Invest or Fall Behind: Maintaining Quality in Hotelling Markets

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Maintaining Quality under Shocks

- “Dain’s Place” vs. “Dimsum Asian Bistro”:
  - Consumers have different tastes over foold options.

- Quality shocks hit unpredictably:
  - Fryer / Oven breakdowns, chef leaves.
  - Costly to restore and maintain quality.

- Common in other industries as well:
  - A TV show loses its leading star.
  - An app suffers from a recently spotted bug.

Placeholder for pictures

Research Questions

Horizontal Differentiation	Vertical Differentiation
Heterogeneous Consumers	Product Quality Differences

- How do firms dynamically invest to maintain product quality, facing heterogeneous consumers?
    - How does adding heterogeneity of consumers affects the quality dynamics?
    - Are horizontal and vertical differentiations separable? If not, how do the two differentiations interact?
    - Do firms invest in quality efficiently, too much, or too little?

First Look at the Model

- Two firms engage in dynamic quality competition (vertical, endogenous) …
- …in Hotelling markets (horizontal, exogenous).
- High quality products face negative shocks from nature.
- Firms need to pay costs to maintain high quality – "product upgrade".

- In each period, each firm decides whether to upgrade, faces nature's potential shocks, and chooses a price.

Preview of the Results

- The upgrading frequency and joint profits are non-monotonic in upgrading costs.
  - Maskin and Tirole (1988a, b, 1987); Rosenkranz (1995); Ericson and Pakes (1995); Doraszelski and Markovich (2007); Doraszelski and Satterthwaite (2010); Besanko et al.,(2010); Board and Meyer-ter Vehn (2013); Abbring et al. (2018).
  - Aghion et al. (2005); Ryan (2012); Gowrisankaran and Rysman (2012); Eizenberg (2014).
- Two upgrading patterns, upgrading deterrence and open competition, from explicitly modeling horizontal differentiations by Hotelling markets. Non-monotonicity does not reply on learning by doing or exit scrap value.

Preview of the Results

- In terms of social welfare, when upgrading costs are lower or higher, firms under-upgrade. When upgrading costs are intermediate, firms over-upgrade.
  - Mankiw and Whinston (1986); Jones and Williams (2000); Bloom et al. (2013); Ahuja and Novelli (2017).
  - Esteban and Shum (2007); Goettler and Gordon (2011).

- Under-investment under competition, generated by either tacit collusion on low quality or failure to internalize consumer surplus.

Preview of the Results

- When upgrading costs are lower, vertical and horizontal differentiations exhibit substitution relations. When upgrading costs are higher, vertical and horizontal differentiations exhibit complementary relations.
    - Shaked and Sutton (1982); Motta(1993); Irmen and Thisse (1998); Gabszewicz and Wauthy (2012).
- Dynamic substitution / complement between two dimensions of differentiations, with clear sign predictions driven by cost.

Outline of the Talk

- The model
- Vertical differentiation only
  - Benchmark: The social planner
  - Strategic competition
  - Welfare implications
- Interaction of two differentiations
  - Benchmark: The social planner
  - Strategic competition
  - Welfare implications

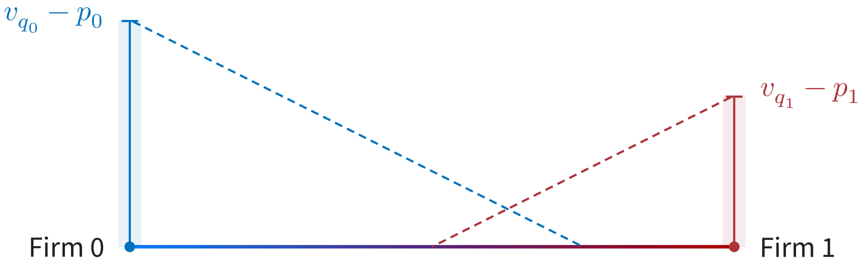
The Model

The Model

- Two long-lived firms, located at two ends of a Hotelling market $[0, 1]$.
- Discrete time, infinite horizon, period length $\mathrm{d}\, t$.
  - Consider the limit of the equilibria at $\mathrm{d}\, t \to 0$.
  - Discrete time naturally models agents' behaviors and avoid technical issues of continuous time.
  - The limit allows cleaner expositions and easier interpretations of the results.
- Each firm produces a product with high or low quality at 0 cost:

$$q_i \in \{L, H\}, \quad v_H = 1, v_L = \alpha \in (0, 1).$$

Stage Game



- At each "moment", there are mass 1 consumers uniformly distributed on $[0, 1]$.
- Consumers are transient, and each consumer purchases at most 1 product.
- Horizontal differentiation: Linear traveling cost k. Assume $k \leqslant 1/3$.

