Topics In Equivariant Cohomology

Or: How I Learned To Stop Worrying And Love Group Actions

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Major Review

27/02/15

Outline

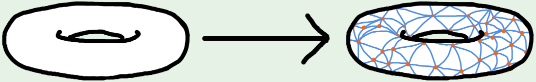
- Introduction
- Basic de Rham Theory
- Equivariant cohomology
- Simplicial methods
- An important example

Algebraic Topology

Algebraic topology is a branch of mathematics that studies spaces by assigning algebraic *invariants* to them.

Example: Euler characteristic of a surface

Take a surface, S, embedded in $\mathbb{R}^3$ and triangulate it. Let v be the number of vertices, e be the number of edges and f be the number of faces of the triangulation. Then $\chi(S) = v - e + f$ is defined to be the Euler characteristic of S.



Cohomology is a *more refined* algebraic invariant than the Euler characteristic.

What is cohomology?

For those without any background in algebraic topology, a cohomology theory can be thought of as assigning a sequence of algebraic objects to a topological space.

$$\text{Topological Spaces} \xrightarrow{H^*} \text{Algebraic Objects}$$

Sometimes it is possible to prove results about topological spaces by reducing it to an algebraic problem.

Example: Singular Cohomology

One example we can consider is *singular cohomology* (with real coefficients).

$$\begin{array}{ccc} \text{Topological Spaces} & \xrightarrow{H^*} & \text{Vector Spaces} \\ & & \\ X & \longmapsto & H^0(X),\ H^1(X), \cdots \end{array}$$

Singular cohomology is more refined than the Euler Characteristic.

$$\chi(X) = dim(H^0(X)) - dim(H^1(X)) + dim(H^2(X)) - \cdots$$

Basic cohomology theory

Let $C = \{C_i\}_{i \in \mathbb{Z}}$ be a sequence of vector spaces (or more generally, *modules*) and $\delta_i : C_i \to C_{i+1}$ be maps such that $\delta_{i+1} \circ \delta_i = 0$. We call this a *cochain complex*.

$$\begin{tikzcd} \cdots \arrow[r] & C_{i-1} \arrow[r, "\delta_{i-1}"] & C_i \arrow[r, "\delta_i"] \arrow[rr, bend left, "0"] & C_{i+1} \arrow[r, "\delta_{i+1}"] & C_{i+2} \arrow[r, "\delta_{i+2}"] & C_{i+3} \arrow[r] & \cdots \end{tikzcd}$$

Note

Saying $\delta_{i+1} \circ \delta_i = 0$ is equivalent to saying that $\mathrm{im}(\delta_i) \subset \ker(\delta_{i+1})$. We say the complex is *exact* if $\mathrm{im}(\delta_i) = \ker(\delta_{i+1})$ for every $i \in \mathbb{Z}$.

Define the q^{th} *cohomology group* of the complex C to be

$$H^q(C) = \frac{\ker(\delta_q : C_q \to C_{q+1})}{\mathrm{im}(\delta_{q-1} : C_{q-1} \to C_q)}.$$

Basic cohomology theory

Sometimes these cochain complexes can be rather abstract

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "\times 0"] \arrow[rr, bend left, "0"] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "\times 0"] & \mathbb{Z} \arrow[r] & \cdots \end{tikzcd}$$

or things we've seen before like *grad*, *curl* and *div*.

$$\begin{tikzcd} \left\{ \begin{matrix} \text{functions} \\ \text{from } \mathbb{R}^3 \text{ to } \mathbb{R} \end{matrix} \right\} \arrow[r, "\nabla"] \arrow[rr, bend left, "\nabla\times \circ \nabla = 0"] & \left\{ \begin{matrix} \text{vector fields} \\ \text{on } \mathbb{R}^3 \end{matrix} \right\} \arrow[r, "\nabla\times"] \arrow[rr, bend right, "\nabla\cdot \circ \nabla\times = 0"'] & \left\{ \begin{matrix} \text{vector fields} \\ \text{on } \mathbb{R}^3 \end{matrix} \right\} \arrow[r, "\nabla\cdot"] & \left\{ \begin{matrix} \text{functions} \\ \text{from } \mathbb{R}^3 \text{ to } \mathbb{R} \end{matrix} \right\} \end{tikzcd}$$

