



Ramsey Numbers of Matchings in Ordered Graphs

Zachary Hammersla¹ Luke Hawranick² Kevin Milans¹

¹West Virginia University

²University of South Carolina

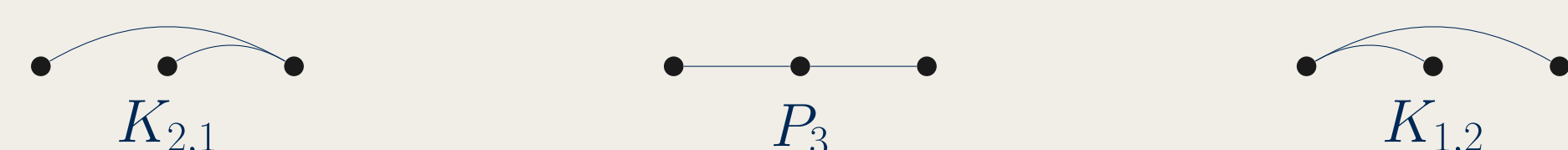


Introduction

Ordered Graph

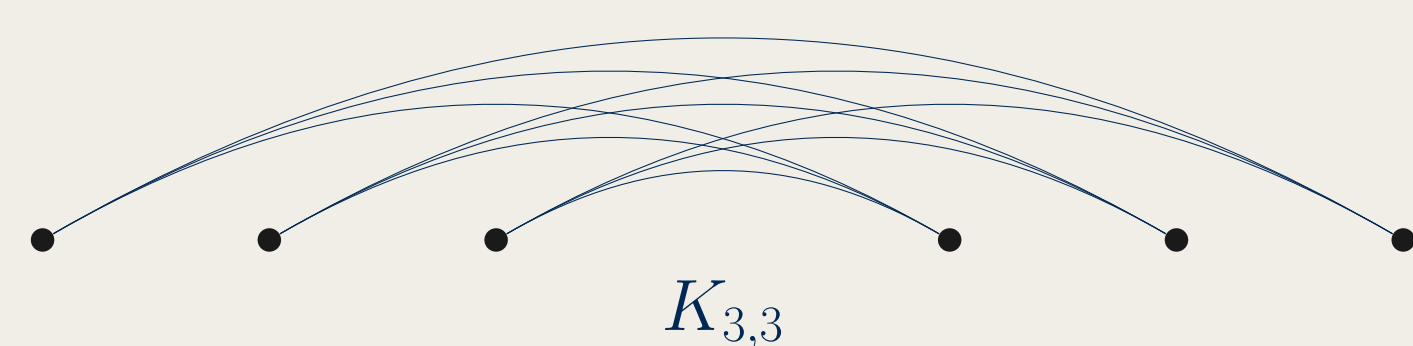
An **ordered graph** G is a graph in which $V(G)$ is linearly ordered.

$K_{2,1}$, P_3 , and $K_{1,2}$ are distinct ordered graphs.



Subgraph Containment

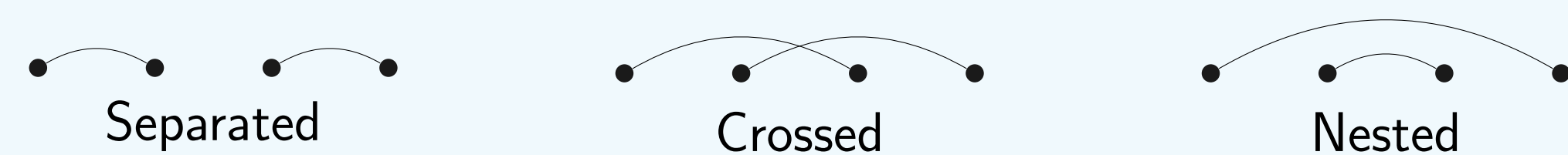
If H and G be ordered graphs, then $H \subseteq G$ means there is an order-respecting injection $f: V(H) \rightarrow V(G)$ such that $uv \in E(H)$ implies $f(u)f(v) \in E(G)$.



