



Ramsey Numbers of Matchings in Ordered Graphs

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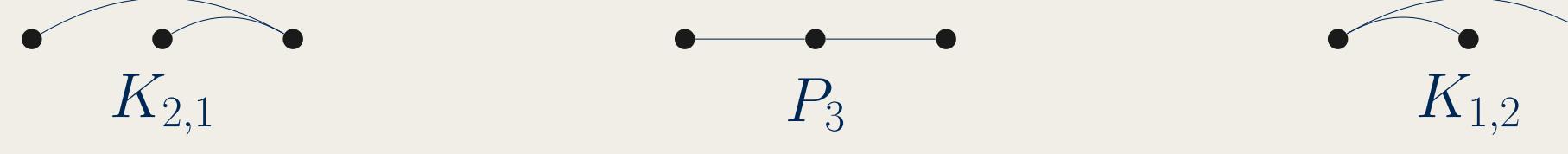


Introduction

Ordered Graph

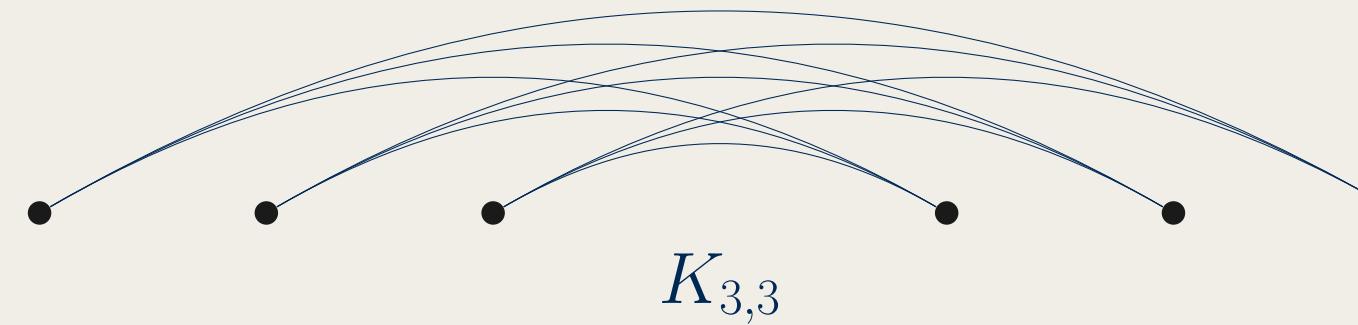
An ordered graph G is a graph in which $V(G)$ is linearly ordered.

$K_{2,1}$, P_3 , and $K_{1,2}$ are distinct ordered graphs.



Subgraph Containment

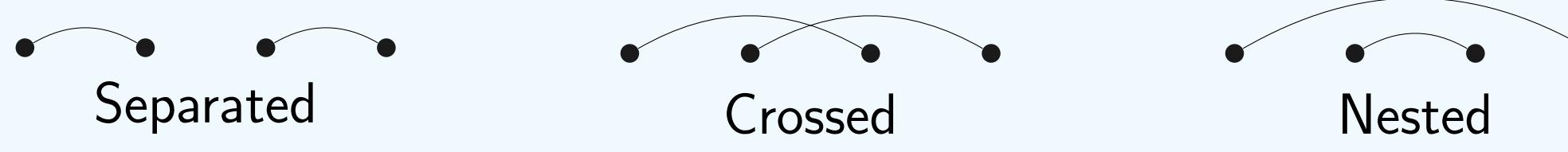
If H and G be ordered graphs, then $H \subseteq G$ means there is an order-respecting injection $f : V(H) \rightarrow V(G)$ such that $uv \in E(H)$ implies $f(u)f(v) \in E(G)$.



Note: $K_{1,2}, K_{2,1} \subseteq K_{3,3}$, but $P_3 \not\subseteq K_{3,3}$.

Separated, Crossed, Nested Edges

A matching of size 2 falls into one of 3 types.



A set of edges S is **separation-free**, **cross-free**, or **nest-free** if no pair of edges in S are separated, crossed, or nested.