Example: Multivariable Calculus

We'll call this complex V.

$$\underbrace{\left\{\begin{array}{c}\text{functions}\\ \text{from } \mathbb{R}^3 \text{ to } \mathbb{R}\end{array}\right\}}_{V_0} \xrightarrow{\nabla} \underbrace{\left\{\begin{array}{c}\text{vector fields}\\ \text{on } \mathbb{R}^3\end{array}\right\}}_{V_1} \xrightarrow{\nabla\times} \underbrace{\left\{\begin{array}{c}\text{vector fields}\\ \text{on } \mathbb{R}^3\end{array}\right\}}_{V_2} \xrightarrow{\nabla\cdot} \underbrace{\left\{\begin{array}{c}\text{functions}\\ \text{from } \mathbb{R}^3 \text{ to } \mathbb{R}\end{array}\right\}}_{V_3}$$

Example

Given a vector field $\mathbf{v}$ on $\mathbb{R}^3$ such that the curl vanishes

$$\nabla \times \mathbf{v} = 0$$

we can solve for a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\nabla \phi = \mathbf{v}.$$

This is the same as saying $H^1(V) = 0$.

The de Rham Complex

We can find a generalisation of grad, div and curl for subsets of $\mathbb{R}^n$. Firstly, let $U \subset \mathbb{R}^n$ and f be a *smooth* function

$$f : U \to \mathbb{R}.$$

We will call this a 0-*form* on U and the collection of all 0-forms will be denoted $\mathbf{\Omega}^0(U)$. We define the *exterior derivative*, d, of a 0-form as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i.$$

More generally, a 1-form will just be anything that is of the form

$$\omega = \sum_{i=1}^{n} f_i\, dx_i \in \mathbf{\Omega}^1(U).$$

The de Rham Complex

In general, a k-form is a formal sum of anything of the form

$$\omega = f\, dx_{i_1} dx_{i_2} \cdots dx_{i_k} \in \mathbf{\Omega}^k(U),\ f : U \to \mathbb{R}$$

on which we define the exterior derivative $d : \mathbf{\Omega}^k(U) \to \mathbf{\Omega}^{k+1}(U)$

$$d\omega = df \cdot dx_{i_1} dx_{i_2} \cdots dx_{i_k}.$$

Notation

Let ω be a form defined on $U \subset \mathbb{R}^n$.

- If $d\omega = 0$ then we call ω *closed*.
- If $\omega = d\phi$, then we call ω *exact*.

de Rham Cohomology

If we impose that

$$dx_i\, dx_j = -dx_j\, dx_i$$

it follows that $d^2\omega = d \circ d(\omega) = 0$ for all $\omega \in \mathbf{\Omega}^n(U)$ and all $n \geq 0$. This means we can form a cochain complex which we denote $\mathbf{\Omega}^*(U)$.

$$\cdots \longrightarrow \mathbf{\Omega}^{k-1}(U) \xrightarrow{d} \mathbf{\Omega}^k(U) \xrightarrow{d} \mathbf{\Omega}^{k+1}(U) \xrightarrow{d} \mathbf{\Omega}^{k+2}(U) \longrightarrow \cdots$$

Since we have a cochain complex we can define *the de Rham cohomology of* $U \subset \mathbb{R}^n$.

$$\begin{aligned} H^q_{dR}(U) &= \frac{\ker(d : \mathbf{\Omega}^q(U) \to \mathbf{\Omega}^{q+1}(U))}{\mathrm{im}(d : \mathbf{\Omega}^{q-1}(U) \to \mathbf{\Omega}^q(U))} \\ &= \frac{\{\text{closed } q\text{-forms}\}}{\{\text{exact } q\text{-forms }\}} \end{aligned}$$

de Rham Cohomology

Let α be an n-form on an open subset $U \subset \mathbb{R}^n$.

