Statistics of Maximal Independent Sets in Grid-like Graphs

Iowa State University, Summer 2024 REU

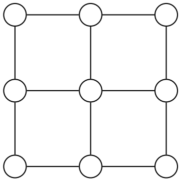
Cole Swain Levi Axelrod Luke Hawranick Nathan Bickel

January 1, 2025

What are Independent sets?

Definition

A set $I \subseteq V(G)$ is called an **independent set** if no two vertices in I are adjacent.

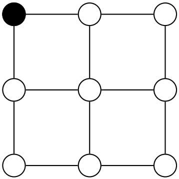


Grid Graph: $G_{3\times 3}$

What are Independent sets?

Definition

A set $I \subseteq V(G)$ is called an **independent set** if no two vertices in I are adjacent.

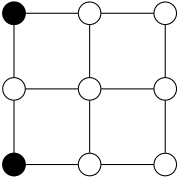


Grid Graph: $G_{3\times 3}$

What are Independent sets?

Definition

A set $I \subseteq V(G)$ is called an **independent set** if no two vertices in I are adjacent.

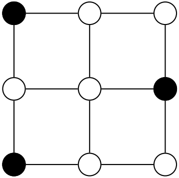


Grid Graph: $G_{3\times 3}$

Maximal Independent Sets

Definition

A **maximal independent set** (**MIS**) I in a graph G is an independent set that is not a proper subset of an independent set.



Grid Graph: $G_{3\times 3}$

Goals

Definition

Let G be a graph. We define

$$\mathrm{MIS}(G) := \{M \subseteq V(G) : M \text{ is a maximal independent set.}\}$$

Goals

Definition

Let G be a graph. We define

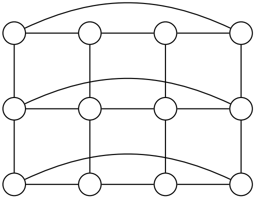
$$\mathrm{MIS}(G) := \{M \subseteq V(G) : M \text{ is a maximal independent set.}\}$$

We are interested in determining statistics of MIS's such as:

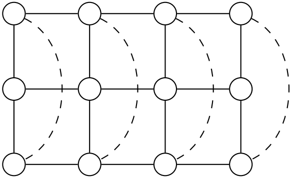
- Enumerating MIS's
- Describing symmetries
- Enumerating non-isomorphic MIS's
- Finding the average size of MIS's

Grid-like Graphs

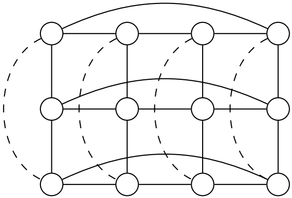
The four graphs are drawn below for $m = 3$ and $n = 4$:



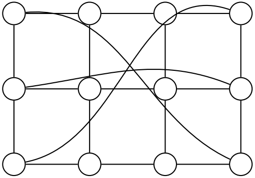
Fat Cylinder: $FC_{m\times n}$



Thin Cylinder: $TC_{m\times n}$



Torus: $T_{m\times n}$



Möbius Strip: $M_{m\times n}$

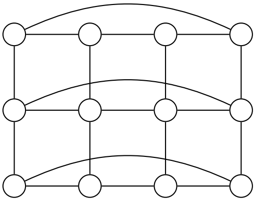
Local and Global Structure

Our grid graphs are Cartesian products of paths and cycles.

Definition

Let G be an $m \times n$ grid-like graph. The **local structure** is the subgraph induced by a vertical slice of the graph, and the **global structure** is the subgraph induced horizontally.

For example, the fat cylinder has local path structure and global cyclic structure:



Fat Cylinder: $FC_{m \times n}$

States

We can build MIS's by concatenating slices one at a time. Let H be a vertical slice of a grid-like graph.

States

We can build MIS's by concatenating slices one at a time. Let H be a vertical slice of a grid-like graph.

Definition

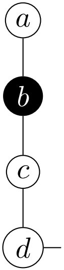
A slice H is in **state** (H, I, D) if I is independent and D is the set of vertices that must be covered by the next slice.