SF_s , CF_s , and NF_s

Let SF_s , CF_s , and NF_s be the family of $2s$ -vertex ordered graphs whose edge sets are separation-free, cross-free, or nest-free matchings of size s .

$$CF_3 = \left\{ \begin{array}{c} \text{Diagram 1: } \text{Three vertices in a row with two edges connecting them.} \\ \text{Diagram 2: } \text{Three vertices in a row with three edges connecting them.} \\ \text{Diagram 3: } \text{Three vertices in a row with four edges connecting them.} \end{array} \right\}$$

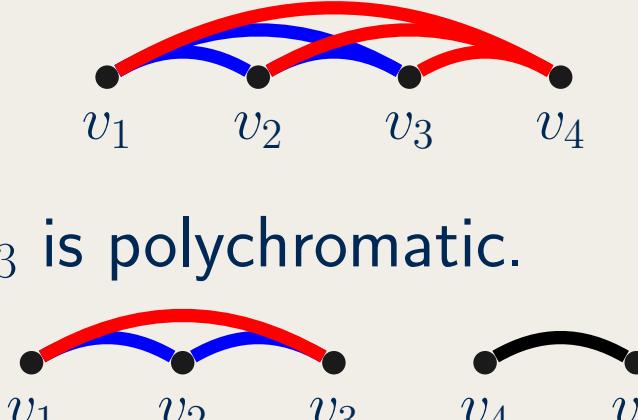
Note: $|CF_s| = s\text{-th Catalan number} = \frac{1}{s+1}\binom{2s}{s}$.

Ramsey Numbers

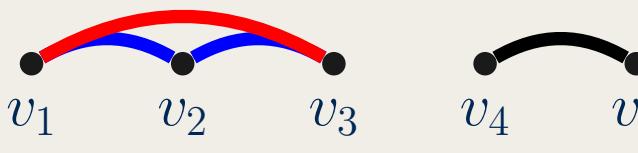
Given families $\mathcal{F}_1, \dots, \mathcal{F}_k$, the **Ramsey Number** $R(\mathcal{F}_1, \dots, \mathcal{F}_k)$ is the minimum n such that every k -edge coloring of K_n contains, for some color i , a color- i copy of some ordered graph in \mathcal{F}_i .

Warm-up: $R(CF_2, CF_2) = 5$.

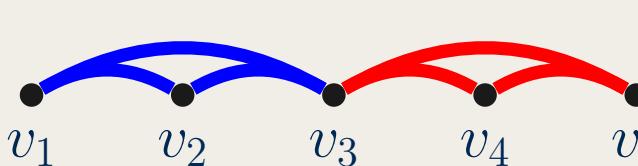
- $R(CF_2, CF_2) > 4$.



- $R(CF_2, CF_2) \leq 5$. Suppose $v_1v_2v_3$ is polychromatic.



Thus, $v_1v_2v_3$ is monochromatic. By symmetry, we have

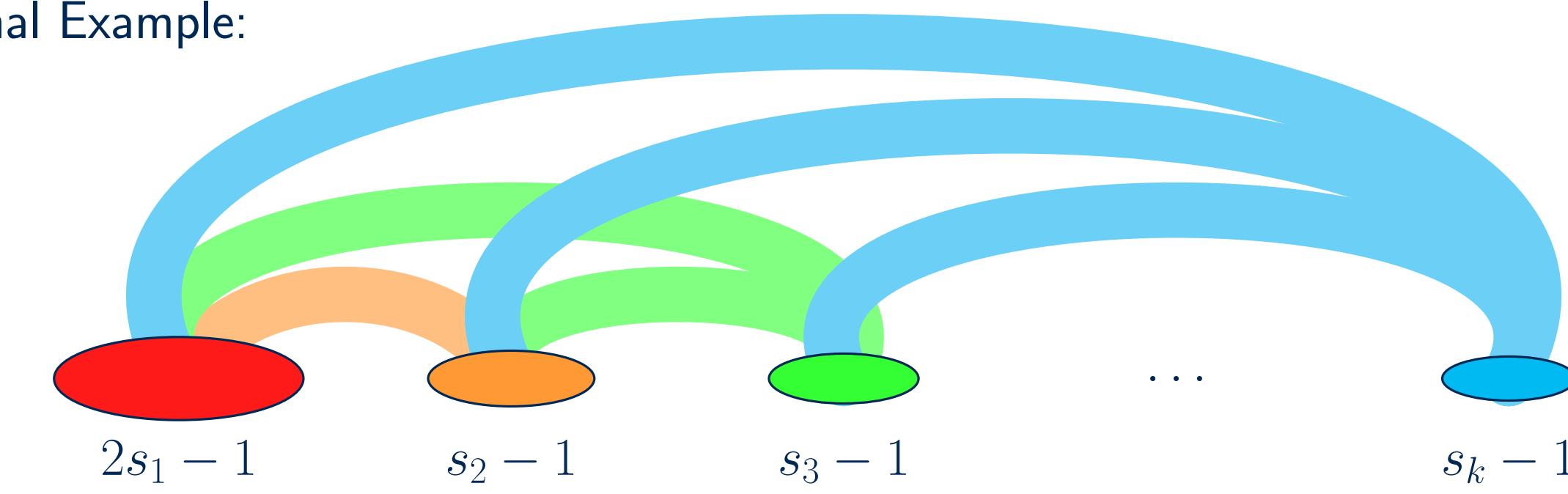


Theorem 1 (Cockayne-Lorimer (1975))

