Invest or Fall Behind: Maintaining Quality in Hotelling Markets

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Abstract

I investigate dynamic duopoly competition in quality and price. The firms face exogenous horizontal differentiation from consumer preferences and endogenous vertical differentiation from evolving product quality. Depending on the cost of quality investment, firms either sustain low quality levels through upgrading deterrence or engage in fierce competition. These two modes of competition generate non-monotonic patterns in upgrading frequency and investment efficiency. I also show that horizontal and vertical differentiation interact in cost-dependent ways: when upgrading is cheap, they are substitutes – greater horizontal differentiation strengthens market power, raises profits, and intensifies quality rivalry while keeping qualities balanced. When upgrading is expensive, they are complements – greater horizontal differentiation makes leadership less profitable, softening quality rivalry and sustaining vertical gaps. My results provide a parsimonious framework for non-monotonic investment incentives and highlight how horizontal and vertical differentiation jointly shape the dynamics of competition and welfare in markets where dynamic quality upgrades are an essential feature of the competitive landscape.

1 Introduction

Competition among firms in markets with heterogeneous consumers is a classic topic in economics (Hotelling, 1929). Firms typically serve distinct market niches to appeal to consumers with specific preferences. Restaurants offer different ethnic foods and dish designs to attract specific types of consumers. Streaming platforms are often categorized by the genres of the programming. Note-taking applications emphasize either efficient organizational tools or refined handwriting interfaces to attract business professionals and students, respectively.

Catering to consumers' heterogenous tastes, however, is just one dimension of competition. Firms also compete and maintain product quality, a task that is subject to unexpected shocks and

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requires constant attention and frequent rounds of investment. For restaurants, an example of a quality shock is equipment malfunction which degrades the flavor or visual appeal of the food. For TV shows, a quality shock can be a leading star departing a series for personal reasons.[1] For apps, a quality shock can be software bugs that reduce its usability.[2] Alternatively, even if a firm maintains its own quality, it may nevertheless fall behind when rivals upgrade their products, thereby altering the relative ranking in the market.

Formally, firms compete along multiple dimensions of differentiation. In the examples above, horizontal differentiation reflects differences in heterogenous consumer tastes, while vertical differentiation captures differences in product quality. In this paper I examine how firms invest in and sustain product quality when facing diverse consumers. More specifically, the questions addressed are as follows: Does consumer heterogeneity intensify or dampen quality competition? How does the interaction between the two dimensions of product differentiation depend on the cost of quality investment? In terms of welfare, when facing heterogeneous consumers, do competing firms invest in quality efficiently, over-invest, or under-invest?

To answer these questions, I develop a dynamic Hotelling model, in which two firms engage in quality and price competition. Horizontal differentiation is represented by the transportation cost of consumers, which is exogenously given, while the firms endogenously determine their quality levels over time. High-quality products may experience unanticipated deterioration due to quality shocks (equipment failure, loss of key personnel , discovery of sofware bugs), whereas low-quality products can be restored to high quality through costly investment, which is referred to as *a product upgrade* hereafter.

My main findings are as follows. First, firms' upgrading frequencies and the joint profits of the firms are non-monotonic in upgrading costs. A large theoretical literature studies dynamic quality investment, often in oligopoly competition settings (Maskin and Tirole, 1987, 1988a,b; Rosenkranz, 1995), frequently incorporating additional features such as firm entry or exit (Ericson and Pakes, 1995; Doraszelski and Markovich, 2007; Doraszelski and Satterthwaite, 2010; Abbring et al., 2018), learning by doing (Besanko et al., 2010), or reputation concerns (Board and Meyer-ter Vehn, 2013). Empirical studies also document dynamic competition with heterogeneous consumers and non-monotonic patterns of investment (Aghion et al., 2005; Ryan, 2012; Gowrisankaran and Rysman, 2012; Eizenberg, 2014). Related to this paper, Besanko et al. (2010) analyzes how firms reduce production costs through higher sales but suffer from organizational forgetting, which together shape their competitive edge. Consumer heterogeneity is modeled by an idiosyncratic preference

[1]As an example, *Two and a Half Men (CBS, 2003 – 2015)* replaced Charlie Sheen with Ashton Kutcher in 2011 and survived for another four seasons.

[2]As an example, *Notability* recognized that they were experiencing performance issues leading to overheating the users' devices and draining battery life. Some other apps experienced more significant issues, such as "the Sonos app fiasco", which even led to Sonos' chief marketing officer, chief product officer, and eventually CEO leaving the company.

shock with Type I extreme value distribution. By contrast, I model consumer heterogeneity with a Hotelling "linear city."footnoteFew papers have studied dynamic Hotelling competition problems, with one example being Lambertini (2012), although that paper investigates the location decisions of the firms; that is, endogenizing the horizontal differentiation., and the transportation cost parameter measures the extent of horizontal differentiation. The model generates non-monotonic dynamics without relying on additional features such as scrap values, learning-by-doing, or reputation. Instead, the non-monotonicity is generated by two competition modes: *upgrading deterrence* when quality investment is relatively cheap, in which the firms agree to maintain low-quality equilibria to avoid costly quality wars, or *direct competition* when quality investment is relatively expensive, in which firms employ mixed investment strategies to manage their rivals' quality upgrading incentives by controlling the rivals' future prospects of investment.

Second, related to the non-monotonicity in the upgrading frequency, I show that firms under-invest at relatively low or relatively high upgrading costs. When the upgrading cost is intermediate, firms over-invest. It is well known that competition can lead to excessive investment, for example through excessive entry (Mankiw and Whinston, 1986), or that investment incentives are shaped by uncertainty, appropriability, and spillovers (Jones and Williams, 2000; Bloom et al., 2013; Ahuja and Novelli, 2017). Over- or under-investment is also documented in empirical research regarding, for example, semiconductor chips (Goettler and Gordon, 2011) and automobiles (Esteban and Shum, 2007). Compared with the existing literature, my model shows that firms may under-invest because of two different incentives: the potential for costly quality wars, mainly at lower investment costs under upgrading deterrence, and the standard failure to internalize consumer surplus, mainly at higher investment costs that prohibit firms' upgrading.

Finally, the model also demonstrates that horizontal differentiation influences the firms' competitive behavior via the upgrading cost, which in turn modifies the levels of vertical differentiation. The dynamic interaction between different dimensions of product differentiation is a topic less studied in the extant literature. Shaked and Sutton (1982) and Motta (1993) examined how quality differentiation is used to attenuate price competition, but the consumers are essentially homogeneous, and quality competition is just one shot. Some papers also employ Hotelling models with additional features to represent the multiple dimensions of differentiation, such as product positioning based on consumers' recognition (Irmen and Thisse, 1998), or non-uniform consumer distributions (Gabszewicz and Wauthy, 2012), but these models are largely static. Along this line, Vanhaecht and Pauwels (2005) considered two competing universities facing both horizontal (location) and vertical (education quality) competition in a one-shot game. In contrast to these static models, in this paper, the interactions between the two dimensions of product differentiation are driven by fully dynamic forces. When upgrades are cheap and firms have similar quality levels, greater horizontal differentiation strengthens market power, raises profits, and intensifies quality competition, thereby reducing vertical differentiation. When upgrades are expensive, however, one firm is more likely to emerge as the quality leader. Greater horizontal differentiation then erodes the leader's profits from market dominance, making leadership less attractive and discouraging rivals from catching up, reinforcing vertical differentiation. In other words, the dynamic effect of horizontal differentiation is dominant-competition-state enhancing: it intensifies the competition if the competition is relatively intense, and it dampens the competition if the competition is relatively weak. That is, the two dimensions of differentiation exhibit substitution effects at lower upgrading costs and complementary effects when upgrading costs are higher.

While the quality shocks are assumed independent between competitors for the majority of the analysis, the model can also accommodate correlated shocks. In an extension, the shocks are assumed to be positively correlated, and the two modes of competition still emerge, albeit in a modified way. More specifically, positively correlated shocks can reset the quality of both firms and offer the opportunity to restart evenly. For upgrading deterrence, this restarting effect can end quality wars faster and make deterrence less effective, which leads to firms choosing a higher upgrading frequency in the quality war to compensate. For direct competition, the restarting effect lowers the competition incentive by reducing the possibility of becoming a market leader, which reduces upgrading frequency.

The remainder of the paper is structured as follows. Section 2 introduces the model. Section 3 analyzes the case without horizontal differentiation, where firms engage in dynamic Bertrand competition in price and quality. Section 4 studies the full model with both horizontal and vertical differentiation. In both cases, the outcomes under duopoly competition are compared with the outcomes under a social planner to illustrate possible efficiency losses. While the main results are presented under symmetric equilibria, Section 5 considers to what extent the results continue to hold under asymmetric equilibria. Section 6 discusses extensions, and Section 7 concludes.

2 The Model

Two firms engage in dynamic competition in a Hotelling market. Time is discrete and infinite, and the period length is Δ. Δ is used to model how fast firms can react to nature or each other.[3] The analysis primarily focuses on the scenarios where firms can react swiftly, and the equilibrium behaviors are described at limit of the equilibria when $\Delta \to 0$. This approach provides an approximation of the equilibria, streamlines the expressions by removing the higher order terms to offer clearer insights, yet still keeps the easier interpretations of a discrete-time model.[4]

[3]In particular, Δ does not affect consumer dynamics. As we will formally establish, consumers are transient and will leave the market by the end of their arrival period. The total mass of consumers are also not affected by Δ.

[4]It may be perceived that we are studying here a continuous time model by taking the limit $\Delta \to 0$. Still, there are a few reasons to consider a discrete time model instead. First, it is more reasonable to believe that firms' strategic

We start by discussing the static competition environment, which is a standard Hotelling competition environment. The two firms are located at the boundaries of the Hotelling market $[0,1]$, thereby referred to as Firm 0 and Firm 1, respectively. Firms' locations are exogenously fixed. Each firm produces a product with quality that can be high (H) or low (L). Let v_q be the value of the product with quality q, and

$$v_H = 1, \quad \text{and} \quad v_L = \alpha \in (0,1).$$

The production costs are normalized to 0.

In each period, there are mass $1 \cdot \Delta$ consumers uniformly distributed on $[0,1]$. Each consumer can choose to purchase from Firm 0, Firm 1, or not to purchase. If the consumer purchases, she pays a traveling cost in addition to the price charged by the firm. The traveling cost is linear in the distance between the consumer and the firm, measured by a coefficient k. If the consumer does not purchase, she gets an outside option normalized to 0. Formally, suppose the quality level and price of Firm i, $i = 0,1$, are q_i and p_i, respectively. Consumers at location $\ell \in [0,1]$ make purchasing decisions to

$$\max \{q_0 - p_0 - k\ell, q_1 - p_1 - k(1-\ell), 0\}.$$

Unless stated otherwise, assume $0 < k \leqslant 1/3$.

The traveling cost coefficient k represents the horizontal differentiation; that is, to what extent the consumers are heterogeneous. From firms' perspective, this also measures how large the market power is. The restriction on the size of k then guarantees that firms engage in meaningful competitions rather than act as separated monopolists when the quality levels are not too low.[5]

We next discuss the dynamics of the model, more specifically, the evolution of the quality. When $q_i = L$, as the quality is already low, there is no further quality decay, but Firm i can upgrade the product to restore the quality to $q_i = H$. The upgrade is instant and requires a lump-sum cost c. When $q_i = H$, as the quality is already high, there is no further upgrade, but a negative shock from nature can cause an instant and permanent quality decay, rendering $q_i = L$. The decay is "permanent" in the sense that q_i remains at L after a shock until the firm pays cost c and upgrades the product. In each period, if the quality is H, nature places a shock with probability b, which is a function of Δ. The shocks are independent between firms. We assume that

$$b = 1 - e^{-\beta\Delta}.$$

In particular, when $\Delta \to 0$, the shock behaves as if a Poission arrival process with parameter $\beta > 0$.

decisions are discrete. Second, starting from a discrete-time model will avoid some possible continuous-time artifacts. Later we will see that there is an equilibrium that is well-defined as a discrete-time limit but not well-defined. In short, the reader should consider the results in discrete time, and the limit is just for the simplicity of expositions.

[5]The choice $1/3$ also involves concerns of static game equilibria when the quality levels are lower. In Appendix A, we will discuss how the size of k and α affect the static Hotelling game equilibria.

The firms play Markov strategies with the quality pair (q_0, q_1) being the quality-relevant state. In general, for a fixed period length Δ, Firm i's strategy is a map:

$$\{(q_i, q_j)\} \to [0, 1] \times \mathbb{R}_+.$$

The first dimension is the upgrading probability. It is only meaningful when $q_i = L$. Alternatively, we can allow Firm i to upgrade when $q_i = H$, although the upgrade does not have any effect on the quality and is still costly. Firm i can play a mixed strategy by choosing an upgrading probability between 0 and 1. The second dimension is the price.

At (an arbitrary) period n, let (q_0^n, q_1^n) be the state at the beginning of the period. If $(q_0^n, q_1^n) \neq (H, H)$, firms first make upgrade decisions, which modify the state to the intermediate $(\tilde{q}_0^n, \tilde{q}_1^n)$. If $(\tilde{q}_0^n, \tilde{q}_1^n) \neq (L, L)$, nature then decides whether or not to place shocks, and the state after nature's shock is (q_0^{n+1}, q_1^{n+1}). This is the state faced by the consumers at period n, the state under which firms make pricing decisions for period n, and the state at the beginning of period $n+1$. This stage game order also allows nature to cancel out the upgrading efforts from firms. The stage-game timeline is pictured at Figure 1. The firms discount future payoffs using a common discount factor $\delta = e^{-r\Delta}$.

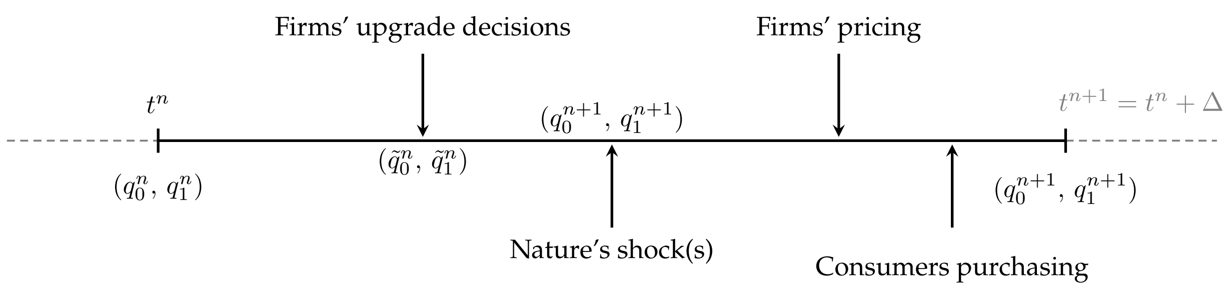


FIGURE 1 Stage Game Timeline

In general, the order in the stage game has strategic impacts on the game outcomes. For instance, it matters whether nature can cancel out firms' upgrading efforts. To see this, consider a social planner who would like to maximize the consumer surplus when $k = 0$. If as opposed to the order above nature moves first, the social planner should never have the incentive to duplicate the high-quality product: The social planner can guarantee a high-quality product for the consumer in each period, by simply keeping one high-quality product after nature's shocks. Given there is no traveling cost, this is sufficient to keep the consumer surplus at the highest possible level. However, if the social planner moves first as stated above, the social planner could duplicate the high-quality products, with the second high-quality product serving as the insurance for nature's shocks. Nevertheless, in the limit as $\Delta \to 0$, the order of moves does not affect the equilibrium outcomes. In particular, this is due to nature's shock probability decreases in Δ and converges to 0 when $\Delta \to 0$. In the social planner example, this means the probability that both products receiving shocks is 0, so that the duplication is not useful again.[6]

Because firms can adjust prices each period after observing the realized state at no cost, pricing is reduced to a *static* problem. Naturally, firms will play the static Nash-equilibrium pricing strategies, which we will briefly discuss in subsection 2.1. For this reason, we will focus mainly on the upgrading strategies, denoted by $\sigma_i : \{(q_i, q_j)\} \to [0, 1]$. For a given σ_j and nature's shock probability b, σ_i then determines the probabilities of state transition. Let

$$m_{(\sigma_i,\sigma_j,b)}\left(q_i', q_j' \mid q_i, q_j\right)$$

denote the probability of the state transitioning from (q_i, q_j) to (q_i', q_j'). Let $\pi_i(q_i, q_j)$ be Firm i's stage payoff at state (q_i, q_j), and Firm i's expected payoff $V_i(q_i, q_j)$ at state (q_i, q_j) can be recursively defined as

$$V_i(q_i, q_j) = \pi_i(q_i, q_j)\Delta + e^{-r\Delta} \sum_{(q_0', q_1')} m_{(\sigma_i,\sigma_j,b)}\left(q_i', q_j' \mid q_i, q_j\right) V_i(q_i', q_j').$$

Both firms are expected-payoff maximizers.

