Ramsey Numbers of Crossing-Free Matchings in Ordered Graphs

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Ordered Graphs

- An ordered graph is a graph G whose vertices $V(G)$ are linearly ordered.

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P_3

$K_{1,2}$

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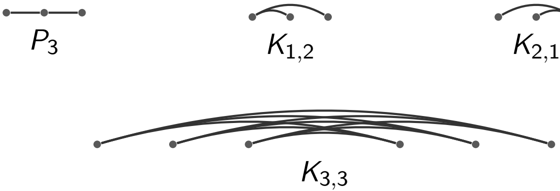

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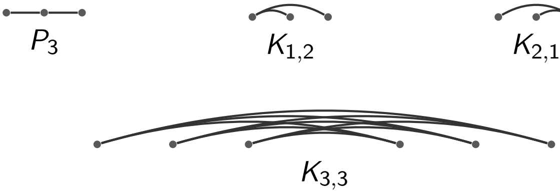

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- Note: $K_{1,2}, K_{2,1} \subseteq K_{3,3}$ but $P_3 \not\subseteq K_{3,3}$.

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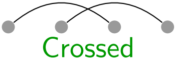

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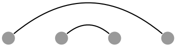

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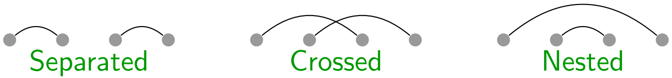

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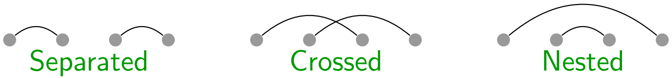

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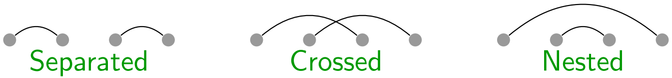

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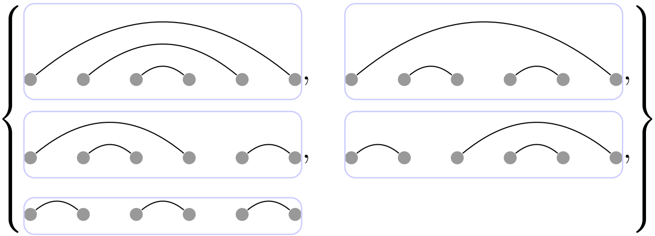

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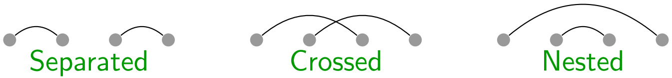

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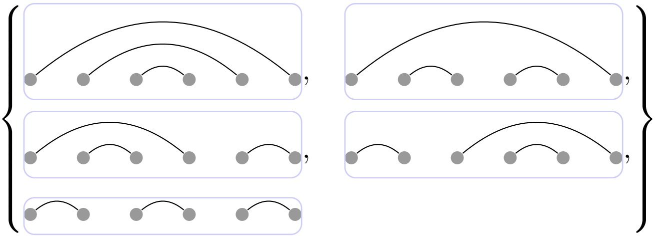
$\mathrm{CF}_3 =$ 

- $|\mathrm{CF}_s| = s$th Catalan number $= \frac{1}{s+1}\binom{2s}{s}$

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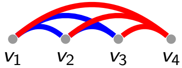

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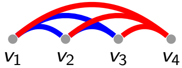

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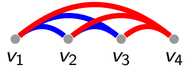

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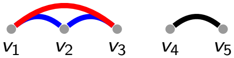

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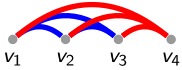

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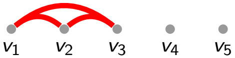

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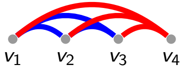

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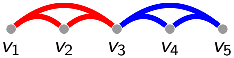

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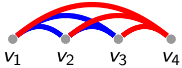

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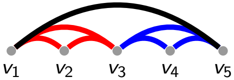

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- Now $v_1 v_5$ and one of $\{v_2 v_3, v_3 v_4\}$ wins.

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Theorem (Alon–Frankl–Lovász (1986))

If $sK_r^{(r)}$ denotes the r-uniform hypergraph consisting of s disjoint edges, then $R(sK_r^{(r)}; k) = (k-1)(s-1) + sr$.