For $s_1 \geq s_2 \geq \dots \geq s_k$, we have

$$R(s_1K_2, \dots, s_kK_2) = \sum_{i=1}^k (s_i - 1) + (s_1 - 1) + 2.$$

Extremal Example:



Let $s_1 \geq s_2 \geq \dots \geq s_k$ and let $\mathcal{F}_{s_i} \in \{NF_{s_i}, SF_{s_i}, CF_{s_i}\}$. Then,

$$\begin{aligned} R(\mathcal{F}_{s_1}, \dots, \mathcal{F}_{s_k}) &\geq R(s_1K_2, \dots, s_kK_2) \\ &= \sum_{i=1}^k (s_i - 1) + (s_1 - 1) + 2. \end{aligned}$$

We call this the **Graph Lower Bound (GLB)**.

Prior Work on Ordered Graphs

- **BGT**: Barát–Gyárfás–Tóth (2023)
- **KS**: Kaiser–Stehlík (2020)
- **KPT**: Károlyi–Pach–Tóth (1997)

	Nest-free	Separation-free	Cross-free
2 colors $k = 2$	Thm (BGT) $R(NF_s, NF_2) = 2s + 1$ $R(NF_s, NF_3) = 2s + 2$	Thm (BGT) $R(SF_{s_1}, SF_{s_2}) = GLB$	Cor (KPT) $R(CF_s, CF_s) = 3s - 1$
2 edges $s = 2$	Thm (BGT) $R(NF_2; k) = k + 3$	Cor (KS) $R(SF_2; k) = k + 3$	Prop (BGT) $R(CF_2; k) > k + 3$
Gen. Case	Conj (BGT) $R(NF_{s_1}, \dots, NF_{s_k}) = GLB$	open	open

Our Results:

- Thm: $R(CF_2; k) = k + \Theta(\sqrt{k})$
- Prop: $R(NF_s, NF_4) = GLB = 2s + 3$ for $s \geq 4$.

Theorem 2 (Hammersla, H., Milans)