The equilibrium concept is symmetric Markov perfect equilibria (MPE), and the following reasons support the focus on symmetry. First, given that the game environment and the firms are perfectly symmetric, the symmetric equilibria are considered more reasonable under *Harsanyi Symmetry-Invariance Criterion*. In particular, this guarantees that the equilibria do not persistently favor one firm over the other. Second, symmetric equilibria are also more robust under many selection standards. For one thing, the model does not introduce a fixed cost to stay in the market at each stage for simplicity. This allows some asymmetric equilibria, such as Chicken-type equilibria, to exist. However, if a fixed cost exists, however small it might be, such asymmetric equilibria will be eliminated as the weaker firm will exits the market, yet the existence of symmetric equilibria is not affected. For another, from an evolutionary perspective, a symmetric equilibrium can be supported when players are randomly drawn from a pool of players of different types. For instance, a restaurant owner may choose to liquidate the restaurant by selling it to the next owner due to personal finance reasons. Luo and Stark (2014) showed that the median of the life spans of a full-service restaurant in the western US is about 4.5 years, and it is even shorter, 3.75 years, for small restaurants. This suggests that a firm may engage in competition with a "renewed" opponent from time to time. Thirdly, as we will discuss in Appendix D, most asymmetric equilibria cannot fully eliminate inefficiencies, do not introduce additional economics insights compared with symmetric equilibria, and are more complicated in terms of computations and expositions. Finally, it is a tradition in strategic investment literature to consider symmetric equilibria (Pakes and McGuire,

[6]In the online appendix (available soon), we considered an alternative stage-game timeline and showed that the results of the paper remain robust under this alternative timeline.

2001; Keller et al., 2005). If there are multiple symmetric MPE, the MPE offers highest joint profit is considered for welfare implications. Naturally, if firms are allowed to communicate before playing the game, given that the two firms and the equilibrium profiles are symmetric, firms should agree on playing the joint-profit maximizing equilibrium.

It may also be a concern regarding firms collusion behaviors, i.e., whether more generic equilibrium concepts, such as subgame perfection, should be selected instead of MPE. While a folk-theorem type equilibrium can possibly achieve highest joint profit, it is a long tradition in both theoretical and empirical research to focus on MPE when considering industry dynamics (Maskin and Tirole, 1987, 1988a,b; Doraszelski and Satterthwaite, 2010; Aguirregabiria et al., 2021). MPE are often favored for the following reasons. First, it is conceptually easy by not relying on the full history of the game. This also suggests that it is easier to compute and to apply in empirical research. Second, collusions are often considerd less plausible in many industries. Collusions may be impossible for policy reasons (i.e., the prohibition of cartels) or the industry structure. For instance, collusions among restaurants can be difficult due to the relatively short life span of the ownership as well as the relatively low entry barrier.

2.1 The Stage Game

Before analyzing the dynamic competitions, we first briefly study the static stage game, which will help illustrate the upgrading incentives in later discussions. In this section, we will state some results and intuitions only, leaving the formal analysis of the stage game to Appendix A.

First consider $k = 0$, where there is no horizontal differentiation. The market is thus reduced to Bertrand, and the products are exactly the same when they are with the same quality. In turn, Bertrand competition implies that

$$\pi_0(H, H) = \pi_0(L, L) = 0.$$

In case that the quality levels are different, the quality leader has the competitive advantage. Consider $(q_0, q_1) = (H, L)$ as an example, where Firm 0 has a higher quality level. As long as $p_0 < v_H - v_L = 1 - \alpha$, Firm 1 cannot gain any consumer even if $p_1 = 0$. Firm 0 then should charge the highest possible price that allows it to occupy the market, leading to

$$\pi_0(H, L) = 1 - \alpha, \quad \pi_0(L, H) = 0$$

in the Nash equilibrium (symmetric for Firm 1). Consumers are indifferent between two firms and choose to purchase from higher quality firm. In particular, notice that the only state that offers positive stage payoff when $k = 0$ is the quality-leader state.

Next, consider $k > 0$, and the market is Hotelling. Under a positive k, firms have a market power and can charge a positive price even at the balanced state (H, H). Even if Firm 1 charges price exactly 0, a consumer at location 0 can get $1-k$ from Firm 1. If Firm 0 instead charges $\varepsilon > 0$, Firm 0 can attract consumers in $[0, (k-\varepsilon)/2k]$ and get a positive payoff. As Firm 0 charges a higher price, Firm 1 also faces weaker competition and has the incentive to increase its price. In the equilibrium,

$$\pi_0(H, H) = \frac{k}{2},$$

and the firms split the market from the middle. Notice that the stage payoff is in fact increasing in k. Intuitively, firms can earn higher profits when their market powers are higher. Of course, this intuition and the corresponding result relies on the assumption that k is not too large. For larger k, firms find it too difficult to attract the middle consumers and become separated small monopolists. At this stage, a higher price leads to fewer consumers and in turn fewer profits.

The situation at $(q_0, q_1) = (H, L)$ is similar if $v_L = \alpha$ is not too small, and the same intuition applies. However, when α is smaller, the forces can be different due to a different equilibrium market structure. When α is small, Firm 0 has an incentive to be a pseudo monopolist and occupies the whole market. The gain in the market share dominates the loss from charging a more aggressive price, $1-\alpha-k$. This leads to

$$\pi_0(H, L) = 1-\alpha-k, \quad \pi_0(L, H) = 0.$$

In this case, notice that the stage profit *decreases* as k increases. Intuitively, as k increases, there is a higher cost to serve far away consumers, and Firm 0 must charge a lower price to keep Firm 1 outside the market.

3 Vertical Differentiation with Homogeneous Consumers

A simpler problem is to consider the firms' upgrading strategies when the consumers are homogeneous. More specifically, this section considers a special case $k = 0$ and removes the horizontal differentiation. This simplification allows clearer interpretations of firms' incentives and provides useful contrasts with the case with both vertical and horizontal differentiation, which is discussed in the next section.

3.1 The Social Planner Benchmark

As a benchmark, a social planner who maximizes the total surplus, defined as

$$\text{Consumer surplus} + \text{Producer surplus} - \text{Upgrading costs},$$

is analyzed first. The social planner is utilitarian: for a given state (q_0, q_1), the allocation of surplus is irrelevant, so that the social planner should always let the consumers purchase from the firm that offers higher trading surplus.[7] One simple price strategy to implement this is to set $p_0 = p_1 = 0$, which allows consumers to choose the product with higher quality. The stage social surplus is then

$$w(q_0, q_1) = \max\{v_{q_0}, v_{q_1}\}.$$

The social planner's optimal policy is as follows.[8]

Proposition 1. The limit of the social planner's optimal policy as $\Delta \to 0$ is

- Upgrading one product at (L, L) and no other upgrade if $c \leqslant (1-\alpha)/(r+\beta)$.
- No upgrade if $c > (1-\alpha)/(r+\beta)$.

The proof is in Appendix B. The threshold has a clear interpretations if the social planner upgrades one product at (L, L), the state will be maintained at state (H, L) (or (L, H)) in the limit. The flow payoff is $v_H = 1$. If the social planner does not upgrade at all, the state will be (L, L) forever, and the flow payoff is $v_L = \alpha$. The threshold, $(1-\alpha)/(r+\beta)$, represents the increase in the present value of social welfare from making the upgrade. If this increase exceeds the cost, the social planner upgrades.

Notice that the social planner never upgrades both products and duplicates high-quality products in the limit. As we mentioned in subsection 2.1, the flow social welfare does not benefit from a second high-quality product: the shock probability is negligible in the limit, consumers face no travel costs, and maintaining an additional high-quality product is costly.

3.2 Duopoly Competition Equilibria

Now consider the duopoly competition. As with the social planner, firms will not upgrade when the upgrading cost is too high. In fact, the threshold at which firms stop upgrading is the same as that of the social planner. To see this, suppose Firm 1 never upgrades. Firm 0's problem is then to decide whether to upgrade at state (L, L). If Firm 0 chooses not to upgrade, the stage payoff is 0. If Firm 0 upgrades, it becomes the pseudo-monopolist that occupies the whole market and enjoys the stage profit $1-\alpha$. The gain from upgrading, in present value, is exactly $(1-\alpha)/(r+\beta)$. If the cost is higher, then even if the opponent chooses not to upgrade, Firm 0 should also refrain from upgrading since the pseudo-monopolist profit is still not sufficient to cover the upgrading cost. This constitutes the no-upgrading MPE.[9]

[7] The trading surplus for each consumer is always positive since $\alpha > 0$ and $k = 0$. In turn, the social planner covers the whole market in any optimal policies.

[8] Unless stated otherwise, the optimal policy and all later results are stated in the limit where $\Delta \to 0$.

[9] Note that the social planner's threshold and the firms' threshold coincide only at $k = 0$. More details are discussed at subsection 3.3.

Lemma 2. If $c > (1-\alpha)/(r+\beta) \equiv \bar{c}$, both firms never upgrading is a limit of an MPE.

The proof is included in Appendix B, which formalizes the argument above.

Low Cost Levels. When costs are lower, firms have incentives to make at least some upgrades. Nevertheless, there is no MPE in which both firms always upgrade whenever possible when $c > 0$. Intuitively, if each firm upgrades whenever a shock occurs and the state remains (H, H), firms earn no stage payoff due to Bertrand competition. Yet upgrades are costly, implying that firms would have strictly negative expected payoffs. Therefore, when $c < \bar{c}$, firms must employ "partial" or "occasional" upgrading strategies in any equilibrium. Specifically, for each firm, there are two upgrading decisions: whether or not to upgrade when the opponent's quality is H, and when the opponent's quality is L. The questions are: (i) in which state should a firm upgrade, and (ii) what should the upgrading frequencies be? Without loss of generality, consider Firm 0:

- At state (L, H), if Firm 0 upgrades, the state becomes (H, H), and Firm 0 receives no immediate profit from upgrading.
- At state (L, L), if Firm 0 upgrades, then as long as Firm 1 does not upgrade for sure, Firm 0 can reach (H, L), where it earns positive profit. Even if Firm 1 upgrades for sure, the state becomes (H, H), the same outcome as upgrading at (L, H).

Thus, the immediate outcome after upgrading at (L, L) is weakly better than upgrading at (L, H). It is therefore natural to conjecture that both firms should upgrade at (L, L) when c is sufficiently low. This leads to the following MPE.

Proposition 3. The following is the limit of an MPE if the upgrading cost

$$0 < c \leqslant \frac{\beta(1-\alpha)}{(2\beta+r)(r+\beta)} \equiv \tilde{c}:$$

- At (L, L), Firm i upgrades for sure.
- At (L, H), Firm i upgrades with rate

$$f(c) = \frac{(1-\alpha)\beta - (r+\beta)(r+2\beta)c}{(r+\beta)c}.$$

Moreover, f is strictly decreasing in c.

The proof is in Appendix B. To understand the upgrading incentives, consider how Firm 0 reaches state (H, L), the only state in which Firm 0 has positive stage profit under this MPE. Given that Firm 1 always upgrades at state (L, L), if Firm 0 also upgrades at (L, L), the state becomes (H, H), which yields no immediate profit. Instead, Firm 0 profits only if nature places a shock on Firm 1's product and changes the state to (H, L). Firm 1's strategy at (H, L) therefore matters, as it determines the expected duration that Firm 0 remains in its only profitable state (H, L).

If Firm 1 upgrades at state (H, L) for sure, the state will transition to (H, H) immediately. As $\Delta \to 0$, the expected duration spent at (H, L) converges to zero. For any positive upgrading cost that remains constant as $\Delta \to 0$, such an aggressive strategy does not provide sufficient incentive for Firm 0 to upgrade at (L, L). Hence, an equilibrium in which both firms upgrade at (L, L) cannot be supported. Conversely, if Firm 1 never upgrades at (H, L), the only way Firm 0 exits state (H, L) is via nature's exogenous shock. The expected duration of (H, L) is then the longest possible, which strictly incentivizes Firm 0 to upgrade for sufficiently low c. But this eliminates Firm 1's incentive to upgrade at (L, L). More specifically, Firm 0's upgrading incentive at (L, H) is the same as at (L, L), since upgrading in either state leads to (H, H) at the same cost c, and staying at (L, H) also yields zero payoff. Thus, if Firm 0 has strict incentive to upgrade at (L, L), it must also have strict incentive to upgrade at (L, H). By symmetry, Firm 1 would then not find upgrading at (L, L) optimal. This again cannot support the candidate equilibrium. Therefore, Firm 1 cannot adopt a pure strategy of never upgrading at (H, L).

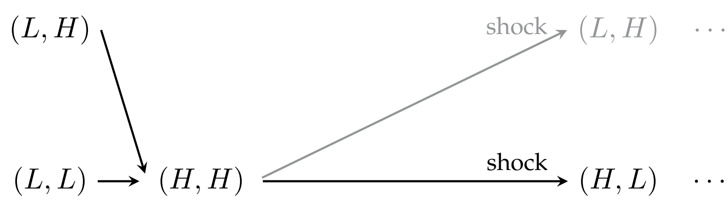


FIGURE 2 State evolution when the opponent upgrades at (L, L)

The only remaining possibility is that Firm 1 plays a mixed strategy at (H, L). The idea is to control Firm 0's upgrading incentive so that Firm 0 does not always upgrade at (L, H), which in turn supports Firm 1's upgrading at (L, L). In other words, Firm 0 must also mix at (L, H). This suggests Firm 1 must choose its mixing probability at (H, L) exactly so that Firm 0 is indifferent between upgrading and not upgrading at (L, L) and (L, H). The indifference condition yields the function $f(\cdot)$ specified in Proposition 3. Given this indifference, Firm 0 upgrades for sure at (L, L) and uses the same mixing strategy at (L, H). Symmetrically, Firm 1 then finds it optimal to upgrade at (L, L) and mix at (H, L), which supports the candidate equilibrium.

One feature of this MPE is worth emphasizing: upgrading incentives are not generated by the immediate payoff from upgrading. In fact, the state (H, H) provides no profit under Bertrand competition. Instead, incentives are derived from expected future profits. In this sense, the vertical-differentiation-only problem is relatively simple: upgrading incentives are determined solely by the expected duration in the profitable state. For Firm 0, the expected positive durration at (H, L) in the future alone provides the incentive for the upgrade at (L, L), and this incentive is fine-tuned by Firm 1's mixed strategy.

As c increases, upgrading at (L, L) becomes more costly. In turn, stronger incentives are needed to induce upgrading. This requires a longer expected duration in the profitable state, which is achieved through a lower upgrading rate. Intuitively, a firm views the opponent's mixed upgrading strategy as an additional "shock" that ends its profitable state, on top of nature's shock. A lower mixing rate reduces this effective shock probability, extending the expected duration. At the boundary $\tilde{c}$,

$$f(\tilde{c}) = 0,$$

so the opponent does not upgrade at the firm's profitable state. The duration in the profitable state is then determined solely by nature's shock rate and is the longest possible.

Middle Cost Levels. Since $\tilde{c} < \bar{c}$, some upgrades are still possible. When $\tilde{c} < c < \bar{c}$, the upgrading incentive must be stronger, yet firms cannot further lower the mixing rate to provide larger incentives to induce upgrading since $f(\tilde{c}) = 0$ already. In turn, if Firms still upgrade at (L, L), there must be another channel to strengthen the upgrading incentive.

To achieve this, observe that state (H, H) is less ideal for each firm. Firms have no immediate profit. Instead, a firm must wait for shocks from nature to land at the profitable state. But this is uncertain: the next shock from nature may strike the firm's own product, benefiting the rival, and the firm still earns no profit after the shocked. Even if the next shock hits the opponent, the probability of an immediate shock is zero, so the firm must wait, and discounting reduces the present value of the resulting profits. Due to the discount, this wait already depreciates the future profits. If the firms can bypass state (H, H) and land at either (H, L) or (L, H), the upgrading incentives can indeed be enlarged. This can be achieved by adopting mixed strategies at (L, L), as shown in Figure 3.

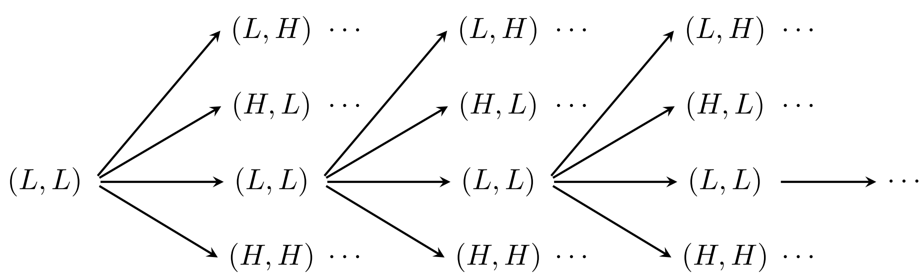


FIGURE 3 Mixing at (L, L)

When firms mix at (L, L), the realized state can be (L, H), (H, L), (L, L) and (H, H). In the first two cases, the corresponding firm avoids the wait and uncertainty at (H, H) and arrives at the preferred state immediately, which provides larger incentive for upgrading. In the case that the mixing realization is (L, L), the firms will mix again (since they play a Markov strategy). Since both play mixed strategies, the firms can still clash and both upgrade, leading to state (H, H).