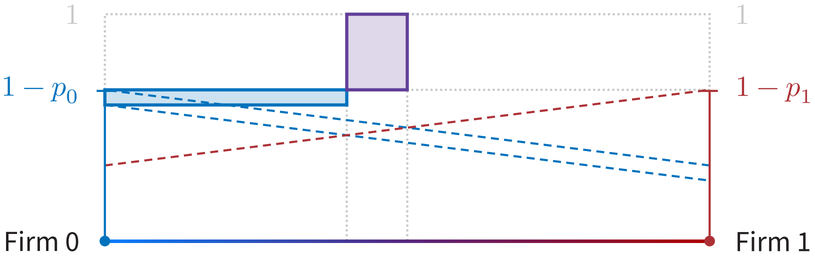
Stage Game: Without Traveling Cost

- Reduce to Bertrand competitions under quality pair (q_0, q_1).
- $\pi_0(H,H) = \pi_0(L,L) = 0$.
- In imbalanced state (H,L):
  - *Firm 0*: Charges $p_0 = 1 - \alpha$ and occupies the market.
  - *Firm 1*: Charges $p_1 = 0$ and does not produce.

$$\pi_0(H,L) = 1 - \alpha, \quad \pi_0(L,H) = 0.$$

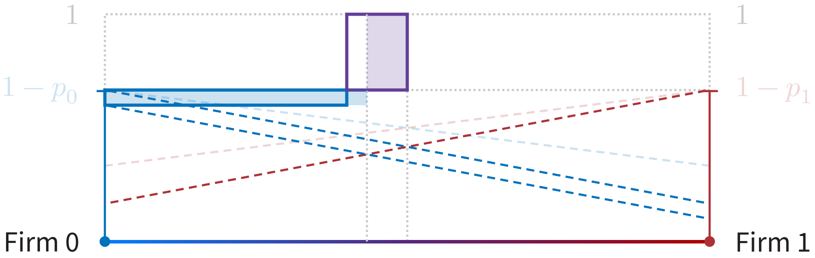
- There is only one profitable state: being the quality leader.

Stage Game: With Traveling Cost



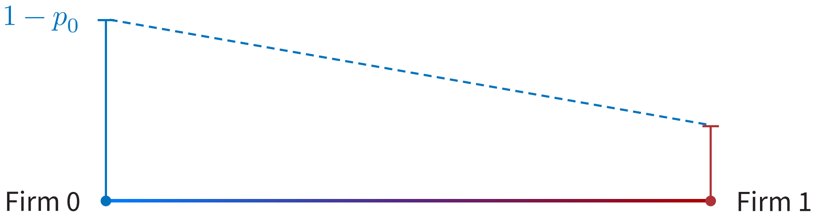
- Hotelling competitions under quality pair (q_0, q_1). Consider $(1, 1)$ first.
- Balancing higher margin and losing demand when raising price.

Stage Game: With Traveling Cost



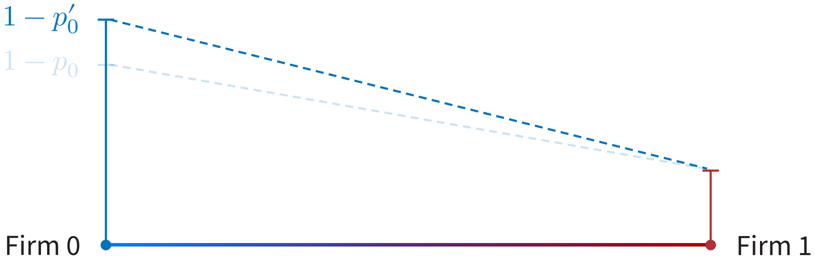
- Hotelling competitions under quality pair (q_0, q_1). Consider $(1, 1)$ first.
- Balancing higher margin and losing demand when raising price.
- $k \Uparrow$: Less competition, less demand loss from raising price.
- $\pi_0(H, H) = k/2$. Increasing in k.