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- We call this the Graph Lower Bound (GLB).

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Our Results

- Thm: $R(\mathrm{CF}_2; k) = k + \Theta(\sqrt{k})$
- Prop: $R(\mathrm{NF}_s, \mathrm{NF}_4) = \mathrm{GLB} = 2s + 3$ for $s \geq 4$.

Lower Bound

Theorem

$R(\mathsf{CF}_2; k) \geq k + (1 - o(1))\sqrt{\frac{4}{\pi} k}$

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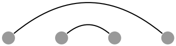
- To prove this, we cover $E(K_n)$ for $n = k + (1 - o(1))\sqrt{(4/\pi)k}$ with k color classes such that each color class avoids graphs in CF_2, where $\mathrm{CF}_2 = \{$  ,  $\}$.

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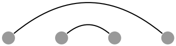
- To prove this, we cover $E(K_n)$ for $n = k + (1 - o(1))\sqrt{(4/\pi)k}$ with k color classes such that each color class avoids graphs in CF_2, where $\mathrm{CF}_2 = \{$ ,  $\}$.
- If e and e' have the same color, then e and e' are not separated and not nested. Hence they are crossed or share an endpoint.

Lower Bound

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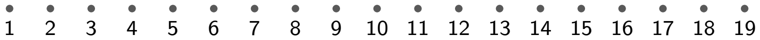

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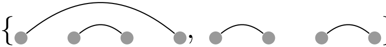
- To prove this, we cover $E(K_n)$ for $n = k + (1 - o(1))\sqrt{(4/\pi)k}$ with k color classes such that each color class avoids graphs in CF_2, where $\mathrm{CF}_2 = \{$  $\}$.
- If e and e' have the same color, then e and e' are not separated and not nested. Hence they are crossed or share an endpoint.
- Our coloring proceeds in stages, starting at stage 1.

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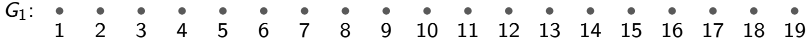

- At the start of stage i, we must color an ordered graph G_i where $V(G_i) = [n_i]$ and $uv \in E(G_i)$ implies $|u - v| \geq i$.

Lower Bound

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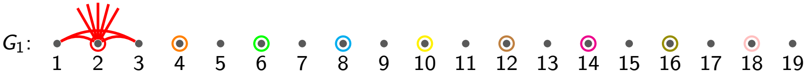

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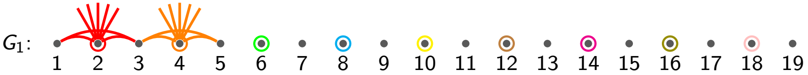

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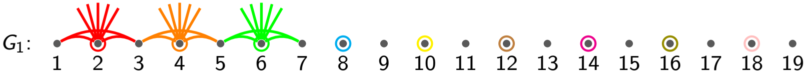

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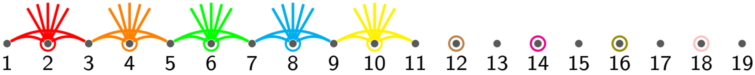

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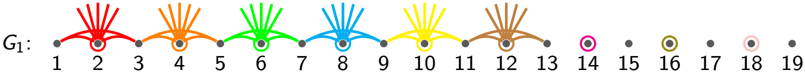

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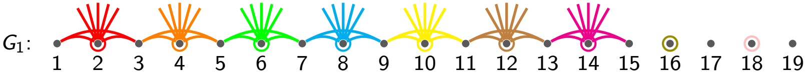

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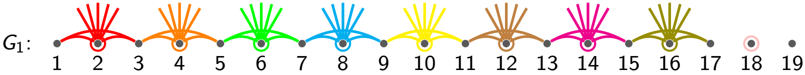

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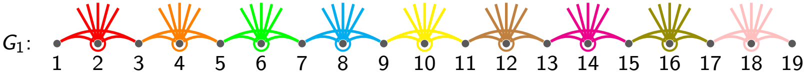

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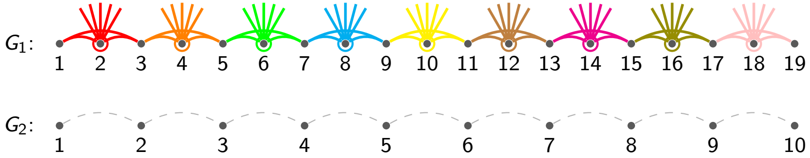

