An Empirical Analysis of Algorithms for Simple Stochastic Games

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Table of Contents

Table of Contents

Table of Contents

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Stochastic Games

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An omniscient adversary and a surveillance drone move between targets on a graph.

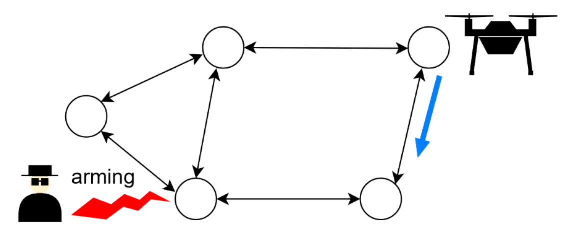

Figure 1: Stackelberg Surveillance

A **stochastic game** is a repeated game with probabilistic transitions played by 2 or more players.

Simple Stochastic Game

A Simple Stochastic Game (SSG), G is a two-player game, defined on a directed multigraph $G(V, E)$. The vertex set V is partitioned into disjoint subsets V_{MAX}, V_{MIN}, V_{AVE}, and $SINK$.

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- If $v \notin SINK$, $d^+(v) = 2$. Otherwise, $d^+(v) = 0$.

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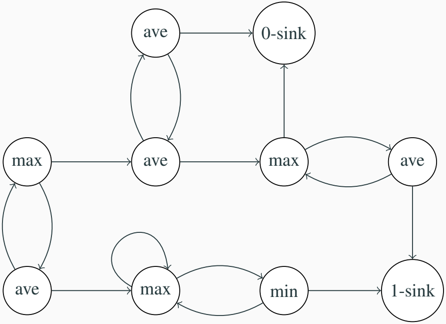
- Players adhere to a strategy that is fixed before the start of the game. A strategy consists of a single edge outgoing from each of a player's vertices.
- **The game ends when the token arrives at a *SINK* vertex**. If it arrives at 1-sink, *MAX* wins. Otherwise, *MIN* wins.

Example 1 - A win for *MAX*



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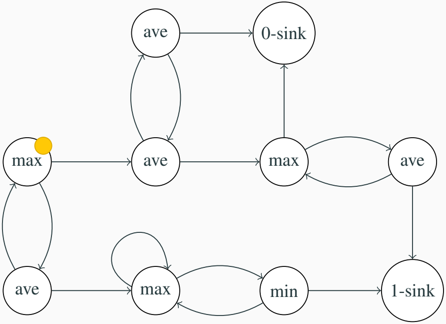

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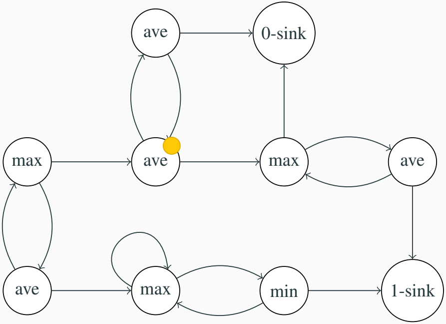

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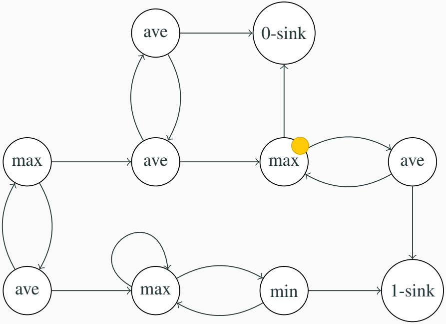

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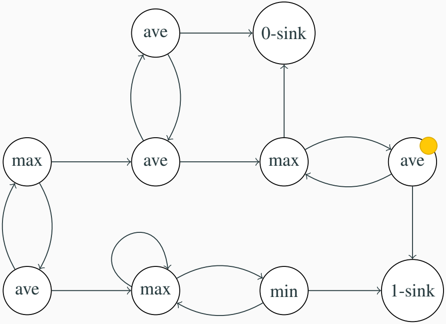