In this state,

$H = P_4$

$I = \{b\}$

$D = \{d\}$



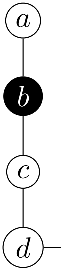
States

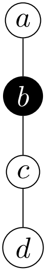
We can build MIS's by concatenating slices one at a time. Let H be a vertical slice of a grid-like graph.

Definition

A slice H is in **state** (H, I, D) if I is independent and D is the set of vertices that must be covered by the next slice.

In this state,
$H = P_4$
$I = \{b\}$
$D = \{d\}$





In this state,
$H = P_4$
$I = \{b\}$
$D = \varnothing$

State Relationships

Definition

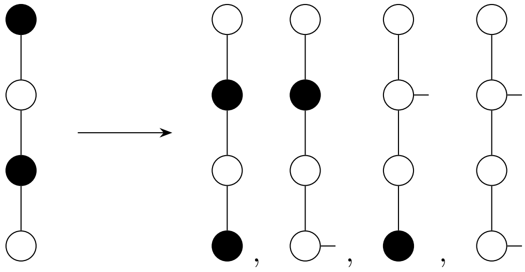
State s_2 **follows** state s_1 if a slice in state s_2 can be pasted to the right of a slice in state s_1 while maintaining an MIS.

State Relationships

Definition

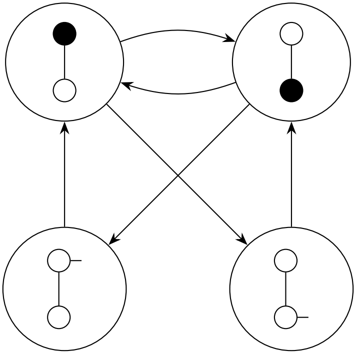
State s_2 **follows** state s_1 if a slice in state s_2 can be pasted to the right of a slice in state s_1 while maintaining an MIS.

For example,



Map Digraph

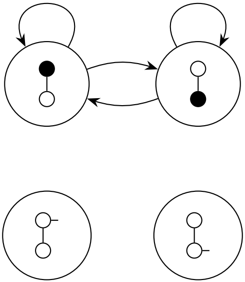
We can construct a directed graph with states as vertices and an arc from s_1 to s_2 if s_2 follows s_1. For example:



P_2 Map Digraph

Ticket Digraph

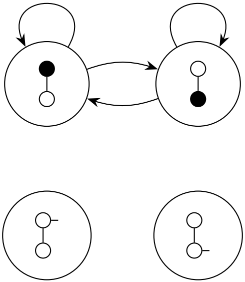
MIS's correspond to walks in the map digraph. We construct the *ticket digraph* to encode whether the endpoints of a walk in the map digraph are valid on a shape with a certain global structure:



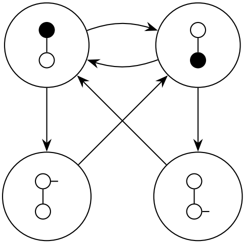
Global Path Structure

Ticket Digraph

MIS's correspond to walks in the map digraph. We construct the *ticket digraph* to encode whether the endpoints of a walk in the map digraph are valid on a shape with a certain global structure:



Global Path Structure



Global Cyclic Structure

Map Digraph

Properties of map digraph we know:

Let P_m or C_m be the slice of the grid-like graph and M be the map digraph.

- **Very** strongly connected
- Number of vertices
- Number of edges
- $d^+(v)$ is a power of 2 for each $v \in M$
- $\Delta^+(M) = 2^{\lceil \frac{m}{2} \rceil}$

Map Digraph

Counting Map Digraph Vertices

Consider some slice, either P_m or C_m. All vertices on a slice are covered by some vertex in the MIS. s is the current state. s_0 is the state which comes before s if it exists. Define the following 'vertex labels' below.

 if $v_i \in I(s)$

 else if $v_{i-1} \in I(s)$

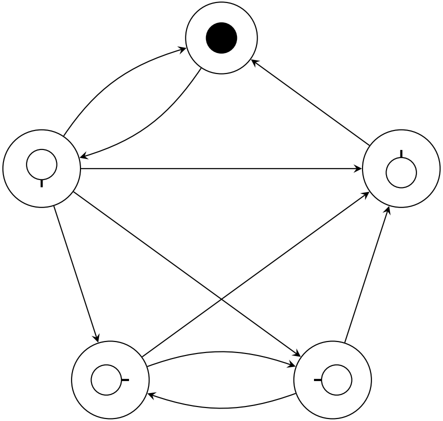
 else if $v_{i+1} \in I(s)$

 else if $v_i \in I(s_0)$

 otherwise

Map Digraph

We can form a digraph to model which vertex labels can lie above each other vertex labels.