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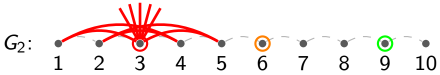

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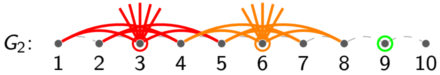

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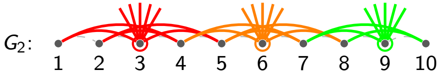

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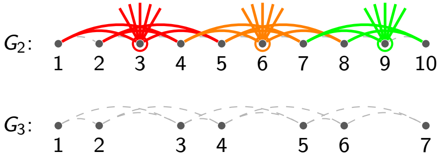

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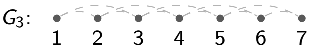

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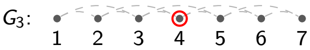
Lower Bound



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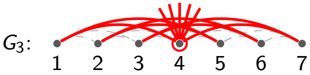

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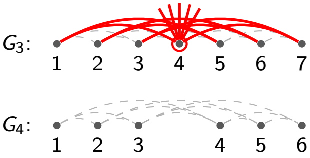

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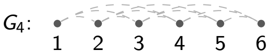

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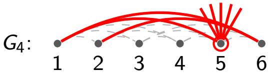

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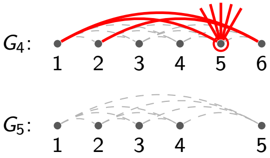

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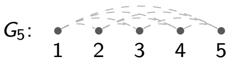

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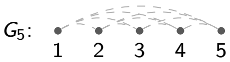

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- The process ends when G_i is empty.
- Num. colors used: $n - |V(G_\ell)|$, where ℓ is the last stage.

Josephus Sieve

Stage:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 $\cdots$

- Start with the list of positive integers and begin stage 1.

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Josephus Sieve

Stage: 1

$1 \quad \cancel{2} \quad 3 \quad \cancel{4} \quad 5 \quad \cancel{6} \quad 7 \quad \cancel{8} \quad 9 \quad \cancel{10} \quad 11 \quad \cancel{12} \quad 13 \quad \cancel{14} \quad 15 \quad \cancel{16} \quad 17 \quad \cancel{18} \quad 19 \quad \cancel{20} \quad 21 \quad \cancel{22} \quad 23 \quad \cancel{24} \quad 25 \quad \cancel{26} \quad 27 \quad \cancel{28} \cdots$

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Josephus Sieve

Stage: 1 2

1 ~~2~~ 3 ~~4~~ ~~5~~ ~~6~~ 7 ~~8~~ 9 ~~10~~ ~~11~~ ~~12~~ 13 ~~14~~ 15 ~~16~~ ~~17~~ ~~18~~ 19 ~~20~~ 21 ~~22~~ ~~23~~ ~~24~~ 25 ~~26~~ 27 ~~28~~ $\cdots$

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Josephus Sieve

Stage: 1 2 3

$1\ \cancel{2}\ 3\ \cancel{4}\ \cancel{5}\ \cancel{6}\ 7\ \cancel{8}\ \cancel{9}\ \cancel{10}\ \cancel{11}\ \cancel{12}\ 13\ \cancel{14}\ 15\ \cancel{16}\ \cancel{17}\ \cancel{18}\ 19\ \cancel{20}\ \cancel{21}\ \cancel{22}\ \cancel{23}\ \cancel{24}\ 25\ \cancel{26}\ 27\ \cancel{28} \cdots$

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Josephus Sieve

Stage: 1 2 3 4

1 ~~2~~ 3 ~~4~~ ~~5~~ ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ ~~11~~ ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ ~~17~~ ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ ~~23~~ ~~24~~ 25 ~~26~~ 27 ~~28~~ $\cdots$

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Josephus Sieve

Stage: 1 2 3 4 5

1 ~~2~~ 3 ~~4~~ ~~5~~ ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ ~~11~~ ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ ~~17~~ ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ ~~23~~ ~~24~~ ~~25~~ ~~26~~ 27 ~~28~~ $\cdots$

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Josephus Sieve

Stage: 1 2 3 4 5

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Stage: 1 2 3 4 5

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- Start with the list of positive integers and begin stage 1.
- At stage i, delete from the list entries at indices divisible by $i + 1$.
- After stage i, the first $i + 1$ entries will never be deleted.
- The numbers that survive form a sequence called the Josephus Sieve: 1, 3, 7, 13, 19, 27, 39, ...

