Contents

Triangle Inequality

$$|x-y| \leq |x-z| + |z-y|$$

[[Screenshot 2025-09-10 at 11.29.49 AM.png]] It suggests the straight distance between two points is a direct line Without loss of generality, suppose $x \leq y$ swap x and y, $|x-y| \leq |x-z| + |z-y| \iff |y-x| \leq |x-z| + |y-z| \iff |x-y| \leq |x-z| + |z-y|$
Proof: (number line) Case 1($z \leq x \leq y$): $|x-y| \leq |z-y| \leq |x-z| + |z-y|$ Case 2($x \leq z \leq y$): $|x-y| = |x-z| + |z-y|$ so $|x-y| \leq |x-z| + |z-y|$ Case 3($x \leq y \leq z$): $|x-y| \leq |x-z| + |z-y|$ ### Triangle Inequality 2 $\forall a, b \in \mathbb{R}, |a+b| \leq |a| + |b|$ Proof: By Triangle Inequality, $|x-y| \leq |x-z| + |z-y|, \forall x, y, z \in \mathbb{R}$ in particular, $x = a, y = -b, z = 0$ becomes $|a+b| \leq |a-0| + |0-(-b)| = |a| + |b|$

Is it true that $|a-b| \leq |a| - |b|, \forall a, b \in \mathbb{R}$? No, let $a = 10, b = -9 \implies |10-(-9)| \nleq |10| - |-9|$

Examples of inequalities: 1) $|x-a| < \delta$ implies $x \in (a-\delta, a+\delta)$ 2) $|x-a| \leq \delta$ implies $x \in [a-\delta, a+\delta]$ 3) $0 < |x-a| \leq \delta$ implies $x \in (a-\delta, a) \cup (a, a+\delta)$

Practice questions:

1) Solve $|-2x+6| < 5 \implies 2|x-3| < 5 \implies |x-3| < 5/2$ Solution interval is $(3-\frac{5}{2}, 3+\frac{5}{2})$
2) Solve $2 < |x+7| \leq 3 \implies 2 < |x-(-7) \leq_3$ Solution interval is $[-10, -9) \cup (-5, -4]$
3) Solve $\frac{|x+2|}{|x-2|} > 5$

$$|x+2| = \begin{cases} x+2, & \text{if } x \geq -2 \\ -x+2, & \text{if } x < -2 \end{cases}$$

$$|x-2| = \begin{cases} x-2, & \text{if } x \geq 2 \\ -x-2, & \text{if } x < 2 \end{cases}$$

Case 1($x < -2$): $\frac{-x-2}{2-x} > 5 \iff -x-2 > 10-5x \iff x > 3 \iff$ impossible Case 2($-2 \leq x < 2$): $\frac{x+2}{2-x} > 5 \iff x+2 > 10-5x \iff \frac{x>4}{3} \implies x \in (\frac{4}{3}, 2)$ Case 3($x > 2$): $\frac{x+2}{x-2} > 5 \iff x+2 > 5x-10 \iff 3 > x \implies x \in (2,3)$ Solution:
$x \in (\frac{4}{3}, 2) \cup (2, 3)$

Infinite Sequence

An infinite sequence is a list of numbers in a definite order

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

where $a_i \in \mathbb{R}$ for i $\in \mathbb{N}$. We use notation $\{a_n\}_{n=0}^{\infty}$

Let $\{a_n\}$ be a sequence of real numbers and $n_1 < n_2 < \dots$ be increasing sequence of natural numbers

$$a_{n_1}, a_{n_2}, a_{n_3} \dots$$

denoted $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

The subsequence $a_k, a_{k+1}, a_{k+2}, \dots$ of $\{a_n\}$ is called the **tail** of $\{a_n\}$ with **cutoff** k

Examples: What is happening to the sequence $\{1/n\}$ and $\{(-1)^n\}$ as n gets larger and larger? 1) Arbitrary close to 0 2) Never arbitrarily close to any single number #### Convergence of Infinite Sequence

IMPORTANT: Let $\{a_n\}$ be a sequence and $L \in \mathbb{R}$. We say that L is the limit** of $\{a_n\}$ if for every $\epsilon > 0, \exists N \in \mathbb{R}$ s.t. if $n > N$, then

$$|a_n - L| < \epsilon$$

If such an L exists, we say $\{a_n\}$ **converges to** L and write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L$$

If no such L exists, then we say $\{a_n\}$ **diverges**.