$$R(CF_2; k) \geq k + (1 - o(1))\sqrt{\frac{4}{\pi}k}$$

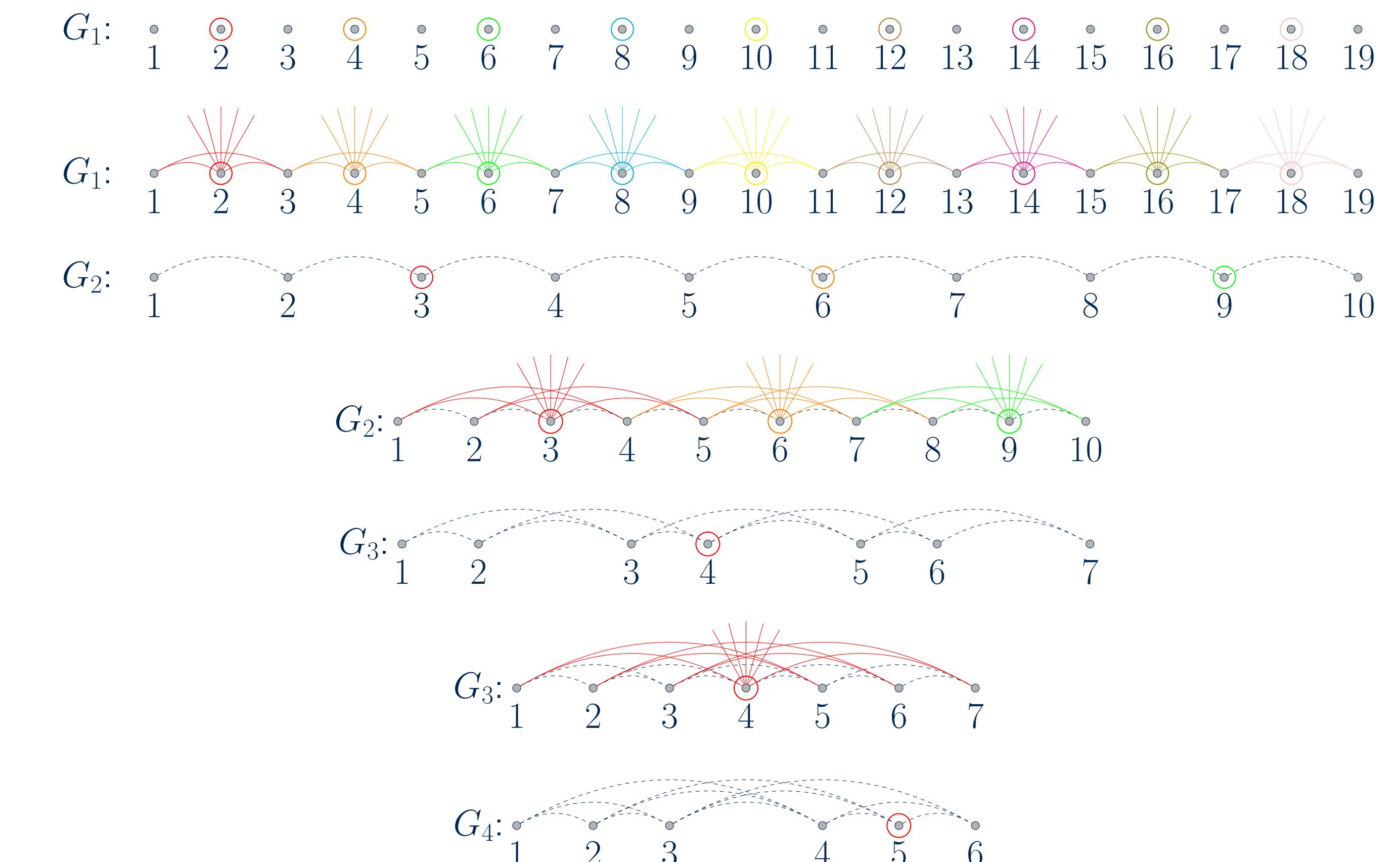
To prove this, we cover $E(K_n)$ for $n = k + (1 - o(1))\sqrt{(4/\pi)k}$ with k color classes such that every color class avoids graphs in CF_2 , where $CF_2 = \{ \text{Diagram 1, 2, 3} \}$.

If e and e' have the same color, they are not separated or nested. Thus, they are crossed or share an endpoint.

Our coloring proceeds in stages.

- At the start of stage i , we must color an ordered graph G_i where $V(G_i) = [n_i]$ and $uv \in E(G_i)$ implies $|u - v| \geq i$.
- Let $S = \{t : (i+1) \mid t\}$. We use a color class L_t for each $t \in S$. In particular, L_t contains $uv \in E(G_i)$ when
 - uv is incident to t , or
 - $|u - v| \in \{i, i+1\}$ with $t \in [u, v]$.
- Let $G_{i+1} = G_i - S - \bigcup L_t$, renaming vertices.
- The process ends when G_i is empty.

Num. colors used: $n - |V(G_\ell)|$, where ℓ is the last stage.



Josephus Sieve

Start with a list of the positive integers. At stage i , delete integers at indices divisible by $i + 1$.

$$\begin{aligned} 1 &\quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \cdots \\ 1 &\quad 3 \quad 5 \quad 7 \quad 9 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad 21 \quad 23 \quad 25 \quad 27 \quad 29 \quad 31 \quad 33 \quad 35 \quad 37 \cdots \\ 1 &\quad 3 \quad 7 \quad 9 \quad 13 \quad 15 \quad 19 \quad 21 \quad 25 \quad 27 \quad 31 \quad 33 \quad 37 \quad 39 \quad 43 \quad 45 \quad 49 \quad 51 \quad 55 \cdots \end{aligned}$$

After stage i , the first $i + 1$ entries will never be deleted. The numbers that survive form a sequence called the **Josephus Sieve**.

- A vertex $u \in V(G_1)$ survives to the end if and only if u appears in the Josephus Sieve.
- Let $\Psi(n)$ be the number of entries in the Josephus Sieve that are at most n .
- Thm: If $k \geq n - \Psi(n)$, then $R(CF_2; k) > n$.
- Thm (Andersson (1998)): $\Psi(n) = \sqrt{(4/\pi)n} + O(n^{1/6})$.