Map Digraph

We can form a digraph to model which vertex labels can lie above each other vertex labels. Call the matrix below A. $A(i,j)$ indicates if vertex label i can lie below label j in a state.

$$
A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0
\end{bmatrix}
$$

Map Digraph

The minimal polynomial of A yields the recurrence relation for the number of walks our vertex label map digraph. We determine the base cases by hand.

Minimal Polynomial:

$$x^4 - x^3 - x^2 - 1$$

Recurrence:

$$s(m) = s(m-1) + s(m-2) + s(m-4)$$

with

$$s(0) = 1, s(1) = 3, s(2) = 4, s(3) = 8.$$

Map Digraph

We use the same technique to count the number of **edges** in the map digraph.

Recurrence:

$$f(m) = 3f(m-2) + 2f(m-3) + 3f(m-4) + 2f(m-5)$$

with

$$f(1) = 4, f(2) = 6, f(3) = 14, f(4) = 30, f(5) = 66, f(6) = 142$$

Metro Pairs

Our method for counting MIS's in grid-like graphs comes down to computing sequences that can be described by walks on the map digraph that satisfy tickets, so we created notation to condense this idea.

Definition

A **metro pair** is a pair of directed graphs (M, T) sharing a vertex set S such that for any sufficiently large n, for any $u, v \in S$, there exists a uv-walk along M of length n, and $E(T)$ is nonempty. Define the **travel sequence** of (M, T) to be $\tau(n) =$ the number of uv-walks of length n such that $(u, v) \in E(T)$.

Metro Pairs

We can express the travel sequence in terms of linear algebra with

$$\tau(n) = A_T \bullet A_M^n,$$

where A_M and A_T are the (transposes of the) adjacency matrices of M and T and $\bullet$ is the vector dot product.

Using the Perron-Frobenius Theorem, we know that

$$\lim_{n\to\infty} \frac{\tau(n)}{r^n} = A_T \bullet \frac{\vec{v}\vec{w}^\top}{\vec{w}^\top \vec{v}},$$

where r is the principle eigenvalue of A_M and $\vec{w}^\top, \vec{v}$ are the principle left and right eigenvectors of A_M.

Globally Cyclic Structures

Theorem

Let H be a graph, and let ϕ be a graph automorphism of H. Let r be the principle eigenvalue of the map digraph M of H. Then

$$\lim_{n\to\infty} \frac{|\operatorname{MIS}((H \square P_{n+1})/\phi)|}{r^n} = 1.$$

In particular, this is a generalization of the result that

$$\lim_{n\to\infty} \frac{|\operatorname{MIS}(M_{m\times n})|}{|\operatorname{MIS}(FC_{m\times n})|} = 1.$$

Globally Cyclic Structures

Proof sketch:

1. The vertices of M are pairs of subsets of H, so an automorphism ϕ of H gives us and automorphism ϕ^* of M.
2. The eigenvectors of A_M are determined the graph structure of M, so if two vectors are sent to each other by an automorphism, their entries of the eigenvectors must be the same.
3. The ticket graph consisting of tickets $(i, \phi^*(i))$ must give the same result as the one consisting of tickets (i, i) when dotted with $\frac{\vec{v}\vec{w}^\top}{\vec{w}^\top\vec{v}}$.
4. $$I \bullet \frac{\vec{v}\vec{w}^\top}{\vec{w}^\top\vec{v}} = \frac{\vec{w}^\top\vec{v}}{\vec{w}^\top\vec{v}} = 1.$$

The Maximal Hard Square Entropy Constant

Seungsang Oh defined and proved the existence of what he called the *maximal hard square entropy constant:*

$$\kappa := \lim_{m,n\to\infty} |\operatorname{MIS}(G_{m\times n})|^{\frac{1}{mn}}.$$

