ELECTRICAL ENGINEERING

QUALS QUESTIONS

2011

HTTP://EE.STANFORD.EDU/PHD/QUALS

$$\boxed{\text{January 2011}}$$

Solutions to R.M. Gray's 2011 qualifying exam problem.

Solutions include more information than was expected from the student during the exam, specific exams could emphasize one thread or another depending on the student's performance on the earlier material.

Several systems are described below by their input/output relations.
For each system:

1. is the system linear?
2. is the system time-invariant?
3. Is the system stable?
4. is the system invertible?

Systems:

- *Input:* $x(t)$; all real t

  Output: $y(t) = \int_{-\infty}^{t} x(\tau)e^{-\alpha(t-\tau)}d\tau$; all real t

- *Input:* $x(t)$; all real t

  Output: $y(t) = [a + mx(t)]\cos(2\pi f_0 t + \theta)$; all real t

- *Input:* $x(t)$; all real t

  Output: $y[n] = x(n)$; all integer n

- *Input:* $x[n]$; all integer n

  Output: satisfies difference equation $y[n] = ay[n-1] + x[n]$; all integer n. The system is assumed to be causal.

Solution

- The system is defined by a convolution, so it is linear and time invariant. The impulse response can be recognized as $h(t) = e^{-\alpha t}u(t)$, where $u(t)$ is the unit step function. The system is stable provided $\alpha > 0$, otherwise it is an integrator (and has a Fourier transform only in the generalized, limiting, sense). Assuming that $\alpha > 0$, The Fourier transform of the impulse response is

  $$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt = \frac{1}{\alpha + j2\pi f}.$$

  The inverse filter will have transform $\alpha + j2\pi f$, which has inverse Fourier transform $\alpha\delta(t) + \delta'(t)$, a scaled identity plus differentiation.

- The system is ordinary amplitude modulation. It is linear if $a = 0$, but only affine otherwise. It is time varying because of the cosine. It is invertible if θ is known provided the positive bandwidth of $x(t)$, say W Hz, satisfies $W < f_0$. E.g., multiply by $\cos(2\pi f_0 t + \theta)$ and low pass filter and DC block. If θ is not known, then demodulation is still possible but takes more work.

- This is a sampling system. It is linear. The system is not time invariant because an input shift of an arbitrary amount does not correspond to an output shift by the same amount (unless the shift is by an integer). The system will be invertible if the sampling theorem holds, which means that the sampling frequency f_s (here 1) satisfies $f_s > 2W$, where W is the maximum frequency in the signal. Thus provided the Fourier transform of the signal is nonzero only for $f \in (-1/2, 1/2)$, the signal can be reconstructed from its samples using the sampling expansion or by low pass filtering the a signal with the samples imbedded on impulses.

- A difference equation defines a linear system, and the system is time invariant since the coefficients are. The system can be characterized as convolving the input with the response to a Kronecker delta $x[n] = \delta(n) = 1$ for $n = 0$, and 0 otherwise. Since the system is assumed causal, $y[n] = 0$ for $n < 0$ and hence $y[0] = 1$, $y[1] = a$, $y[2] = a^2$, etc. so that the response to a Kronecker delta is $h[n] = a^n$ for $n = 0, 1, 2, \ldots$ and 0 otherwise. The inverse filter has Kronecker delta response $g[n] = \delta[n] - a\delta[n-1]$, the discrete time convolution of h and g is δ. Alternatively, from the geometric series (assuming $|a| < 1$)

$$H(f) = \sum_{n=0}^{\infty} a^n e^{-j2\pi fn} = \frac{1}{1 - ae^{-j2\pi f}}$$

and the inverse filter is $1/H(f)$.

Alternatively, taking the transform of the difference equation and changing variables (or using the delay theorem)

$$\begin{aligned} Y(f) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j2\pi fn} \\ &= \sum_{n=-\infty}^{\infty} ay[n-1]e^{-j2\pi fn} + \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn} \\ &= \sum_{n'=-\infty}^{\infty} ay[n']e^{-j2\pi f(n'+1)} + X(f) \\ &= e^{-j2\pi f}Y(f) + X(f) \end{aligned}$$

so that

$$Y(f) = \frac{X(f)}{1 - e^{-j2\pi f}}$$

If $a = 1$, then the first argument still works and the system is still invertible, but the transform arguments get more complicated.