Stage Game: With Traveling Cost



- At (H, L), for α not too large, Firm 0 occupies the market.

Stage Game: With Traveling Cost



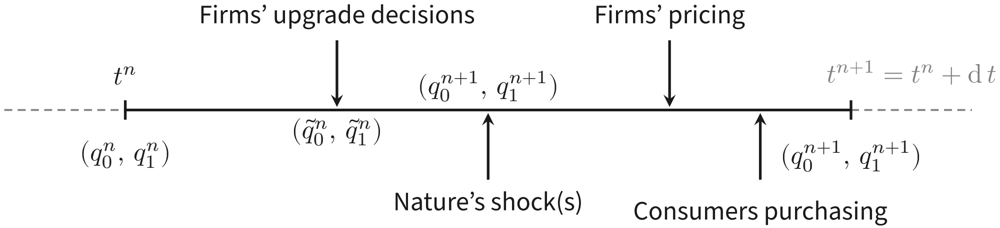
- At (H, L), for α not too large, Firm 0 occupies the market.
- $k \Uparrow$: Harder to reach consumers far away, lowering the price.
- $\pi_0(H, L) = 1 - \alpha - k$. Decreasing in k.

Quality

In each period:

- When $q_i = H$, nature can place a shock leading to quality decay, $q_i = L$.
  - Independent shocks between products.
- When $q_i = L$, firm i can upgrade q_i to H by paying a (lump-sum) cost c.
  - No further shocks from nature if $q_i = L$.

Stage Timeline



- The limiting results as $\mathrm{d}\, t \to 0$ are also robust under alternative stage timelines.

Firms' Strategies

- Markov strategies with payoff-relevant state (q_0, q_1).
- Firm i's strategy:
  - Upgrading: when $q_i = L$, making contigent upgrading decisions:

$$\sigma_i : \{(q_i = \alpha, q_j)\} \to [0, 1].$$

  - Pricing (static NE pricing employed):

$$p_i : \{(q_i, q_j)\} \to \mathbb{R}_+.$$

Equilibrium Concept

- Symmetric Markov Perfect Equilibrium (S-MPE).
- In case of multiplicity, we consider the joint-profit maximizing S-MPE.

- Mainly concerned with the limit of the equilibrium when period length $\mathrm{d}t \to 0$.
  - Common discount factor $\delta = e^{-r\,\mathrm{d}t}$.
  - Shock probability $b = 1 - e^{-\beta\,\mathrm{d}t}$. In the limit, the shock is a Poisson arrival process with arrival rate $\beta > 0$.

Vertical Differentiations Only

Social Planner Benchmark

- Suppose $k = 0$.
- A utilitarian social planner: max trading surplus.
  - Would like the consumer to choose the higher quality product.
  - Set $p_0 = p_1 = 0$ and let the consumers freely choose which product to purchase.
  - Stage social surplus is $\max\{q_0, q_1\}$.

- No duplication of high quality: At (L, L), should the social planner upgrade to (H, L)?

Social Planner's Problem

No Upgrade at (L, L)

$$W_N(L,L) = \underbrace{\alpha \, \mathrm{d}t}_{\text{flow}} + \overbrace{e^{-r\,\mathrm{d}t}}^{\text{discount}} \underbrace{W_N(\alpha,\alpha)}_{\text{continuation}} \xrightarrow{\mathrm{d}t \to 0} W_N(L,L) = \frac{\alpha}{r}$$

- Stay at (L, L) forever and get the perpetuity of the flow payoff $q_L = \alpha$.

Social Planner's Problem

Upgrade at (L, L)

$$W_U(L,L) = -c \quad \text{upgrading cost}$$

$$\text{no shock} \quad +e^{-\beta \, \mathrm{d}t}\left[1 \, \mathrm{d}t + e^{-r \, \mathrm{d}t} W_U(H,L)\right]$$

$$\text{shock} \quad +\left(1 - e^{-\beta dt}\right)\left[\alpha dt + e^{-rdt} W_U(\alpha,\alpha)\right]$$

$$W_U(H,L) = e^{-\beta \, \mathrm{d}t}\left[1 \, \mathrm{d}t + e^{-r \, \mathrm{d}t} W_U(H,L)\right] + \left(1 - e^{-\beta dt}\right)\left[\alpha dt + e^{-rdt} W_U(\alpha,\alpha)\right]$$

$$\xrightarrow{\mathrm{d}t \to 0} \quad W_U(L,L) = \frac{1}{r} - \frac{(\beta + r)c}{r}$$

- Stays at (H, L) forever, gets the perpetuity of the flow payoff 1, and pays the costs.