This constant plays a large role in all of the limits we are trying to compute, as the above limit yields the same result for each structure we are studying. In particular, if r_H is the principle eigenvalue of the adjacency matrix of the map digraph of H, we have

$$\lim_{m\to\infty} (r_{P_m})^{\frac{1}{m}} = \kappa = \lim_{m\to\infty} (r_{C_m})^{\frac{1}{m}}.$$

The Maximal Hard Square Entropy Constant

Oh proved that the limit existed by showing it was equal to

$$\sup_{m,n} |\,\mathrm{MIS}(G_{m\times n})|^{\frac{1}{(m+1)(n+1)}},$$

which he bounded above by 16. He also estimated the value of κ to be a little over

$$|\,\mathrm{MIS}(G_{8\times 380})|^{\frac{1}{(8+1)(380+1)}} \approx 1.225084.$$

The Maximal Hard Square Entropy Constant

We have improved on both of these bounds.
For a lower bound, we calculated

$$\kappa = \sup_{m,n} |\operatorname{MIS}(G_{m\times n})|^{\frac{1}{(m+1)(n+1)}} \geq \lim_{n\to\infty} |\operatorname{MIS}(G_{10\times n})|^{\frac{1}{(10+1)(n+1)}}$$

$$= \sqrt[11]{r_{P_{10}}} \approx 1.230538.$$

For an upper bound, we calculated first that

$$\kappa = \lim_{m,n\to\infty} |\operatorname{MIS}(G_{m\times n})|^{\frac{1}{mn}} \leq \lim_{m,n\to\infty} |\mathcal{P}(V(G_{m\times n}))|^{\frac{1}{mn}} = 2.$$

The Maximal Hard Square Entropy Constant

We then improved again on this upper bound by more tightly bounding the number of MIS's total.

In an MIS, each slice must be in a state, so given some local structure with s states, there are at most s^n MIS's of that structure with n slices. This gives $\kappa \leq (s^n)^{\frac{1}{mn}} = (s)^{\frac{1}{m}}$.

If there are s_l walks in the map digraph of length l, that that would similarly give us $\kappa \leq (s_l^{\frac{n}{l}})^{\frac{1}{mn}} = (s)^{\frac{1}{lm}}$.

We have found a way to compute s_l for large local structures and small values of l, and as such were able to give an upper bound with $l = 7$ of $\kappa < 1.311534$.

MIS Parity

Theorem

$|\operatorname{MIS}(G_{m\times n})|$ *is even for* $m, n \geq 2$

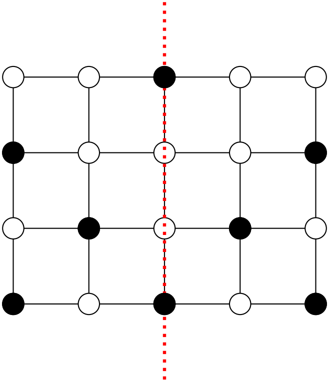
Consider the set of MISs on some $G_{m\times n}$.

MIS Parity

Theorem

$|\,\mathrm{MIS}(G_{m\times n})|$ *is even for* $m, n \geq 2$

Consider the set of MISs on some $G_{m\times n}$. Some of these MIS's have symmetry.

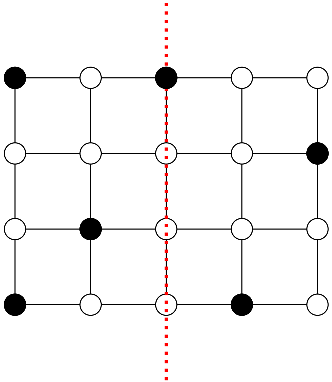


MIS Parity

Theorem

$|\operatorname{MIS}(G_{m\times n})|$ *is even for* $m, n \geq 2$

Consider the set of MISs on some $G_{m\times n}$. Some of these MIS's have symmetry. Some do not.