Josephus Sieve

Stage: 1 2 3 4 5

1 ~~2~~ 3 ~~4~~ ~~5~~ ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ ~~11~~ ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ ~~17~~ ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ ~~23~~ ~~24~~ ~~25~~ ~~26~~ 27 ~~28~~ ···

- A vertex $u \in V(G_1)$ survives to the end if and only if u appears in the Josephus Sieve.

Josephus Sieve

Stage: 1 2 3 4 5

1 ~~2~~ 3 ~~4~~ ~~5~~ ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ ~~11~~ ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ ~~17~~ ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ ~~23~~ ~~24~~ ~~25~~ ~~26~~ 27 ~~28~~ ···

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Josephus Sieve

Stage: 1 2 3 4 5

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Josephus Sieve

Stage: 1 2 3 4 5

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Josephus Sieve

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- Cor: $R(\mathrm{CF}_2; k) \geq k + (1 - o(1))\sqrt{(4/\pi)k}$.

Upper bound

- An edge-coloring of an ordered graph G is good if every color class avoids subgraphs in CF_2.

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Lemma (Compression)

Let G be a good k-edge-coloring of an n-vertex ordered graph with m non-edges. If $k > 0$, then there is a good $(k-1)$-edge-coloring of an $(n-1)$-vertex ordered graph with at most $m+1$ non-edges.

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If $\binom{n-k}{2} > k$, then $R(\mathrm{CF}_2; k) \leq n$.

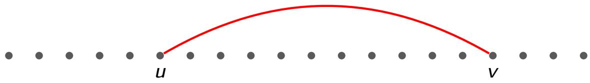
Corollary

$R(\mathrm{CF}_2; k) \leq k + 1 + \left\lceil \sqrt{2k} \right\rceil$

Compression Lemma Illustration

- Let uv be a minimal edge in G, meaning G has no other edge with both endpoints in $[u, v]$.

Compression Lemma Illustration



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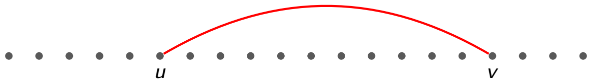

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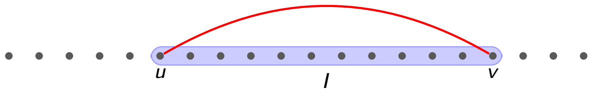

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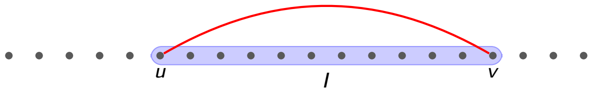

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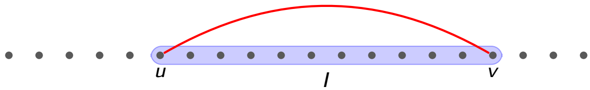
- Let α be the color of uv, let S be the set of edges with color α, and let $I = [u, v]$.
- Since S has no separated or nested edge pairs, every edge in S besides uv has exactly one endpoint in I.

Compression Lemma Illustration



- After perhaps recoloring some edges with color α, we may assume $N_\alpha(w)$ is a nonempty subinterval of I for each $w \notin I$.

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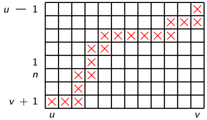

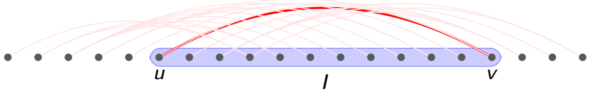

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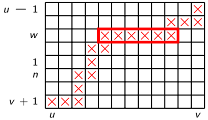

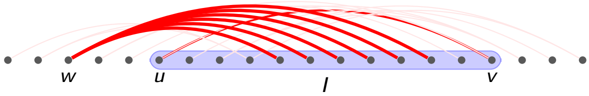

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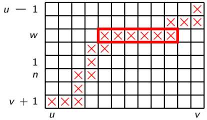