Note: $K_{1,2}, K_{2,1} \subseteq K_{3,3}$, but $P_3 \not\subseteq K_{3,3}$.

Separated, Crossed, Nested Edges

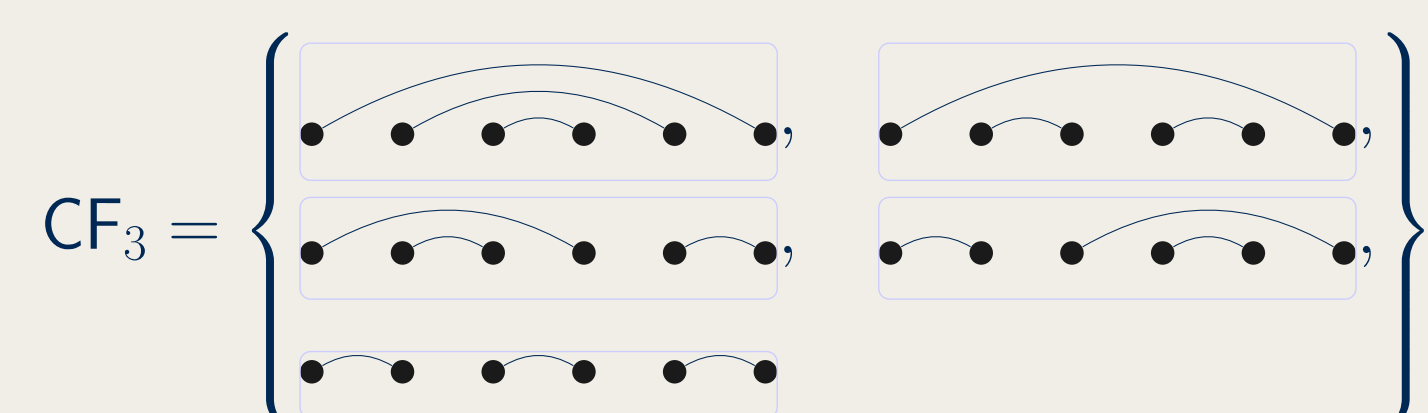
A matching of size 2 falls into one of 3 types.



A set of edges S is **separation-free**, **cross-free**, or **nest-free** if no pair of edges in S are separated, crossed, or nested.

SF_s, CF_s, and NF_s

Let SF_s, CF_s, and NF_s be the family of $2s$ -vertex ordered graphs whose edge sets are separation-free, cross-free, or nest-free matchings of size s .



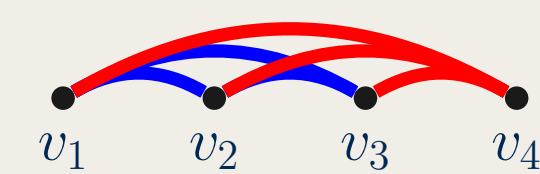
Note: $|\text{CF}_s| = s$ -th Catalan number $= \frac{1}{s+1} \binom{2s}{s}$.

Ramsey Numbers

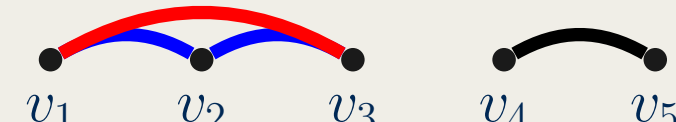
Given families $\mathcal{F}_1, \dots, \mathcal{F}_k$, the **Ramsey Number** $R(\mathcal{F}_1, \dots, \mathcal{F}_k)$ is the minimum n such that every k -edge coloring of K_n contains, for some color i , a color- i copy of some ordered graph in \mathcal{F}_i .

Warm-up: $R(\text{CF}_2, \text{CF}_2) = 5$.