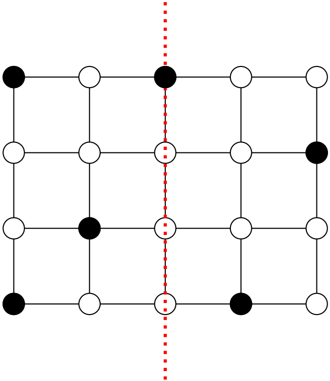
MIS Parity

Theorem

$|\operatorname{MIS}(G_{m\times n})|$ *is even for* $m, n \geq 2$

Asymmetric Case

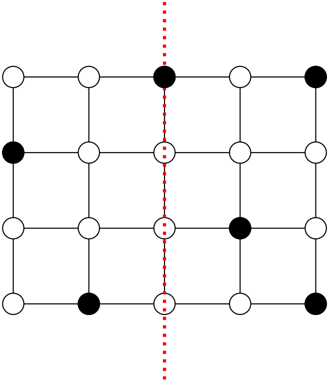
Consider some MIS which does **not** have symmetry over this axis.



MIS Parity

Asymmetric Case

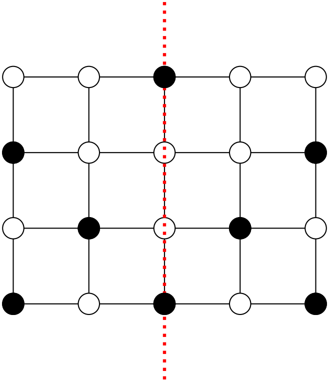
Consider some MIS which does **not** have symmetry over this axis.



MIS Parity

Symmetric Case - odd n

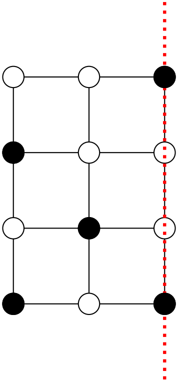
Otherwise, consider a symmetrical MIS. If n is odd,



MIS Parity

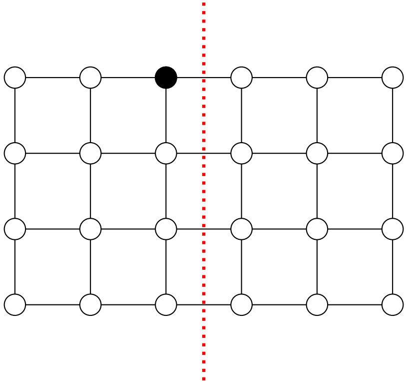
Symmetric Case - odd n

Otherwise, consider a symmetrical MIS. If n is odd,



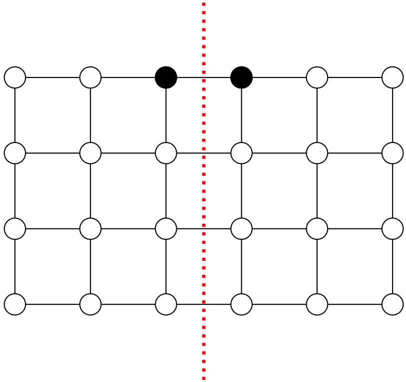
MIS Parity

Symmetric Case - even n



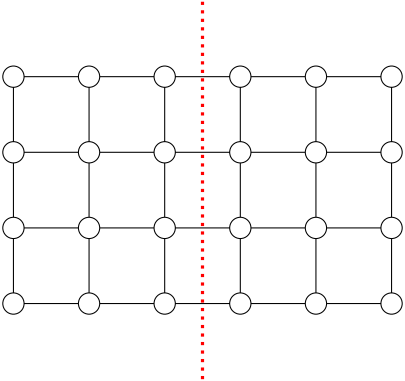
MIS Parity

Symmetric Case - even n



MIS Parity

Symmetric Case - even n



There is no way for an MIS on $G_{m \times n}$ to contain two empty slices.

What is a Non-isomorphic MIS?

Definition

The **orbit** of a maximal independent set A written as $\mathcal{O}(A)$ is the set of all maximal independent sets that can be obtained from A by a series of rotations or reflections.

What is a Non-isomorphic MIS?

Definition

The **orbit** of a maximal independent set A written as $\mathcal{O}(A)$ is the set of all maximal independent sets that can be obtained from A by a series of rotations or reflections.

Definition

Let G, H be two graphs. $\mathrm{MIS}(G)$ is **non-isomorphic** to $\mathrm{MIS}(H)$ if $\mathcal{O}(\mathrm{MIS}(G)) \neq \mathcal{O}(\mathrm{MIS}(H))$.