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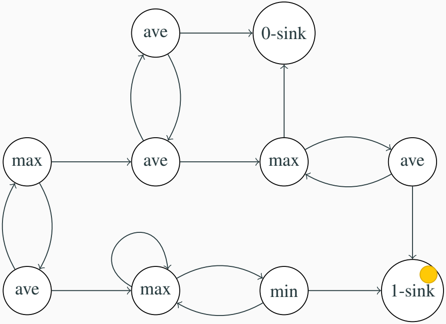
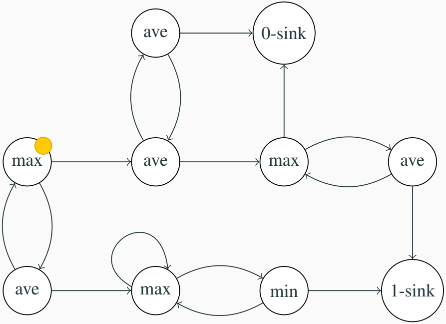


A Simple Stochastic Game

Example 2 - A win for *MIN*



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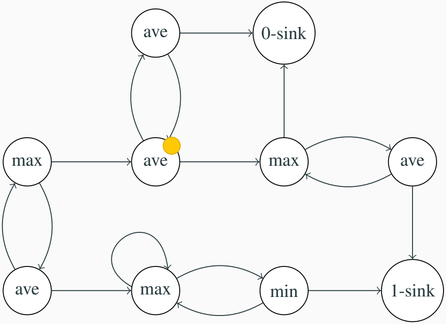

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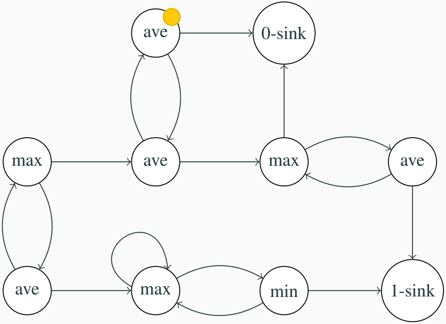

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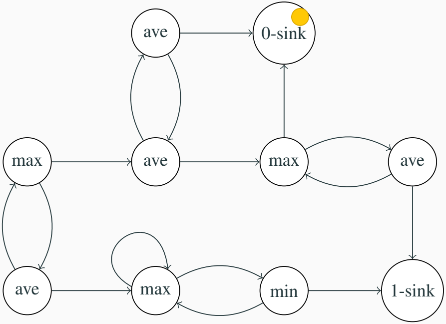

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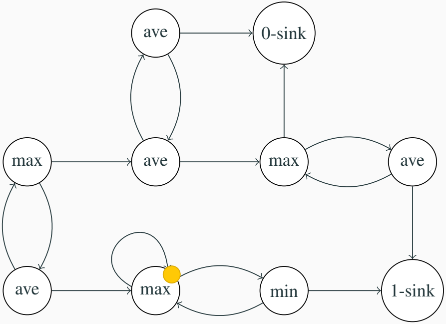
Example 3



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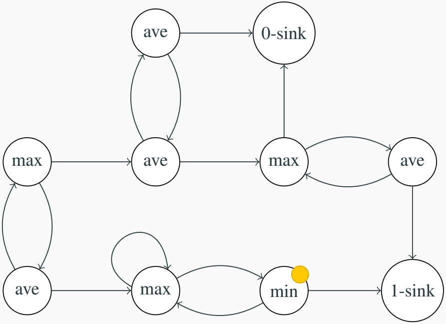

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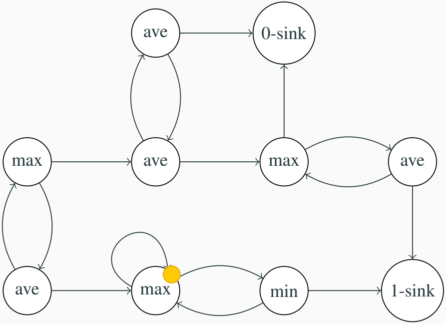

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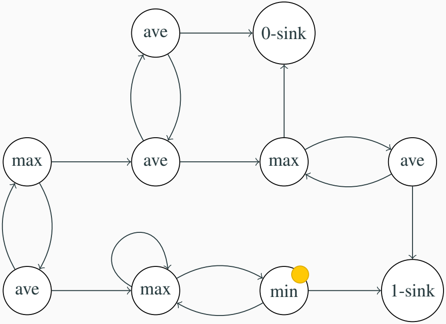