We can also define the limit of a sequence in terms of tails!

Theorem(Equivalent Definition of the Limit of a Sequence)

1) $\lim_{n\to\infty} a_n = L$
2) For every $\epsilon > 0$, the interval $(L-\epsilon, L+\epsilon)$ contains a tail of $\{a_n\}$
3) For every , the number of elements of $\{a_n\}$ that do not lie in $(L-\epsilon, L+\epsilon)$ is finite
4) Every interval (a, b) containing L contains a tail of $\{a_n\}$
5) Given any interval (a, b) containing L, the number of elements of $\{a_n\}$that do not lie in (a, b) is finite

Example 1: We want to show that $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$. For now, we suppose $\epsilon = \frac{1}{1000}$ Side work: $|\frac{1}{\sqrt[3]{n}} - 0| < \epsilon \iff |\frac{1}{\sqrt[3]{n}}| < \epsilon \iff \frac{1}{\sqrt[3]{n}} < \epsilon,$since $n > 0 \iff \sqrt[3]{n} > \frac{1}{\epsilon} \iff n > \frac{1}{\epsilon^3}$

Actual proof work Let $\epsilon > 0$ be given, choose $N = \frac{1}{\epsilon^3}$. Then if $n > N$, $|a_n - L| = |\frac{1}{\sqrt[3]{n}} - 0| = |\frac{1}{\sqrt[3]{n}}|$ Since $n > N \implies \sqrt[3]{n} > \sqrt[3]{N}, \frac{1}{\sqrt[3]{n}} < \frac{1}{\sqrt[3]{N}} = \frac{1}{\sqrt[3]{\frac{1}{\epsilon^3}}} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$

Example 2: Prove that $\lim_{n\to\infty} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$ using the formal definition of the limit of a sequence Side work: we want N s.t. for $n > N, |a_n - L| < \epsilon$ $|\frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4}| \iff |\frac{12n^2+8n}{16n^2+4n+4} - \frac{12n^2+3n+3}{16n^2+4n+4}| = |\frac{5n-3}{16n^2+4n+4}| = \frac{5n-3}{16n^2+4n+4}$(to get a common factor to cancel, make top bigger and make bottom smaller)

$\frac{5n-3}{16n^2+4n+4} < \frac{5n}{16n^2+4n} = \frac{5}{\frac{16}{4}} < \epsilon \iff 16n+4 > \frac{5}{\epsilon} \iff n > \frac{5}{16\epsilon} - \frac{1}{4}$

Actual proof work: Let $\epsilon > 0$ be given, choose $N = \frac{5}{16\epsilon} - \frac{1}{4}$. Then $|a_n - L| = |\frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4}| = |\frac{12n^2+8n}{16n^2+4n+4} - \frac{12n^2+3n+3}{16n^2+4n+4}| = |\frac{5n-3}{16n^2+4n+4}| = \frac{5n-3}{16n^2+4n+4} < \frac{5n}{16n^2+4n} = \frac{5}{16n+4} = \frac{5}{16N+4} = \frac{5}{16(\frac{5}{16\epsilon} - \frac{1}{4})+4} = \frac{5}{\frac{5}{\epsilon} - 4 + 4} = \epsilon$ Idea, we need to find some n in terms of ϵ to be N Then we Let $\epsilon > 0$ be given, choose $N =$ result in terms of ϵ. Then continue