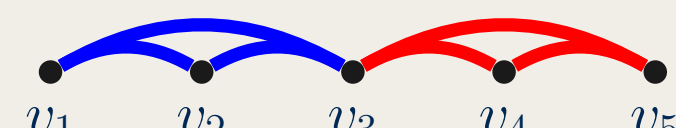
- $R(\text{CF}_2, \text{CF}_2) > 4$.



- $R(\text{CF}_2, \text{CF}_2) \leq 5$. Suppose $v_1v_2v_3$ is polychromatic.



Thus, $v_1v_2v_3$ is monochromatic. By symmetry, we have

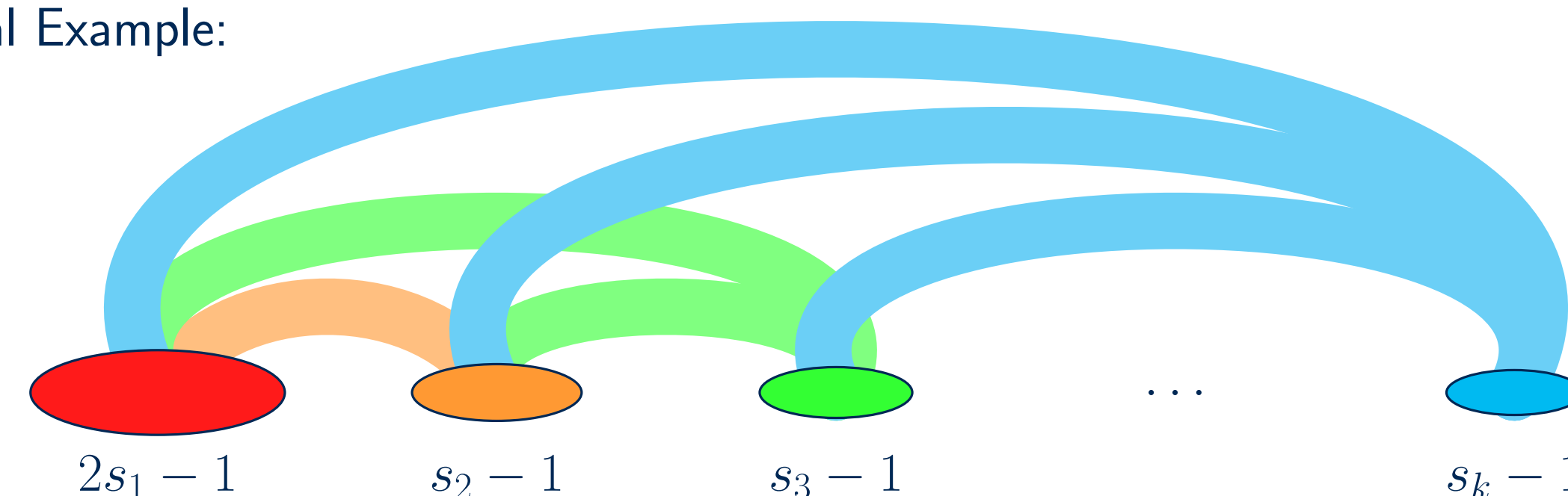


Theorem 1 (Cockayne-Lorimer (1975))

For $s_1 \geq s_2 \geq \dots \geq s_k$, we have

$$R(s_1K_2, \dots, s_kK_2) = \sum_{i=1}^k (s_i - 1) + (s_1 - 1) + 2.$$

Extremal Example:



Let $s_1 \geq s_2 \geq \dots \geq s_k$ and let $\mathcal{F}_{s_i} \in \{\text{NF}_{s_i}, \text{SF}_{s_i}, \text{CF}_{s_i}\}$. Then,

$$R(\mathcal{F}_{s_1}, \dots, \mathcal{F}_{s_k}) \geq R(s_1K_2, \dots, s_kK_2) = \sum_{i=1}^k (s_i - 1) + (s_1 - 1) + 2.$$

We call this the **Graph Lower Bound (GLB)**.