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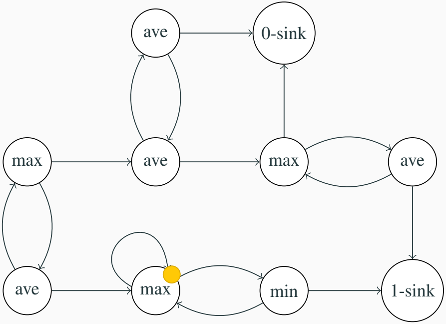

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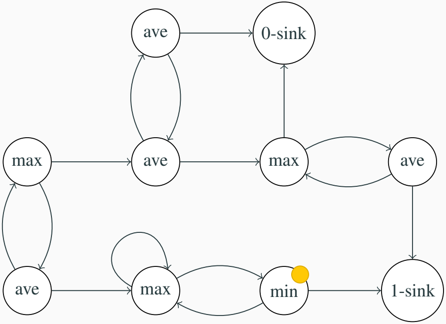

Complexity and Theoretical Motivations

Probability Decision Problem

Given an SSG with a particular starting vertex, is there a strategy for player *MAX* that guarantees victory with a probability of at least $\frac{1}{2}$ regardless of the strategy of *MIN*?

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This problem is in $\mathbf{NP} \cap \mathbf{coNP}$ with no known solution in $\mathbf{P}$.

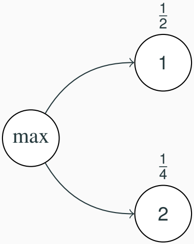
Given the graph G, the most common approach is to compute the optimal strategies for both players.

Strategies and Probability

Propogating Probabilities

V_{MAX}, V_{MIN} **Probabilities**

The probability of player *MAX*'s victory for *max* and *min* vertices with one outgoing edge is equal to the probability of victory at the vertex connected by that edge.



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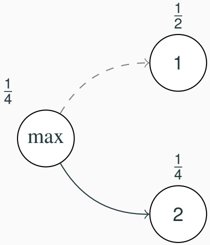

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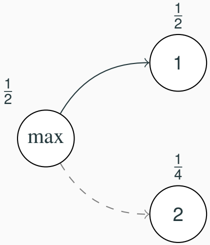

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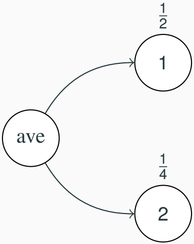

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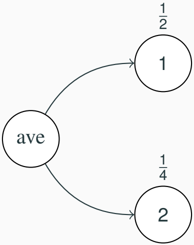
The probability of player *MAX*'s victory for *ave* vertices is equal to the average probability of the vertices connected by the outgoing edges.

$$v(ave) = \frac{1}{2}v(1) + \frac{1}{2}v(2)$$

$$v(ave) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$$



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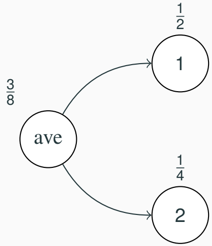

Optimal Strategies

$v \in V_{MAX} \cup V_{MIN}$ is $\tilde{v}$**-switchable** iff $v(i)$ is not locally optimal.