The continuous time Fourier transform (CTFT) of a continuous time signal $x(t)$ is
$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

The discrete time Fourier transform (DTFT) of a discrete time signal $x[n]$ is
$X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$
For the same set of systems, describe how the Fourier transform of the output can be determined from that of the input (using the appropriate type of Fourier transform)
Systems:

- *Input:* $x(t)$; all real t

  Output: $y(t) = \int_{-\infty}^{t} x(\tau)e^{-\alpha(t-\tau)}d\tau$; all real t

- *Input:* $x(t)$; all real t

  Output: $y(t) = [a + mx(t)]\cos(2\pi f_0 t + \theta)$; all real t

- *Input:* $x(t)$; all real t

  Output: $y[n] = x(n)$; all integer n

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Solution: Some of these may have been derived in answering the first question, in which case they were skipped or just rephrased.

- $Y(f) = X(f)H(f)$ as before.

- $Y(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))e^{j\theta} + \frac{m}{2}[X(f-f_0) + X(f+f_0)]$

- If the signal is bandlimited to $[-1/2, 1/2]$, then DTFT $Y(f)$ of the sampled waveform is the same as that of the original waveform for $f \in [-1/2, 1/2]$, the only range of f needed for inversion. If f is allowed to range over the entire real line, then the DTFT has periodic replicas of $X(f)$ with period 1.

  A careful proof was not expected, I was more interested in either memory or intuition. A short proof is the following: If $X(f)$ is nonzero only in $[-1/2, 1/2]$, then in that region it can be expanded as a Fourier series in f as

$$X(f) = \sum_{k=-\infty}^{\infty} a_k e^{-j2\pi kf}; f \in [-1/2, 1/2]$$

  with

$$a_k = \int_{-1/2}^{1/2} X(f)e^{j2\pi kf}df$$

where the signs are reversed from the usual convention because this is in the frequency domain. But from the bandlimited assumption,

$$a_k = \int_{-\infty}^{\infty} X(f) e^{j2\pi kf} df = x(k) = y[k]$$

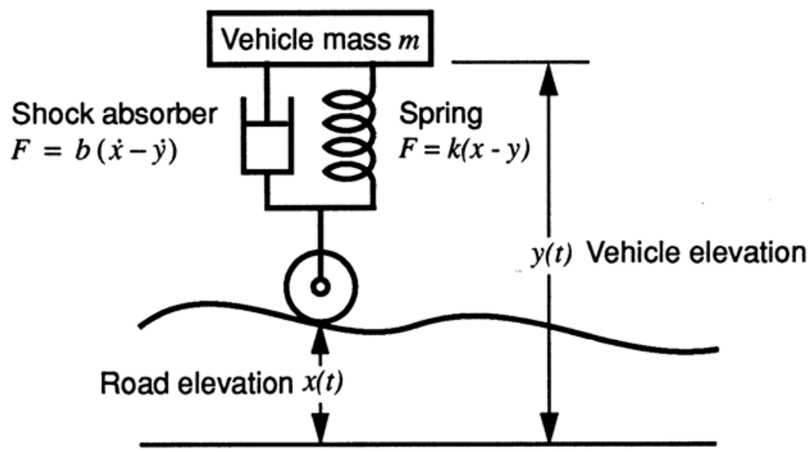
so that

$$X(f) = \sum_{k=-\infty}^{\infty} y[k] e^{-j2\pi kf} = Y(f)$$

- As earlier,

$$Y(f) = \frac{X(f)}{1 - e^{-j2\pi f}}$$

Stanford University, Department of Electrical Engineering
Qualifying Examination, Winter 2010-11
Professor Joseph M. Kahn