What is a Non-isomorphic MIS?

Definition

The **orbit** of a maximal independent set A written as $\mathcal{O}(A)$ is the set of all maximal independent sets that can be obtained from A by a series of rotations or reflections.

Definition

Let G, H be two graphs. $\mathrm{MIS}(G)$ is **non-isomorphic** to $\mathrm{MIS}(H)$ if $\mathcal{O}(\mathrm{MIS}(G)) \neq \mathcal{O}(\mathrm{MIS}(H))$.

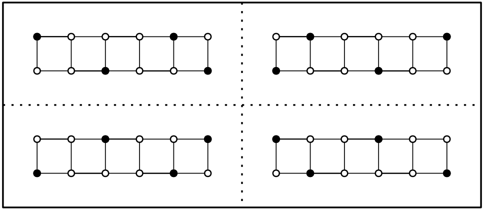


Figure: $\mathrm{MIS}\, G_{2\times 6}$

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- $|\,\mathrm{MIS}\, G_{2\times n}| = 2 * F(n)$ (proven by Euler in 2005)

Enumerating $\text{NIMIS}(G_{2\times n})$

We found:

$$|\text{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- $|\text{MIS}\, G_{2\times n}| = 2 * F(n)$ (proven by Euler in 2005)
- Each $\text{MIS}(G_{2\times n})$ can be reflected vertically to form a pair of distinct isomorphic MIS's.

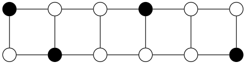
Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- $|\,\mathrm{MIS}\,G_{2\times n}| = 2 * F(n)$ (proven by Euler in 2005)
- Each $\mathrm{MIS}(G_{2\times n})$ can be reflected vertically to form a pair of distinct isomorphic MIS's.



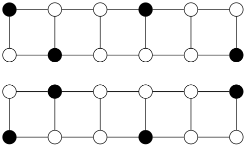
Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- $|\,\mathrm{MIS}\, G_{2\times n}| = 2 * F(n)$ (proven by Euler in 2005)
- Each $\mathrm{MIS}(G_{2\times n})$ can be reflected vertically to form a pair of distinct isomorphic MIS's.



Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- We now have $F(n)$ pairs of MIS's

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- We now have $F(n)$ pairs of MIS's
- Label each pair with bit string. Each column with a vertex shaded is a 1. Each column with no vertices shaded is a 0.

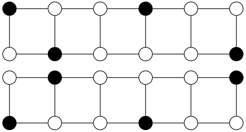
Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- We now have $F(n)$ pairs of MIS's
- Label each pair with bit string. Each column with a vertex shaded is a 1. Each column with no vertices shaded is a 0.
- This pair of MIS's in $G_{2\times 6}$ maps to 110101:



Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- Note, since the bit strings label the pairs we have $F(n)$ possible bit strings.

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- Note, since the bit strings label the pairs we have $F(n)$ possible bit strings.
- The symmetrical strings i.e. 101, 11011, 1111,... correspond to MIS's with orbit of size 2.

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- Note, since the bit strings label the pairs we have $F(n)$ possible bit strings.
- The symmetrical strings i.e. 101, 11011, 1111,... correspond to MIS's with orbit of size 2.
- The non-symmetrical strings i.e. 1011, 1101, 11101,... correspond to MIS's with orbit of size 4.

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- Note, since the bit strings label the pairs we have $F(n)$ possible bit strings.
- The symmetrical strings i.e. 101, 11011, 1111,... correspond to MIS's with orbit of size 2.
- The non-symmetrical strings i.e. 1011, 1101, 11101,... correspond to MIS's with orbit of size 4.
- We count the amount of symmetrical strings, add them to $F(n)$, and then divide by 2 to get rid of the double counting.