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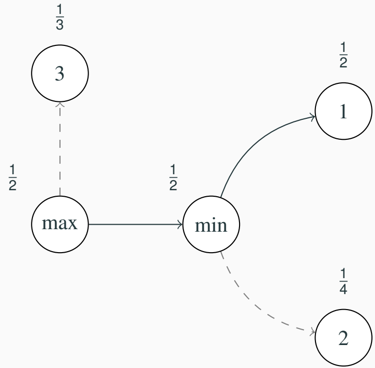
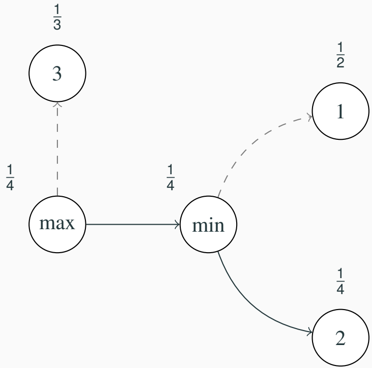
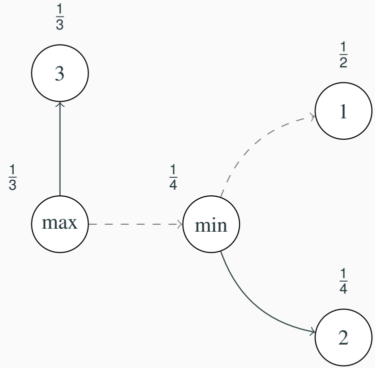
$v \in V_{MAX} \cup V_{MIN}$ is $\tilde{v}$**-switchable** iff $v(i)$ is not locally optimal. If a vertex is not $\tilde{v}$-switchable, then it is $\tilde{v}$*-stable*.


$\frac{1}{3}$
3
$\frac{1}{2}$
1
$\frac{1}{2}$
$\frac{1}{2}$
max
min
$\frac{1}{4}$
2



1/3
3
1/2
1
1/4
1/4
max
min
1/4
2

Now, both the min and max vertices are $\tilde{v}$-stable, meaning that the current strategies are optimal.



Optimal Strategies

A probability vector is considered stable if all player vertices are $\tilde{v}$-stable.

The strategy pair σ, τ is considered *optimal* if it results in the stable vector.

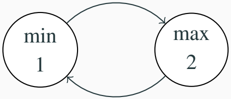
Computing Probabilities

Given σ, τ, the following function computes $v(i)$: the corresponding probability at vertex i, generating a linear system.

$$v_{\sigma,\tau}(i) = \begin{cases} \frac{1}{2}(v_{\sigma,\tau}(j) + v_{\sigma,\tau}(k)) & \text{if } i \in V_{AVE} \text{ for } G_{\sigma,\tau} \text{ with outgoing edges } (i,j), (i,k). \\ v_{\sigma,\tau}(j) & \text{if } i \in V_{MAX} \text{ or } i \in V_{MIN} \text{ for } G_{\sigma,\tau} \text{ with outgoing edge } (i,j). \\ 0 & \text{if } i \text{ is the 0-sink} \\ 1 & \text{if } i \text{ is the 1-sink} \end{cases}$$

Edge Case

Consider the linear system produced by the following vertices: $v(1) = v(2)$

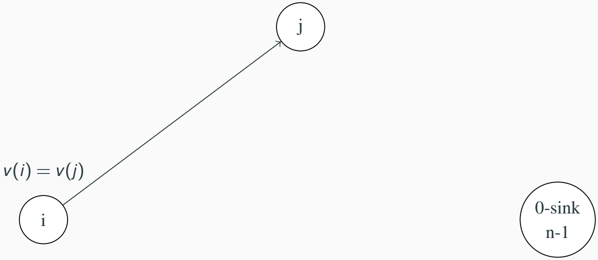


The system is satisfied by any values for the vertices so long as they are the same, but they should have value 0 by the definition of the SSG.

To address this issue, we can reduce an SSG to one where the token always reaches a sink.

Reduction to a Stopping Game

By replacing all edges between two vertices of $V_{MAX} \cup V_{MIN}$ in an SSG with a chain of *ave* vertices, it may become a stopping game with probabilities arbitrarily close to that of the original game.



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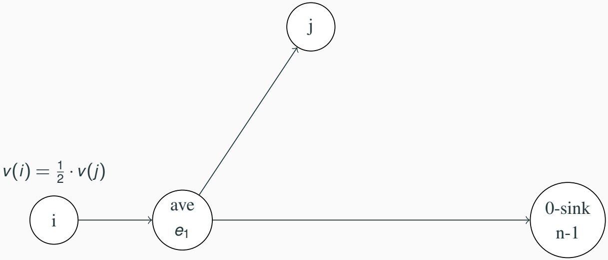