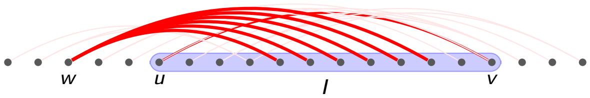

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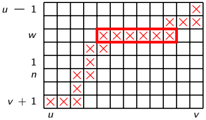

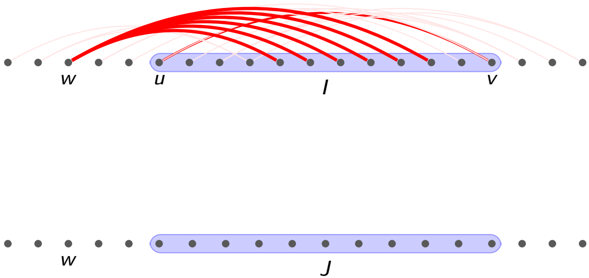

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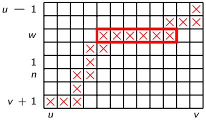

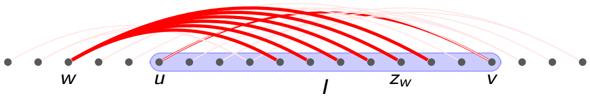

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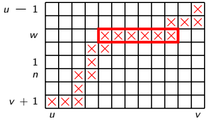

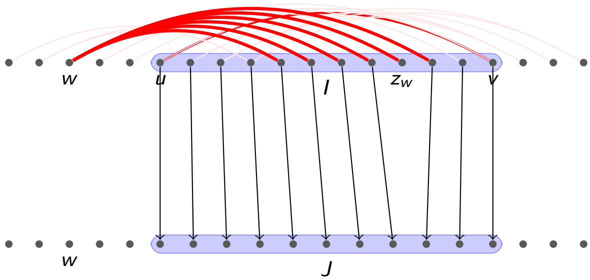

- After perhaps recoloring some edges with color α, we may assume $N_\alpha(w)$ is a nonempty subinterval of I for each $w \notin I$.
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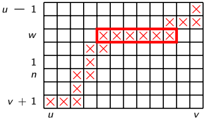

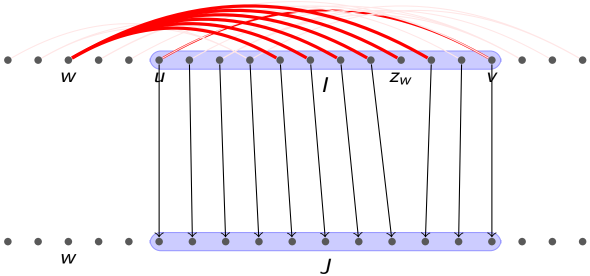

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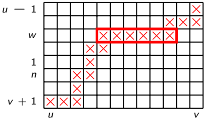

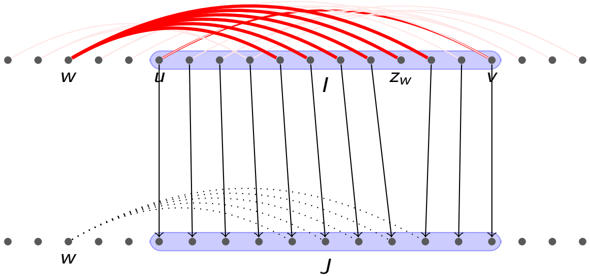

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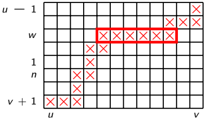

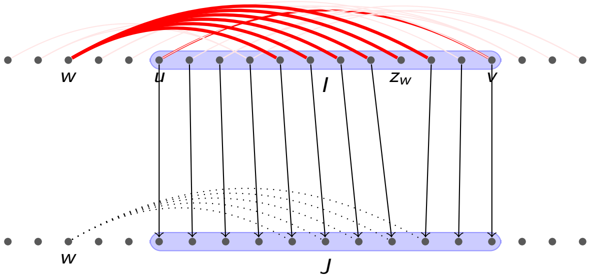

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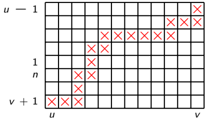