One particular issue is how to interpret the mixed strategy at the limit when $\Delta \to 0$. Traditionally, if one takes a complete continuous-time perspective, a mixed strategy is expressed by a cumulative distribution function (CDF), which under Markov strategies corresponds to a constant hazard rate. This is exactly what happens in Proposition 3, when firms play mixed strategies at (H, L) and (L, H). By contrast, that reduction to a constant rate does *not* apply here. Let $\hat{g}(\Delta, c)$ be the mixing probability before taking the limit, and let's temporarily suppress the argument c for the ease of notation. If the mixed strategy here also converges to a rate, this means there must exist $\hat{g} > 0$ such that

$$\lim_{\Delta \to 0} \hat{g}(\Delta) = 0 \quad \text{and} \quad \lim_{\Delta \to 0} \frac{\hat{g}(\Delta)}{\Delta} = \hat{g}.$$

However, such mixed strategy cannot be supported in an MPE. To illustrate the reason, consider Firm 0's upgrading incentives at (L, L), when Firm 1 mixes in the rate fashion. When Δ is small, the probability that Firm 1 upgrades at (L, L) is then small. Then instead of playing a rate mixing as well, if Firm 0 upgrades at (L, L) for sure, Firm 0 can land at (H, L) with probability almost 1 (and exactly 1 in the limit). This means not only is (H, H) perfectly bypassed, but Firm 0 also lands at its profitable state (H, L) for sure. Given the candidate profile in which Firm 1 does not upgrade at (H, L), Firm 0 can enjoy the positive profit state for the longest possible duration in expectation. This represents a profitable deviation from the rate-mixing strategy, which breaks the candidate equilibrium. In other words, in a rate-mixing profile, firms have a first-mover advantage and an incentive to front-load upgrading probability, so that such profile cannot be an MPE.

The mixed strategy that sustains an MPE satisfies

$$\lim_{\Delta \to 0} \hat{g}(\Delta) = g \in [0, 1].$$

In other words, the mixing probability converges to a positive number in the limit. This will keep the possibility that two firms clash and get to state (H, H), or the possibility that the state lands at the opponent's favorite state. These possibilities balance out the first-mover advantage of upgrading at (L, L), restoring indifference and sustaining mixing rather than certain upgrading. The MPE is summarized in Proposition 4 below.

Proposition 4. The following is the limit of an MPE if the upgrading cost $\tilde{c} \leqslant c < \bar{c}$:

- At (L, L), Firm i upgrades with probability $g(c)$, where,

$$g(c) = \frac{r + 2\beta}{r + \beta} \frac{(1 - \alpha) - (r + \beta)c}{1 - \alpha}.$$

- At (L, H), Firm i does not upgrade.

Moreover, g is strictly decreasing in c.

The proof is in Appendix B. Some technical details regarding the probabilistic mixing at (L, L) are in order. First, this should be considered as a limit of discrete-time MPE instead of a purely continuous-time MPE. At continuous-time setup, it is difficult to define probabilistic mixing without encountering a measurability issue[10], unless one is willing to employ a non-standard strategy space. Interpreted as a discrete-time limit, the result approximately describes the mixing behavior when Δ is sufficiently small but positive. For a positive Δ, the mixing must be probabilistic by nature, and the measurability issue does not arise. Second, under probabilistic mixing, the outcome of the game in the limit has the following two features:

- (L, L) is not on-path. The remaining three states are all on-path.
- g determines the relative duration of states on path.

To better understand these two features, consider Figure 3 again. For Δ close to 0, for any positive time duration at (L, L), the firms would need to mix "almost infinitely many" times during any positive time spent at (L, L) if neither upgrades. But as $\Delta \to 0$, the law of large numbers implies that some firm(s) must have upgraded. In the limit, this implies (L, L) cannot be sustained on the equilibrium path. In each round, the probability mass on (L, L) is redistributed across the remaining three states. Therefore, the outcome distribution of (H, H), (H, L) and (L, H) is then

$$\frac{g^2}{g^2 + 2g(1-g)}, \quad \frac{g(1-g)}{g^2 + 2g(1-g)}, \quad \text{and} \quad \frac{g(1-g)}{g^2 + 2g(1-g)},$$

respectively.

From the perspective of this outcome distribution, the MPE profile in Proposition 4 provides stronger incentives than the profile in Proposition 3 by replacing part of the probability mass of landing at (H, H) with the probability mass of landing at $(H, L)/(L, H)$. Taking Firm 0 for example, replacing (H, H) with (L, H) does not hurt Firm 0 since neither state provides any profits, and replacing (H, H) with (H, L) is strictly beneficial. As g decreases, the probability of landing at (H, H) decreases, so that the incentives is increasing, which helps to counter the increasing upgrading cost. Hence, decreasing g shifts probability mass from (H, H) to (H, L), lengthening the expected tenure in the profitable state and offsetting higher costs.

The aforementioned three MPE in Lemma 2, Proposition 3 and Proposition 4 cover the cost range, with the upgrading frequency gradually decreasing. At the boundaries of two MPE, the upgrading profile transitions smoothly. That is, the two MPE at the corresponding boundary coincide. The three MPE and the corresponding upgrading rates/probabilities are illustrated in Figure 4, and it turns out these are the only symmetric MPE for postive upgrading cost.

Theorem 5. In the limit, there is a unique symmetric MPE for each c when $k = 0$.

[10]In a simplified argument, imagine that each firm may potentially mix many rounds before one firm finally upgrades. In particular, as long as no one upgrades, the mixing must be repeated indefinitely. As $\Delta \to 0$, the law of large numbers indicates that each firm should have an upgrading probability 1 instead of a positive number strictly between 0 and 1.

The proof is in Appendix B. Conceptually, since there are only four states, the proof enumerates all possible profiles and checks, for each, the cost ranges under which it can be an MPE. It turns out that most profiles cannot be an MPE in any positive cost level.

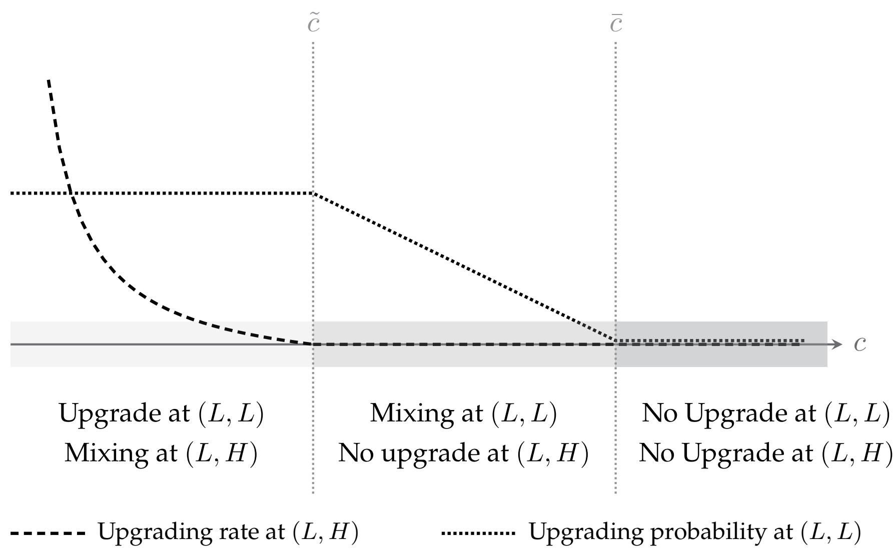


FIGURE 4 MPE with Vertical Differentiation Only

The two types of MPE discussed above has a common feature: firms need to act simultaneously when the state hit (L, L). This may become problematic: if one firm just simply hesitate for a short time period, the symmetric simultaneous product upgrade may be broken. However, by considering the discrete-time setup, firms do not need to upgrade exactly simultaneously. Instead, it is sufficient if they upgrade within a short period. More realistically, this means both firms choose to invest when they observe the opportunity of becoming a quality leader, and both obtain the quality advancement within a reasonably short time period. Many real-world examples have this feature: PlayStation and Xbox, Covid vaccines from Pfizer and Moderna, AI race among OpenAI, Google, and Meta, etc. Duplications of investment are also observed in some other innovation papers, such as Akcigit and Liu (2016).

Additional MPE. Apart from the MPE above, there are some other symmetric MPE at $c = 0$, and other asymmetric MPE when $c > 0$. When $c = 0$, there are multiple MPE possible. It is straightforward to verify that always upgrading when possible is an MPE given the trivial upgrading cost. It turns out that the corresponding firm upgrades at (L, H) and (H, L) while no firm upgrades at (L, L) is an MPE as well. Such MPE and related multiplicity are discussed later in subsubsection 4.2.2. The game also has asymmetric MPE, which will be discussed in section 5.

3.3 Investment Efficiency

In this section, we consider firms' investment efficiency. In particular, compared with the social planner who maximizes the social surplus, do the firms upgrade too frequently, not frequent enough, or just at the efficient level? Following from the MPE profiles, it is clear that when $0 < c < \bar{c}$, firms competition outcome is a distribution over three states, (H, H), (H, L), and (L, H). Moreover, (H, H) will happen with strictly positive duration. The social planner, however, never duplicates high-quality products.[11] When (H, H) is on-path (and (L, L) is not), firms over-upgrade relative to the social planner.

Corollary 6. If $0 < c < \bar{c}$, firms over-upgrade and never under-upgrade.

As in the prior literature, the over-upgrading (or over-investment in general) is driven by the competitions of the firms. As upgrading cost increases, the competition is softened, and the upgrading frequency gradually decreases, reducing the extent of over-upgrading. When c reaches $\bar{c}$, the firms stop upgrading and attain efficiency.

In addition, in this vertical-differentiation-only case, the social planner and the firms stop upgrading at the same bound $\bar{c}$. This is specific to the vertical-differentiation only case. For the social planner, let $w(q_0, q_1)$ be the stage social welfare, and the highest cost that supports upgrading is

$$\frac{w(H,H) - w(H,L)}{r + \beta}.$$

For firms, let $\pi_0(q_0, q_1)$ be the stage profits of Firm 0, and the highest cost that supports upgrading is

$$\frac{\pi_0(H,H) - \pi_0(H,L)}{r + \beta}.$$

When $k = 0$, $w(H, H) - w(H, L) = \pi_0(H, H) - \pi_0(H, L)$. When there is horizontal differentiation, the boundaries in general do not coincide. Fundamentally, with vertical differentiation only ($k = 0$), consumers always buy from the higher-quality firm even under duopoly, so competition does not generate stage-game inefficiency. As shown in the next section, this is no longer true when $k > 0$, and the two thresholds differ.

3.4 Firms' Long-Run Average Joint Profits

The social planner can also be considered as a consumer-surplus maximizer. From this perspective, it is also useful to consider firms' welfare in terms of joint profit. While this can be measured

[11]The social planner's policy may appear asymmetric, since the high-quality product is never duplicated. Nevertheless, this outcome can also be achieved by a symmetric policy: Each time nature places a shock and the state reaches (L, L), the social planner can randomize with equal probability to upgrade Firm 0's product or Firm 1's product. This is a symmetric policy that never duplicates high-quality products. In fact, this shows clearly that the inefficiency is generated by the lack of coordination between firms in duopoly competition.

using the value function directly, the value functions depend on the initial state. For example, consider the MPE profile in Proposition 3, in which case both firms upgrade at the moment the state hits (L, L). The value function of either firm at (L, L) contains an immediate upgrading cost c compared with the value function at (H, H). The significant impact of the upgrading cost may obscure the insights regarding joint profits. To avoid this issue, this section considers the firms' long-run average profits. Specifically, the long-run average joint profit measures the expected joint profit per unit of time in the far future, when the state distribution has converged and no longer depends on the initial state. This can be called the "steady state", characterized by the steady-state distribution. At this unit of time, in either the MPE described in Proposition 3 or the MPE described in Proposition 4, part of the time τ_B is spent at the balanced high state (H, H), while the remaining time τ_I is spent at the imbalanced states (H, L) or (L, H)[12]. The shock rate and the MPE determines the transition probability between states, and the frequency at which the firms pay the upgrading costs.

A further and arguably more important reason to consider joint profits in this way is to obtain τ_B and τ_I. τ_B measures the time spent in state $(H, H)$when the firms' products are not vertically differentiated, while $\tau_I = 1 - \tau_B$ measures the time duration of states that feature vertical differentiation. τ_B in turn is an indicator that reflects the extent of vertical differentiation, which is particularly useful when considering the interactions of the two differentiation in the next section.

As an example, we calculate τ_B and the joint profit in MPE profile in Proposition 3. It is useful to temporarily distinguish the time spent at (H, L) and (L, H), denoted by τ_{HL} and τ_{LH}, for clearer illustration. For state (H, L), under the MPE considered, it emerges when nature places the corresponding shock at (H, H) (with rate β). State (H, L) exits either when Firm 1 upgrades the product (with rate $f(c)$) or when nature places another shock (with rate β). In the steady state, the inflow and the outflow balance, implying that

$$\beta\tau_B = [f(c) + \beta]\,\tau_{HL}.$$

The expression of τ_{LH} is similar. Given that $\tau_B + \tau_I = 1$ and $\tau_I = \tau_{HL} + \tau_{LH}$,

$$\tau_B = \frac{f(c) + \beta}{f(c) + 3\beta} \quad \text{and} \quad \tau_I = \frac{2\beta}{f(c) + 3\beta}.$$

This gives the distribution of time spent at each state in the long-run steady state. In terms of upgrading frequency, notice that at state (H, H), there are no upgrades. At (H, L) and (L, H), either the corresponding firm upgrade according to $f(c)$, or both firm upgrades when nature places a shock that the state hits (L, L). The upgrading frequency, or in other words, the frequency at which the firms pay the cost is

$$\tau_{HL}\,[f(c) + 2\beta] + \tau_{LH}\,[f(c) + 2\beta] = \frac{2\beta\,[f(c) + 2\beta]}{f(c) + 3\beta}.$$

[12]For joint profits, the two states provide the same joint profit and are considered as just one state.

The firms' long-run average joint profit is then

$$[\pi_0(H,H)+\pi_1(H,H)]\underbrace{\frac{f(c)+\beta}{f(c)+3\beta}}_{\tau_B}+[\pi_0(H,L)+\pi_1(H,L)]\underbrace{\frac{2\beta}{f(c)+3\beta}}_{\tau_I}-\frac{2\beta\,[f(c)+2\beta]}{f(c)+3\beta}c. \quad (1)$$

We can also calculate the distribution and the joint profits under the MPE in Proposition 4. The results are summarized as follows.

Proposition 7. For $0 < c < \bar{c}$, firms' long-run average joint profit is increasing in c.

The proof is in Appendix B. To see the intuition, recall that as c increases, the mixing rate/probability decreases so as to provide sufficient upgrading incentives. As the upgrading becomes less frequent, firms incur upgrading costs less often, and τ_I increases, allowing them to enjoy positive profits for longer durations. Both factors contribute to higher joint payoffs. However, observe that the joint profits will be 0 once $c > \bar{c}$ since both firms in the long run stay at (L, L), and the profits are exhausted by Bertrand competition. Figure 5 shows the time distribution and joint profits as a function of c.

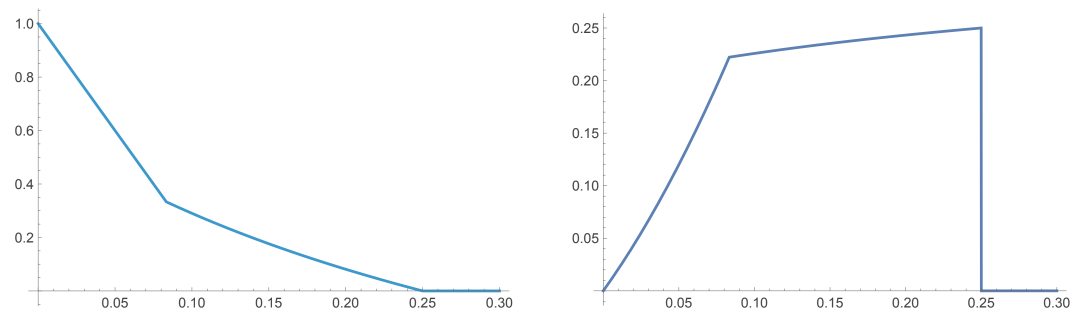


FIGURE 5 Time Distribution and Joint Profits

Left: τ_B as a function of c. Right: Joint profit as a function of c. Calculated at $r = 1, \beta = 1, \alpha = 0.5$.

Remark Although the two MPE in Proposition 3 and Proposition 4 are technically different, they can be and should be considered as a unified equilibrium structure. First, the driving force of the upgrading dynamics is to provide upgrading incentives that match the upgrading costs through the appropriate structuring of upgrading behavior. In turn, the resulting profiles are continuous, coinciding at $c = \tilde{c}$. Second, the equilibrium outcomes exhibit continuous trends in upgrading frequency, social efficiency, steady-state time distribution, and long-run average joint profits. For these reasons, we will hereafter use the term **direct competition** to refer to these two MPE.

4 Interactions of Vertical and Horizontal Differentiation

This section studies the general model that features both vertical and horizontal differentiation. As a benchmark, the social planner's optimal policy is first analyzed.