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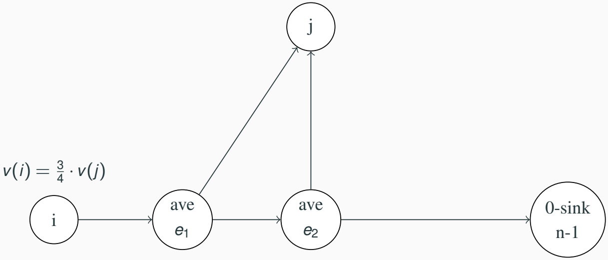

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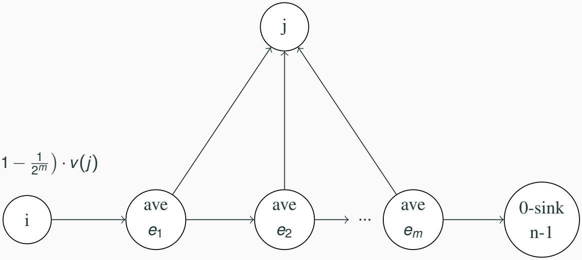
$v(i) = \left(1 - \frac{1}{2^m}\right) \cdot v(j)$



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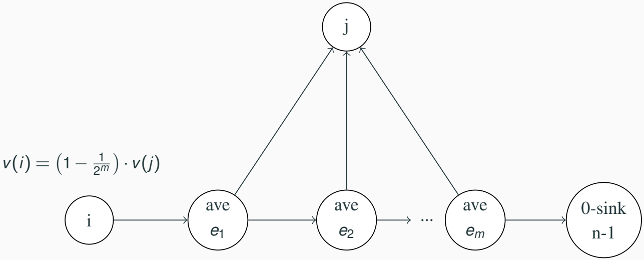
Each movement of the token has a probability of at least $\beta = 1/2^m$ of moving to the 0-sink.



Indirectly Reducing to a Stopping Game

Instead of adding numerous ave vertices to introduce the small probability β, the piece-wise function used previously may be modified.

$$v_{\sigma,\tau}(i) = \begin{cases} \frac{1}{2}(v_{\sigma,\tau}(j) + v_{\sigma,\tau}(k)) & \text{if } i \in V_{AVE} \text{ for } G_{\sigma,\tau} \text{ with outgoing edges } (i,j), (i,k). \\ v_{\sigma,\tau}(j) & \text{if } i \in V_{MAX} \text{ or } i \in V_{MIN} \text{ for } G_{\sigma,\tau} \text{ with outgoing edge } (i,j). \\ 0 & \text{if } i \text{ is the 0-sink} \\ 1 & \text{if } i \text{ is the 1-sink} \end{cases}$$

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Overview of Algorithms

Finding an Optimal Response

The following two algorithms are used to find stable (optimal) responses, given a fixed strategy for their opponent.

1. Derman's LP
2. Naive Stable Response

Finding a stable vector

The following algorithms will return a stable vector of strategies given a graph G. Three are deterministic:

- Hoffman-Karp
- Non-Convex Quadratic Program (Condon, 1990)
- Converge-From-Below (Condon, 1992)

Two are randomized:

- Tripathi Algorithm (Tripathi, Valkanova, Kumar, 2010)
- Ludwig Algorithm (Ludwig, 1995)

Results

Implementation

Each of the algorithms discussed were implemented in C++ using the tools Gurobi and Eigen.