Social Planner's Optimal Policy

Proposition. The social planner's optimal policy is to upgrade one product at (L, L) and no upgrade elsewhere if

$$\frac{1-\alpha}{r+\beta} \equiv \bar{c} \geqslant c.$$

Otherwise, the social planner's optimal policy is to never upgrades any product.

- The social planner upgrades at (L, L) if the present value of gain is greater than the upgrading cost.

Firms' Problem

- There is only one state offering positive flow profit: (H, L).
- Naturally, if the upgrading cost c is large, firms should not upgrade.
  In fact, firms will not upgrade if $c \geqslant \bar{c}$, just as the social planner.
- How about lower upgrading costs?
  - For any $c > 0$, "always upgrading when possible" is never an equilibrium.
  - Pay cost to upgrade, but never get positive profits.

Upgrading at (L, L)

- Consider sufficiently low cost c.
- Upgrading at (L, L) has a stronger “temptation”:

$$\begin{array}{ccc} 0 & \xrightarrow{\quad c \quad} & 1-\alpha \\ \pi_0(L, L) & & \pi_0(H, L) \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{\quad c \quad} & 0 \\ \pi_0(L, H) & & \pi_0(H, H) \end{array}$$

- Conjecture: Firms upgrade at (L, L).
- No always upgrade: Not upgrade for sure at (L, H).

Upgrading Strategy at (L, H)

- Suppose that firms always upgrade at (L, L), and c is small.
- If firm 1 always upgrades at (H, L)

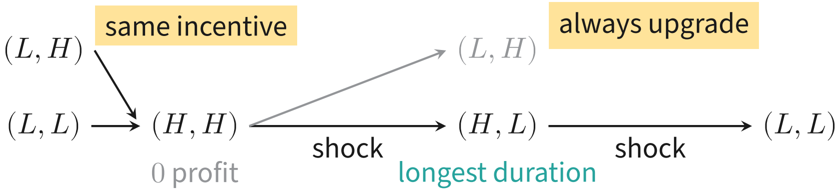
$$
\begin{tikzcd}
 & & (L, H) & \\
(L, L) \arrow[r] & (H, H) \arrow[ru] \arrow[r, "\text{shock}"'] & (H, L) \arrow[r] & (H, H)
\end{tikzcd}
$$

0 profit — 0 duration

- Not BR Firm 0 never gets positive profits from upgrading at (L, L).

Upgrading Strategy at (L, H)

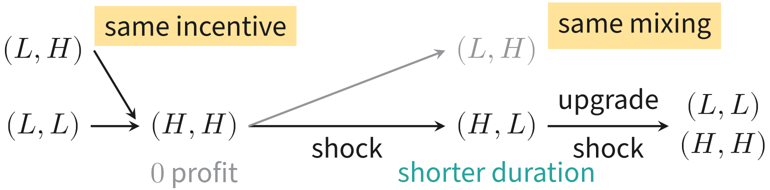
- Suppose that firms always upgrade at (L, L), and c is small.
- If firm 1 does not upgrade at (H, L)



- BR Firm 0 gets large profits from upgrading at (L, L).
- Not BR Firm 1 never gets positive profits from upgrading at (L, L).

Upgrading Strategy at (L, H)

- Suppose that firms always upgrade at (L, L), and c is small.
- If firm 1 mixes at (H, L)



- BR Firm 0 gets just enough profits from upgrading at (L, L).
- BR Firm 1 gets just enough profits from upgrading at (L, L).