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- If n is even, we have:

$$111111$$

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- If n is even, we have:

$$111111$$

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- If n is even, we have:

$$111|\underbrace{111}_{F(\frac{n}{2})}$$

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- If n is even, we have:

$$111|\underbrace{111}_{F(\frac{n}{2})}$$

- if n is odd, we have:

$$11111 \qquad\qquad 11011$$

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- If n is even, we have:

$$111|\underbrace{111}_{F(\frac{n}{2})}$$

- if n is odd, we have:

$$11|111 \qquad\qquad 11|011$$

Enumerating $\mathrm{NIMIS}(G_{2\times n})$

We found:

$$|\,\mathrm{NIMIS}(G_{2\times n})| = \begin{cases} \frac{F(n)+F(\frac{n}{2})}{2} & \text{if } n \text{ is even} \\ \frac{F(n)+F(\frac{n+3}{2})}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Sketch:

- If n is even, we have:

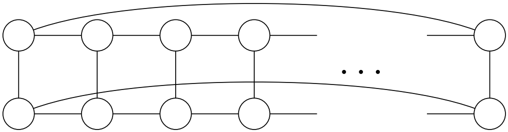
$$111|\underbrace{111}_{F(\frac{n}{2})}$$

- if n is odd, we have:

$$11\underbrace{|111}_{F(\frac{n+1}{2})} \qquad\qquad 110|\underbrace{11}_{F(\frac{n-1}{2})}$$

Counting $|\mathrm{NIMIS}(FC_{2\times n})|$

Consider the set of MISs on $FC_{2\times n}$ below.



For each MIS on $FC_{2\times n}$, generate the corresponding cyclic binary string.

- no consecutive 0s
- even number of 1s

Each string corresponds to a pair of isomorphic MISs, which can be obtained by reflection over horizontal axis between row 1 and 2.

Counting $|\mathrm{NIMIS}(FC_{2\times n})|$

The problem is now translated to finding the number of distinguishable cyclic strings in our set.

Example:

011010111 is not distinguishable among:

$$101101011, 110110101, \ldots \text{(rotational symmetry)}$$

$$110101101, 101011011, \ldots \left(\text{reflection symmetry over index } \frac{n+1}{2}\right)$$

$\square$

Counting $|\mathrm{NIMIS}(FC_{2\times n})|$

Collapsed Strings

To make life easier, each binary string can be collapsed into a smaller string.

$$\{011010111, 101101011, 110110101\} \to \text{‘213’}$$

Counting $|\mathrm{NIMIS}(FC_{2\times n})|$

How many collapsed strings can we have?

The number of digits of the collapsed string + the sum of the digits must be n. Let $P_n(k)$ be the number of nonnegative integer partitions of n into k parts.

Counting $|\mathrm{NIMIS}(FC_{2\times n})|$

How many collapsed strings can we have?

For an even non-collapsed string b of length n, b must have an even number $2k$ of 0s. Each index of a collapsed string must be at least 1. Thus, for each value of $2k$, the set of collapsed strings is $P_{n-4k}(2k)$.

$$\begin{array}{c|c|c|c|c|} \geq 1 & \geq 1 & \geq 1 & \cdots & \geq 1 \end{array}$$

$$0 \qquad 0 \qquad 0 \qquad\qquad 0 \qquad 0$$

The group of symmetries of a cyclic string of length n is equivalent to the group of symmetries of an n-gon: D_n.

Counting $|\mathrm{NIMIS}(FC_{2\times n})|$

Burnside's Lemma

Theorem

Let G be a finite group acting on X, and let X/G be the set of orbits of X. Then,

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

where X^g is the set of fixed points of X with respect to $g \in G$.

Counting $|\mathrm{NIMIS}(FC_{2\times n})|$

Therefore, with the help of Burnside's Lemma,

$$|\mathrm{NIMIS}(FC_{2\times n})| = \begin{cases} 1 + \sum_{k=1}^{\lfloor \frac{n}{4} \rfloor} |P_{n-4k}(2k)/D_{4k}| & \text{if } n \text{ is even} \\ 1 + \sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} |P_{n-4k-2}(2k+1)/D_{4k+1}| & \text{if } n \text{ is odd} \end{cases}$$