4.1 The Social Planner Benchmark

Unlike the case with vertical differentiation only, the social planner now has an incentive to duplicate high-quality products when there is horizontal differentiation, i.e., $k > 0$. Consider the upgrade decision at state $(q_0, q_1) = (H, L)$. Consumers near Firm 1 either settle for the low-quality product or incur a high traveling cost to purchase the high-quality product from Firm 0. If the social planner upgrades to (H, H), these consumers can enjoy the high-quality product without high transportation costs. Thus, the social planner prefers to keep both products at high quality, at least when upgrading costs are sufficiently low.

Proposition 8. Define

$$c_H^{\text{sp}} = \begin{cases} \dfrac{k(2-k) - 2\alpha^2}{2k(\beta + r)} & \text{if } \alpha < \dfrac{k}{2}, \\ \dfrac{4 - 4\alpha - k}{4(\beta + r)}, & \text{if } \dfrac{k}{2} \leqslant \alpha < 1 - k, \\ \dfrac{(1-\alpha)^2 + 2k(1-\alpha)}{4k(\beta + r)} & \text{if } 1 - k \leqslant \alpha. \end{cases}$$

and

$$c_L^{\text{sp}} = \begin{cases} \dfrac{k}{4(\beta + r)}, & \text{if } \alpha < 1 - k, \\ \dfrac{-(1-\alpha)^2 + 2k(1-\alpha)}{4k(\beta + r)} & \text{if } 1 - k \leqslant \alpha. \end{cases}$$

The social planner's optimal policy is:

- Never upgrade if $c \geqslant c_H^{\text{sp}}$;
- Upgrade one product at (α, α) and do not upgrade elsewhere if $c_L^{\text{sp}} \leqslant c < c_H^{\text{sp}}$;
- Always upgrade when possible if $0 \leqslant c < c_L^{\text{sp}}$.

The proof is in Appendix C. Proposition 8 shows that the social planner's optimal policy is defined by three cost regions, low, middle, and high, separated by two boundaries c_L^{sp} and c_H^{sp}. At the low cost range, the social planner keeps both products at high quality. At the middle cost range, although there are benefits duplicating high-quality products, the higher costs prevent the social planner from doing so, and the social planner can only afford to keep one product at high quality. At the high cost range, product upgrading becomes too expensive, and the social planner stops upgrading at all. The optimal policy is shown in Figure 6.

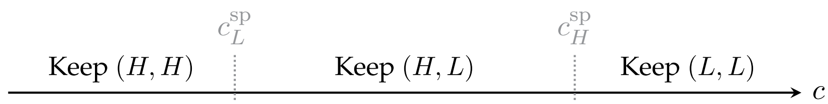


Figure 6 The Social Planner's Optimal Policy

It is also easy to verify that the optimal policy is continuous at $k = 0$. That is,

$$c_H^{\mathrm{sp}} = \frac{1-\alpha}{\beta + r} \quad \text{and} \quad c_L^{\mathrm{sp}} = 0 \quad \text{when } k = 0.$$

When $k = 0$, the policy in Proposition 8 collapses to the policy in Figure 5.

Notice that there are policies that are never optimal, as shown in Proposition 8. In general, the planner has six pure policy options:[13] at (L, L), upgrade none, one, or both products; and at $(H, L)/(L, H)$, upgrade the low-quality product. The policies that are never optimal are internally inconsistent. For example, consider the policy that upgrades both products at (L, L) but not at (H, L). Upgrading both products at (L, L) implies that the marginal benefit of a second high-quality product exceeds the upgrading cost, which implies the choice not to upgrade at (H, L) is not optimal. Similarly, upgrading one product at (L, L) and also at (H, L) is internally inconsistent: upgrading only one product at (L, L) suggests that the marginal benefit of a second high-quality product is below cost, while upgrading at (H, L) suggests the opposite.

To see why upgrading at (H, L) but not at (L, L) is never optimal, suppose the gain in social welfare from upgrading (L, L) to (H, H) is 2δ. If the upgrade is step-by-step, the intermediate state is (H, L). If consumers in $[0, 1/2]$ always purchase from Firm 0 and the rest consumers from Firm 1, the gain from each step must be δ. But consumer demand will re-distribute in the stage-game equilibrium: upgrading Firm 0's product first draws some consumers from Firm 1, so the gain from (L, L) to (H, L) is strictly larger than δ, and the gain from (H, L) to (H, H) is strictly less than δ.[14] Thus, if the planner is willing to upgrade from (H, L) to (H, H), she must also be willing to upgrade from (L, L) to (H, H).

4.2 Duopoly Competition Equilibria

I now consider the quality evolution under firms' duopoly competition when there is horizontal product differentiation. It can be verified that the no-upgrading MPE in Lemma 2 and the two direct competition MPE in Proposition 3 and Proposition 4 continue to exist as k increases, although there are changes under positive k. First, the boundaries of the range of upgrading cost c that allow for direct competition MPE change in k. In particular, the lower bound that allows for

[13] Since this is the social planner's decision problem, it suffices to consider pure policy options. As before, any seemingly asymmetric policy has an outcome-equivalent symmetric version via randomization.

[14] In the extreme case, when k is not too large, all consumers purchase from Firm 0. When the two firms split the market, the indifference consumer is still in $(1/2, 1)$.

direct competition MPE is some $c_3(k) > 0$ for $k > 0$. Second, the mixing behaviors of the firms also change in k. Such changes lead to the interactions of the two dimensions of product differentiation.

The game also admits a second competition mode, upgrading deterrence, when there is horizontal product differentiation, at even lower cost range $c_1(k) < c < c_3(k)$. Under such MPE, the firms realize that the opponents can frequently upgrade their products when lagging behind due to the low upgrading costs, and choosing to upgrade their own products is to start a quality war. Such quality wars are costly and act as a deterrence of product upgrade, so that firms can agree on maintaining low quality levels together.

The two competition modes, along with the no-upgrading MPE when the cost is sufficiently high and the always-upgrading MPE when the cost is sufficiently low, are summarized in Figure 7. I now discuss the two competition modes in details.

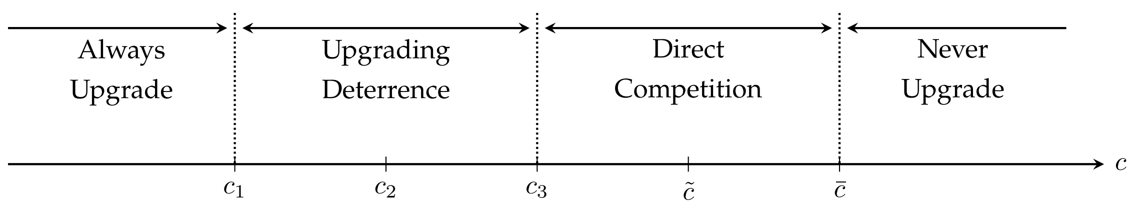


FIGURE 7 Duopoly Competition MPE with Horizontal Product Differentiation

4.2.1 Direct Competition: Interaction between Product Differentiation

Consider the direct competition region, $c_3 < c < \bar{c}$. As discussed in section 3, firms can in states (H, H), (H, L), or (L, H) in the steady state under equilibria, and the distribution of time among different states, τ_B and τ_I, represents the levels of vertical differentiation. This section considers the changes of mixing behaviors, and in turn, the changes in τ_B (or equivalently, τ_I), as k increases. Such changes then reflect how vertical differentiation changes as horizontal differentiation increases.[15]

As a preparation, recall that firms now can get positive stage payoffs in multiple states instead of just the quality-leader state. In particular, both firms receive positive profits at state (H, H), and the profits increase as k increases, while the profits of the quality leader firm at $(H, L)/(L, H)$ decreases when α is small. Still, firms still benefit more from being the quality leader. That is,

$$\pi_0(L, H) < \pi_0(L, L) \leqslant \pi_0(H, H) < \pi_0(H, L).$$

To simplify the analysis, attention is restricted to the case where $0 < k \leqslant 2/9$ and $0 < \alpha \leqslant 1-3k$. The relatively small α makes the shock meaningful by creating a relatively significant divergence

[15]As shown later, the steady state under MPE other than direct competition ones are mostly (H, H) or (L, L) only. The firms' competition are in extreme: either fiercely compete with each other under low upgrading cost, or avoid competition with each other under high upgrading cost. Intuitively, the interaction of product differentiation exist when the upgrading cost is intermediate, which corresponds to the direct competition region when considering symmetric MPE.

between the value of a high-quality product and the value of a low-quality one, and it also satisfies the monotonicity of the stage profits stated above. This is the more interesting and meaningful case: For larger α, firms under state $(H, L)/(L, H)$ still split the market, only not from exactly the middle. The competition still proceeds in familiar fashion, and the shock only has quantitative rather than qualitative effects. Under smaller α, shocks from nature are more significant by pushing the quality-follower outside the market and firms have to cope with the shocks with behaviors that are qualitatively different, which shows, for example, in how the stage payoff changes in k.

Under this parameter range, there exists $\hat{c}$ such that

- The two dimensions of differentiation exhibit substitution relations in equilibria if $c_3 \leqslant c < \hat{c}$.
- The two dimensions of differentiation exhibit complementary relations in equilibria if $\hat{c} \leqslant c < \bar{c}$.

Formally, the substitution and complementary relations are represented by how τ_B changes as k changes. This is summarized below.

Theorem 9. Suppose that $0 < k \leqslant 2/9$ and $0 < \alpha \leqslant 1 - 3k$. There exists $\hat{c} \in (c_3, \bar{c})$ such that for a given upgrading cost c,

$$\frac{\partial \tau_B}{\partial k} > 0 \text{ if } c \in (c_3, \hat{c}) \quad \text{and} \quad \frac{\partial \tau_B}{\partial k} < 0 \text{ if } c \in (\hat{c}, \bar{c}).$$

And $\partial \tau_B / \partial k = 0$ only at $c = \hat{c}$.

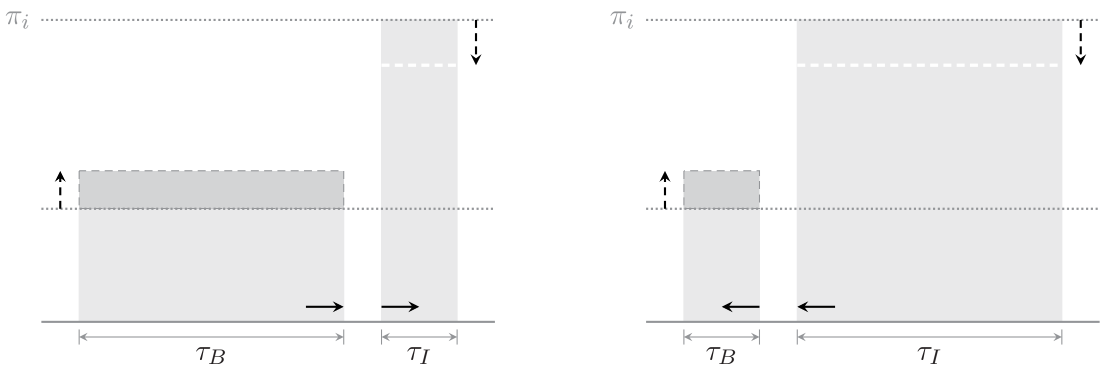
The proof is in Appendix C. To interpret this result, note that k measures the degree of horizontal differentiation. When k increases, the horizontal differentiation increases. As discussed in the previous section, τ_B is the time spent at the balanced (H, H) state, where there is no vertical differentiation. Thus, an increase in τ_B corresponds to a decrease in vertical differentiation. In turn, if τ_B increases in k, this means the vertical differentiation is reduced when there is more horizontal differentiation, hence the two dimensions of differentiation exhibit substitution relations.

To trace out the driving force of the substitution/complementary relations, consider how the upgrading incentive changes as k increases. Unlike the case where horizontal differentiation is absent, both (H, H) and (H, L) states[16] can provide upgrading incentives for Firm 0. If the mixing rate/probability does not change, so that τ_B does not change, the upgrading incentives in general will change given that $\pi_0(H, L)$ decreases in k while $\pi_0(H, H)$ increases. Since c is fixed, the upgrading incentives then must be restored by adjusting the distribution between τ_B and τ_I, which changes the mixing rate/probability. This then manifests as the substitution or complementary relations.

More specifically, when c is close to c_3, the mixing rate at $(H, L)/(L, H)$ is higher, and τ_B is

[16]Under the parameter setting of Theorem 9, Firm 0 still gets flow payoff 0 at state (L, H).

significantly larger than τ_I. (For a more concrete observation, see Figure 5.) This means most of the upgrading incentives are provided by $\pi_0(H, H)$. As k increases, this also increases, and the upgrading incentives are larger than needed. To restore the upgrading incentives, firms must increase the time spent at the state with lower incentive provision. Therefore, τ_B must be further enlarged since $\pi_0(H, L) > \pi_0(H, H)$.[17] This is shown in Panel (a) of Figure 8.



(a) Smaller c **(b)** Larger c

FIGURE 8 Interactions of Differentiation and Upgrading Incentives

The width of each bar is the corresponding time distribution. The height of each bar represents the upgrading incentive, or the benefits from upgrading. The total gray area represents the overall incentive of upgrading. The dashed lines, borders, and arrows show how the stage payoffs change as k increases. For example, as k increases, $\pi_0(H, H)$ increases, and it is represented by the darker area within the dashed boarder. The solid arrow shows how the time distributions change as k increases.

When c is closer to $\bar{c}$, τ_I is significantly larger, and the upgrading incentive is provided mainly by the inbalanced quality-leader state. As k increases, the overall incentive decreases. In turn, τ_I must be further enlarged since $\pi_0(H, L) > \pi_0(H, H)$, to restore the upgrading incentives. This is shown in Panel (b) of Figure 8.

More fundamentally, when the upgrading cost is lower, firms compete more aggressively, and they are more likely to stay in the comparable quality levels. If the market power increases, firms are able to charge a higher price and earn a higher profit. This in turn fuels quality competition, making it even harder to differentiate with each other. When the upgrading cost is higher, firms' upgrading frequencies are lower, and a quality leader is more likely to emerge and dominates the market. If the market power increases, the weaker firm has a stronger incentive to enter, and the leader must give up more profits to maintain the market dominance. This in turn dampens the competitions in quality, reducing the upgrading frequencies, and a quality leader is even more

[17]Note that the direction of the change of the incentives are determined by $\partial\pi_i(q_0, q_1)/\partial k$, while the overall size of incentives depend on $\pi_i(q_0, q_1)$.

likely to emerge instead of both upgrading to the high-quality level.

4.2.2 Upgrade Deterrence

There are new MPE when $c \leqslant c_3$. Before discussing upgrading deterrence MPE, it is customary to introduce the always-upgrading-when-possible MPE. As k increases, the firms gain market power even at balanced quality states. In particular, when k is small as assumed, the firms profit at (H, H) increases in k. This means the balanced high-quality state (H, H) can now also offer upgrading incentives for a quality follower. As upgrading cost c drops below the threshold c_3, the upgrading incentive from (H, H) can be already sufficient for the quality follower to invest in quality and catch up. This leads to the always-upgrading-when-possible MPE.

Proposition 10. Always upgrading when possible is a limit of a symmetric MPE if

$$0 \leqslant c \leqslant \frac{\pi_0(H,H) - \pi_0(L,H)}{r+\beta} \equiv c_3.$$

The proof, which involves solving the value function and ruling out profitable deviations, is relatively straightforward and hence omitted. Instead, notice that the condition in Proposition 10 is exactly the no-profitable-deviation condition at (L, H). The RHS is the present value of upgrading gain, and if the upgrading cost is lower, firms prefer to upgrade at the quality-follower state.

Interestingly, there is not a second condition that corresponds to the no-profitable-deviation condition at (L, L). The reason is easy to see at the limit, where $\Delta \to 0$. Given that Firm 1 always upgrades (at least at (L, L)), if Firm 0 follows the strategy and upgrades, they move to (H, H) immediately. If Firm 0 does not upgrade at (L, L), they move to (L, H) instead. But then the condition in Proposition 10 indicates that Firm 0 should upgrade at (L, H), leading to (H, H) as well. Given $\Delta \to 0$, the two paths have no payoff difference. For this reason, the no-profitable-deviation condition at (L, L) is always satisfied. This is an example of self-fulfilling: both firms believe that the opponent will upgrade at (L, L), which renders (L, L) not on-path, so that they might as well upgrade at (L, L).

Such self-fulfilling features usually indicate equilibrium multiplicity. Indeed, there are additional MPE that overlaps with the always-upgrading MPE. Define

$$c_1 = \frac{\pi_0(H,H) - \pi_0(L,L)}{r+\beta} \quad \text{and} \quad c_2 = \frac{1}{r+\beta}\left[\pi_0(H,H) - \frac{\beta}{r+\beta}\pi_0(L,L) - \frac{r}{r+\beta}\pi_0(L,H)\right],$$

and it is easy to see that $0 \leqslant c_1 < c_2 < c_3$.

Proposition 11. The following is the limit of a symmetric MPE if $c_1 \leqslant c < c_2$:

- Firm i does not upgrade at (L, L).
- Firm i upgrades for sure at (L, H).