MPE at Low Upgrading Costs

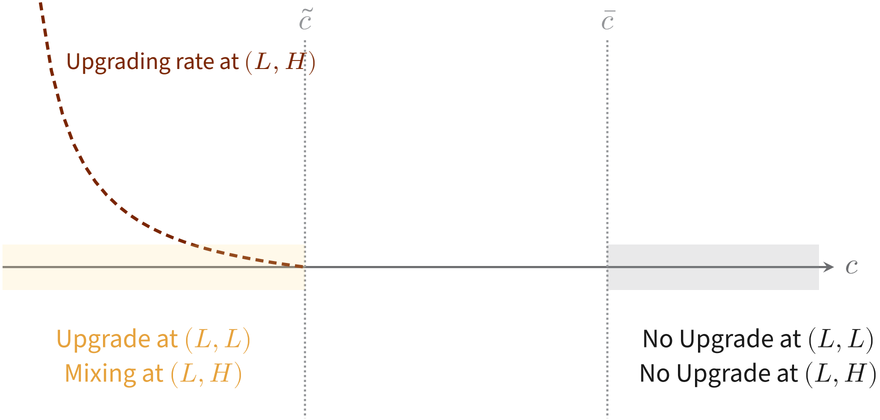
Proposition. If $c \leqslant \tilde{c}$, the following is the limit of a symmetric MPE:

- Firm i upgrades at (L, L) for sure.
- Firm i upgrade at (L, H) at a rate $f(c)$.

Moreover, f decreases in c.

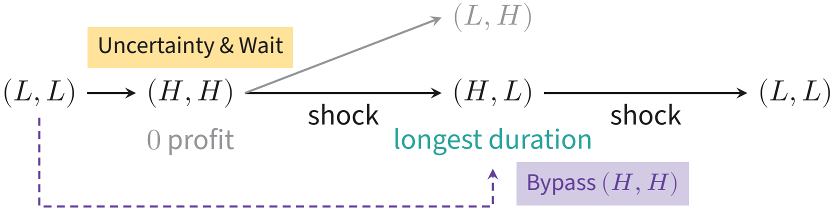
- The upgrading incentive is provided by the potential future profits only.
- More incentives needed as $c \Uparrow$. Then, decreasing f gives longer duration at (H, L).
- $f(\tilde{c}) = 0$: no upgrade at (L, H) at the boundary.

MPE at Low Upgrading Costs



MPE at Intermediate Upgrading Costs

- Larger incentives needed for $c > \tilde{c}$:



- Firms can (partially) bypass (H, H) by mixing at (L, L).

Limiting Behavior of the Mixed Strategy

Let $\hat{g}(\mathrm{d}\,t)$ be the upgrading probability.

- Can $\hat{g}(\cdot)$ converge to a rate?

$$\lim_{\mathrm{d}\,t\to 0} \hat{g}(\mathrm{d}\,t) = 0 \quad \text{and} \quad \lim_{\mathrm{d}\,t\to 0} \frac{\hat{g}(\mathrm{d}\,t)}{\mathrm{d}\,t} = \hat{g} > 0.$$

- Suppose firm 1 upgrades with a rate in the limit:
  - At the moment when the state hits (L, L), firm 1 upgrade with 0 probability.
  - Firm 0 should then upgrade for sure, and get to (H, L) for sure.

Limiting Behavior of the Mixed Strategy

Let $\hat{g}(\mathrm{d}\,t)$ be the upgrading probability.

- $\hat{g}(\cdot)$ must converge to a probability:

$$\lim_{\mathrm{d}\,t \to 0} \hat{g}(\mathrm{d}\,t) = g \in [0, 1].$$

- In the limit, the state (L, L) immediately transitions to (H, H), (H, L), or (L, H) with probabilities

$$\frac{g^2}{g^2 + 2g(1-g)}, \quad \frac{g(1-g)}{g^2 + 2g(1-g)}, \quad \frac{g(1-g)}{g^2 + 2g(1-g)}.$$

- Larger g: More likely to land at (H, H).

MPE at Intermediate Upgrading Costs

Proposition. If $\tilde{c} < c \leqslant \bar{c}$, the following is the limit of a symmetric MPE:

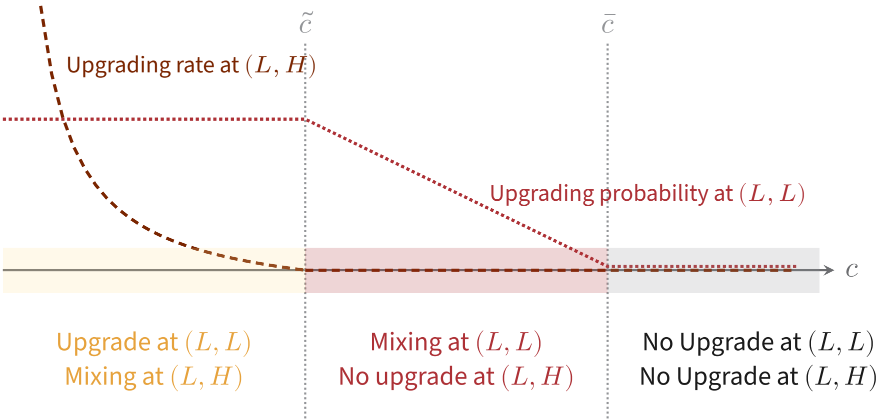
- Firm i upgrades at (L, L) with probability $g(c)$.
- Firm i does not upgrade at (L, H).