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Question:

If $d\alpha = 0$ can we solve for β?

This is precisely what cohomology is measuring. If there is a form that satisfies

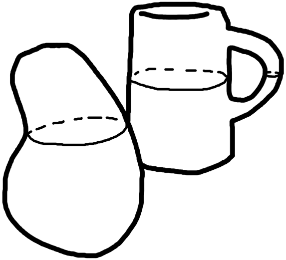
$$d\alpha = 0$$

and we can't solve $\alpha = d\beta$ then it will have a non-zero equivalence class in

$$H^n_{dR}(U) = \frac{\{\text{closed } n\text{-forms}\}}{\{\text{exact } n\text{-forms }\}}.$$

Manifolds

Often we're interested in studying *manifolds*.



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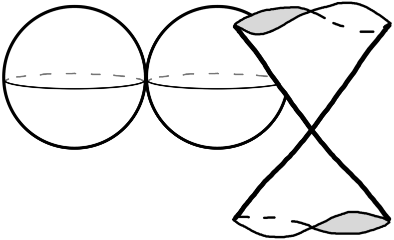

Not Manifolds

Because a manifold M locally *looks like* $\mathbb{R}^n$ we can define a similar complex which we denote $\mathbf{\Omega}^*(M)$.

$$\cdots \longrightarrow \mathbf{\Omega}^{k-1}(M) \xrightarrow{d} \mathbf{\Omega}^{k}(M) \xrightarrow{d} \mathbf{\Omega}^{k+1}(M) \xrightarrow{d} \mathbf{\Omega}^{k+2}(M) \longrightarrow \cdots$$

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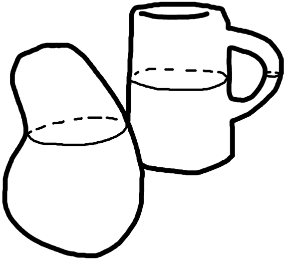

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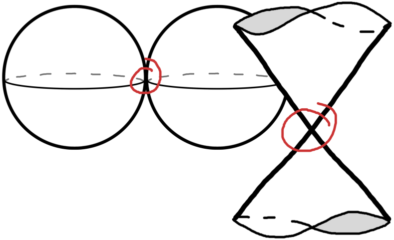

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de Rham's Theorem

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From the de Rham complex of a manifold M, $\mathbf{\Omega}^*(M)$, we can also define the cohomology groups analogously

$$H^q_{dR}(M) = \frac{\ker(d : \mathbf{\Omega}^q(M) \to \mathbf{\Omega}^{q+1}(M))}{\operatorname{im}(d : \mathbf{\Omega}^{q-1}(M) \to \mathbf{\Omega}^q(M))}$$

for which we have the classical theorem of de Rham's.

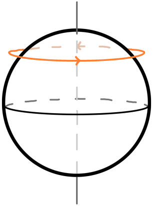
de Rham's Theorem

Let M be a smooth manifold. Then there is an isomorphism

$$H^*_{dR}(M) \cong H^*(M).$$

Group action on a manifold

Sometimes we're interested in studying a group G *acting* on a manifold M.



Question:

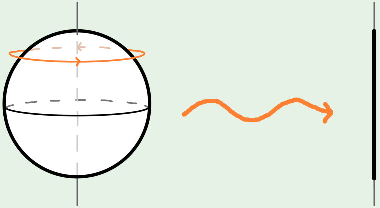
Can we define a cohomology theory for M that reflects the geometry of M *and* the action of the group G?

Let M/G denote the orbit space – that is, impose the relation $x \sim y \iff x = y \cdot g$ for some $g \in G$. What does the cohomology of M/G tell us?