Prior Work on Ordered Graphs

- **BGT**: Barát–Gyárfás–Tóth (2023)
- **KS**: Kaiser–Stehlík (2020)
- **KPT**: Károlyi–Pach–Tóth (1997)

	Nest-free	Separation-free	Cross-free
2 colors $k = 2$	Thm (BGT) $R(\text{NF}_s, \text{NF}_2) = 2s + 1$ $R(\text{NF}_s, \text{NF}_3) = 2s + 2$	Thm (BGT) $R(\text{SF}_{s_1}, \text{SF}_{s_2}) = \text{GLB}$	Cor (KPT) $R(\text{CF}_s, \text{CF}_s) = 3s - 1$.
2 edges $s = 2$	Thm (BGT) $R(\text{NF}_2; k) = k + 3$.	Cor (KS) $R(\text{SF}_2; k) = k + 3$	Prop (BGT) $R(\text{CF}_2; k) > k + 3$
Gen. Case	Conj (BGT) $R(\text{NF}_{s_1}, \dots, \text{NF}_{s_k}) = \text{GLB}$	open	open

Our Results:

- Thm: $R(\text{CF}_2; k) = k + \Theta(\sqrt{k})$
- Prop: $R(\text{NF}_s, \text{NF}_4) = \text{GLB} = 2s + 3$ for $s \geq 4$.

Theorem 2 (Hammersla, H., Milans)

$$R(\text{CF}_2; k) \geq k + (1 - o(1))\sqrt{\frac{4}{\pi}k}$$

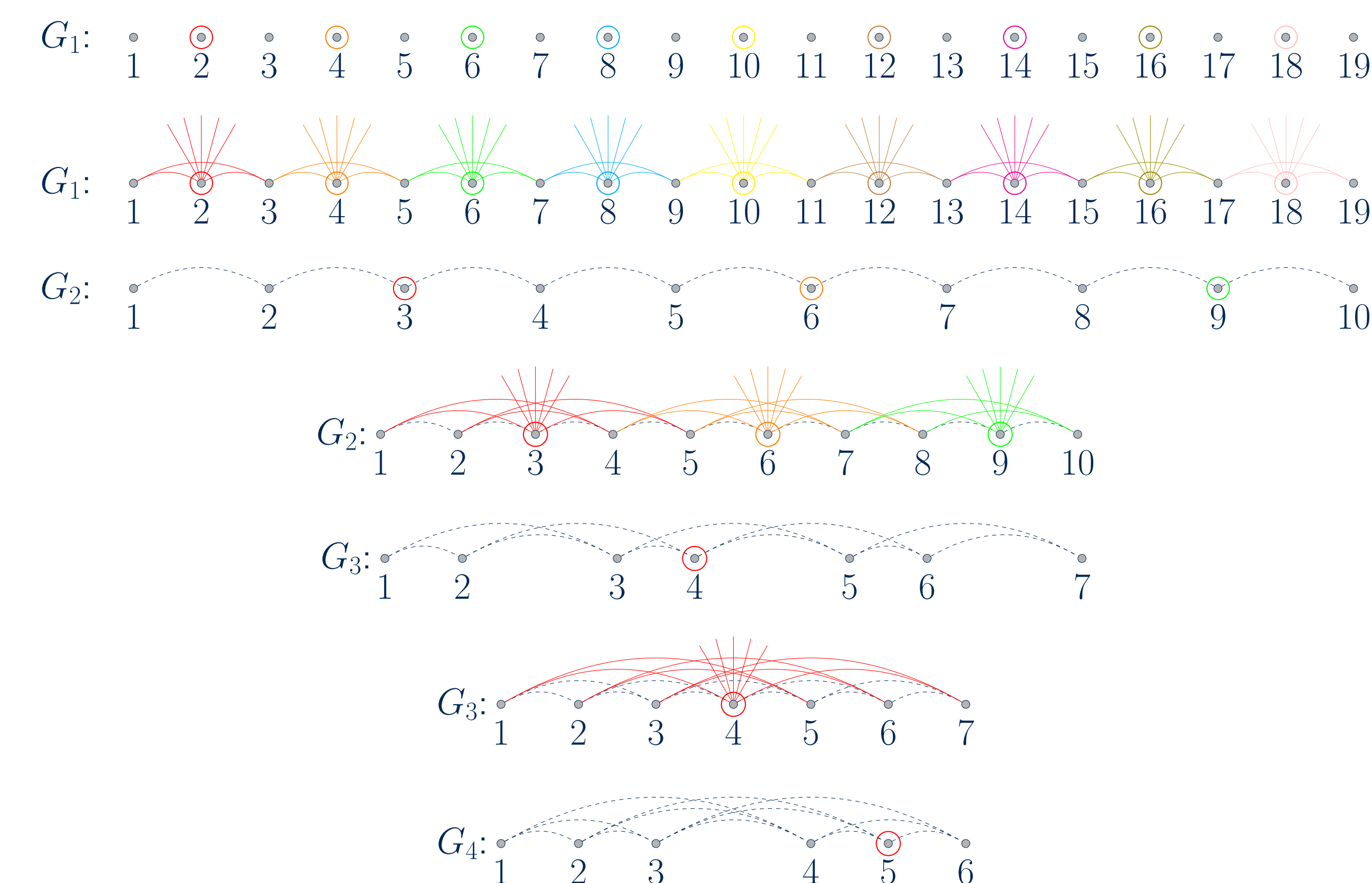
To prove this, we cover $E(K_n)$ for $n = k + (1 - o(1))\sqrt{(4/\pi)k}$ with k color classes such that every color class avoids graphs in CF_2 , where $\text{CF}_2 = \{\text{crossed}, \text{nested}\}$.