Moreover, g decreases in c.

- More incentives to keep indifference at (L, L) as $c \Uparrow$. More likely to land at (H, L) for smaller g.
- $g(\bar{c}) = 0$: no upgrade at the boundary.

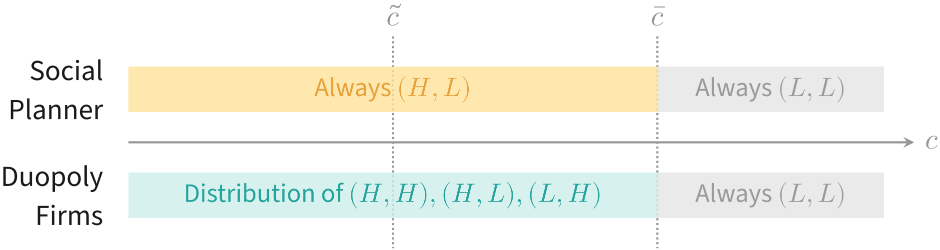
MPE: Vertical Differentiation Only

Theorem. There is a unique S-MPE for each $c > 0$ at the limit $\mathrm{d}\,t \to 0$.



Over-Upgrading

Corollary. Firms over-upgrade if $c \leqslant \bar{c}$ and never under-upgrade.



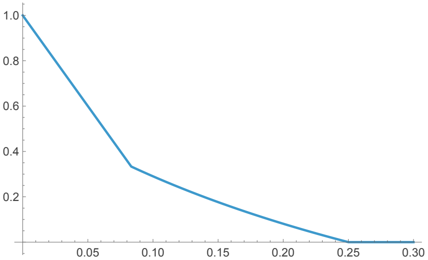
Firms' Long Run Average Joint Profits

- In each mixing MPE, firms' mixing profile generates an ergodic distribution of time duration of states in the long run.
- Within a unit time:
  - τ_B: the proportion of time spent at the balanced state (H, H).
  - τ_I: the the proportion of time spent at the imbalanced states $(H, L)/(L, H)$.
- Long run average profit is defined as

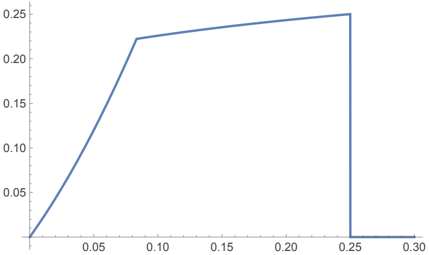
$$2\tau_B \pi_i(1,1) + \tau_I \left[\pi_i(1,\alpha) + \pi_i(\alpha,1)\right] - \mathbb{E}(\text{upgrading cost}).$$

Firms' Long Run Average Joint Profits

Proposition. Firms' long run average joint profits is 0 if $c = 0$ or $c > \bar{c}$. At $0 < c < \bar{c}$, firm's long run average joint profits is increasing in c.



Time spent at (H, H)



Firms' joint profits

Vertical and Horizontal Differentiations

Social Planner Benchmark

- Now consider $k > 0$.
- There are merits to keep both products at high quality.
  - From (H, L) to (H, H), no consumer is worse off, and consumers near location 1 strictly benefit.
- Conjecture: The social planner should upgrade both products if c is sufficiently low.

Social Planner's Optimal Policy

Proposition. There exists $c^A < c^N$ such that social planner's optimal policy is

c^A c^N

Keep (H, H) | Keep (H, L) | Keep (L, L) $\rightarrow c$

- Policies are never optimal: not consistent when comparing marginal cost and marginal benefit of upgrading.

