On a Rainbow Extremal Number in the Hypercube

Kenny Adeogun[1] Luke Hawranick[2] Muchen Ju[3] Henry Zhan[4]

[1]Carnegie Mellon University

[2]West Virginia University

[3]Fudan University

[4]Middlebury College

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Preliminaries

Definition (Hypercube Q_k)

The **hypercube** is a graph with vertex set $\{0, 1\}^k$ such that two k-tuples are adjacent if and only if they differ in exactly one position.

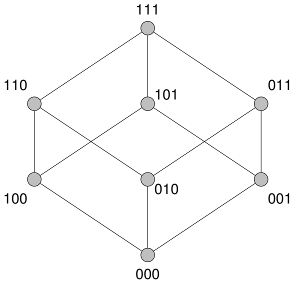

Figure: Q_3.

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Definition (Proper Edge-coloring)

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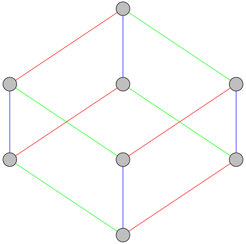

Figure: Proper Edge-coloring of Q_3.

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Figure: Rainbow Edge-colored P_4.

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Given a graph H, let $\mathrm{ex}^*(n, F)$ be maximum number of edges in an n-vertex graph which admits a proper edge coloring with no rainbow copy of F.

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Theorem (Keevash–Mubayi–Sudakov–Verstraëte, 2007)

Let G be a fixed graph. Then,

$$\mathrm{ex}(n, G) \leq \mathrm{ex}^*(n, G) \leq \mathrm{ex}(n, G) + o(n^2).$$

Relative Rainbow Extremal Numbers

Definition (The Rainbow Extremal Number of F with respect to G)

Given a graph F, let $\mathrm{ex}^*(G, F)$ be the maximum number of edges in a subgraph of G which admits a proper coloring with no rainbow copy of F.

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Question

Determine $\mathrm{ex}^*(Q_n, P_{n+1})$.

Existing Bounds

Theorem (Rombach, unpublished)

$ex^*(Q_n, P_{n+1}) < n2^{n-1} = |E(Q_n)|$.

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Proof (sketch).

- Let $s(v_i, v_j) = v_i + v_j \in \{e_i\}_{i \in [n]}$ for $(v_i, v_j) \in E(Q_n)$.

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- Greedily build a rainbow path with an arbitrary start $v_0 v_1$.
- At v_k, there are at least $n - k$ incident colors not on the path. Choose v_{k+1} such that $s(v_k, v_{k+1})$ is distinct from the previous $\leq (n-k)$-many choices.

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- Greedy choice does not induce a cycle.

$\square$

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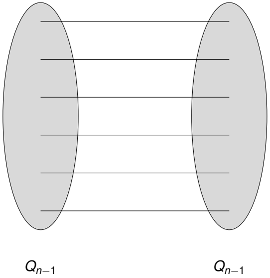

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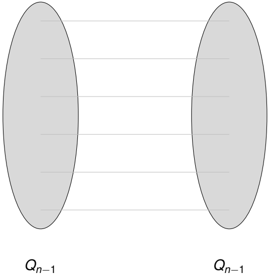

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- Find a rainbow P_{n+1} using Rombach's Algorithm starting from uv' for $v' \neq v$.

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- Restore the edge and assign it a color not used in the coloring. This yields a properly edge-colored Q_n.
- Find a rainbow P_{n+1} using Rombach's Algorithm starting from uv' for $v' \neq v$.
- The path will not include the edge uv, as the algorithm doesn't create a cycle by construction.

$\square$

Open Problems

Conjecture

$ex^*(Q_n, P_{n+1}) = (n\text{-}1)2^{n-1}$.

- Verified for $n = 3, 4$. Confirmed independently by Crawford, King, and Spiro (2025).

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Thank You