easily generated by appending leaf nodes to graphs. The following theorem provides a method for generating non-backtracking cospectral connected graphs with minimum degree 2. This method was originally developed in [2] to generate graphs which are adjacency, Laplacian, sign-less Laplacian, and normalized Laplacian cospectral. We have an alternate proof showing that they are also non-backtracking cospectral. Note that these graphs may be isomorphic.

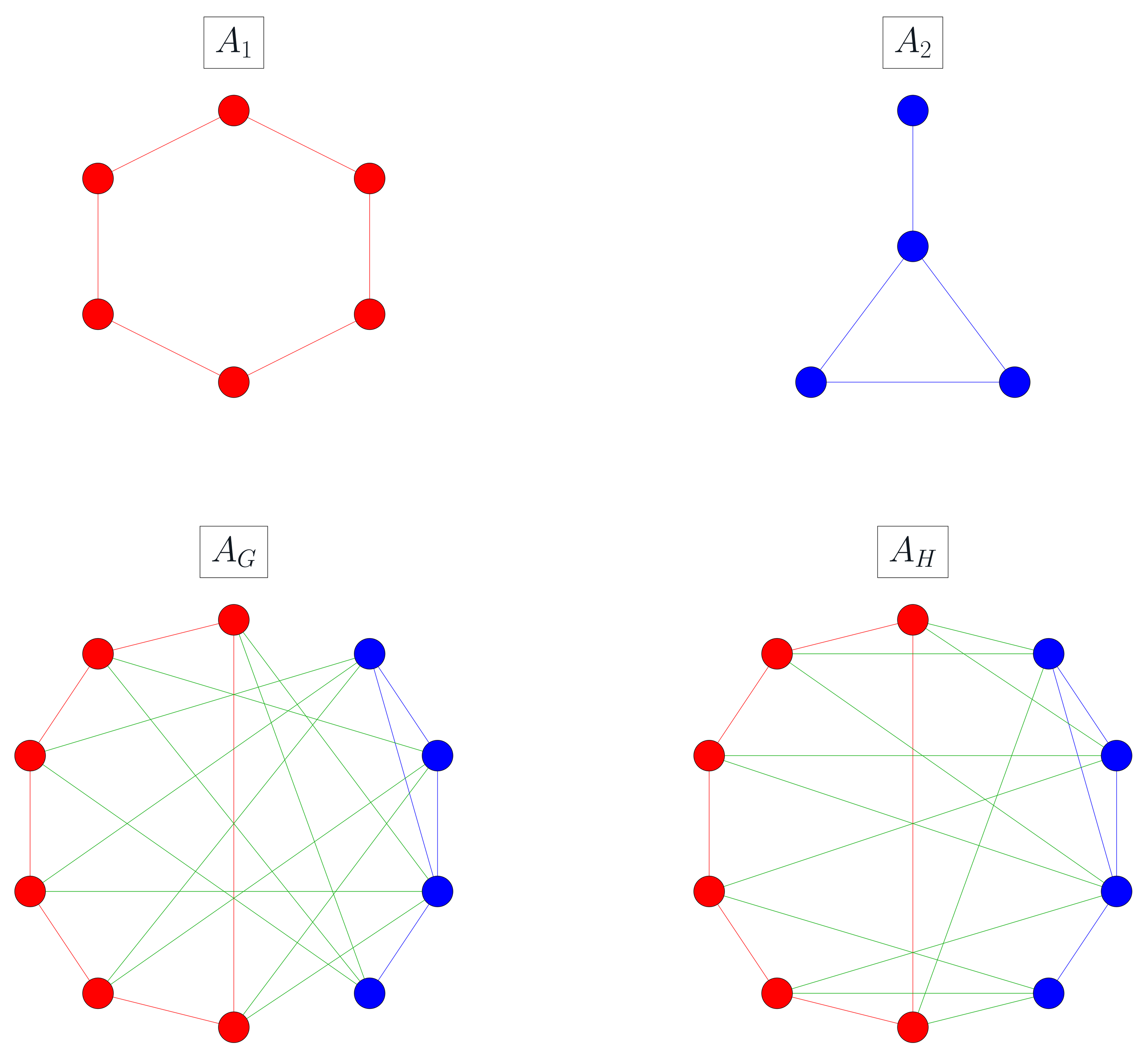
The Main Theorem

Let G_1 be a d-regular graph with adjacency matrix A_1, and G_2 be an arbitrary graph with adjacency matrix A_2. Suppose the order m (resp. k) of G_1 (resp. G_2) is even. Let C be an $m \times k(0,1)$- matrix such that $Ce_m = \frac{k}{2}e_m$ and $Ce_k = \frac{m}{2}e_k$, where e_m (resp. e_k) is all-one vector of order m (resp. k). Let the adjacency matrices of graphs G and H be given as follows:

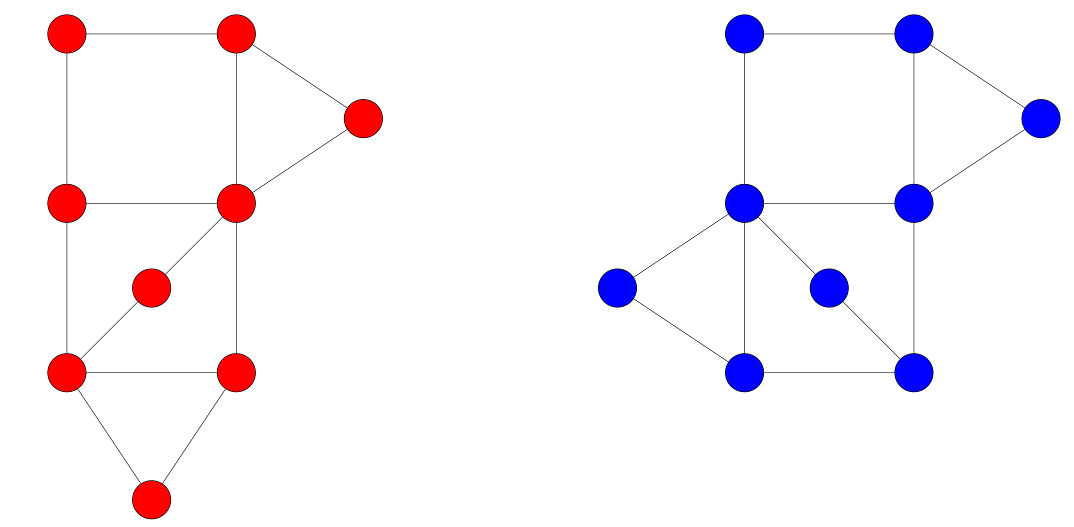
$$A_G = \begin{bmatrix} A_1 & C \\ C^T & A_2 \end{bmatrix}, \qquad A_H = \begin{bmatrix} A_1 & J - C \\ J^T - C^T & A_2 \end{bmatrix},$$

where J is an m by k all-one matrix. Then A_G and A_H are cospectral in regards to their non-backtracking matrices.

Non-Isomorphic Example of Special Case of GM-Switch



Does Not Generate All NB-Cospectral Pairs



Up to 9 vertices, there are only three pairs of non-backtracking cospectral connected minimum degree 2 graphs which are neither adjacency, Laplacian, sign-less Laplacian, nor normalized Laplacian cospectral. Therefore, these pairs of graphs could not have been produced using our specialized case of GM-switching. Shown above is an example of a pair of graphs with this property.

Same Degree Sequence

As seen in our first example, without the assumption of connected minimum degree 2, not every non-backtracking cospectral pair of graphs have the same degree sequence. It is possible, however, that this added assumption may imply same degree sequence. The method described in **The Main Theorem always generates graphs with the same degree sequence** [2]. The three pairs of graphs mentioned above that were not generated using The Main Theorem also have the same degree sequence.

Open Questions

- Does non-backtracking cospectral imply same degree sequence for connected minimum degree 2 graphs?
- Does being cospectral with respect to adjacency, Laplacian, sign-less Laplacian, and normalized Laplacian matrices imply non-backtracking cospectral?
- Does being cospectral with regards to two of either adjacency, Laplacian, sign-less Laplacian, or normalized Laplacian matrix imply cospectral with regards to all four?

References

[1] Cory Glover and Mark Kempton. "Spectral properties of the non-backtracking matrix of a graph". In: (2020). arXiv: 2011.09385 [math.CO].

[2] Hongliang Lu Wei Wang Feng Li and Zongben Xu. "Graphs determined by their generalized characteristic polynomials". In: *Linear Algebra and its Applications* 434.5 (2011), pp. 1378–1387. ISSN: 0024-3795.