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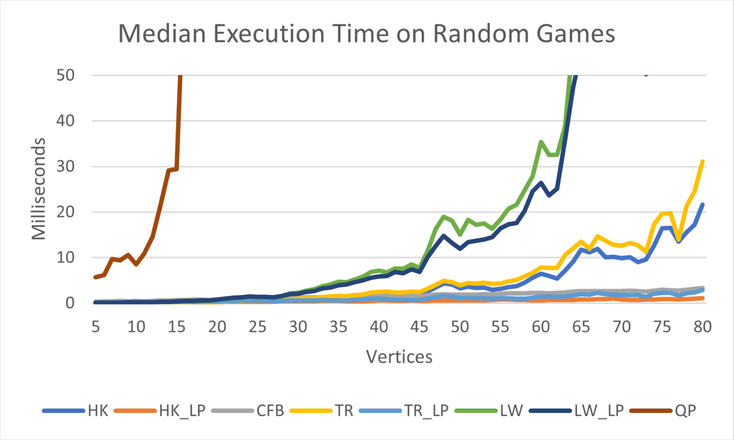
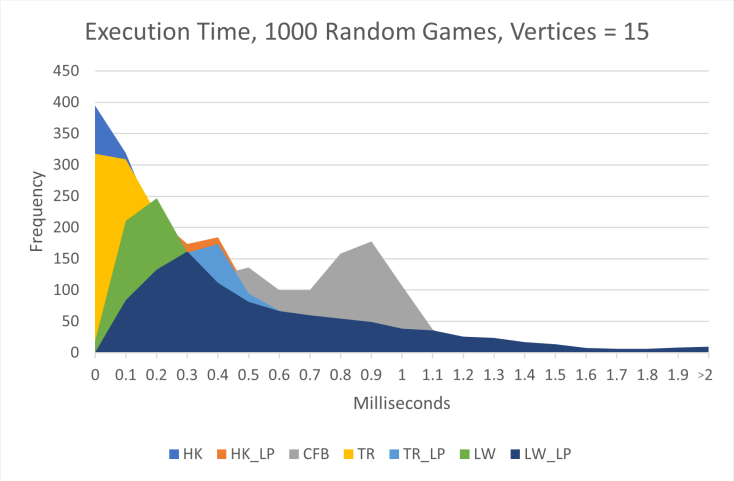
The algorithms that compute the optimal response as an intermediate step were split into two variations, one using Derman's LP and the other using the naive approach.

Empirical Results



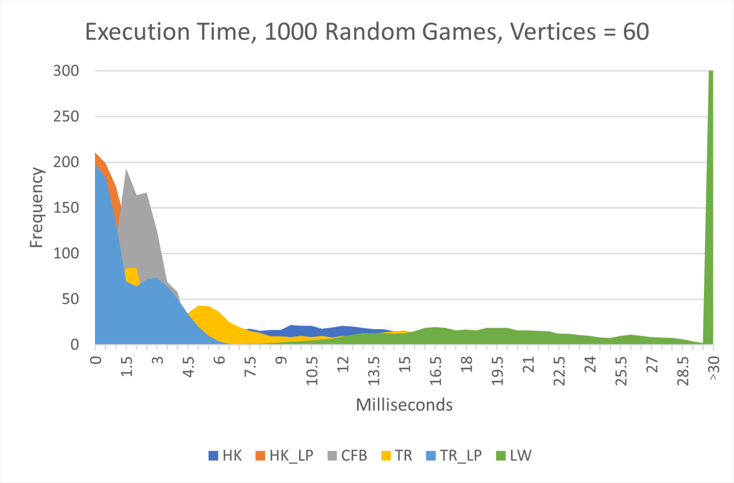
- Quadratic Program is incomparable.
- Converge-From-Below, Tripathi with Derman's LP, and Hoffman-Karp with Dermans' LP are contenders.
- Algorithms with Derman tend to have smaller median execution time compared to those with Naive stable response.

Empirical Results



- On a small instance, all algorithms except QP have comparable runtimes.
- Converge-From-Below has a small spike at a longer execution time.

Empirical Results



- On a larger instance, Ludwig's algorithm becomes incomparable

Searching for Difficult Cases

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The Hoffman-Karp algorithm is among the best performing, but its worst case is not known.

We counted the iterations needed by the Hoffman-Karp algorithm on

- Hundreds of thousands of random games
- Random games without easily solved vertices
- Games found by locally searching the problem space
- Every possible game up 12 vertices

Max-chain Case

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Max-chain Case

The Hoffman-Karp algorithm is among the best performing, but its worst case is not known.

The max-chain only takes $n-1$ iterations of the Hoffman-Karp algorithm to solve.

