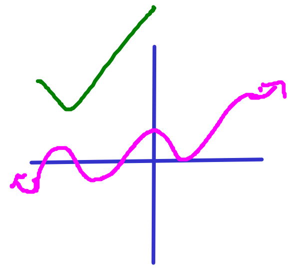
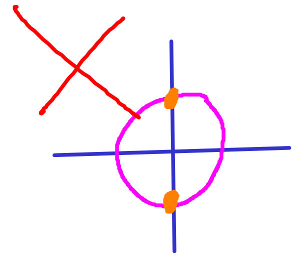
Let $X$ and $Y$ be sets. A **function** $f : X \to Y$ is a mapping that assigns to each $x \in X$ exactly one $y = f(x) \in Y$.

vertical line test:

a function intersects any vertical line at most once





The **domain** of a function, written as $\mathrm{Dom}(f)$ is the set of values for which it is defined. The **range** of a function, written as $\mathrm{Ran}(f)$ is the set of values of $Y$ that $f$ assigns some point of $X$ to.

real number $2, -3, -\sqrt{5}, \pi$

In MATH 137, $Y = \mathbb{R}$ always, $X \subseteq \mathbb{R}$

* no need to worry about complex numbers

Interval notation can be used to describe the domain and range of a function.

$(a, b)$ = all points between a and b, not including a and b

$[a, b]$ = " " including a and b

$[a, b)$ = " " including a but not b

$(a, b]$ = " " including b but not a

$(a, \infty)$ = all values $> a$

always be round bracket, $\infty$ is not a number

$[a, \infty)$ = all values $\geq a$

$(-\infty, b)$ = all values $< b$

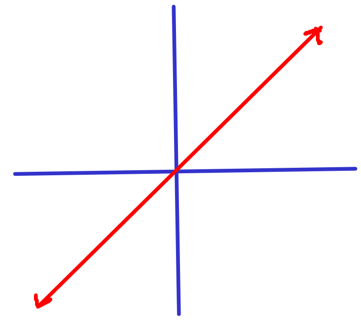
$(-\infty, b]$ = all values $\leq b$

$(-\infty, \infty) = \mathbb{R}$

Let's look at some examples of functions that you've (hopefully) seen before. $y = x$:

domain: $\mathbb{R}$

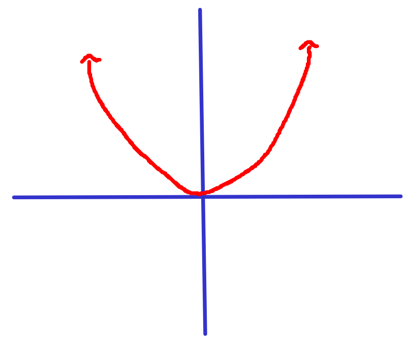
range: $\mathbb{R}$



$y = x^2$:

domain: $\mathbb{R}$

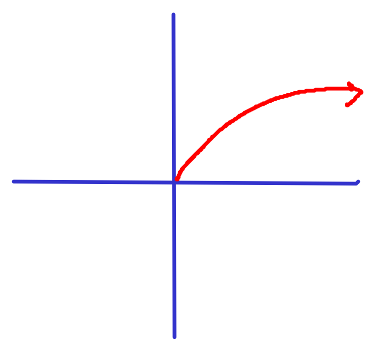
range: $[0, \infty)$



What's the opposite of squaring something? Square rooting it!
$y = \sqrt{x}$:

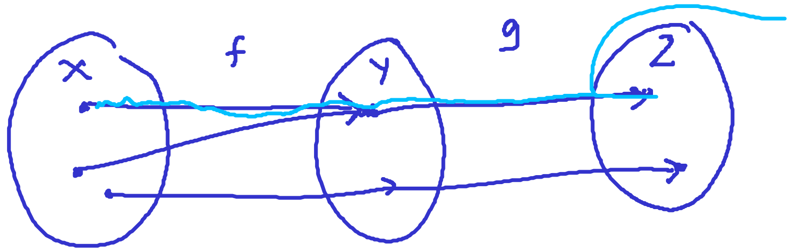
domain: $[0, \infty)$

range: $[0, \infty)$



Since taking the square root "undoes" the process of squaring something, these functions are said to be **inverses** of one another. To formalize this, we need to talk about composition of functions.

Let $f : X \to Y$ and $g : Y \to Z$ be functions. The **composition of** $f$ **and** $g$, denoted by $g \circ f$, is the function $g \circ f : X \to Z$ given by $(g \circ f)(x) = g(f(x))$.




$g \circ f(x)$

always apply function closer to x first!

When we multiply two numbers, the order does not matter: $2 \times 3 = 3 \times 2 = 6$. Does this work for composition? Is $g \circ f = f \circ g$?