If e and e' have the same color, they are not separated or nested. Thus, they are crossed or share an endpoint.

Our coloring proceeds in stages.

- At the start of stage i , we must color an ordered graph G_i where $V(G_i) = [n_i]$ and $uv \in E(G_i)$ implies $|u - v| \geq i$.
- Let $S = \{t : (i+1) \mid t\}$. We use a color class L_t for each $t \in S$. In particular, L_t contains $uv \in E(G_i)$ when
 - uv is incident to t , or
 - $|u - v| \in \{i, i+1\}$ with $t \in [u, v]$.
- Let $G_{i+1} = G_i - S - \bigcup L_t$, renaming vertices.
- The process ends when G_i is empty.

Num. colors used: $n - |V(G_\ell)|$, where ℓ is the last stage.



Josephus Sieve

Start with a list of the positive integers. At stage i , delete integers at indices divisible by $i + 1$.

1 ~~2~~ 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 10 11 12 13 14 15 16 17 18 19 ...
 1 3 ~~5~~ 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 ...
 1 3 7 ~~9~~ 13 15 19 21 25 27 31 33 37 39 43 45 49 51 55 ...

After stage i , the first $i + 1$ entries will never be deleted. The numbers that survive form a sequence called the **Josephus Sieve**.

- A vertex $u \in V(G_1)$ survives to the end if and only if u appears in the Josephus Sieve.
- Let $\Psi(n)$ be the number of entries in the Josephus Sieve that are at most n .
- Thm: If $k \geq n - \Psi(n)$, then $R(\text{CF}_2; k) > n$.
- Thm (Andersson (1998)): $\Psi(n) = \sqrt{(4/\pi)n} + O(n^{1/6})$.

