Solutions to Douglas B. West's

# Combinatorial Mathematics

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# Chapter 1

# Combinatorial Arguments

## 1.1 Classical Models

1.1.1 (−) When rolling $n$ dice, what is the probability that the sum is even?

*Solution.* The event that the sum of all rolls is even is equivalent to the event that there was not an odd number of odd rolls. The probability of this adverse event is

$$\sum_{\text{odd } k} \binom{n}{k} \left(\frac{1}{2}\right)^n \tag{1.1}$$

That is, fix an odd $k$. Select the indices of dice to assign the odd rolls to. They must roll odd. The probability of this event is $\left(\frac{1}{2}\right)^k$. The rest must roll even. The probability of this event is $\left(\frac{1}{2}\right)^{n-k}$. We claim that

$$\sum_{\text{odd } k} \binom{n}{k} = 2^{n-1}$$

By the Binomial Theorem,

$$\begin{aligned} 0 = (1-1)^n &= \sum \binom{n}{k} (-1)^k \\ &= \sum_{\text{even } k} \binom{n}{k} - \sum_{\text{odd } k} \binom{n}{k} \end{aligned}$$

Thus,

$$\sum_{\text{even } k} \binom{n}{k} = \sum_{\text{odd } k} \binom{n}{k} = 2^{n-1}$$

From Expression 1.1, the desired probability is

$$1 - \left(\frac{2^{n-1}}{2^n}\right) = \frac{1}{2}$$

□

1.1.2 (−) Count all the rectangles with positive area formed by segments in a grid of $m$ horizontal lines and $n$ vertical lines.

*Solution.* Index the vertical lines from left to right and the horizontal lines from bottom to top. A rectangle consists of a left (L) and right (R) boundary of vertical lines, as well as a top (T) and bottom (B) boundary consisting of horizontal lines. Select two vertical lines and then two horizontal lines to constrain our rectangle. In every pair, there is one way to assign labels to obtain a geometrically valid rectangle. That is, T > B and R > L. We assume this is what the author means by positive area, excluding the case of a negative length and width. Additionally, with every choice of pairs, exactly one rectangle is formed. Thus, by the multiplication principle, there are

$$\binom{m}{2}\binom{n}{2}$$

geometrically valid rectangles in the input grid. □

1.1.3 (−) The roman alphabet has 21 consonants and 5 vowels. How many strings can be formed using $r$ consonants and $s$ vowels?

*Solution.* We will always form a string of length $r+s$. First, choose the indices to place each vowel. There are $\binom{r+s}{s}$ such choices. In each of the $s$ slots, we independently choose a vowel to place. Thus, for this fixed indexing, there are $5^s$ many ways to assign vowels. Lastly, there are $21^r$ many ways to assign consonants to their corresponding indices of the string. By the multiplication principle, there are

$$\binom{r+s}{s} \cdot 5^s \cdot 21^r$$

many such strings. □

1.1.4 (−) Count the possible outcomes of an election with 30 voters and four candidates. Count those in which no candidate receives more than half the votes.

*Solution.* We assume an "outcome" to be a 4-tuple indicating the distribution of votes to the different candidates. The first exercise asks for the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

with the restriction that $x_i \geq 0$ for each $i$. By the model of dots and bars, there are $\binom{33}{3}$ distinct 4-tuples. The second exercise adds further restrictions to the $x_i$ such that for each $i$, $0 \leq x_i \leq 15$. Because only one candidate can have more than 15 votes at once, for each $i$ we remove the solutions in which $x_i > 15$. For $i = 1$, we count the number of nonnegative integer solutions to

$$x_1' + x_2 + x_3 + x_4 = 14$$

in which $x_1' = x_1 - 16$. For each $i$, there are $\binom{17}{3}$ distinct solutions. Thus, the number of nonnegative integer solutions with increased restrictions is

$$\binom{33}{3} - 4 \cdot \binom{17}{3}$$

□

1.1.5 (−) Prove that $(n^5 - 5n^3 + 4n)/120$ is an integer for every $n \in \mathbb{N}$.

*Proof.* The factored form of the polynomial is

$$\frac{(n+2)_{(5)}}{5!} = \binom{n+2}{5}$$

□