$f(x) = x^2, \quad g(x) = x^2+1$

$g \circ f(1) = 2$

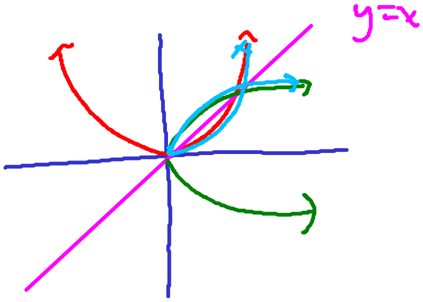
$g \circ f(x) = g(x^2) = (x^2)^2 + 1 = x^4 + 1$

$f \circ g(1) = 4$

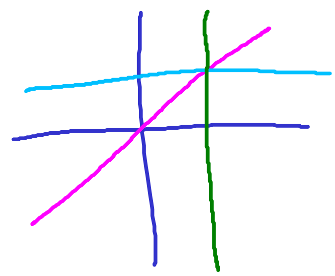
$f \circ g(x) = f(x^2+1) = (x^2+1)^2 = x^4 + 2x^2 + 1$

We saw that the inverse of a function "undoes" what the original function did. Formally, we say that $f^{-1} : Y \to X$ is the **inverse** of $f : X \to Y$ if $f^{-1} \circ f(x) = x$ for all $x \in X$ and $f \circ f^{-1}(x) = x$ for all $y \in Y$.

An inverse essentially swaps the role of $x$ and $y$, so geometrically we can view it as a reflection through the line $y = x$. But <u>be careful</u>, the domain might not be the same.



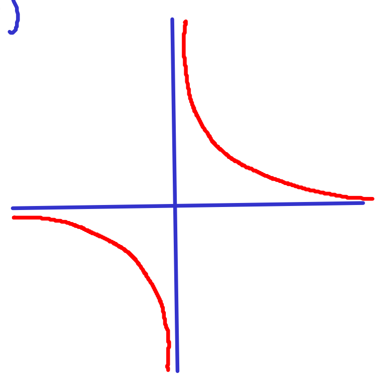

The reflection of a horizontal line through the line $y = x$ is vertical, so in order for $f^{-1}$ to pass the vertical line test, $f$ must pass the horizontal line test.



Back to functions you've seen before, consider $y = 1/x$:

domain: $\overbrace{\mathbb{R}}^{\text{everything but } 0}\setminus\{0\} = (-\infty, 0) \underbrace{\cup}_{\text{union = both}} (0, \infty)$

range: $\mathbb{R}\setminus\{0\}$



**Example:** If $f(x) = 1/x$ and $g(x) = \sqrt{x}$, what is the domain of $g \circ f(x)$?

$f$ has domain $\mathbb{R} \setminus \{0\}$

$g$ has domain $[0, \infty)$

what inputs to $f$ produce an output in the domain of $g$?

$f$ maps $+$ to $+$, $-$ to $-$

$g \circ f$ has domain $(0, \infty)$

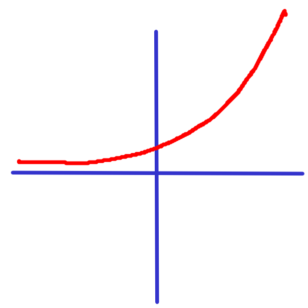
For another function you've seen before, check out $y = e^x$:

domain: $\mathbb{R}$

range: $(0, \infty)$

$e \sim 2.71828$

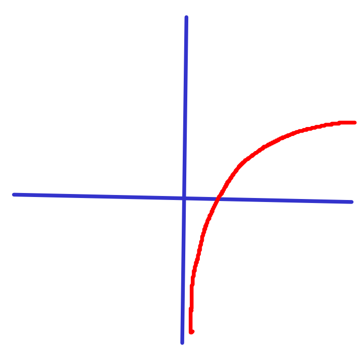
important constant in calculus



The opposite of exponentiating is the **logarithm**. $\log_b(x)$ answers the question of what power we have to raise $b$ to in order to get $x$. We write $\log_e(x)$ as $\ln(x)$, called the **natural logarithm of** $x$.

domain: $(0, \infty)$

range: $\mathbb{R}$



**Example:** Let $f(x) = e^x$ and $g(x) = \ln(x)$. What is the domain and range of $g \circ f(x)$? How about $f \circ g(x)$?

f takes everything, g takes positive things

always maps to positive

$g \circ f$ domain: $\mathbb{R}$

range: $\mathbb{R}$

g takes positive things, maps to everything

f takes anything, map to positive things

$f \circ g$ domain: $(0, \infty)$

range: $(0, \infty)$