Duopoly Competition: Stage Payoffs

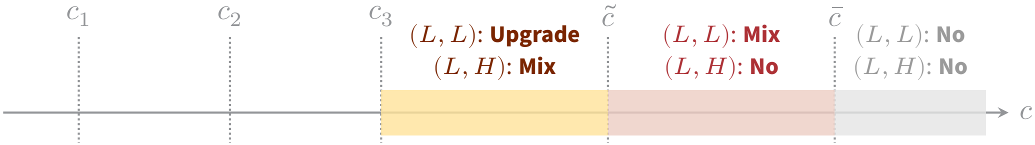
- Duopoly firms can benefit from product differentiation and gain market power.
- Firms get positive profits in balanced states. The flow payoffs are ranked as follows:

$$\pi_0(L, H) < \pi_0(L, L) \leqslant \pi_0(H, H) < \pi_0(H, L).$$

- $\pi_0(H, H)$ is increasing in k.
- $\pi_0(H, L)$ is decreasing in k for α not too large.

MPE

- Previous MPE still exist ...



...with some changes:

- Mixing rates / probabilities are affect by k: interactions of differentiations.
- Start from $c_3 > 0$.

Interaction of Differentiations

Propositions. Suppose $0 < k \leqslant 2/9$ and $0 < \alpha \leqslant 1 - 3k$. There exists $\hat{c} \in (c_3, \bar{c})$ such that for a given upgrading cost c,

$$\frac{\partial \tau_B}{\partial k} \geqslant 0 \text{ if } c \in (c_3, \hat{c}], \quad \text{and} \quad \frac{\partial \tau_B}{\partial k} < 0 \text{ if } c \in (\hat{c}, \bar{c}).$$

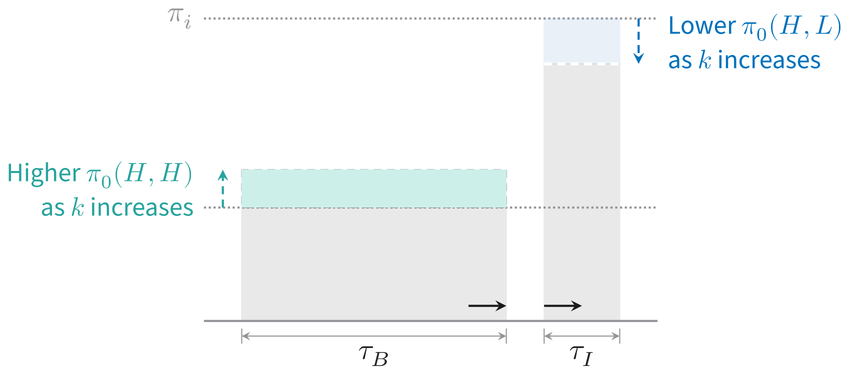
- For lower c, two differentiations exhibit substitution relations in equilibria.
- For higher c, two differentiations exhibit complementary relations in equilibria.

Interaction of Differentiations

- When $k > 0$, both (H, L) and (H, H) provide upgrading incentives:
  - (H, L) still provide stronger incentives: $\pi_0(H, L) > \pi_0(H, H)$.
  - $\pi_0(H, L)$ decreases in k: harder to attract far-away buyers.
  - $\pi_0(H, H)$ increases in k: market power reduces competitions.

- Mixing rates / probabilities determine the split of upgrading incentives provided between the two states.

Substitution at Lower Costs



- Overall incentive is larger since $\tau_B > \tau_I$. Relocate more time to τ_B since $\pi_0(H, L) > \pi(H, H)$.

Always Upgrade

Proposition. Always upgrade when possible is the limit of a symmetric MPE if

$$c \leqslant \frac{\pi_i(H,H) - \pi_i(L,H)}{r+\beta} \equiv c_3.$$

- Firms stay at (H,H) at this MPE.
- The upgrading cost is smaller than the present value of gain from upgrade.
  - And this gain is positive since $k > 0$.

State (L, L)

- If firm 1 always upgrade at (L, L):
  - If firm 0 also upgrades:
$$(L, L) \to (H, H).$$
  - If firm 0 does not upgrade:
$$(L, L) \to (L, H) \to (H, H).$$
- Self-fulfilling: Firms might just upgrade as well at (L, L), as they believe their opponent will upgrade. Suggesting possible multiplicity.

Upgrade Deterrence: Lower Cost

- Can firms agree on not upgrading at (L, L)?

Proposition If $c_1 < c \leqslant c_2$, the following is the limit of a symmetric MPE:

- Firm i does not upgrade at (L, L).
- Firm i upgrades at (L, H) for sure.

- Grim-trigger style of quality war once someone upgrades.