Theorem 3 (Hammersla, H., Milans)

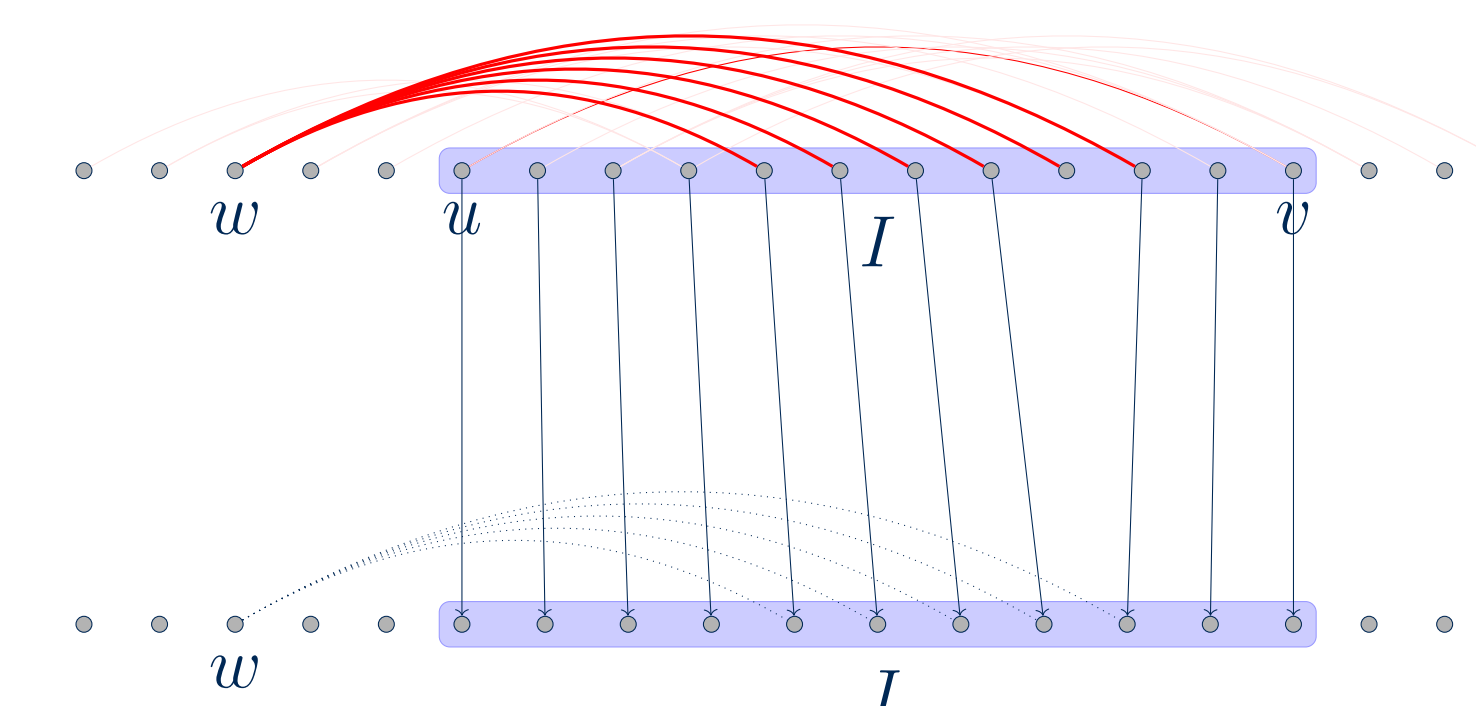
$$R(\text{CF}_2; k) \leq k + 1 + \lceil \sqrt{2k} \rceil.$$

Let an edge-coloring of an ordered graph be **good** if every color class avoids subgraphs in CF_2 .

Lemma 4: (Compression)

Let G be a good k -edge coloring of an n -vertex ordered graph with m non-edges. If $k > 0$, then there is a good $(k - 1)$ -edge coloring of an $(n - 1)$ -vertex ordered graph with at most $m + 1$ non-edges.

Compression Illustration:



Thm: If $\binom{n-k}{2} > k$, then $R(\text{CF}_2; k) \leq n$.