Naive approach

Answer: Not much.

Calculating $H^*(S^2/S^1)$



The quotient of the sphere by the circle is isomorphic to the interval $[-1, 1]$ which is homotopy equivalent to a point.

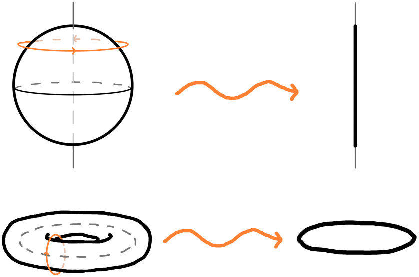
$$S^2/S^1 \simeq [-1, 1] \simeq \{\mathrm{pt}\}.$$

Thus, $H^*(S^2/S^1)$ is trivial.

Where are we losing so much information?

Naive approach

One problem is that the group G is not acting *freely* on the manifold M and so the quotient M/G is losing too much information.



Free action of a group on a manifold

Let G be a group acting on a topological space X. We say that the action of G is *free* if, for every $x \in X$

$$x \cdot g = x \iff g \text{ is the identity.}$$

What should we do?

Proposition

Let X and Y be G-sets such that G acts freely on X. Then G acts freely on $X \times Y$.

Solution: Find a suitable space E on which G acts freely and calculate the cohomology of the orbit space of the product

$$H^*((M \times E)/G).$$

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More problems:

1. What is E?
2. Is this still related to the original problem?
3. Is this an easy calculation?

A suitable space

There *is* a *topological space*, EG, for every group G such that

- G acts freely on EG.
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It is also possible to show that, for any topological space X with G-action and contractible space E on which G acts freely, the cohomology group

$$H^*((X \times E)/G)$$

is independent of the choice of E.

We have a model

We will define the *homotopy quotient*

$$M//G := (M \times EG)/G$$

and define the *equivariant cohomology* of a manifold M by

$$H_G^*(M) := H^*(M//G).$$

Equivariant de Rham theory

Question:

Can we come up with an analogue of de Rham's theorem for equivariant cohomology?

To be able to form a de Rham complex on $M//G$,

$$\mathbf{\Omega}^*(M//G),$$

we would require $(M \times EG)/G$ to be a manifold. Instead we can look at the *"Cartan model"*

$$\mathbf{\Omega}^*_G(M)$$

which is an *equivariant version* of the de Rham complex.

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$$\mathbf{\Omega}^*_G(M) = (\mathbf{\Omega}^*(M) \otimes S^*(\mathfrak{g}^*))^G$$

which is an *equivariant version* of the de Rham complex.

A classical theorem

From the Cartan Model we have the following theorem:

Equivariant de Rham theorem

Let G be a compact, connected Lie group and M be a manifold with G-action. Then there is an isomorphism

$$H_G^*(M) \cong H^*(\mathbf{\Omega}_G^*(M)).$$

Goal

The goal of my project will be to realise this isomorphism explicitly using simplicial methods.

Simplicial Methods

The action of a Lie group on a manifold can also be captured in a *simplicial manifold*, $M_\bullet$. Consider the sequence of manifolds

$$M \;\underset{s_0}{\overset{d_0,\, d_1}{\rightleftarrows}}\; M \times G \;\underset{s_0,\, s_1}{\overset{d_0,\, d_1,\, d_2}{\rightleftarrows}}\; M \times G \times G \;\underset{s_0,\, s_1,\, s_2}{\overset{d_0,\, d_1,\, d_2,\, d_3}{\rightleftarrows}}\; \cdots$$

with *face* and *degeneracy* maps, d_i and s_j respectively. We will form a cohomology theory for this *simplicial manifold* that computes the equivariant cohomology of M with G-action.