A one-wheeled vehicle rolls along a road, supported by a suspension that includes a spring and a shock absorber. At time t, the road elevation is $x(t)$ and the vehicle elevation is $y(t)$. The suspension can be considered as a linear time-invariant system H with input $x(t)$ and output $y(t)$, $H\{x(t)\} = y(t)$, which is governed by the differential equation:

$$m\ddot{y} + b\dot{y} + ky = b\dot{x} + kx\,,$$

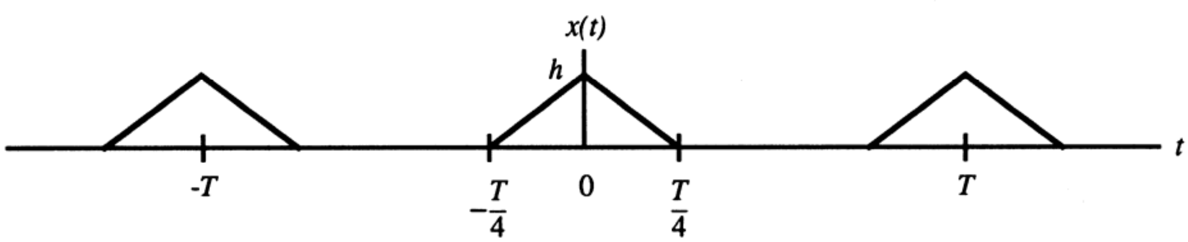
where m, b and k are positive real constants.

1. Find the frequency response $H(j\omega)$ of the system, which satisfies:

$$H\{e^{j\omega t}\} = H(j\omega) \cdot e^{j\omega t}\,.$$

2. Assume $m = 1$, $b = 1$ and $k = 1$. Make a sketch of the magnitude response of the system, $|H(j\omega)|$, and describe the system qualitatively.

3. The vehicle rolls along an infinitely long road with regularly spaced triangular speed bumps, so the road elevation $x(t)$ is the periodic signal indicated below. Give an exponential Fourier series representation of this $x(t)$.



4. Assuming general values of m, b and k, find an expression for the vehicle elevation $y(t)$ when the vehicle rolls over the road pictured above.

Answers

1. Taking the Fourier transform of the differential equation and using the property

$$\dot{x}(t) \leftrightarrow (j\omega)X(j\omega),$$

we obtain:

$$\left[m(j\omega)^2 + b(j\omega) + k\right]X(j\omega) = \left[b(j\omega) + k\right]Y(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{b(j\omega)+k}{m(j\omega)^2 + b(j\omega) + k}.$$

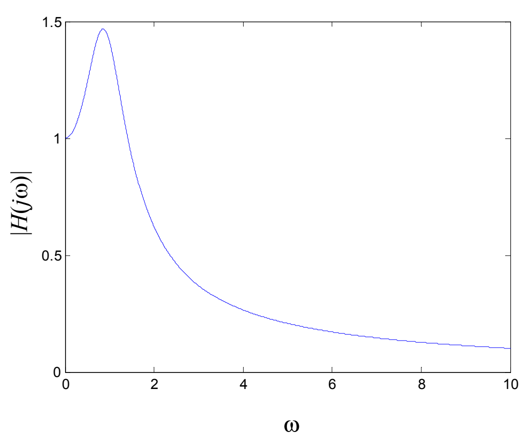
2. Setting $m = b = k = 1$:

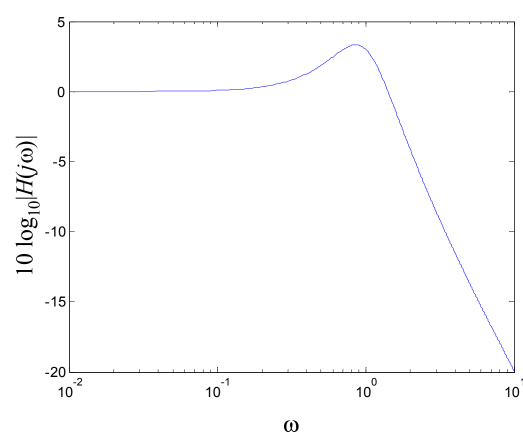
$$H(j\omega) = \frac{j\omega+1}{j\omega+1-\omega^2}$$

$$\left|H(j\omega)\right| = \sqrt{\frac{\omega^2+1}{\omega^2+\left(1-\omega^2\right)^2}}$$

We evaluate $\left|H(j\omega)\right|$ for several values of ω:

ω	$\left\|H(j\omega)\right\|$
0	1
1	$\sqrt{2}$
∞	0





3. Let $q(t)$ denote one period of $x(t)$. Note that $q(t) = h\Lambda\left(\frac{t}{T/4}\right)$, and use the Fourier transform pair

$$\Lambda\left(\frac{t}{\tau}\right) \leftrightarrow \tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$$

to obtain

$$q(t) \leftrightarrow Q(j\omega) = \frac{Th}{4}\operatorname{sinc}^2\left(\frac{\omega T}{8\pi}\right).$$

Representing $x(t)$ by a Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \ ,$$

where the fundamental frequency is $\omega_0 = \frac{2\pi}{T}$, the Fourier series coefficients are given by:

$$a_n = \frac{1}{T} Q(j\omega)\Big|_{\omega = n\omega_0} = \frac{h}{4}\operatorname{sinc}^2\left(\frac{n\omega_0 T}{8\pi}\right) = \frac{h}{4}\operatorname{sinc}^2\left(\frac{n}{4}\right).$$

4. Given an input $x(t)$, the output is:

$$y(t) = H\{x(t)\}.$$

Representing $x(t)$ as a Fourier series, and using the fact that complex exponentials are eigenfunctions of any LTI system, so that $H\{e^{jn\omega_0 t}\} = H(jn\omega_0)e^{jn\omega_0 t}$, the output is:

$$\begin{aligned}
y(t) &= H\left\{\frac{h}{4}\sum_{n=-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{n}{4}\right) e^{jn\omega_0 t}\right\} \\
&= \frac{h}{4}\sum_{n=-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{n}{4}\right) H\{e^{jn\omega_0 t}\} \\
&= \frac{h}{4}\sum_{n=-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{n}{4}\right) H(jn\omega_0) e^{jn\omega_0 t} \\
&= \frac{h}{4}\sum_{n=-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{n}{4}\right) \frac{b(jn\omega_0) + k}{m(jn\omega_0)^2 + b(jn\omega_0) + k} e^{jn\omega_0 t}
\end{aligned}$$

CS Quals 2011 (Networking)

Nick McKeown

Question 1

Why does the Internet use packet switching?

Question 2

How do we define statistical multiplexing gain?

How could we define it for a network?

Question 3

What do you think the statistical multiplexing gain is at different points in the Internet? E.g. WAN, enterprise, home.

How could we measure it?

Question 4

In the 1980s and 90s, the telecommunication companies proposed an alternative to TCP/IP called "ATM" or "B-ISDN". It had the following characteristics:

1. Virtual circuits were established end-to-end.
2. A separate control plane for establishing circuits
3. Virtual circuits carried small, fixed length packets (called "cells")
4. Virtual circuits could have different qualities of service.

What are the pros and cons of such a design?

Why do you think it didn't succeed?

Qualifying Exam 2011
Engineering Physics, Shanhui Fan

As a reminder, the time-dependent Schodinger equation of electrons:

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V(x)\phi$$

and the time-independent Schodinger equation of electrons:

$$E\phi = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V(x)\phi$$

(a) Suppose an electron is confined in an infinite potential well

$$V(x) = \begin{cases} 0,\ 0 < x < a \\ \infty,\ \text{everywhere else} \end{cases}$$

sketch the ground state $\phi_0(x)$ and the first excited state $\phi_1(x)$ for the electron in the potential well. Provide the eigen-energy of these two states.

(b) Suppose at $t = 0$, the electron has a wavefunction $\phi_0(x)$, what is the electron wavefunction at a time t later?

(c) Suppose at $t = 0$, the electron has a (un-normalized) wavefunction of $\phi_0(x) + \phi_1(x)$, could you sketch the shape of the electron probability density distribution as a function of time?

(d) Instead, consider the potential well:

$$V(x) = \begin{cases} \infty, x < 0 \\ 0, 0 < x < a \\ U > 0, a < x < a + b \\ 0, x > b \end{cases}$$

suppose at $t = 0$ the electron has a wavefunction $\phi_0(x)$, how does the probability of finding the electron in the potential well vary as a function of time?