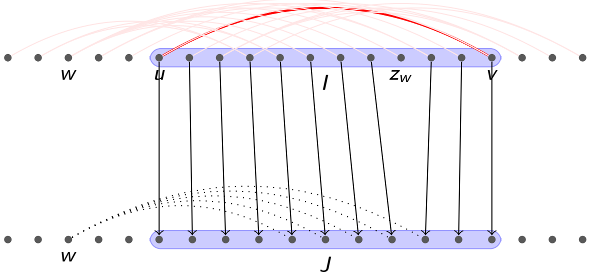

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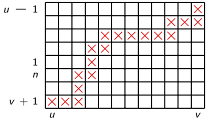

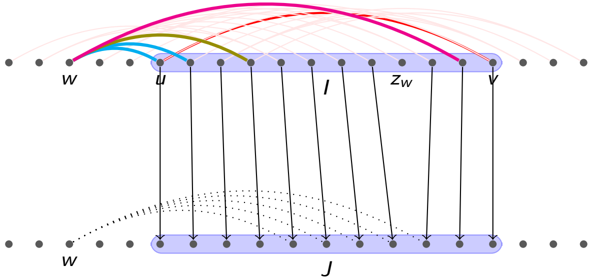

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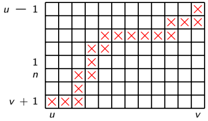

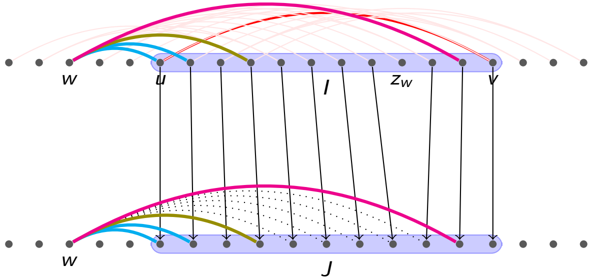

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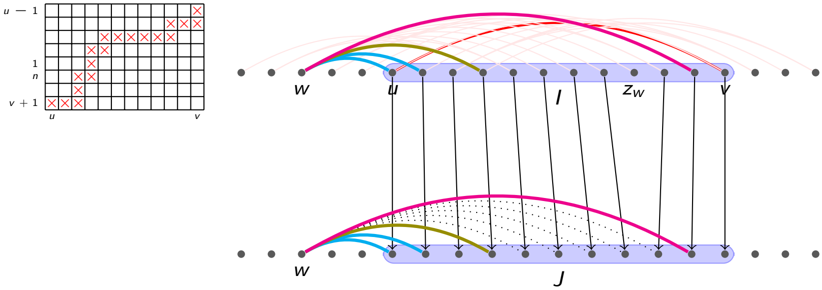

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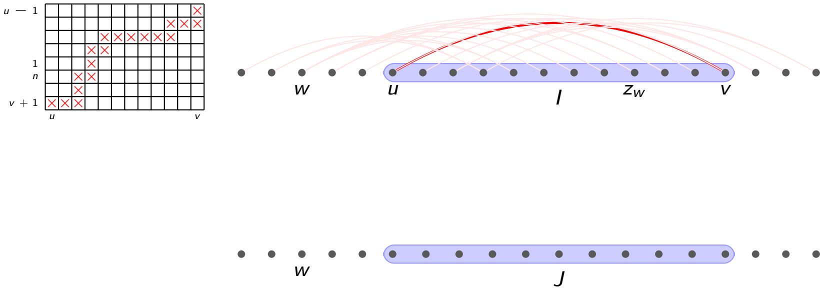

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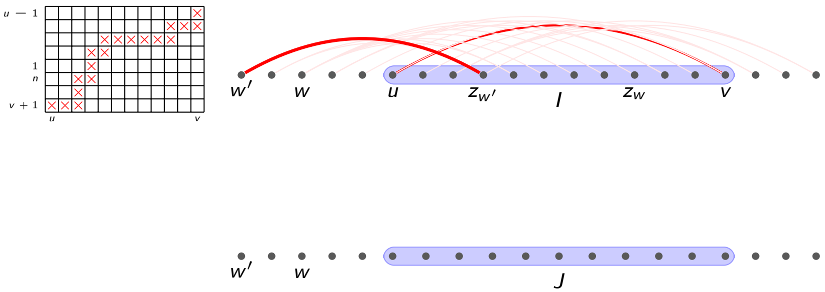