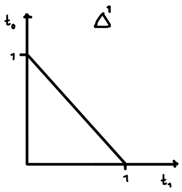
A simplex

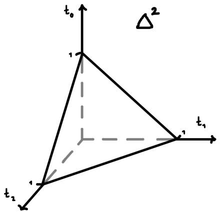
The standard topological *n-simplex* is the set of points

$$\Delta^n = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} : \ t_0 + t_1 + \cdots + t_n = 1, t_i \geq 0\}.$$

This topological space is special because we can decompose the boundary of Δ^n into components that are homeomorphic to Δ^{n-1}. Another way to think about this is that there are *canonical inclusions and projections*

$$d^i : \Delta^n \to \Delta^{n+1}, \ s^i : \Delta^n \to \Delta^{n-1}.$$





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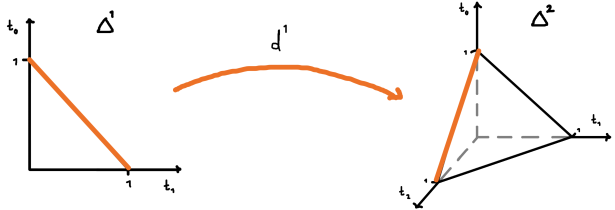

Simplicial Manifolds

Let $X_\bullet = \{X_q\}_{q \geq 0}$ be a sequence of manifolds

$$\begin{tikzcd}[column sep=huge]
X_0 \arrow[r, "s_0"', shift right=3] & X_1 \arrow[l, shift right=1, "d_0,\, d_1"'] \arrow[l, shift right=2] \arrow[r, shift right=3] \arrow[r, shift right=4, "s_0,\, s_1"'] & X_2 \arrow[l, shift right=1, "d_0,\, d_1,\, d_2"'] \arrow[l, shift right=2] \arrow[l, shift right=3] \arrow[r, shift right=3] \arrow[r, shift right=4] \arrow[r, shift right=5, "s_0,\, s_1,\, s_2"'] & \cdots \arrow[l, shift right=1, "d_0,\, d_1,\, d_2,\, d_3"'] \arrow[l, shift right=2] \arrow[l, shift right=3] \arrow[l, shift right=4]
\end{tikzcd}$$

with *face* and *degeneracy* maps, d_i and s_j respectively. We call $X_\bullet$ a simplicial manifold if the face and degeneracy maps are smooth and satisfy

- $d_i\, d_j = d_{j-1}\, d_i$ if $i < j$
- $s_i\, s_j = s_{j+1}\, s_i$ if $i \leq j$
- $d_i\, s_j = \begin{cases} s_{j-1}\, d_i & \text{if } i < j \\ \mathrm{id}_{X_i} & \text{if } i = j, i = j+1 \\ s_j\, d_{i-1} & \text{if } i > j+1 \end{cases}$

Simplicial Methods

Let's look again at the simplicial manifold we are interested in. In particular, we want to define the maps

$$M \times G^{n-1} \xleftarrow{d_i} M \times G^n \xrightarrow{s_i} M \times G^{n+1}.$$

for $0 \leq i \leq n$ where $M \times G^n = M \times G \times G \times \cdots \times G$. An element of $M \times G^n$ will be of the form

$$(m, g_0, g_1, \cdots, g_{n-1}), \qquad m \in M,\ g_i \in G$$

so we could define the maps d_i and s_i by

$$\begin{aligned} d_i(m, g_0, \cdots, g_{n-1}) &= (m, g_0, \cdots, g_{i-2}, g_{i-1} \cdot g_i, g_{i+1}, \cdots, g_{n-1}) \\ s_i(m, g_0, \cdots, g_{n-1}) &= (m, g_0, \cdots, g_{i-1}, 1, g_i, \cdots, g_{n-1}) \end{aligned}$$

to form a simplicial manifold.