The following is the limit of a symmetric MPE if $c_2 \leqslant c < c_3$:

- Firm i does not upgrade at (L, L).
- Firm i upgrades at (L, H) with a rate $h(c)$, where

$$\begin{aligned} h(c) =& \frac{(\beta + r)\left[\pi_0(H,H)(\beta + r) + \beta\pi_0(H,L)\right]}{(\beta + r)^2 c - \pi_0(H,H)(\beta + r) + \pi_0(L,H)r + \beta\pi_0(L,L)} \\ &- \frac{(\beta + r)\left[\pi_0(L,H)(\beta + r) + \beta\pi_0(L,L) + (\beta + r)(2\beta + r)c\right]}{(\beta + r)^2 c - \pi_0(H,H)(\beta + r) + \pi_0(L,H)r + \beta\pi_0(L,L)}. \end{aligned}$$

Moreover, $h(c)$ is decreasing in c. Also, in the corresponding cost range, the MPE here yield higher joint profits than the always-upgrading MPE.

The proof is in Appendix C. Instead of utilizing the low costs and always upgrading, firms play "upgrade deterrence" in these MPE. Consider the first MPE in Proposition 11. Both firms "agree" to stay at the low quality level as long as no one upgrades. However, once a firm breaks the agreement and upgrades, they play a quality war forever by always upgrading since (L, L) is reached with 0 probability. To support this MPE, the upgrading cost cannot be too low. In particular, $c \geqslant c_1$, so that quality war is a meaningful deterrence. When $c < c_1$, the upgrading incentive provided by moving from (L, L) to (H, H) is already large enough compared with the upgrading cost, and no deterrence can sustain (L, L). The upgrading cost also cannot be too high, since fighting an indefinite quality war following upgrading at (L, L) is also costly, and the deterrence is not credible when the cost is too high. $c = c_2$ is the highest cost where the forever quality war can be sustained.

As the upgrading cost further increases, the forever quality war becomes too costly, and the firms switch to weaker deterrence using mixed upgrading strategies. Unlike the MPE above in which the quality state is maintained at (H, H) once a quality war starts, the steady state under this MPE is (L, L): in a quality-war phase such as (H, L), given enough time, nature will eventually place a shock before Firm 1 upgrades, restoring the absorbing state (L, L). For this reason, the quality war for $c_2 < c < c_3$ is a finite-length quality war (in expectation), which is weaker deterrence. Still, this weaker deterrence is already sufficient for the equilibrium: for a firm who is tempted to upgrade, it faces a higher upgrading cost that can be deterred by a finite-length quality war. For a firm who is supposed to fight the quality war after the opponent's upgrade, a finite-length quality war is also more affordable compared with the infinite-length quality war under the higher upgrading cost. At $c = c_3$, even the mixed-strategy punishment becomes too costly to sustain.

We call the MPE in Proposition 11 **upgrade deterrence**. Just as the two MPE in the direct competition case, although they are technically two different MPE, the driving forces are similar. These two equilibria are also similar to the trigger punishment in repeated games. For example, when $c_1 < c < c_2$, the MPE is similar to grim trigger: a deviation leads to quality war forever.

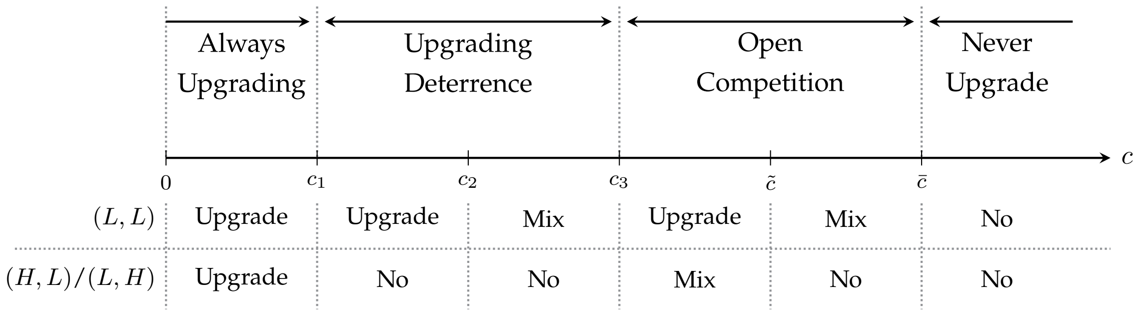
However, it should be clarified that such equilibria are still MPE, and the strategic incentives in these MPE are not the same as the incentives in repeated games. In repeated games, a punishment is based on the sunk behaviors that are not payoff-relevant. For example, in a repeated prisoners' dilemma, players engage in punishment if a player deviates to defect in any history. But the deviation itself does not change the payoffs of the current and future games. If the players have sufficient beliefs that no one will deviate again, the best option can be to restore cooperation rather than implement costly punishment. This is not the case here. By deviating to upgrading at (L, L), the deviation firm changes the payoff relevant state for both firms, namely, its quality. The incentives of the firms thereafter are still based on payoff-relevant states only. In other words, firms play quality wars not to punish the deviations in the history. Instead, they play quality wars because it is the best option since they are already there, even if they both believe for sure that no one would upgrade if they could go back to (L, L) now. For this reason, the seemingly-trigger strategy can be better named as "competition trigger".

Such equilibria can also exist under $k = 0$, but only trivially, as they require $c = 0$ to be sustained. To see this, consider the first upgrading-deterrence (forever quality war) profile. There is no profitable deviation at (L, H) or (H, L) since the game is played exactly the same as always upgrading. At (L, L), no upgrade leads to (L, L) forever, in which case both firms yield payoff 0. If Firm 0 deviate and upgrades, the state is (H, H) forever, and the payoff is still 0. Given the upgrade is free, Firm 0 is exactly indifferent between upgrading or not and might as well choose not to upgrade.[18] This is rather delicate and cannot be maintained for $c > 0$. It is the existence of market power that allows deterrence-type equilibria to exist under non-trivial parameter values: the upgrading incentives from (L, L) to (H, H) are smaller than the upgrading incentives from (L, H) to (H, H), which creates a non-trivial range of c to support forever or finite-length quality war MPE.

Summary of MPE. It turns out that the six MPE (Lemma 2, Proposition 3, Proposition 4, Proposition 10, Proposition 11) constitutes the joint-profit-maximizing symmetric MPE, which is summarized in Figure 9. Although there are many cases, there are only two meaningful types (if we do not count the relatively obvious and less interesting always-upgrading and never-upgrading MPE at the extreme cost levels): upgrade deterrence, by either a grim trigger strategy or finite-length punishment strategy, and direct competitions, by playing appropriate mixed strategies to control the opponents' upgrading incentives.

There is another symmetric MPE, under which both firms engage in probabilistic mixing at (L, L) while upgrade for sure at $(H, L)/(L, H)$. Although this is a technically different MPE, it is outcome equivalent to always upgrading in the limit. We leave the details to Appendix C, where we also compare the joint profits for the MPE in Figure 9 in case of multiplicity.

[18] Under $k = 0$ and $c = 0$, the finite-length quality war MPE converges to the forever quality war MPE.



Above axis: Competition modes. Both the upgrading-deterrence mode and the open-competition mode contains two types of MPE. Below axis: upgrading strategies of firms when at low quality.

Figure 9 The Summary of Joint-Profit-Maximizing Symmetric MPE

It can be of interest regarding how the bounds above changes as parameters r, β or k change. Regarding r and β on bounds c_1, c_3 and $\bar{c}$, the effect is straightforward: as r or β increases, all three bounds decreases. A larger discount or a higher shock rate means the upgrading today is relatively more expensive. When the firms discount the future more, the flow profits from future are lower compared with the current cost of upgrading. When the shock rate is higher, the expected duration of the product maintaining high quality after an upgrade is shorter, leading to again lower profits per upgrade. In turn, upgrading regions shrink, and no-upgrading region expands.

For k, under the condition in Theorem 9, both c_1 and c_3 increases in k, while $\bar{c}$ decreases in k. To see why this is the case, first consider c slightly above c_1, where the firms in the upgrading-deterrence equilibria are tempted to deviate to (H, H). As k increases, the flow payoff at (H, H) is higher, so that upgrading deterrence is harder to be maintained, leading to the increase in c_1. For c slightly below c_3, the increased flow payoff at (H, H) means that firms can afford more costly quality war, leading to the increase in c_3. However, at c below $\bar{c}$ where firms have low upgrading frequency and mostly sustain the leader-follower states, the flow payoff at (H, L) (or (L, H)) is now lower, offering lower upgrading incentives, so that firms stop upgrading even sooner, leading to lower $\bar{c}$.

4.3 Investment Efficiency

As in the case of $k = 0$, the firms in general do not invest in quality in an efficient fashion. Nevertheless, the firms can now under-invest, which is not present when $k = 0$. As before, we consider the long-run state distributions. There is one complication though – the first upgrading-deterrence MPE is not ergodic, since its steady state depends on the initial state, no matter how far into the future. For this reason, this profile is not formally considered in the efficiency and joint profit analysis. Still, this may not be a serious issue. Since the steady state is either (H, H)

or (L, L), the steady state of this MPE effectively aligns with either that of the always-upgrading equilibrium or that of the finite-length punishment equilibrium.[19]

It should also be clear that the finite-length punishment will have (L, L) as the steady state. This should be straightforward if the initial state is (L, L). When the initial state is not (L, L), given long enough time, nature eventually will place a shock sooner than a firm upgrading its product, and the state transitions to (L, L), which is absorbing.

> **Corollary 12.** Assume that $0 < k \leqslant 2/9$ and $0 < \alpha \leqslant 1 - 3k$. The firms over-upgrade if $c_3 < c < \bar{c}$. There also exist costs $c < c_3$ and $c > \bar{c}$ such that firms under-upgrade.

The proof is in Appendix C, which is simply showing that $c_L^{\text{sp}} < c_3$ and $\bar{c} < c_H^{\text{sp}}$. The intuition of the over-upgrading region is the same: the competition between the firms enforces too many investments. What makes this result a little surprising is that the efficiency itself is not monotone: under-upgrading can happen at lower and higher cost levels, though driven by different forces. At the cost level just below c_3, firms under-upgrade due to the *de facto* collusion achieved by the upgrading deterrence. Firms do not upgrade as frequently as the social planner because the quality war is too costly. In a way, under-upgrading is possible only because the potential competitions are too severe. At the high cost level above $\bar{c}$, firms do not upgrade and there is no dynamic quality competition. Firms stop upgrading sooner than the social planner because they do not internalize the consumer surplus. While the social planner finds it still worth upgrading considering the social surplus, the joint profits are not sufficient to cover the upgrading cost to maintain quality competitions.

Notice that the existence of both under-upgrading regions requires horizontal differentiation. In the low-cost region where under-upgrading is generated by upgrading deterrence, deterrence relies on the fact that the flow gain from (L, H) to (H, H) strictly exceeds that from (L, L) to (H, H), which does not hold if there is no horizontal differentiation. In the high-cost region where under-upgrading is generated by failure of internalizing consumer surplus, firms' competition reallocates consumers in an inefficient way only when there is horizontal differentiation.

4.4 Firms' Long-Run Average Joint Profits

While the firms' long-run average joint profits are increasing in c when $k = 0$ (until c is too large), the joint profits under $k > 0$ can be non-monotonic in two regions. First, when c is sufficiently small and the MPE is firms always upgrading, the joint profit is decreasing in c. Second, when c is close to $\bar{c}$ and $c < \bar{c}$, the joint profit is decreasing in c if β is large.

[19] Note that (H, H) may be a more reasonable choice for the steady state as it is stable under trembling. Any trembles at (L, L) will lead to (H, H) due to the grim trigger strategy.

To understand the reasons, note that increasing c in general has two effects. The direct effect is the cost effect; that is, firms must pay higher costs to upgrade the products, which decreases the joint profits. There is also an indirect competition effect: Higher costs dampen the competitions between the firms, a firm is more likely to become quality leader and enjoy higher profits, and the firms also pay the upgrading cost less frequently. This increases the joint profits.

When c is small and firms always upgrading, only the direct effect is present. As c is sufficiently low, firms always upgrade, and the upgrade frequency is determined by the shock frequency alone. Slightly increasing c does not dampen the competitions (for any interior c). In the case of direct competition, the indirect competition effect usually dominates, which is indeed the case when $k = 0$. When $k > 0$, the indirect effect may be weakened when c is close to $\bar{c}$, where the two dimensions of differentiation are complementary. As k increases, τ_B decreases, and firms stay at imbalanced quality states for a longer period. In such states, whenever a shock happens, at least one firm needs to upgrade. If shocks occur frequently enough, the direct cost effect dominates the indirect competition effect, leading to decreasing joint profits.

5 Asymmetric Equilibria

While focusing on symmetric equilibria is reasonably supported, as discussed in section 2, it can be of concern regarding how much the results relying on the symmetry assumption. This section briefly discusses the asymmetric MPE of the game. It will be shown that (i) there still exists non-monotonicity of upgrading frequencies in the upgrading cost c, (ii) all but one asymmetric MPE still induce inefficiencies, and (iii) there are still interactions between the two dimensions of product differentiations in asymmetric MPE.

There are three types of asymmetric MPE in this game:

- *Case A.* In the upgrading-deterrence region, there exist the following MPE:
  - One firm mixes at (L, L) with a rate and upgrades at the quality-follower state for sure.
  - The other firm does not upgrade at (L, L) and upgrades at the quality-follower state with a rate.
- *Case B.* In the first direct-competition region ($c_3 \leqslant c < \tilde{c}$), there exist the following MPE:
  - One firm upgrades at (L, L) for sure and mixes at the quality-follower state with a rate.
  - The other firm mixes at (L, L) with a probability and mixes at the quality-follower state with a rate.
- *Case C.* In the second direct-competition region ($\tilde{c} < c < \bar{c}$), there exist the following MPE:
  - One firm upgrades at (L, L) for sure.
  - No other upgrade.

The formal statements of the asymmetric MPE above are delegated to Appendix D. Here, I briefly discuss the upgrading incentives of the firms in the asymmetric equilibria. In Case A, the upgrading deterrence feature is mainly preserved, especially for the less-active firm who chooses not to upgrade at state (L, L). This MPE reflects the competition scenario where one firm is more inclined to start a quality competition, but this inclination is partially inhibited by potentially fierce quality war. In turn, (L, L) is a state remain on path in the steady state. In Case B, one firm chooses the same strategy as in the symmetric MPE, so that the opponent is indifferent at (L, L). In the symmetric MPE, the opponent has the equally-strong upgrading inclination and chooses the same upgrading strategy. Here, the opponent has a weaker upgrading inclination and chooses a lower upgrading frequency at (L, L). In Case C, the competition is the classic Chicken.[20]

Naturally, the specific upgrading frequencies, inefficiencies, and the modes of interactions of the two dimensions of product differentiations are different under asymmetric equilibria. However, as argued below, the results under symmetric equilibria can be qualitatively extend to asymmetric ones.

Non-monotonicity of the upgrading frequency. It should be easy to see that this is still not monotonic even if we consider the asymmetric equilibria. In Case A, as discussed above, (L, L) is on path in the steady state. But in Case B, (L, L) is no-longer on path, since at least one firm will upgrade at the moment the state hits (L, L). In other words, while the upgrading frequencies are relatively low for $c < c_3$ to maintain the lowest quality state (L, L) on path, the firms' upgrading frequencies are higher to remove (L, L) from the steady state when $c > c_3$.

Inefficiencies. In Case A, for c not too low, the social planner's optimal policy is to keep the quality state at (H, L). Case A instead generate a steady state where four possible quality states are all present. In particular, both (L, L) and (H, H) are possible. In Case B, notice that (H, H) is on path just as the symmetric equilibria, while (L, L) is not on path. This suggests firms' upgrading behaviors are not dynamically efficient, since the social planner would like to keep one and only one product at high quality. The Chicken MPE in Case C is the only asymmetric MPE that is dynamically efficient. To compare this with Corollary 12, notice that in Corollary 12, there is exactly one cost $\bar{c}$ at which the firms are efficient, i.e., where the firms transitions from over-upgrading to under-upgrading.[21] If we consider Case C, the transition range of c is enlarged from a point, $\bar{c}$, to an interval, $(\tilde{c}, \bar{c}]$.

This shows for all but the Chicken MPE, asymmetric MPE cannot avoid dynamic inefficiencies. Moreover, the sources of the inefficiencies are often similar to the symmetric MPE. In Case A, there still exists under-upgrading due to the partial deterrence feature. In Case B, the over-investment

[20] This is the MPE that will be eliminated if there is a (small) flow cost of staying in the market, especially under small k and α, as assumed in Theorem 9.

[21] Under the symmetric direct competition equilibria, as $c \to \bar{c}$, the firms' mixing probability at (L, L) decreases and converges to 0. This means as c close to $\bar{c}$, both firm upgrading almost never happens, and almost always one firm will emerge as the quality leader of the market. In the limit, $\bar{c}$, this gives the dynamic efficiency.

is again derived from firms' competition incentives, exactly the same as the symmetric MPE.

Interactions of two dimensions of product differentiation. The discussion needs to be limited to $c_1 < c < \tilde{c}$ (i.e., Case A and Case B) for the following reasons:

- There are possible interactions now in the originally upgrading-deterrence region $c_1 < c < c_3$, i.e., Case A, since the outcome distribution is not trivially (L, L) only. However, since (L, L) is still on path, there are two states in which firms have no vertical differentiation, τ_B now include the time spent at (H, H) *and* (L, L).
- In the second direct-competition region $\tilde{c} < c < \bar{c}$, i.e., Case C, there is no longer interaction due to the asymmetric Chicken MPE, in which case the outcome is now always a quality-leader-follower state, suggesting $\tau_B = 0$.