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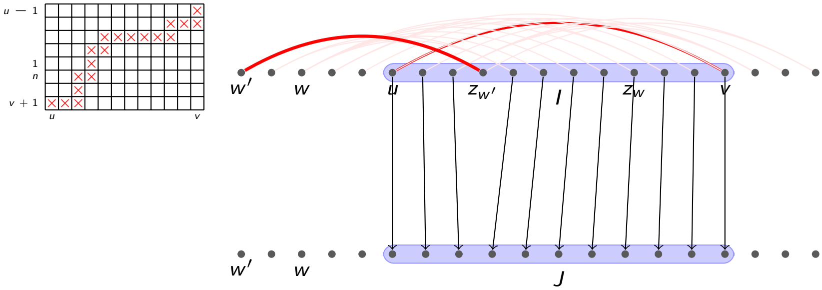

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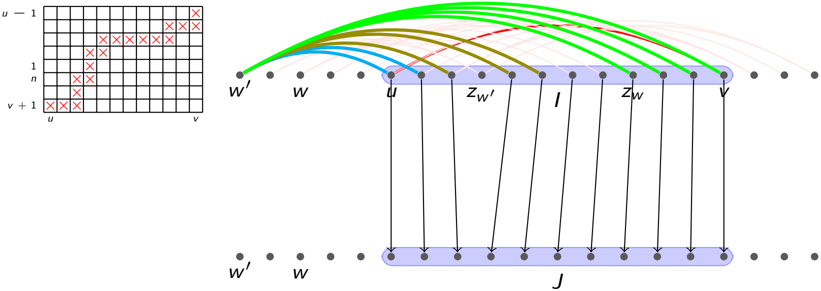

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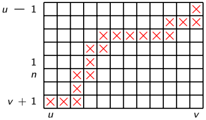

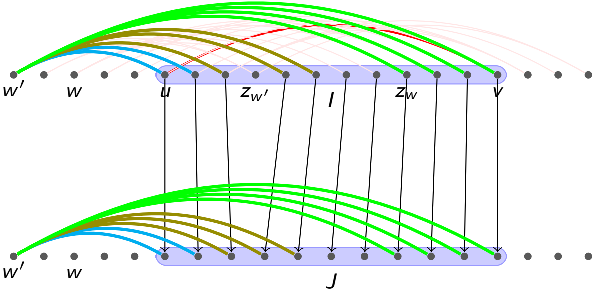

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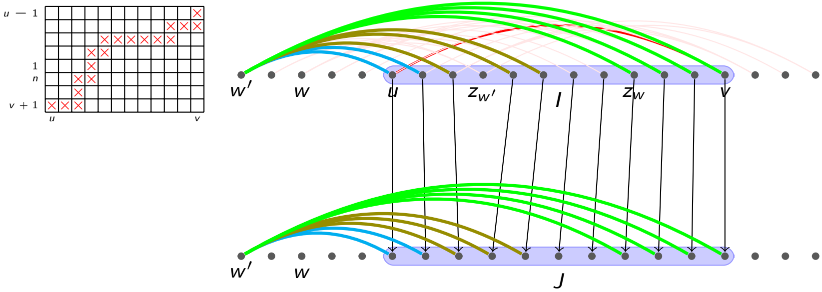

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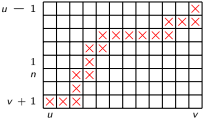

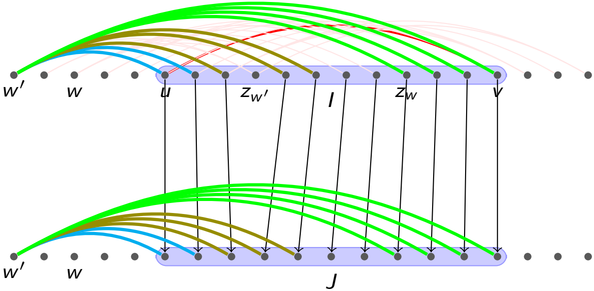

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- All color classes in G' still avoid subgraphs in CF_2.
- New non-edges in G' corresponding to α edges in G are almost offset by replacing the near-independent set I in G with the independent set J in G' since $|J| = |I| - 1$.

Open Problems

- Improve the constant on the $\sqrt{k}$ term in

$$k + (1 - o(1))\sqrt{(4/\pi)k} \leq R(\mathrm{CF}_2; k) \leq k + \left\lceil \sqrt{2k} \right\rceil + 1.$$

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Thank You