Average Size of MIS's

Definition

Let G be a graph. Then the **total MIS size** is

$$T(G) := \sum_{I \in \mathrm{MIS}(G)} |I|.$$

Average Size of MIS's

Definition

Let G be a graph. Then the **total MIS size** is

$$T(G) := \sum_{I \in \mathrm{MIS}(G)} |I|.$$

Definition

The **average MIS size** is

$$A(G) := \frac{T(G)}{|\,\mathrm{MIS}(G)|}.$$

$2 \times n$ Case

We found that

$$T(G_{2\times n}) = 2\sum_{i=1}^{n} F(i)F(n+1-i).$$

$2 \times n$ Case

We found that

$$T(G_{2\times n}) = 2\sum_{i=1}^{n} F(i)F(n+1-i).$$

Proof Sketch:

$2 \times n$ Case

We found that

$$T(G_{2\times n}) = 2\sum_{i=1}^{n} F(i)F(n+1-i).$$

Proof Sketch:

- Sum entry-by-entry over all valid bit strings

$2 \times n$ Case

We found that

$$T(G_{2\times n}) = 2\sum_{i=1}^{n} F(i)F(n+1-i).$$

Proof Sketch:

- Sum entry-by-entry over all valid bit strings
- Note that each 1 splits the string into two valid substrings
- Example:

$$\mathbf{1}01011\mathbf{1}01\mathbf{1}$$

$2 \times n$ Case

We found that

$$T(G_{2\times n}) = 2\sum_{i=1}^{n} F(i)F(n+1-i).$$

Proof Sketch:

- Sum entry-by-entry over all valid bit strings
- Note that each 1 splits the string into two valid substrings
- Example:

$$\mathbf{1}01011\mathbf{1}01\mathbf{1}$$

- Use that each substring is counted by $F(i)$

$2 \times n$ Case

Using a "convolution", we showed that

$$A(G_{2\times n}) \sim \frac{2}{5} + \left(\frac{\varphi}{\sqrt{5}}\right) n.$$

$2 \times n$ Case

Using a "convolution", we showed that

$$A(G_{2\times n}) \sim \frac{2}{5} + \left(\frac{\varphi}{\sqrt{5}}\right) n.$$

With a similar method in $FC_{2\times n}$, we showed that

$$A(FC_{2\times n}) \sim \left(\frac{\varphi}{\sqrt{5}}\right) n.$$

$2 \times n$ Case

Using a "convolution", we showed that

$$A(G_{2\times n}) \sim \frac{2}{5} + \left(\frac{\varphi}{\sqrt{5}}\right) n.$$

With a similar method in $FC_{2\times n}$, we showed that

$$A(FC_{2\times n}) \sim \left(\frac{\varphi}{\sqrt{5}}\right) n.$$

Barbossa showed that the MIS sizes in $FC_{2\times n}$ range from $\frac{n}{2}$ to n. Since $\frac{\varphi}{\sqrt{5}} \approx 0.724$, the average MIS is a bit under the midpoint.

General Case

Let G be an $m \times n$ grid-like graph. We can use transfer matrices to obtain $T(G)$. The reasoning is similar to in the $G_{2\times n}$ case.

General Case

Let G be an $m \times n$ grid-like graph. We can use transfer matrices to obtain $T(G)$. The reasoning is similar to in the $G_{2\times n}$ case.

Let H be the local structure of G, $s_1, \ldots, s_N \in S(H)$ the states, A the transfer matrix of H, and T the ticket matrix of G. Then,

$$T(G) = \sum_{\overrightarrow{ij} \in E(T)} \sum_{k=1}^{N} |I(s_k)| \sum_{m=1}^{n} \left(A^{m-1}\right)_{(k,i)} \left(A^{n-m}\right)_{(j,k)}.$$

References

Reinhardt Euler.
The Fibonacci number of a grid graph and a new class of integer sequences.
Journal of Integer Sequences, 8, 2005.

Reinhardt Euler, Paweł Oleksik, and Zdzisław Skupień.
Counting maximal distance-independent sets in grid graphs.
Discussiones Mathematicae Graph Theory, 33:531–557, 2013.

Mordecai J. Golin, Yiu-Cho Leung, Yajun Wang, and Xuerong Yong.
Counting structures in grid graphs, cylinders and tori using transfer matrices: Survey and new results.
In *ALENEX/ANALCO*, 2005.

Seungsang Oh.
Maximal independent sets on a grid graph.
Discrete Mathematics, 340(12):2762–2768, 2017.