Due to complexity of the calculations under asymmetric equilibria, simplifications of the range of the parameters are adopted. For example, under small k, it can be shown that $\partial\tau_B/\partial k > 0$ when $c < c_3$ (under Case A), while for c close to c_3 and $c > c_3$, $\partial\tau_B/\partial k < 0$, so that the two dimensions of product differentiation can still exist first substitution then complementary relations. See Appendix D for details.

It should be noted that the interactions between the two dimensions of product differentiation are less clear and more parameter-dependent in general. The sign of the derivative $\partial\tau_B/\partial k$ will depend k, r, β, and α, and the first substitution then complementary relations can be violated. However, the economics intuition of the interactions is still the same: as k increases, the firms experience strengthened or a weakened upgrading incentives triggered by the changes of stage payoffs, and the time distribution τ_B in the steady state under the corresponding MPE must adapt to restore the correct upgrading incentives. This in turn increases or decreases the product differentiation in the equilibria, giving the substitution or complementary relations between the two dimensions of product differentiation.

6 Extensions

This section considers an extension where the shocks are no longer independent. Instead, if nature places a shock on one firm, it is more likely to place a shock on the other firm as well. This can be reasonable in the real world: if the fryer breaks at one restaurant, removing fries from the restaurant's menu, the consumers may order more fries from competitors, which can overload and possibly break the fryers of competitors.

More geneally, a symmetric correlation between two shocks as shown in Table 1 is assumed. Specifically, in each period, the shocks keep the same marginal distribution, i.e., a shock happens with probability b, but now the shocks are correlated with correlation coefficient $\rho \in [0, 1]$. Note that this correlation is only in effect at state (H, H). At (H, L) or (L, H), only one product can be shocked, and the shock probability is still the marginal probability b.

Table 1 The Distribution of Correlated Shocks

	Shock to Firm 1	No Shock to Firm 1
Shock to Firm 0	$b^2 + \rho b(1-b)$	$b(1-b)(1-\rho)$
No Shock to Firm 0	$b(1-b)(1-\rho)$	$(1-b)^2 + \rho b(1-b)$

Also, the main concern of this section is to verify if the two major competition modes, upgrading deterrence and direct competition, still hold under correlated shocks, and if so, how they changes due to the correlation.[22]

As the results above mainly study the limiting behavior of the equilibria, it is worth mentioning that ρ is assumed to be a constant as $\Delta \to 0$.[23] If $\rho \to 0$, there is no change from the result above.

We start with the two upgrading-deterrence equilibria.

Proposition 13. The two upgrading-deterrence MPE still exist for $\rho \in (0, 1)$. Also,

- c_1, c_3 do not change with ρ.
- c_2 increases in ρ and coincides with c_3 when $\rho = 1$.
- $h(\cdot)$ increases in ρ.

The proof is in Appendix E. For some intuitions, notice that correlated shocks make upgrading deterrence more difficult. For the upgrading deterrence MPE with finite expected length ($c_2 < c < c_3$), if the shocks are not correlated, (H, H) can only fall to (H, L) or (L, H) as $\Delta \to 0$. The corresponding firms upgrade at such states with a rate and potentially extend the quality-war phase. With correlated shocks, (H, H) can fall to (L, L) directly with positive probability. In other words, the correlation has a *restarting effect*: it resets the quality levels to a balanced low-low state, restarting the competition of the firms. Once the state reaches (L, L), the quality-war phase terminates immediately for sure. Therefore, the correlation between shocks effectively reduces the expected length of quality-war. In order to keep sufficient deterrence, firms then must upgrade with a higher rate at (L, H) or (H, L). This reduces the payoff of the opponent by not only reducing the expected duration spent at their preferred state, but also lowering the probability that a shock arrives sooner than the upgrading at (L, H) or (H, L), which extends the expected duration of the quality-war phase. In turn, as ρ increases, both c_2 and $h(\cdot)$ increase continuously in ρ, and at $\rho = 1$, only the competition-trigger MPE survives.[24]

[22]It is also straightforward to verify that the always-upgrading MPE and the never-upgrading MPE still exist under the same range, and c_1 and $\bar{c}$ do not change with ρ.

[23]More generally, it at least needs to be assumed that ρ has a positive limit.

[24]We should note that "competition trigger" here is no longer competition trigger any more with positive correlations.

We next move to the direct-competition equilibria.

Proposition 14. The two direct-competition MPE still exist for $\rho \in (0,1)$. Also,

- $c_3, \bar{c}$ do not change with ρ.
- $\tilde{c}$ decreases in ρ and coincides with c_3 when $\rho = 1$.
- $f(\cdot)$ and $g(\cdot)$ decrease in ρ.

The proof is in Appendix E. To understand this result, recall that the mixing rates or probabilities need to provide correct upgrading incentives in direct-competition MPE. As the shocks become more correlated, the expected durations at imbalanced (H, L) and (L, H) states are reduced. The correlation again has the restarting effect, by reducing the firms' opportunities of becoming the future quality leader. Since (H, L) and (L, H) states provide major upgrading incentives (especially if k is small), mixing rates and probabilities must decrease to increase the expected duration at the imbalanced states, compensating the loss from correlated shocks. As $\rho \to 1$, the incentives provided by such imbalanced states are smaller as such states are less likely to emerge. When $\rho = 1$, (H, L) and (L, H) are no longer on path, and the only way to provide upgrading incentives is to reduce the upgrading probabilities at (L, L), so that the first direct-competition MPE disappears Figure 10 summarizes the changes of MPE under correlated shocks.

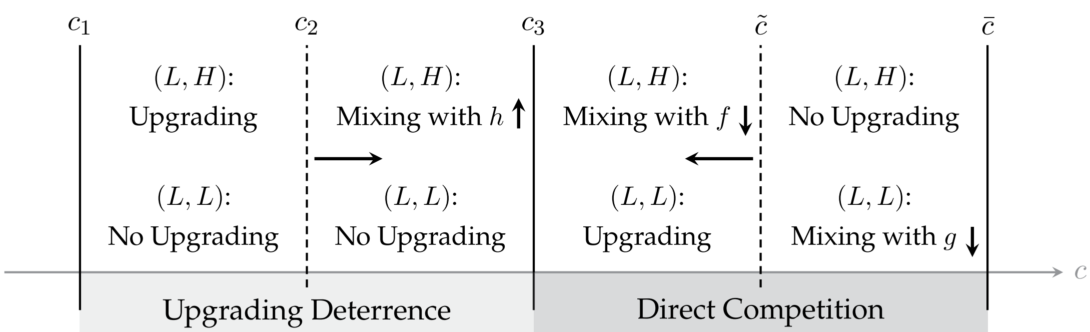


Figure 10 MPE with Correlated Shocks

The black arrows on the graph show how the MPE change as ρ increases.

While the discussion above focuses on positive correlations, the results and intuitions are symmetric under negative correlations as well. The negative correlation increases the expected durations of imbalanced (H, L) and (L, H) states, leading to the opposite adjustments as in Proposition 13 and Proposition 14.

Again, (H, H) can transition into (L, L) directly, and firms stop upgrading at (L, L), so that the quality-war phase does not necessarily last forever. We still keep the term "competition trigger" for consistency.

7 Conclusion

This paper has examined how firms compete dynamically in quality and price in the presence of horizontally heterogeneous consumers. We showed that horizontal differentiation can significantly alter firms' quality-investment behavior, joint profits, and social welfare.

More specifically, while investment frequency decreases monotonically with investment costs when consumers are homogeneous, heterogeneous consumers enable upgrading deterrence, allowing firms to sustain lower investment frequencies even when upgrading is relatively cheap. When investment becomes more expensive, deterrence is no longer sustainable, and firms switch to direct competition, in which the upgrading frequency once again declines with higher costs. This non-monotonicity in upgrading frequency also creates non-monotonicity in investment efficiency, leading to under-investment at both low and high cost levels. The quality evolution patterns are robust under correlated quality shocks.

It is also further demonstrated that horizontal and vertical differentiation can exhibit substitution or complementarity depending on the cost of investment. Higher horizontal differentiation can either fuel or dampen quality competition. When investment is inexpensive and firms tend to have similar quality levels, stronger horizontal differentiation fuels competition by enhancing market power. When investment is costly and firms are more vertically differentiated, stronger horizontal differentiation dampens quality competition because maintaining a leadership position becomes less profitable.

Future research could further explore dynamic competition along two dimensions by considering asymmetric firms, correlated quality shocks, or settings with more than two quality levels. Since consumers are transient in our model, it would also be valuable to examine durable goods markets where past consumers periodically return.

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A The stage game

A.1 The Social Planner

We first solve the social planner's stage game.

Stage games $(1,1)$ **and** (α,α)**.** We consider (q,q) for some generic q. If k is not too large, consumers in $[0,1/2)$ purchase from firm 0, consumers in $(1/2,1]$ purchase from firm 1, and the consumer at $1/2$ is indifferent. For the indifference consumer, she purchases if $q-(k/2)\geqslant 0$, or $k\leqslant 2q$. The social surplus is

$$2\times\int_0^{\frac{1}{2}}(q-kx)\mathrm{d}x=q-\frac{k}{4}.$$

Note that here we utilize the fact that the environment is symmetric.

If $k>2q$, some buyers in the middle chooses not to purchase. Let $\underline{x}$ be the location of the last consumer who purchases from firm 0. Then,

$$q-k\underline{x}=0 \quad\Rightarrow\quad \underline{x}=\frac{q}{k}.$$

The social surplus is

$$2\times\int_0^{\frac{q}{k}}(q-kx)\mathrm{d}x=\frac{q^2}{k}.$$

Stage game $(1,\alpha)$**.** There are three cases:

- Firm 0 and firm 1 split the market at some x^* (and the whole market is covered).
- Firm 0 dominates the market.
- Firm 0 serves the consumers at location $[0,\underline{x}]$, firm 1 serves the consumers at location $[\overline{x},1]$, and $\underline{x}<\overline{x}$.

Let's consider the first case. The location x^* of the indifference consumer is defined by

$$q_0-kx^*=q_1-k(1-x^*) \quad\Rightarrow\quad x^*=\frac{1}{2}+\frac{1-\alpha}{2k}.$$

It is clear that $x^*>1/2$, and $x^*\leqslant 1$ if $1-\alpha\leqslant k$, or $\alpha\geqslant 1-k$. The indifference consumer gets

$$1-kx^*=\frac{1+\alpha-k}{2}.$$

This consumer gets nonnegative payoff if $1+\alpha\geqslant k$, or $\alpha\geqslant k-1$. Therefore, if $\alpha\geqslant|1-k|$, the two firms split the market at x^*. The social welfare is

$$\int_0^{x^*}(1-kx)\mathrm{d}x+\int_{x^*}^1[\alpha-k(1-x)]\mathrm{d}x=\frac{1+\alpha}{2}+\frac{(1-\alpha)^2}{4k}-\frac{k}{4}.$$

The second case happens if $x^*>1$, or $1-\alpha>k$. To verify that firm 0 indeed dominates the market, consider the consumer at location 1. Purchasing from firm 0 gets $1-k$. Since $1-\alpha>k$, the consumer gets more than α (so that more than 0), which shows that firm 0 indeed dominates the market. The social surplus is

$$\int_0^1 (1 - kx)\mathrm{d}x = 1 - \frac{k}{2}.$$

The last case happens if the consumer at x^* has negative payoff. Formally, $\underline{x} = 1/k$ as we defined above. For $\overline{x}$, it is defined by

$$\alpha - k(1 - \overline{x}) = 0 \quad \Rightarrow \quad \overline{x} = 1 - \frac{\alpha}{k}.$$

We need $\underline{x} < \overline{x}$, which implies $\alpha < k - 1$. Notice that this is indeed the condition for the consumer at location x^* gets negative payoff if purchasing. The social surplus is

$$\int_0^{\underline{x}} (1 - kx)\mathrm{d}x + \int_{\overline{x}}^1 [\alpha - k(1 - x)]\mathrm{d}x = \frac{1 + \alpha^2}{2k}$$

A.2 Duopoly Firms

We first state the best responses of Firm 0, in terms Firm 1's quality q_1 and price p_1:

1. If $q_1 - p_1 \geqslant k$:
   (a) If $q_0 > 3k + q_1 - p_1$: $BR_0 = p_0^D = q_0 - q_1 + p_1 - k$.
   (b) If $-k + q_1 - p_1 \leqslant q_0 \leqslant q_1 - p_1 + 3k$: $BR_0 = p_0^S = (q_0 - q_1 + p_1 + k)/2$.
   (c) If $q_0 < -k + q_1 - p_1$: BR_0 is any price higher than $q_0 - q_1 + p_1 - k$ so that firm 0 has zero demand.
2. If $0 \leqslant q_1 - p_1 < k$:
   (a) If $q_0 > 3k + q_1 - p_1$: $BR_0 = p_0^D = q_0 - q_1 + p_1 - k$.
   (b) If $3k - 3(q_1 - p_1) \leqslant q_0 \leqslant 3k + q_1 - p_1$: $BR_0 = p_0^S = (q_0 - q_1 + p_1 + k)/2$.
   (c) If $2k - 2(q_1 - p_1) \leqslant q_0 \leqslant 3k - 3(q_1 - p_1)$: $BR_0 = p_0^F = q_0 + q_1 - p_1 - k$.
   (d) If $q_0 < 2k - 2(q_1 - p_1)$: $BR_0 = p_0^M = q_0/2$.
3. If $q_1 - p_1 < 0$:
   (a) If $q_0 \geqslant 2k$: $BR_0 = q_0 - k$.
   (b) If $q_1 < 2k$: $BR_0 = p_0^M = q_0/2$.

In general, there are four possible cases: (i) one firm dominates the market, (ii) two firms split the market with FOC pricing, (iii) two firms split the market with corner solutions, and (iv) two firms are separated small monopolists. Case (iii) may not be obvious: FOC pricing may not always leave the middle consumer positive consumer surplus. If this happens, and the quality levels are not too low, firms will still compete, since the profits of covering a larger market is still higher than the profits of charging a higher price and losing some markets.

We can now use the best responses above to formulate the stage game equilibria. The cases where the quality levels are the same are relatively easy and we state the result directly in Table 2.

TABLE 2 Stage Game Equilibria in (q, q)

	Each Firm's Price	Each Firm's Profit
$3k/2 \leqslant q$	k	$k/2$
$k \leqslant q < 3k/2$	$q - (k/2)$	$(q/2) - (k/4)$
$q < k$	$q/2$	$(q^2)/4k$

Stage game $(1, \alpha)$**.** Since $\alpha \in (0, 1)$, there are three possible cases in this subgame

- Firm 0 dominates the market.
- The two firms split the market. This includes two scenarios: the indifference consumer has positive surplus, or the indifference consumer has zero surplus.
- The two firms leave some buyers not served.

We should note that it is impossible that firm 1 dominates the market, given that firm 0 has higher quality. Also, it is not possible that for some $\underline{x}$, consumers in $[0, \underline{x}]$ purchase from firm 0, and the remaining consumers choose not to purchase. Here, firm 1 can always charge $\varepsilon < \alpha$ and attract consumers close to 1 to purchase, which is a profitable deviation.

Let's start from the case that firm 0 dominates the market. We argue that $p_1 = 0$ in such equilibria. Since firm 0 dominates the market and $q_1 = \alpha > 0$, we must be in Case 1(a) or Case 2(a) regarding firm 0's best response, meaning that $p_0 = p_0^D(p_1) = q_0 - q_1 + p_1 - k$. That is, the consumer at location 1 is indifferent. Suppose that $p_1 = \tilde{p}_1 > 0$. Given $p_0 = p_0^D(p_1)$, firm 1 can charge $\tilde{p}_1/2$ instead, so that consumers closer to 1 will purchase from firm 1, leading to a positive profit. This is a profitable deviation exists for any positive p_1. Therefore, in any equilibrium that firm 0 dominates the market, firm 1's price p_1 must be 0. Then, following the best response function of firm 0, firm 0 dominates the market if $q_0 - q_1 = 1 - \alpha > 3k$. Firm 0 charges $1 - \alpha - k$, which is also firm 0's equilibrium profit.

Next, consider the case that they split the market. Depending on parameters, we may be in Cases 1(b), 2(b), or 2(c) of firm 0's best response. For Cases 1(b) and 2(b), the indifference consumer has (weakly) positive surplus, and firms chooses the FOC price. This leads to

$$p_0 = k + \frac{q_0 - q_1}{3}, \quad \text{and} \quad p_1 = k + \frac{q_1 - q_0}{3}.$$

Note that if $1 - \alpha > 3k$, firm 0 will dominate the market. Therefore, the prices here are nonnegative. The indifference consumer is

$$x^* = \frac{1}{2} + \frac{q_0 - q_1}{6k}.$$

Again, if $1 - \alpha \leqslant 3k$, x^* is between $1/2$ and 1. And the indifference consumer gets

$$q_0 - p_0 - kx^* = \frac{q_0 + q_1 - 3k}{2}.$$

Therefore, the indifference consumer has nonnegative surplus if $1 + \alpha \geqslant 3k$. In this case, both firms split the market at x^*.