Geometric Realisation

Encoded into the simplicial manifold $M_\bullet$ is precisely the group action of G on the manifold M.

$$d_0(m, g) = m \cdot g$$

We can stick all this information into a single topological space which we call the *geometric realisation* $|M_\bullet|$ which leaves us with a familiar topological space.

$$|M_\bullet| \;=\; \coprod_{n \geq 0} \Delta^n \times M \times G^n / \sim \;\;=\; M//G$$

This means if we can form a de Rham theory for the simplicial manifold $M_\bullet$ we may be able to relate it back to the equivariant cohomology of M with G-action.

Simplicial de Rham complex

Since G is a Lie group, each of the spaces $M \times G^n$ is a manifold. This means that we can form de Rham complexes of differential forms

$$\mathbf{\Omega}^*(M \times G^n).$$

The smooth face maps, $d_i : M \times G^n \to M \times G^{n-1}$ induce pullback maps

$$d_i^* : \mathbf{\Omega}^*(M \times G^{n-1}) \to \mathbf{\Omega}^*(M \times G^n).$$

From this we can construct a complex

$$\cdots \longrightarrow \mathbf{\Omega}^k(M \times G^{n-1}) \xrightarrow{\delta} \mathbf{\Omega}^k(M \times G^n) \xrightarrow{\delta} \mathbf{\Omega}^k(M \times G^{n+1}) \longrightarrow \cdots$$

where $\delta : \mathbf{\Omega}^*(M \times G^{n-1}) \to \mathbf{\Omega}^*(M \times G^n)$ is defined to be

$$\delta = \sum_{i=0}^{n} (-1)^i d_i^*.$$

Double complex

Because we have two differential operators defined on $\mathbf{\Omega}^p(M \times G^q)$, we can form a double complex.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
\mathbf{\Omega}^2(M) \arrow[r, "\delta"] \arrow[u] & \mathbf{\Omega}^2(M \times G) \arrow[r, "\delta"] \arrow[u] & \mathbf{\Omega}^2(M \times G^2) \arrow[r, "\delta"] \arrow[u] & \mathbf{\Omega}^2(M \times G^3) \arrow[r] \arrow[u] & \cdots \\
\mathbf{\Omega}^1(M) \arrow[r, "\delta"] \arrow[u, "d"] & \mathbf{\Omega}^1(M \times G) \arrow[r, "\delta"] \arrow[u, "d"] & \mathbf{\Omega}^1(M \times G^2) \arrow[r, "\delta"] \arrow[u, "d"] & \mathbf{\Omega}^1(M \times G^3) \arrow[r] \arrow[u, "d"] & \cdots \\
\mathbf{\Omega}^0(M) \arrow[r, "\delta"] \arrow[u, "d"] & \mathbf{\Omega}^0(M \times G) \arrow[r, "\delta"] \arrow[u, "d"] & \mathbf{\Omega}^0(M \times G^2) \arrow[r, "\delta"] \arrow[u, "d"] & \mathbf{\Omega}^0(M \times G^3) \arrow[r] \arrow[u, "d"] & \cdots
\end{tikzcd}
$$

Total complex

We want to pull a cohomology theory out of this double complex, so we are going to try to reduce this to a regular complex.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
\boldsymbol{\Omega}^2(M) \arrow[r, "\delta"] \arrow[u] & \boldsymbol{\Omega}^2(M \times G) \arrow[r, "\delta"] \arrow[u] & \boldsymbol{\Omega}^2(M \times G^2) \arrow[r, "\delta"] \arrow[u] & \boldsymbol{\Omega}^2(M \times G^3) \arrow[r] \arrow[u] & \cdots \\
\boldsymbol{\Omega}^1(M) \arrow[r, "\delta"] \arrow[u, "d"] & \boldsymbol{\Omega}^1(M \times G) \arrow[r, "\delta"] \arrow[u, "d"] & \boldsymbol{\Omega}^1(M \times G^2) \arrow[r, "\delta"] \arrow[u, "d"] & \boldsymbol{\Omega}^1(M \times G^3) \arrow[r] \arrow[u, "d"] & \cdots \\
\boldsymbol{\Omega}^0(M) \arrow[r, "\delta"] \arrow[u, "d"] & \boldsymbol{\Omega}^0(M \times G) \arrow[r, "\delta"] \arrow[u, "d"] & \boldsymbol{\Omega}^0(M \times G^2) \arrow[r, "\delta"] \arrow[u, "d"] & \boldsymbol{\Omega}^0(M \times G^3) \arrow[r] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