Upgrade Deterrence: Lower Cost

- How to determine the lower bound c_1?
  - The upgrading cost cannot be too low: Otherwise the quality war does not provide sufficient deterrence.

- How to determine the upper bound c_2?
  - The upgrading cost cannot be too high: Otherwise grim trigger punishment is too costly to be credible.

Upgrade Deterrence: Higher Cost

Proposition If $c_2 < c \leqslant c_3$, the following is the limit of a symmetric MPE:

- Firm i does not upgrade at (L, L).
- Firm i upgrades at (L, H) with a rate $h(c)$.

Moreover, $h(c)$ decreases in c.

- Switch to finite length punishment in the quality war.

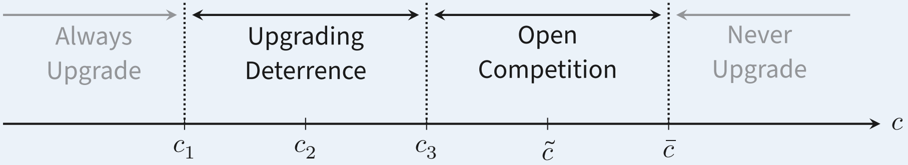
Upgrade Deterrence: Higher Cost

- At higher cost levels:
  - Higher self cost: Grim-trigger is too costly to implement.
  - Higher opponent cost: Grim-trigger offers more than necessary deterrence.
- As c increases, less punishment is required and desired. At c_3, deterrence is too costly to maintain.

- Upgrade deterrence offers higher joint profits compared with always upgrading when possible.

MPE

Theorem. The joint-profit maximizing S-MPE in the limit is



Over- and Under-Upgrading

- While the social planner keeps (H, L):

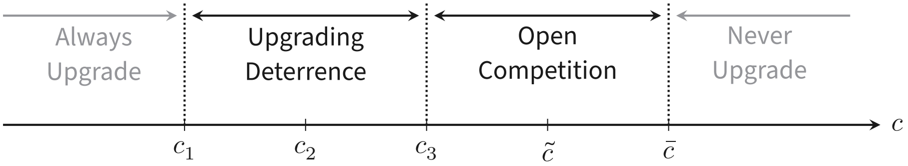
Under-upgrade	Over-upgrade	Under-upgrade
Upgrade Deterrence	Open Competition	No Upgrade

- Under-upgrade at lower cost level: Tacit collusion by upgrade deterrence.
- Under-upgrade at higher cost level: Failure to internalize consumer surplus.

Firms' Long Run Joint Profits

- A higher c has two effects:
  - (Direct) cost effect: joint profits $\Downarrow$.
    It is more expensive for firms to maintain a high quality.
  - (Indirect) competition effect: joint profits $\Uparrow$.
    When firms compete to upgrade, higher costs soften the competitions by reducing the upgrading frequency.
- Indirect effect is the dominant effect when $k = 0$: joint profits increase in c until c is too high and no one upgrades.

Firms' Long Run Joint Profits



- Decrease in c initially:
  - Firms always upgrade with market powers and small enough c.
  - Only direct cost effect presents.
- Can decrease in c for larger c, if shocks are frequent enough:
  - Firms can maintain this MPE for more frequent shocks due to complementarity.
  - Direct cost effect is stronger when shocks are frequent enough.

Conclusion

Conclusion

- With market powers, firms prefer entry deterrence and compete less aggressively, until the deterrence becomes too costly, where they have to upgrade more frequently to not fall behind.
  - Non-monotone upgrading frequency and joint profits.
  - Under-upgrade first, then over-upgrade, then under-upgrade.

- Firms' upgrading incentives are jointly provided by the flow profit jump at balanced and imbalanced high states. A change in horizontal differentiations leads to time reallocation at the two states to adjust upgrading incentives.
  - Substitution of the differentiations first, then complement.

Appendix

Symmetry

- Harsanyi Symmetry-Invariance Criterion.
- Robustness considerations:
  - Fixed costs: Asymmetric equilibria, such as Chicken, cannot survive if there is a (small) fixed cost every period.
  - Evolutionary stableness: In each round, a new player is drawn from a large population to take the role.
- Efficiency: Firms are still not efficient in most of the asymmetric equilibria. Examples come later.
- Traditions in the literatures: Pakes and McGuire (2001), Board and Meyer-ter Vehn (2013).