

## Example File

### 1 [Finite $T$ correction for fermion chemical potential in $D$ -space]

(1) By a quantum-classical correspondence argument, the total number of quantum states for a single particle whose energy is less than or equal to  $\varepsilon$  is given by

$$\int_0^\varepsilon d\varepsilon D_D(\varepsilon) = \frac{1}{h^3} \int_{\mathbf{q} \in V} d\mathbf{q} \int_{|\mathbf{p}| < \sqrt{2m\varepsilon}} d\mathbf{p} \propto \frac{V}{h^3} \int_0^{\sqrt{2m\varepsilon}} p^D dp.$$

Note the integral in the last expression represents the volume of a  $D$ -sphere in momentum space with radius  $\sqrt{2m\varepsilon}$  and hence scales as  $\varepsilon^{D/2}$ , while the LHS scales as  $\varepsilon[D_D(\varepsilon)]$ . Therefore, we must have  $D_D(\varepsilon) \propto \varepsilon^{D/2-1}$  and

$$\int_0^\varepsilon d\varepsilon D_D(\varepsilon) = \frac{2}{D} D_D(\varepsilon).$$

When this is substituted into the expression of  $PV$  [Eq. (21.45)], it simply replaces the usual  $2/3$  with  $2/D$  and we obtain

$$PV = \frac{2}{D} E,$$

which is the universal  $P$ - $E$  relation in  $D$ -dimensional space.

(2) At  $T = 0$ , with Gibbs relation and the  $P$ - $E$  relation derived above, we have

$$E = TS - PV + \mu N = -\frac{2}{D} E + \mu(0)N.$$

Rearrange, we obtain  $E/\mu(0)N = D/(D+2)$ .

(3) From the formula for  $E$ , we get  $S = C_V = \alpha_D NT$  since  $S(0) = 0$ . Thus,

$$\mu(T) = \frac{G}{N} = \frac{1}{N}(E - TS + PV),$$

which upon substitution of  $E$ ,  $PV = 2E/D$  and  $S = \alpha_D NT$  becomes

$$\mu(T) = \frac{1}{N} \left[ \left(1 + \frac{2}{D}\right) \left(E_0 + \frac{1}{2} \alpha_D NT^2\right) - \alpha_D NT^2 \right] = \frac{1}{N} \left(1 + \frac{2}{D}\right) E_0 + \left(\frac{1}{D} - \frac{1}{2}\right) \alpha_D T^2.$$

(4) The continuum approximation for non-condensate population  $N_1$  of an ideal boson system goes as

$$N_1(T, \mu) \equiv \gamma V \int_0^\infty d\varepsilon \frac{\varepsilon^{D/2-1}}{e^{\beta(\varepsilon-\mu)} - 1}.$$

Note that this is an increasing function of both  $T$  and  $\mu$ . Now we decrease  $T$ , but to maintain  $N = N_1$ , we must increase  $\mu$ , where  $\mu = 0$  sets the upper limit. Considering that  $N_1(T, \mu) \leq N_1(T, 0)$ , we must have

$$\gamma V \int_0^\infty d\varepsilon \frac{\varepsilon^{D/2-1}}{e^{\beta(\varepsilon-\mu)} - 1} \leq \gamma V (k_B T)^{D/2} \int_0^\infty dz \frac{z^{D/2-1}}{e^z - 1}.$$

The integral on the RHS is finite, where the equality holds when  $\mu = 0$ . Therefore,

$$\boxed{N_1(T, 0) \propto T^{D/2}}.$$

## 2 [Adiabatic free expansion]

(F1) Because the process is an adiabatic free expansion, the internal energy  $E$  remains constant. Hence, by  $PV = 2E/3$ , we must have  $P_f/P_i = \boxed{10/11}$ .

(F2) Since  $E$  is constant, increasing  $V$  implies decrease of the one particle level spacings. If this happens at a very low temperature, then the particles must be excited to go beyond the Fermi energy. Therefore,  $T$  must increase.

(B1) Due to the fact that  $PV = 2E/3$  is universal, we have  $P_f/P_i = \boxed{10/11}$ , the same as that of the fermi case.

(B2 & 3) Recall that at  $T \leq T_c$  with  $\mu = 0$ , we have

$$E = \int_0^\infty d\varepsilon D(\varepsilon) \frac{\varepsilon}{e^{\beta\varepsilon} - 1} \propto V \int_0^\infty d\varepsilon \varepsilon^{1/2} \frac{\varepsilon}{e^{\beta\varepsilon} - 1} \propto V (k_B T)^{5/2} \int_0^\infty dz \frac{z^{3/2}}{e^z - 1} \propto VT^{5/2},$$

while

$$N = \int_0^\infty d\varepsilon D(\varepsilon) \frac{1}{e^{\beta\varepsilon} - 1} \propto V (k_B T_c)^{3/2} \int_0^\infty dz \frac{z^{1/2}}{e^z - 1} \propto VT_c^{3/2}.$$

Because the system is isolated and the free expansion process adiabatic, both  $E$  and  $N$  are constant. Hence, the above relations tells us that

$$T \propto \left(\frac{E}{V}\right)^{2/5}, \text{ and } T_c \propto \left(\frac{N}{V}\right)^{2/3}.$$

Therefore, as  $V$  increases, the BEC temperature  $T_c$  decreases. If we suppose that  $T_i = T_c$ ,  $T_c$  decreases at a faster rate than  $T_f$ . Consequently, the Bose-Einstein condensate state cannot be maintained after the expansion.

## 3 [Quantum Ideal Gas]

(1) To study the entropy change due to a volume increase at constant temperature is equivalent to examining the sign of the partial derivative  $(\partial S/\partial V)_T$ . Thermodynamics tells us:

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{\partial(S, T)}{\partial(V, T)} = \frac{\partial(S, T)}{\partial(-P, V)} \frac{\partial(-P, V)}{\partial(V, T)} = 1 \cdot \frac{\partial(-P, V)}{\partial(V, T)} = \left(\frac{\partial P}{\partial T}\right)_V.$$

But thermodynamics alone is not enough. The universal state function  $PV = 2E/3$  supplies the additional information we need by

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{2}{3V} \left(\frac{\partial E}{\partial T}\right)_V = \frac{2}{3V} C_V > 0.$$

Therefore, the entropy of the system increases.

(2) To determine whether we should increase or decrease the system volume to lower the

temperature at constant pressure, it suffices to examine the sign of the partial derivative  $(\partial V/\partial T)_P$ . Again, thermodynamics indicates that

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{\partial(V, P)}{\partial(T, P)} = \frac{\partial(V, P)}{\partial(E, P)} \frac{\partial(E, P)}{\partial(T, P)} = \left(\frac{\partial V}{\partial E}\right)_P \left(\frac{\partial E}{\partial T}\right)_P = \left(\frac{\partial V}{\partial E}\right)_P C_P.$$

Note that  $C_P > 0$  is always true due to stability criterion. By using the relation  $PV = 2E/3$ , we also have

$$\left(\frac{\partial V}{\partial E}\right)_P = \frac{2}{3P} \left(\frac{\partial E}{\partial E}\right)_P = \frac{2}{3P} > 0.$$

Therefore, the partial derivative  $(\partial V/\partial T)_P$  is positive, and we must decrease the system volume.