Ph.D. Quals Question
January 10-14, 2011
A.C. Fraser-Smith
Department of Electrical Engineering
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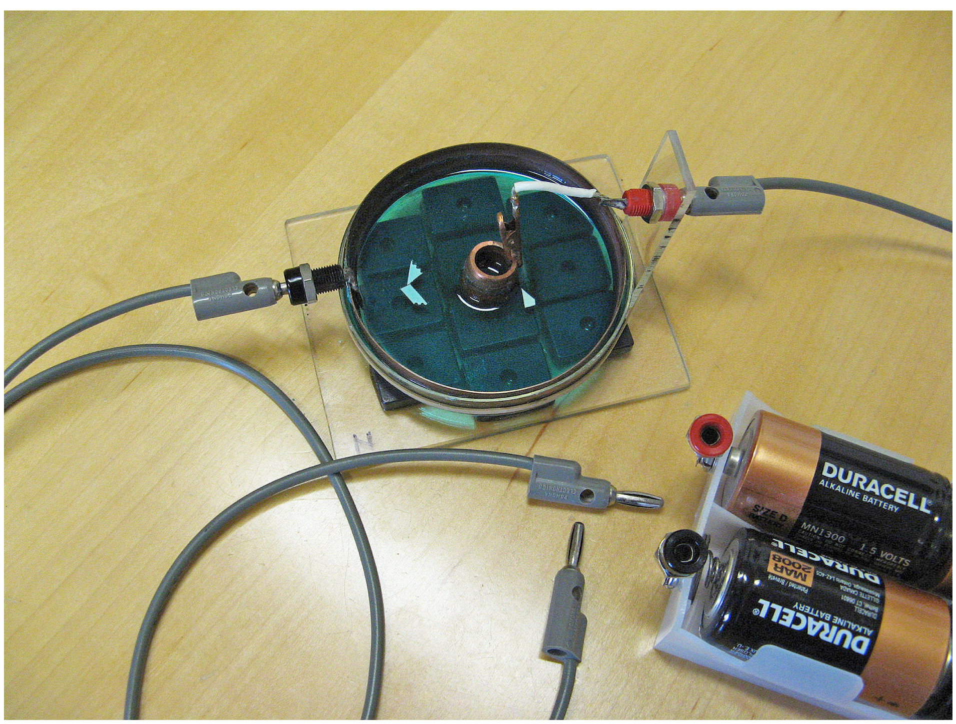
Liquid Cyclotron

The figure below shows the liquid cyclotron that was shown to each student. It consisted of a little plastic dish (actually a petri dish) containing distilled water, "laced" with some copper sulphate to make it electrically conducting, and with a copper ring inside its outer circumference and a much smaller copper ring at its center. Both of these copper rings are immersed the copper sulphate solution and both can be separately connected to a battery pack (4 D-cell batteries, giving 6 V), part of which can be seen at the bottom right (the connections to the inner and outer copper rings can also be seen). The little dish sits on top of a number of black rectangular items with holes in their centers that are glued to an underlying piece of flat plastic sheet. The examiner assembled this demonstration equipment and, originally for his own information, marked the enigmatic "N" on the plastic sheet, which can be seen in the photo.

For the next step in the exam process the examiner connected the wires to the battery pack, thus driving a current between the copper rings, which we might now refer to as electrodes. Lo and behold, the copper sulphate solution begins to rotate. This rotation is made more obvious by some small pieces of white plastic floating on top of the solution, which can be seen in the photo and which rotate along with the fluid. It appears that the fluid rotates more quickly near the central electrode. Next, the examiner swaps the wires to the battery pack. The rotating solution slows down, stops, and reverses its direction of rotation. At this stage the examiner asks the students what is going on and states that he is prepared to answer any questions about the equipment.



Liquid cyclotron demonstration

One key question that the examiner was looking for was "what are the black rectangular items?" This generally indicated that the student was already on the right track. The answer was that they were common refrigerator magnets and they were glued down with their poles all oriented in the same direction. The "N" indicated that all the N poles were pointing upward, although that is not an important issue regarding how the system works. In a few cases the gluing issue was addressed: unless the magnets are glued down they immediately clump up, due to the attraction between their various poles, and will not lie flat. This is actually an extremely important feature rarely discussed in textbooks: magnets are not held together just by the magnetic fields of their component structures (e.g., "domains").

Once the magnet situation was understood it was usually deduced that there must be a largely vertical magnetic field penetrating through the solution.