$$\boldsymbol{\Omega}^0(M \times G^0) \longrightarrow \bigoplus_{p+q=1} \boldsymbol{\Omega}^p(M \times G^q) \longrightarrow \bigoplus_{p+q=2} \boldsymbol{\Omega}^p(M \times G^q) \longrightarrow \bigoplus_{p+q=3} \boldsymbol{\Omega}^p(M \times G^q) \longrightarrow \cdots$$

Total complex

$$\cdots \to \bigoplus_{p+q=n-1} \mathbf{\Omega}^p(M \times G^q) \xrightarrow{D} \bigoplus_{p+q=n} \mathbf{\Omega}^p(M \times G^q) \xrightarrow{D} \bigoplus_{p+q=n+1} \mathbf{\Omega}^p(M \times G^q) \to \cdots$$

We can define a differential operator on this sequence

$$D : \bigoplus_{p+q=n} \mathbf{\Omega}^p(M \times G^q) \longrightarrow \bigoplus_{p+q=n+1} \mathbf{\Omega}^p(M \times G^q)$$

defined by

$$D = \delta + (-1)^p d.$$

Since $D^2 = 0$ we can construct a new *total complex* $\mathbf{\Omega}^*_{total}(M_\bullet)$.

Theorem

Let M be a manifold with smooth G-action for G a compact Lie group. Then there is an isomorphism

$$H^*(\mathbf{\Omega}^*_{total}(M_\bullet)) \cong H^*(|M_\bullet|)$$

An important example

The case where we take our manifold M to be a point may seem trivial after this discussion, but this is precisely the work we wish to extend. In his monograph *Curvature and Characteristic Classes*, Dupont uses this model to calculate the equivariant cohomology of a point and shows that

$$S^*(\mathfrak{g}^*)^G \cong H^*(\mathbf{\Omega}^*_{total}(\{pt\}_\bullet)) \cong H^*_G(\{pt\}).$$

By extending this work we wish to show an explicit isomorphism from the Cartan model of equivariant cohomology to the simplicial case.

$$H^*(\mathbf{\Omega}^*_{total}(M_\bullet)) \xrightarrow{\cong} H^*((\mathbf{\Omega}^*(M) \otimes S^*(\mathfrak{g}^*))^G)$$

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Questions?

Extra: Geometric Realisation

$$|M_\bullet| \;=\; \coprod_{n\geq 0} \Delta^n \times M \times G^n / \sim$$

Where for $t \in \Delta^n$, $m \in M$ and $(g_0, \cdots, g_{n-1}) \in G^n$ we have the identifications

- $(d^i t, (m, g_0, \cdots, g_{n-1})) \sim (t, d_i(m, g_0, \cdots, g_{n-1})$
- $(s^i t, (m, g_0, \cdots, g_{n-1})) \sim (t, s_i(m, g_0, \cdots, g_{n-1}))$

Extra: $D^2 = 0$

We defined

$$D = \delta + (-1)^p d.$$

Note that δ is the sum of pullback maps d_i and hence commutes with d. The task is then showing that $D^2 = 0$ for each element in the sum

$$\bigoplus_{p+q=n} \mathbf{\Omega}^p(M \times G^q)$$

$$\begin{aligned} D^2 &= \delta^2 + \delta(-1)^p d + (-1)^{p-1} d\delta + (-1)^{2p} d \\ &= 0 + \delta(-1)^p d + \delta(-1)^{p-1} d + 0 \\ &= 0 \end{aligned}$$