Theorem 3 (Hammersla, H., Milans)

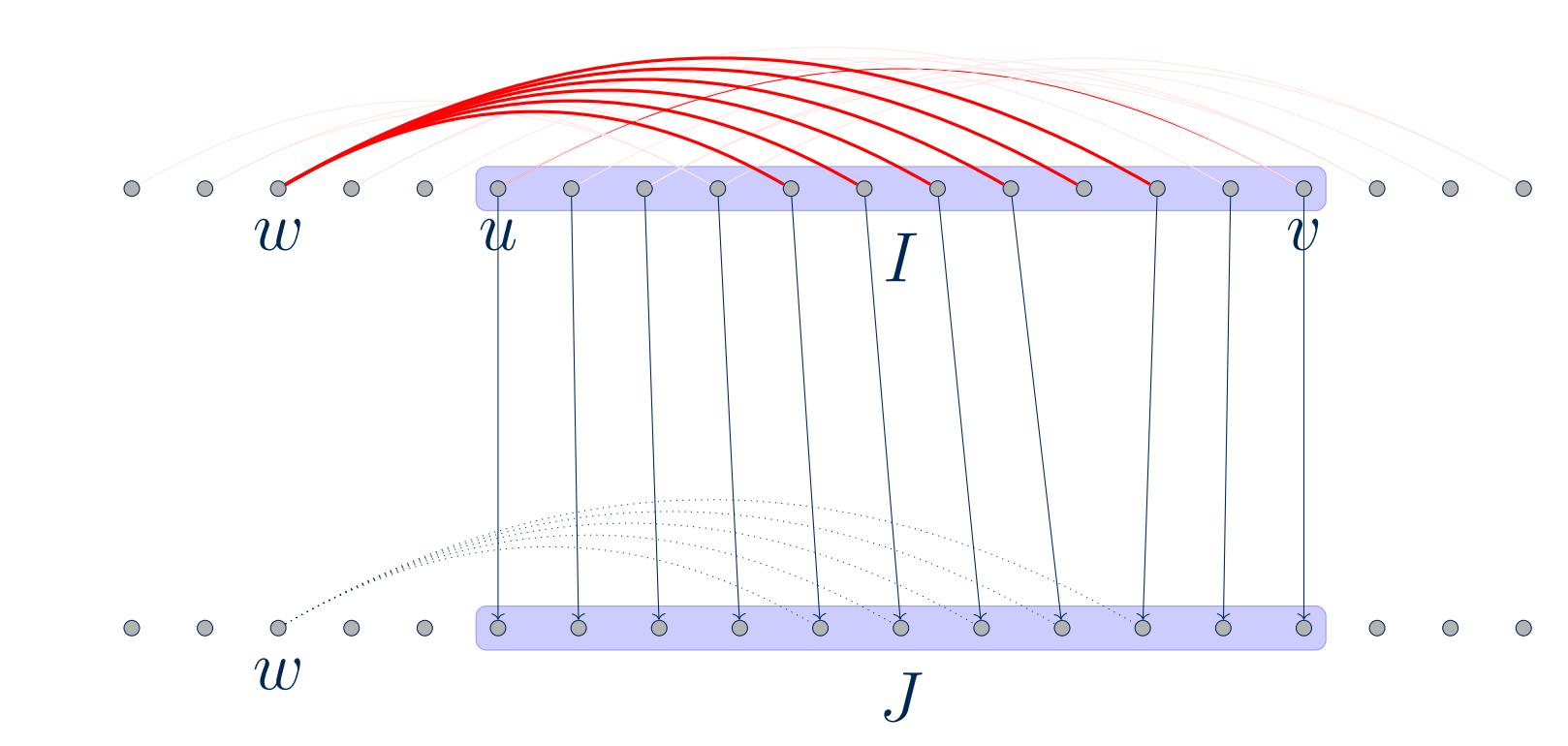
$$R(CF_2; k) \leq k + 1 + \lceil \sqrt{2k} \rceil.$$

Let an edge-coloring of an ordered graph be **good** if every color class avoids subgraphs in CF_2 .

Lemma 4: (Compression)

Let G be a good k -edge coloring of an n -vertex ordered graph with m non-edges. If $k > 0$, then there is a good $(k - 1)$ -edge coloring of an $(n - 1)$ -vertex ordered graph with at most $m + 1$ non-edges.

Compression Illustration:



Thm: If $\binom{n-k}{2} > k$, then $R(CF_2; k) \leq n$.