The examiner rather expected that the students would be knowledgeable about the blue tint of the solution, but students nowadays seem to have had less exposure to chemistry and thus are not aware that copper sulphate is the only common salt giving a blue solution. At all events, the examiner would volunteer this information and mention that its presence made the ordinary distilled water it was dissolved in electrically conducting (unlike distilled water itself). Because it was not an important issue for understanding how the solution rotated, it was only occasionally pointed out that the reason copper sulphate was used instead of ordinary salt, or any other salt, was to minimize the electrolytic interactions between the copper electrodes and the conducting solution.

The next step in understanding the liquid cyclotron was for the students to make use of the force equation:

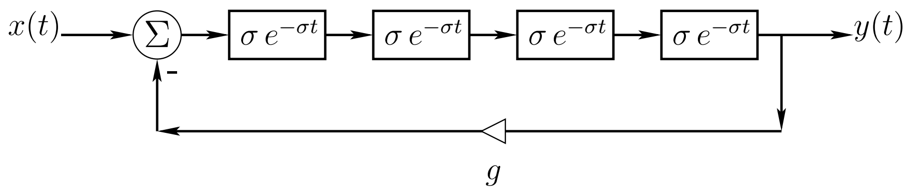
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Where **F** is the force on charge q, **E** is the electric field, **v** is the charge velocity, and **B** is the magnetic field. Obviously **E** will be radially directed and, with a little more thought, strongest near the central electrode. With the resulting v's of the charge carriers all radially directed, the **v x B** forces will lead to circular motion – in the same direction for both positive and negative charges. The rate of circular motion will be strongest near the central electrode. Finally, interaction (collisions) between the charge carriers and neutral particles will produce circular motion of the fluid itself. Reversing the voltage supply will reverse the direction of **E** and thus reverse the direction of motion of the fluid.

There are a few tricky points in comparing this liquid cyclotron with the conventional cyclotron discussed in, say, physics classes describing particle accelerators. In particular, there are both positive and negative charges being driven around inside the region of magnetic field, and then there are their interactions with the neutral fluid, making the circular motion of the charged particles visible but also slowing it down.

EE Quals Problem
January 10-14, 2011
Julius Smith

Consider the following system:



1. Where are the *open-loop poles and zeros?*
2. What values of σ yield a *stable open-loop system?*
3. Find the *transfer function* of the system from its input x to its output y.
4. Sketch the *root locus* of the system as g grows positively from zero.
5. What is the range of $g \geq 0$ for which the system is stable?
6. What is the resonance frequency ω_0 at the stability limit?
7. Describe how to *digitize* this system, assuming σ and g will be used as variable controls.

Birds independently land on an infinitely long telegraph wire at random positions. Each bird has a right neigbor and a left neighbor. The distance Δ between neighboring birds is exponentially distributed according to the pdf $f_{\Delta}(\Delta) = e^{-\Delta}$ for $\Delta \geq 0$.

(a) What is the pdf of the distance between a bird and its nearest neighbor?

(b) If bird B' is the nearest neighbor of B, what is the probability that B is also the nearest neighbor of B'?

(c) Paint the interval between B and its nearest neighbor B' yellow. If we do this for each bird B, what fraction of the real line will be painted yellow?

EE Qualifying Exam (2011)
S.E. Harris

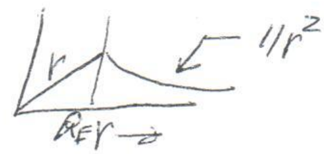
1 Note the similarity of coulomb's law and the gravitational law; i.e. $\bar{E} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q}{r^2}\,\bar{a}_r$ and $\bar{g} = -G\frac{m_1}{r^2}\,\bar{a}_r$

what is $\bar{g}$ at the center of a (spherical) earth? why? ans: $g=0$

2 Derive a Gauss's law equvalent for gravity

ans: $\bar{\nabla}\cdot\bar{g} = -4\pi G\rho_m$; ρ_m = mass density

3 Find the functional form of $\bar{g}$ versus distance from earth center ans:



4 If the earth were an ellipsoid of revolution how would you do the problem

ans: direct (3D) vectorial integration

5 write the functional form for the escape velocity as a function of ~~...~~ position above the earth's surface. ans $V_{escape} \sim \sqrt{\frac{1}{r}}$

6 for a few very fast students, what law would you use to study a racket where the mass is changing? ans: conservation of momentum.