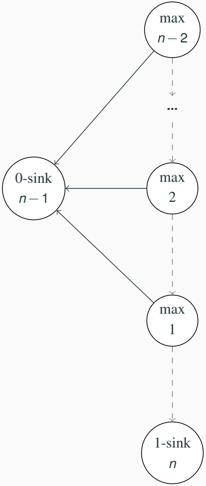


Figure 2: Initialization of max-chain

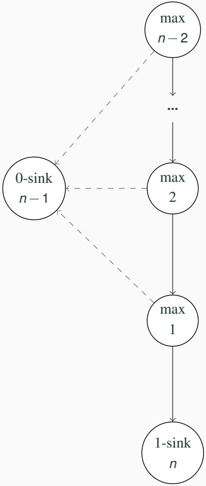
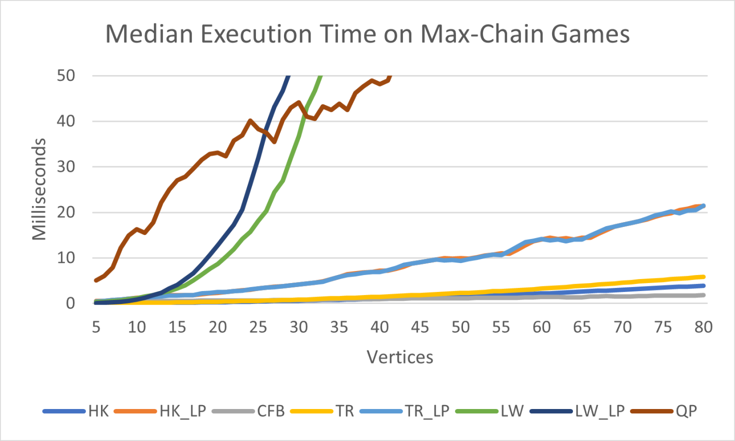


Figure 3: Solved max-chain

Max-chain Case



- On this instance, Ludwig's algorithm performs very badly. QP overcomes it.
- Hoffman-Karp with naive stable response, Tripathi's algorithm, and Converge-From-Below champion this instance.

Modifying the max-chain to include a chain of min vertices yields the following game:

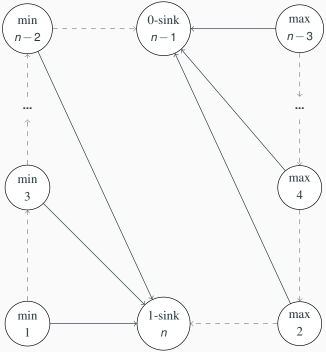


Figure 4: Double-Chain initialization

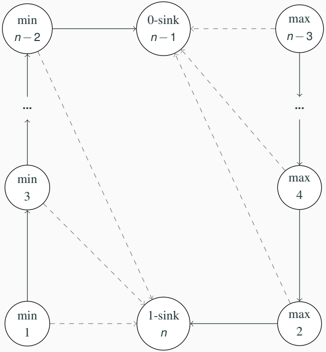
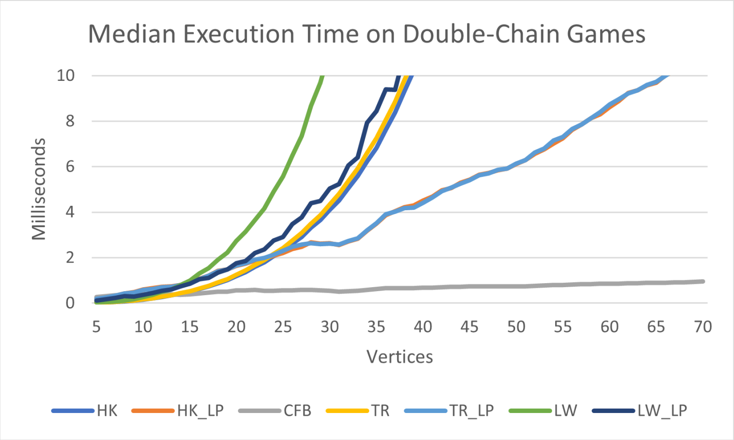


Figure 5: Solved Double-Chain

Double-chain Case



- Converge-From-Below champions this instance.

Conclusion

- The Converge-From-Below algorithm most consistently performs well for the game sizes considered despite being thought to be exponential in the worst case.

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- Many other exponential algorithms, including those by Hoffman-Karp and Tripathi, also perform well in these cases.
- This paper hints that the current complexity results for these algorithms warrant further research.