If $1 + \alpha < 3k$, the firms cannot split the market at the FOC prices. We consider the last case first, that is, the case where two firms act as monopolists. Firm 0 charges $q_0/2$ and serves consumers in $[0, \underline{x}]$, where $\underline{x} = q_0/2k$, while firm 1 charges $q_1/2$ and serves consumers in $[\overline{x}, 1]$, where $\overline{x} = 1-(q_1/2k)$. If both firms act as monopolists, we must have $\underline{x} < \overline{x}$, or $q_0+q_1 = 1+\alpha < 2k$.

Therefore, the case left is $2k \leqslant 1 + \alpha < 3k$. This is the case that two firms still split the market at some x^*, but this indifference consumer has zero surplus. In other words, firm 0's best response at Case 2(c), and similar for firm 1. We should note that there are many equilibria. Both firms are now restricted by the same constraint – the boundary consumer must have zero consumer surplus, which leads to multiplicity. It is possible to pick a natural equilibrium pricing profile, for $1 + \alpha = mk$, $m \in [2, 3)$,

$$p_0 = \frac{m-1}{m} \quad \text{and } p_1 = \frac{(m-1)\alpha}{m},$$

which take the uniform functional form of the equilibria at both boundaries, and the equilibrium pricing is also continuous. Still, as the condition assumed in the paper, this case is not used in the analysis.

We can summarize the results in Table 3.

TABLE 3 Stage Game Equilibria in $(1, \alpha)$

	p_0	p_1	Firm 0's profit	Firm 1's profit
$1 - \alpha > 3k$	$1 - \alpha - k$	0	$1 - \alpha - k$	0
$1 - \alpha \leqslant 3k$, $1 + \alpha \geqslant 3k$	$k + \dfrac{1-\alpha}{3}$	$k - \dfrac{1-\alpha}{3}$	$\dfrac{(3k+1-\alpha)^2}{18k}$	$\dfrac{(3k+\alpha-1)^2}{18k}$
$1 - \alpha \leqslant 3k$, $2k \leqslant 1 + \alpha < 3k$	$1 - \dfrac{k}{1+\alpha}$	$\alpha - \dfrac{k\alpha}{1+\alpha}$	$\dfrac{1+\alpha-k}{(1+\alpha)^2}$	$\dfrac{(1+\alpha-k)\alpha^2}{(1+\alpha)^2}$
$1 - \alpha \leqslant 3k$, $1 + \alpha < 2k$	$\dfrac{1}{2}$	$\dfrac{\alpha}{2}$	$\dfrac{1}{4k}$	$\dfrac{\alpha^2}{4k}$

B Proof of Results in Section 3

Proof of Proposition 1

If the social planner chooses not to upgrade at (L, L), the value function is

$$W_N(L, L) = \alpha\Delta + e^{-r\Delta} W_N(\alpha, \alpha).$$

This gives $W_N(L,L) = \alpha/r$ in the limit. If the social planner chooses to upgrade instead, the value function is

$$\begin{aligned} W_U(L,L) &= -c + e^{-\beta\Delta}\left[1dt + e^{-r\Delta}W_U(1,\alpha)\right] + \left(1 - e^{-\beta\Delta}\right)\left[\alpha dt + e^{-r\Delta}W_U(\alpha,\alpha)\right] \\ W_U(H,L) &= e^{-\beta\Delta}\left[1dt + e^{-r\Delta}W_U(1,\alpha)\right] + \left(1 - e^{-\beta\Delta}\right)\left[\alpha dt + e^{-r\Delta}W_U(\alpha,\alpha)\right] \end{aligned}$$

In the limit, this gives

$$W_U(L,L) = \frac{1}{r} - \frac{(\beta + r)c}{r}.$$

The result is obtained by comparing $W_N(L,L)$ and $W_U(L,L)$. ■

Proof of Lemma 2

For this and the next two proofs, we consider the more general $k > 0$ since such MPE also exist under $k > 0$. The results under $k = 0$ are obtained by setting $k = 0$ in the proofs.

Define

$$\delta = e^{-r\Delta}, \quad b = 1 - e^{-\beta\Delta}, \quad \text{and} \quad \xi = \frac{1 - e^{-r\Delta}}{r}.$$

In particular, ξ acts as Δ, only it has discount baked in.

Also, notice that firms always have no action at (H,H), so that the value function at (H,H) is always the same.

$$\begin{aligned} V_0(H,H) = (1-b)^2(\pi_0(H,H)\xi + \delta V_0(H,H)) &+ b(1-b)(\pi_0(H,L)\xi + \delta V_0(H,L)) \\ &+ b(1-b)(\pi_0(L,H)\xi + \delta V_0(L,H)) \\ &+ b^2(\pi_0(L,L)\xi + \delta V_0(L,L)). \end{aligned} \tag{2}$$

In case that firms never upgrades, the remaining value functions of Firm 0 are

$$\begin{aligned} V_0(H,L) &= (1-b)(\pi_0(H,L)\xi + \delta V_0(H,L)) + b(\pi_0(L,L)\xi + \delta V_0(L,L)), \\ V_0(L,H) &= (1-b)(\pi_0(L,H)\xi + \delta V_0(L,H)) + b(\pi_0(L,L)\xi + \delta V_0(L,L)), \\ V_0(L,L) &= \pi_0(L,L)\xi + \delta V_0(L,L). \end{aligned}$$

Since this is a symmetric profile, it is sufficient to consider firm 0 only. We omit the solutions of the value functions as they are usually too long (by including many higher order terms).

No profitable deviations at (L,H) means

$$V_0(L,H) - [-c + V_0(H,H)] \geqslant 0,$$

which is just the application of one-shot deviation principle. Substitute the value function solutions to the LHS, then take the limit to get

$$\frac{2\beta^2 c + cr^2 + 3\beta cr - \pi_0(H,H)(\beta + r) - \beta\pi_0(H,L) + \pi_0(L,H)(\beta + r) + \beta\pi_0(L,L)}{(\beta + r)(2\beta + r)},$$

so that we need

$$c(\beta + r) > \frac{(\pi_0(H,H) - \pi_0(L,H))(\beta + r)}{2\beta + r} + \frac{\beta(\pi_0(H,L) - \pi_0(L,L))}{2\beta + r}. \tag{3}$$

No profitable deviations at (L, L) means

$$V_0(L,L) - [-c + V_0(H,L)] \geqslant 0.$$

The procedure of simplication is the same, and we acquire the condition

$$c(r + \beta) > \pi_0(H,L) - \pi_0(L,L).$$

Notice that the RHS is larger than the RHS of (3), so that this is the condition needed. In particular, notice that this gives $\bar{c} = [\pi_0(H,L) - \pi_0(L,L)]/(r + \beta)$. It is also easy to verify that the value functions are always positive under this condition. ■

Proof of Proposition 3

Let x_h be the upgrade probability of Firm 1. Since Firm 0 is indifferent between upgrading or not at (L, H), we can first assume Firm 0 does not upgrade at (L, H) to solve the value function, then use the indifference condition to solve for x_h.

Recall that $V_0(H, H)$ is given in (2). The value remaining functions are

$$\begin{aligned}
V_0(H,L) &= x_h V_0(H,H) + (1 - x_h)\left[(1-b)(\pi_0(H,L)\xi + \delta V_0(H,L)) + b(\pi_0(L,L)\xi + \delta V_0(L,L))\right] \\
V_0(L,H) &= (1-b)(\pi_0(L,H)\xi + \delta V_0(L,H)) + b(\pi_0(L,L)\xi + \delta V_0(L,L)), \\
V_0(L,L) &= -c + V_0(H,H).
\end{aligned}$$

And the indifference condition is

$$V_0(L,H) = -c + V_0(H,H).$$

This gives the expression of x_h. (The exact expression contains many terms due to the discrete-time setup and is omitted.) Substitute this back to the value functions, we get the expression of the value functions under the equilibrium. In the limit, $x_h \to 0$, so that it sufficient to check that the limit of x_h/t,

$$\frac{-2\beta^2 c - cr^2 - 3\beta cr + \pi_0(H,H)(\beta + r) + \beta\pi_0(H,L) - \pi_0(L,H)(2\beta + r)}{c(\beta + r) - \pi_0(H,H) + \pi_0(L,H)}$$

is positive to ensure x_h is a well-defined probability when Δ is sufficiently small. This is one of the conditions to establish MPE. The other two conditions are no-profitable deviation condition, but they are most trivial here. Notice that the incentives of upgrading at (L, L) and (L, H) are the same, which reduces the two conditions to just one and is guaranteed by the indifference condition. The same upgrading incentive does not rely on $k = 0$. To see this:

- At (L, L), if Firm 0 upgrades, the state before nature's decision is (H, H). If Firm 0 does not upgrade, Firm 1 upgrades, and the state before nature's decision is (L, H).
- At (L, H), if Firm 0 upgrades, the state before nature's decision is (H, H). If Firm 0 does not upgrade, Firm 1 does not have any action, and the state before nature's decision is (L, H).

Since the interim states before nature's shock(s) are the same, the distributions of states after nature's decision are also the same, meaning that the stage payoffs and continuation states are the same, which completely aligns upgrading incentives.

Therefore, it suffices to consider that x_h/t has a positive limit, which requires

$$\pi_0(H, H) - \pi_0(L, H) < c(\beta + r) < \frac{\pi_0(H, H)(\beta + r)}{2\beta + r} + \frac{\beta\pi_0(H, L)}{2\beta + r} - \pi_0(L, H). \quad \blacksquare$$

Proof of Proposition 4

The idea is similar to the proof of Proposition 3: start by assuming that Firm 1 mixes at (L, L) with probability x_l. Formulate the value functions under the assumption that Firm 0 does not upgrade. The value functions are

$$\begin{aligned}
V_0(H, L) &= (1 - b)(\pi_0(H, L)\xi + \delta V_0(H, L)) + b(\pi_0(L, L)\xi + \delta V_0(L, L)), \\
V_0(L, H) &= (1 - b)(\pi_0(L, H)\xi + \delta V_0(L, H)) + b(\pi_0(L, L)\xi + \delta V_0(L, L)), \\
V_0(L, L) &= x_l V_0(L, H) + (1 - x_l)(\pi_0(L, L)\xi + \delta V_0(L, L)).
\end{aligned}$$

The indifference condition is

$$V_0(L, L) = -c + x_l V_0(H, H) + (1 - x_l)V_0(H, L).$$

This gives x_l.[25] After taking the limit,

$$x_l = \frac{(2\beta + r)(-c(\beta + r) + \pi_0(H, L) - \pi_0(L, H))}{(\pi_0(H, L) - \pi_0(H, H))(\beta + r)}.$$

Observe that x_l does not converge to 0, so that the strategy is a probabilistic mixing instead of rate mixing. This also means the limit must be between 0 and 1. This gives

$$\frac{\pi_0(H, H)(\beta + r)}{2\beta + r} + \frac{\beta\pi_0(H, L)}{2\beta + r} - \pi_0(L, H) < c(\beta + r) < \pi_0(H, L) - \pi_0(L, H).$$

There is no profitable deviation at (L, L) due to the indifference condition. At (L, H), the no profitable deviation is that

$$V_0(L, H) - (-c + V_0(H, H)) \geqslant 0.$$

[25] Technically, this is a quadratic equation of x_l and has two solutions. Nevertheless, the other solution is 0.

After taking the limit, this simplifies to

$$\frac{\pi_0(H,H)(\beta+r)}{2\beta+r}+\frac{\beta\pi_0(H,L)}{2\beta+r}-\pi_0(L,H)<c(\beta+r).$$

Therefore, the profile is an MPE in the limit if

$$\frac{\pi_0(H,H)(\beta+r)}{2\beta+r}+\frac{\beta\pi_0(H,L)}{2\beta+r}-\pi_0(L,H)<c(\beta+r)<\pi_0(H,L)-\pi_0(L,H). \quad \blacksquare$$

Proof of Theorem 5

This is shown by exhausting all possibilities of symmetric MPE. Here, we discuss the pure strategy symmetric profiles only and leaving the mixed strategy symmetric profiles to online appendix so that the proof is not overly long.

As we have discussed after Theorem 5, there is no always-upgrading MPE. Next, consider that both firms upgrade at (L,L) but not at $(H,L)/(L,H)$. It can be verified that the limits of the value functions of Firm 0 are

$$\begin{aligned}
V_0(H,H) &= -\frac{\beta(\alpha+2\beta c-1)}{r(3\beta+r)},\\
V_0(H,L) &= -\frac{\alpha\beta^2+3\alpha\beta r+\alpha r^2+2\beta^3c+3\beta^2cr-\beta^2+\beta cr^2-3\beta r-r^2}{r(\beta+r)(3\beta+r)},\\
V_0(L,H) &= -\frac{\alpha\beta^2+2\beta^3c+3\beta^2cr-\beta^2+\beta cr^2}{r(\beta+r)(3\beta+r)},\\
V_0(L,L) &= -\frac{\alpha\beta+2\beta^2c+3\beta cr-\beta+cr^2}{r(3\beta+r)}.
\end{aligned}$$

We can then verify that the two no-profitable-deviation conditions result to the direct opposite signs. The only c that maintain this MPE is

$$c=\frac{\beta-\alpha\beta}{(\beta+r)(2\beta+r)},$$

but this is the cost level at the boundary of the two MPE in Proposition 3 and Proposition 4, and this is exactly the limiting MPE at the boundary.

We have also discussed the upgrading deterrence MPE after Theorem 5, which also only exists at $c=0$. The final pure symmetric profile is always upgrading, which is covered in Lemma 2. $\blacksquare$

Proof of Proposition 7

We have shown the calculation of joint profits under the MPE in Proposition 3 before Proposition 7. Substitute flow payoffs and mixing rates into (1), the joint profit is

$$\frac{2\beta cr(-\alpha+c(\beta+r)+1)}{-\alpha\beta+\beta+c\left(\beta^2-r^2\right)}.$$

We now consider the MPE in Proposition 4. Let

$$y(c) = \frac{g(c)(1-g(c))}{2g(c)(1-g(c)) + g^2(c)}$$

be the probability that the realized outcome is a quality differentiated state. The state transition must satisfy

$$\beta\tau_B + \beta y(c)\tau_{LH} = [\beta(1-2y(c)) + \beta y(c)]\,\tau_{HL}.$$

And the expression regarding τ_{LH} is symmetric. This gives

$$\tau_B = \frac{1-2y(c)}{3-2y(c)}, \quad \text{and} \quad \tau_{HL} = \tau_{LH} = \frac{1}{3-2y(c)}.$$

The upgrading cost is paid at rate

$$\beta(\tau_{HL} + \tau_{LH})\,[y(c) + y(c) + 2(1-2y(c))]\,.$$

Together, we get the joint profit

$$\frac{2(1-\alpha)r(\alpha - c(\beta+r) - 1)}{2\beta(\alpha-\beta c-1) + 3r(\alpha-\beta c-1) - cr^2}.$$

The remaining part is to verify that both joint profits are increasing in c and is continuous at the boundary, which are just algebra and omitted here. ■

C Proof of Results in Section 4

Proof of Proposition 8

We show the case that $0 < \alpha < k/2$. For the case that α is larger, the method is the same. The only change is the stage social surplus at (L, L) and (L, H), as shown in Appendix A.

Consider the never-upgrading policy. The social planner's value function is formulated in the similar fashion. For example,

$$\begin{aligned} W(H,H) = (1-b)^2\,[w(H,H)\xi + \delta W(H,H)] + 2b(1-b)\,[w(H,L)\xi + \delta W(H,L)] \\ + b^2\,[w(L,L)\xi + \delta W(L,L)]\,. \end{aligned}$$

At (L, L), no profitable deviation to (H, L) requires

$$W(L,L) - (-c + W(H,L)) \geqslant 0,$$

which, in the limit, simplifies to

$$c > \frac{-2\alpha^2 - k^2 + 2k}{2k(\beta+r)}.$$

Similarly, no profitable deviation to (H,H) requires, in the limit,

$$c > \frac{-4\alpha^2(3\beta+r)+\beta k(12-5k)-(k-4)kr}{8k\left(2\beta^2+r^2+3\beta r\right)}$$

At (H,L), no profitable deviation to (H,H) requires, in the limit,

$$c > \frac{-4\alpha^2\beta+k^2(r-\beta)+4\beta k}{4k(\beta+r)(2\beta+r)}.$$

All three conditions must be satisfied, and the RHS of the three conditions need to be compared. It turns out the the RHS of the first condition is the largest, which means it is the condition needed.

The procedure for the other policies are similar. Here, we state the limit of the value functions under the other two optimal policies. For the policy that upgrades one product at (L,L) and does not upgrade at (H,L), the value functions $W(H,H), W(H,L)$, and $W(L,L)$ in the limits are

$$-\frac{4\beta(2\beta c+k-2)+(k-4)r}{4r(2\beta+r)}, \quad -\frac{2\beta c+k-2}{2r}, \quad \text{and} \quad -\frac{2c(\beta+r)+k-2}{2r}.$$

For the always upgrading policy, the same value functions, in the limits, are

$$-\frac{8\beta c+k-4}{4r}, \quad -\frac{4c(2\beta+r)+k-4}{4r}, \quad \text{and} \quad -\frac{8c(\beta+r)+k-4}{4r}.$$

For each policy, the cost range is determine by three no-profitable-deviation conditions at two states, just as the argument above. We also checked the policies that are not consistent, as mentioned after Proposition 8, and they are indeed never optimal. ■

Proof of Theorem 9

The expressions of the time distributions and rate of upgrading costs are shown in the proof of Proposition 7. We here state the time distributions and long-run joint profits directly (using the values under $k>0$). In the MPE where both firms upgrade at (L,L) and mixing at the quality-follower state,

$$\tau_B = \frac{k(r-2\beta)-2\left((\alpha-1)\beta+c(\beta+r)^2\right)}{-2\beta(\alpha+2k-1)+2\beta^2c+r(k-2cr)}.$$

And the long-run joint profit is

$$\frac{r\left(-4\beta c(-\alpha+c(\beta+r)+1)+2ck(5\beta+r)-k^2\right)}{2\beta(\alpha+2k-1)-2\beta^2c+r(2cr-k)}.$$

In the MPE that both firm probabilistically mixing at (L,L) and do not upgrade at the quality-follower states,

$$\tau_B = -\frac{(2\beta+r)(a+c(\beta+r)+k-1)}{-(a+k-1)(2\beta+3r)+c(2\beta+r)(\beta+r)-2k(\beta+r)}$$

And the joint profits are

$$-\frac{r(2\alpha+3k-2)(a-c(\beta+r)+k-1)}{2\beta(\alpha-\beta c+2k-1)+r(3\alpha-3\beta c+5k-3)-cr^2}.$$

What remains is taking derivatives. It turns out the same factor, $1-\alpha-3c(r+\beta)$, determines the sign of the derivatives $\partial\tau_B/\partial k$, which is a strictly decreasing function of c, positive at the lower bound of the MPE c_3, and negative at the upper bound $\bar{c}$. More specifically, at lower bound c_3, $(r+\beta)c_3=\pi_i(H,H)-\pi_i(L,H)=k/2$, and the factor is

$$1-\alpha-\frac{3k}{2}\geqslant 3k-\frac{3k}{2}>0,$$

where the second inequality follows from the assumption that $\alpha\leqslant 1-3k$. At the upper bound, $(r+\beta)\bar{c}=\pi_i(H,L)-\pi_i(L,L)$. Depending on α, this is

$$\begin{cases} -2+2\alpha+\frac{9k}{2}, & \text{if } \frac{3k}{2}\leqslant\alpha\leqslant 1-3k,\\ -2+\frac{7\alpha}{2}+\frac{9k}{4}, & \text{if } k\leqslant\alpha<\frac{3k}{2},\\ -2+2\alpha+\frac{3\alpha^2}{4k}+3k, & \text{if } 0<\alpha<k. \end{cases}$$

In the first case, since $-2+2\alpha\leqslant -6k$, it is negative. In the second case, notice that it is increasing in α and k, so that the maximum is obtained at $\alpha=k$ and $k=2/9$, which is $-1/3<0$. At the last case, first observe that it is increasing in α, so that it is less than $-2+23k/4$, by setting $\alpha=k$. Then, this is increasing in k, and the maximum is obtained at $k=2/9$, which gives $-13/18<0$. The existence of $\hat{c}$ follows from the intermediate value theorem, and the uniqueness of $\hat{c}$ follows from the fact that $1-\alpha-3(r+\beta)c$ is strictly decreasing in c. ■

Proof of Proposition 11

We show the simpler first MPE here, and leaving the second mixed strategy deterrence to the online appendix. The value functions are

$$V_0(H,L)=V_0(H,H),\quad V_0(L,H)=-c+V_0(H,H),\quad \text{and}\quad V_0(L,L)=\pi_0(L,L)\xi+\delta V_0(L,L).$$

The no-profitable-deviation condition at (L,L) simplifies to $c>(\pi_0(H,H)-\pi_0(L,L))/(r+\beta)$, which is just $c>c_1$. The no-profitable-deviation condition at (L,H) involves higher order terms (as the first order terms converge to exactly 0)[26] and simplifies to

$$c(\beta+r)<\pi_0(H,H)-\frac{\pi_0(L,H)r}{\beta+r}-\frac{\beta\pi_0(L,L)}{\beta+r}.$$

[26]Checking higher order terms when the first order terms converge to 0 is necessary: this guarantees that the equilibrium in consideration is indeed supported in a discrete time setup, namely, when Δ is small but strictly positive.

Notice that the RHS is $(r+\beta)c_2$.

In the mixed-strategy deterrence MPE, the procedure still involves first solving the value function (by assuming that Firm 0 does not upgrade), and then use the indifference condition of Firm 0 to solve the mixing probability of Firm 1 in the equilibrium. This is similar to the procedure in the proof at Appendix B. It is then shown that the probability converges to a rate in the limit, and one need to check (i) the rate is weakly positive, and (ii) there is no profitable deviation at (L,L) (state (L,H) is covered by the indifference condition). ■

Discussions of Symmetric MPE

The remaining symmetric MPE mentioned at the end of subsubsection 4.2.2 is the following:

- At (L,L), both firms upgrade with probability

$$\frac{\pi_0(H,L)-\pi_0(H,H)-\pi_0(L,H)+\pi_0(L,L)}{2(\pi_0(H,L)-\pi_0(H,H))} - \frac{\sqrt{\begin{Bmatrix} 4(\pi_0(H,L)-\pi_0(H,H))(-\beta c - cr + \pi_0(H,H)-\pi_0(L,L)) \\ +(-\pi_0(H,H)+\pi_0(H,L)-\pi_0(L,H)+\pi_0(L,L))^2 \end{Bmatrix}}}{2(\pi_0(H,L)-\pi_0(H,H))}$$

- At $(L,H)/(H,L)$, the corresponding firm upgrades for sure.

This is an MPE if $\pi_0(H,H)-\pi_0(L,L)\leqslant c(r+\beta)<\pi_0(H,H)-\pi_0(L,H)$. Observe that this range is fully covered by the alway-upgrading MPE. Also, this MPE is outcome equivalent to the always-upgrading MPE, given that the upgrade is probabilistic. For this reason, it suffices to consider the always-upgrading MPE.

We also need to note that the joint profits in joint-profit-maximizing MPE selection standard are not the same as the long-run average joint profits. In the previous case, it can be considered as a firm's decision problem. As an equilibrium refinement, it considers the possible choice of the firms, and firms make decisions based on the value functions. The joint-profit maximization standard there means that state by state, the sum of the two firms value functions is the largest among all such sums. Importantly, firms discount the future under this refinement. In this sense, the front-load upgrading costs at lower-quality states can have a significant impact on the firms' value functions. It then can be verified that when there is a multiplicity, firms always prefer the equilibria with fewer upgrades.

In the latter case, the term "long-run average joint profits" is a welfare consideration. It is considered from the planner's perspective when the initial state does not matter any longer. It utilize the *Abel-Cesaro equivalence* so that it seems that there is no discount. In particular, the upgrading costs are not front loaded.

We now consider firms' standard (i.e., value function) to resolve multiplicity. Multiplicity exists under three regions: two deterrence region, and the higher-cost no upgrade region. We here consider the first deterrence region, where the two MPE in considerations are: (i) Always upgrading, and (ii) Pure-punishment upgrading deterrence in Proposition 11. We will use c_1, which indicates that in this multiplicity region, $(r+\beta)c \geqslant \pi_0(H,H) - \pi_0(L,L)$.

Notice that the two MPE coincide as long as the initial state is not (L,L), which suggests that it suffices to consider the difference of $V_0(L,L)$. Below, we consider

$$V_0^D(L,L) - V_0^A(L,L), \tag{4}$$

where the superscript D means "deterrence" and the superscript A means "always upgrading". The difference is evaluated at $\Delta \to 0$. Since $\pi_0(L,L)$ depends on α, we need to consider three cases:

- $3k/2 < \alpha < 1-3k$. The difference in (4) is $2c(r+\beta)/r$. This is always positive.
- $k < \alpha < 3k/2$. The difference in (4) is

  $$\frac{2\alpha - 3k + 4c(r+\beta)}{2r},$$

  which is increasing in c. Substitute the lower bound $(r+\beta)c \geqslant \pi_0(H,H) - \pi_0(L,L)$, and since $\pi_0(L,L) = (\alpha/2) - (k/4)$ in this case, the difference is exactly 0 at the lower bound of c and become positive in the multiplicity region.
- $k < \alpha < 3k/2$. The difference in (4) is

  $$\frac{\alpha^2 - 2k(k - 2c(\beta + r))}{2kr}$$

  which is again increasing in c. Given that $\pi_0(L,L) = \alpha^2/4k$ in this case, the difference is again exactly 0 at the lower bound of c and become positive in the multiplicity region.

The comparisons for the remaining two regions are similar and mostly algebraic, which we omit here.

Proof of Corollary 12

For the ease of notations, instead of considering c directly, we consider $c(r+\beta)$ instead. We start with the upper boundaries. The firms stop upgrading at $\pi_0(H,L) - \pi_0(L,L)$. To avoid too many case-by-case discussions involved at $\pi_0(L,L)$, we consider a large $\pi_0(H,L) - \pi_0(L,H) = 1 - \alpha - k$, and show this is still lower than the cost where the social planner stops upgrading. If $\alpha < k/2$, the social planner's boundary is

$$\frac{-2\alpha^2 - k^2 + 2k}{2k}$$

and the difference between the social planner's boundary and the firms' boundary is

$$\alpha - \frac{\alpha^2}{k} + \frac{k}{2},$$

which is positive under the given range. When $\alpha > k/2$, the social planner's boundary is $1-\alpha-k/4$, which is obviously higher.

For the lower bound, notice that the social planner starts to upgrade at $k/4$, while the firms' threshold is $(r+\beta)c_3 = k/2$. ■

D Discussions of Asymmetric MPE

There are also asymmetric MPE in this game, which are summarized in section 5. In this section, I first give the specific mixing probabilities or rates and then discuss the details of the substitution and complementary relations presented in section 5.

Mixing behaviors in asymmetric MPE. In Case A, for the firm who upgrades at (L, L) with a rate, the rate is

$$\frac{\beta^2 c + cr^2 + 2\beta cr - \pi_0(H,H)(\beta+r) + \pi_0(L,H)r + \beta\pi_0(L,L)}{-c(\beta+r) + \pi_0(H,H) - \pi_0(L,H)}.$$

For the firm who upgrades at the quality-follower state with a rate, the rate is

$$\frac{(\beta+r)((\pi_0(H,L) - \pi_0(L,L)) - c(\beta+r))}{c(\beta+r) - (\pi_0(H,H) - \pi_0(L,L))}.$$

In Case B, let z_{lp} be a constant between 0 and 1. an MPE as described in Case B exists if

$$\pi_0(H,H) - \pi_0(L,H) < c(\beta+r) < \frac{\pi_0(H,H)(\beta+r)}{\beta+r+\beta z_{lp}} + \frac{\pi_0(H,L)(\beta z_{lp})}{\beta+r+\beta z_{lp}} - \pi_0(L,H).$$

For the firm upgrades at (L, L) for sure, the mixing rate at the quality-follower state is

$$\frac{-2\beta^2 c - cr^2 - 3\beta cr + \pi_0(H,H)(\beta+r) + \beta\pi_0(H,L) - \pi_0(L,H)(2\beta+r)}{c(\beta+r) - \pi_0(H,H) + \pi_0(L,H)}.$$

For the firm who plays a mixed strategy at (L, L), the mixing probability is z_{lp}. The rate of upgrading at the quality follower state is

$$\frac{\beta^2 c + cr^2 + 2\beta cr + \beta cr z_{lp} + \beta^2 c z_{lp} - \pi_0(H,H)(\beta+r) - \beta\pi_0(H,L)z_{lp} + \pi_0(L,H)(\beta+r+\beta z_{lp})}{-c(\beta+r) + \pi_0(H,H) - \pi_0(L,H)}.$$

In this case, one firm plays the exact same strategy as in the symmetric MPE, leaving the opponent firm indifferent between upgrading or not at (L, L). In the symmetric MPE, the opponent firm has the strongest inclination $z_{lp} = 1$ to upgrade, but in general an MPE can exist for the opponent with any upgrading inclination.

In Case C, firms play pure strategies, so that the mixing behaviors are irrelevant.

Substitution and complementary relations. For Case A, in the case that $0 < \alpha < k$ and $k \to 0$, the sign of $\partial\tau_B/\partial k$ is determined by

$$-\frac{4\beta(\beta+r)(\beta c-(1-\alpha))}{\alpha^2 r^2}.$$

Since $r > 0$, $(r+\beta)c < 1-\alpha = \pi_0(H, L)$, this is positive. Alternatively, pick any $k \in (0, 2/9)$, and let $\alpha \to 0$, the sign of the same derivative is determined by

$$-8k^4(\beta+r)^3(3c(\beta+r)-1)(k-2c(\beta+r))^2,$$

and this is again positive since $3c(\beta+r) < 3k/2 < 1$, given the range of c under Case A. Therefore, under Case A, for small enough α, or small enough k (and $\alpha < k$), the two dimensions of product differentiation exhibits substitution relations.[27]

For Case B, pick the simple case such that the firm mixes at (L, L) will not choose to upgrade at (L, H). Let this firm be firm 0, without loss of generality. The mixing probability z_{lp} is now

$$z_l \equiv \frac{r+\beta}{\beta}\frac{c(r+\beta)-(\pi_0(H,H)-\pi_0(L,H))}{\pi_0(H,L)-\pi_0(L,H)-c(r+\beta)}.$$

Firm 1's strategy is the same as in the symmetric direct-competition MPE. τ_B is defined in the same way: the time spent at (H, H) in the equilibrium. It turns out that for $c(r+\beta)$ close to $k/2$, the lower bound, $\partial\tau_B/\partial k < 0$.[28]

To see why the sign is opposite to the sign under the symmetric MPE, observe that $\partial z_l/\partial k < 0$. That is, while firm 1 still upgrades more as k increases, firm 0 now upgrade less as k increases. This is because firm 0 is the disadvantageous player who is dominated in competition. As firm 1 becomes more aggresive given the higher k, firm 0 is closer to a chicken player.

E Proof of Results in Section 6

Proof of Proposition 13

The proof uses the same idea: solving the value functions and check the one-shot deviations. The value function $V(H, H)$ is different though because of correlations. In particular, one needs to use the probabilities in Table 1 to replace the probabilities under independence.

This procedure gives the same c_1 and c_3, and

$$(r+\beta)c_2 = \pi_0(H,H) - \frac{\pi_0(L,H)(\beta\rho+r)}{\beta+r} - \frac{\pi_0(L,L)(\beta(1-\rho))}{\beta+r}.$$

[27]The general expressions of $\partial\tau_B/\partial k$ are rather complicated, and it is difficult to determine the exact sign.

[28]And for larger c, the sign in general depends on the relative size of r and β.

Then it is easy to see that c_2 is increasing in ρ and becomes c_3 when $\rho = 1$, since $(r+\beta)c_3 = \pi_0(H,H) - \pi_0(L,H)$. $h(c)$ is a bit longer, but the derivative $\partial h/\partial\rho$ has the following relatively simple form

$$\frac{\beta(\beta+r)^2(c(\beta+r) - \pi_0(H,H) + \pi_0(L,H))(c(\beta+r) - \pi_0(H,L) + \pi_0(L,H))}{\left(c(\beta+r)^2 - \pi_0(H,H)(\beta+r) + \pi_0(L,H)(\beta\rho+r) - \beta\pi_0(L,L)\rho + \beta\pi_0(L,L)\right)^2}$$

The denominator is clearly positive. The two factors containing $(r+\beta)c$ are both negative since $c \leqslant c_3$. It follows that the derivative is positive. ■

Proof of Proposition 14

The proof starts with the same idea again by solving the value functions and check the indifference condition, just as the case where $\rho = 0$. This gives the same c_3 and $\bar{c}$, and

$$(r+\beta)\tilde{c} = \frac{\pi_0(H,H)(\beta+r)}{-\beta\rho+2\beta+r} + \frac{\pi_0(H,L)(\beta-\beta\rho)}{-\beta\rho+2\beta+r} - \pi_0(L,H).$$

It is easy to see that $\tilde{c}$ is decreasing in ρ and becomes c_3 when $\rho = 1$.

We also have

$$\frac{\partial f}{\partial\rho} = \frac{\beta(c(\beta+r) - \pi_0(H,L) + \pi_0(L,H))}{c(\beta+r) - \pi_0(H,H) + \pi_0(L,H)}.$$

The denominator is positive since $c > c_3$, and the numerator is negative since $c < \bar{c}$, so that the derivative is negative.

$$\frac{\partial g}{\partial\rho} = \frac{\beta(c(\beta+r) - \pi_0(H,L) + \pi_0(L,H))}{(\pi_0(H,L) - \pi_0(H,H))(\beta+r)}$$

The denominator is positive, and the numerator is negative since $c < \bar{c}$, so that the derivative is negative. ■