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## Definitions

- 1) **Proposition** is a true statement has to be proved to be true
- 2) **Theorem** is a significant proposition
- 3) **Lemma** is a subsidiary proposition
- 4) **Corollary** is a proposition that follows almost immediately from a theorem

## Proving Universally Quantified Statements

- 1) Choose a representative mathematical object  $x \in S$  (e.g. Let the object be an arbitrary in  $S$ )
- 2) Show the open sentence must be true for representative  $x$ , using known facts about the elements of  $S$   
 Example: Proof that  $\forall x, y \in \mathbb{R}, x^4 + x^2y + y^2 \geq 5x^2y - 3y^2$  Discovery: if  $x^4 + x^2y + y^2 \geq 5x^2y - 3y^2 \implies x^4 + x^2y + y^2 - 5x^2y + 3y^2 \geq 0$  then  $(x^2 - y)^2 \geq 0$  However this is not a proof, it assumes the statement to be true already **Actual Proof:** Let  $x$  and  $y$  be arbitrary real numbers. Since  $x, y \in \mathbb{R}$ , then  $(x^2 - y)$  is a real number. Which means that  $(x^2 - y)^2 \geq 0$ , as all real numbers squared will be non-negative. Thus,  $x^4 - 4x^2y + 4y^2 \geq 0$ . Then,  $x^4 + x^2y - 5x^2y + y^2 + 3y^2 \geq 0$ . So,  $x^4 + x^2y + y^2 \geq 5x^2y - 3y^2$ . Therefore, we have proved that  $\forall x, y \in \mathbb{R}, x^4 + x^2y + y^2 \geq 5x^2y - 3y^2$  ##### Disprove Universally Quantified Statement Find an element  $x \in S$  for which the open sentence is false; that is proving an existentially quantified statement. Example: Proof that  $\forall x \in \mathbb{R}, x^2 = 5$  Proof: Let  $x = 0$  Thus  $x^2 = 0$ , which is not 5. Therefore  $\exists x \in \mathbb{R}, x^2 \neq 5$ , namely  $x = 0$ . So it is false  $\forall x \in \mathbb{R}, x^2 = 5$

## Prove Existentially Quantified Statement

Find an element  $x \in S$  for which the open sentence is true.

Example 1: Prove that  $\exists \in \mathbb{Z}$  s.t.  $\frac{m-7}{2m+4} = 5$  Discovery:  $m-7 = 5(2m+4) \iff m-7 = 10m+10 \iff -27 = 9m \iff m = -3$  **Actual Proof:** Let  $m = -3$ , clearly  $-3 \in \mathbb{Z}$  and  $\frac{m-7}{2m+4} = \frac{-3-7}{2(-3)+4} = \frac{-10}{-6+4} = \frac{-10}{-2} = 5$  Therefore, we have shown that  $m \in \mathbb{Z}, \frac{m-7}{2m+4} = 5$  Example 2: Prove that there exists a perfect square  $k$  s.t.  $k^2 - \frac{31}{2}k = 8$  Proof: Let  $k = 16$ , clearly,  $k = 16 = 4^2$  Also,  $k^2 - \frac{31}{2}k = 16^2 - \frac{31}{2}(16) = 16(16 - \frac{31}{2}) = \frac{16}{2} * 1 = 8$  Therefore, we have shown that there exists a perfect square  $k$  s.t.  $k^2 - \frac{31}{2}k = 8$

**Disprove Existentially Quantified Statement** Example: Disprove that  $\exists x \in \mathbb{R}, \cos(2x) + \sin(2x) = 3$  we need to prove  $\forall x \in \mathbb{R}, \cos(2x) + \sin(2x) \neq 3$  Let  $x \in \mathbb{R}$  Notice that  $-1 \leq \cos 2x \leq 1$  and  $-1 \leq \sin 2x \leq 1$  So we can add the inequalities to get:  $-2 \leq \cos 2x + \sin 2x \leq 2$  Clearly,  $\cos(2x) + \sin(2x) \neq 3$ , since  $3 \notin [-2, 2]$   $\therefore \forall x \in \mathbb{R}, \cos(2x) + \sin(2x) \neq 3$  i.e.  $\neg(\exists x \in \mathbb{R}, \cos(2x) + \sin(2x) = 3)$

## Prove/Disprove Nested Quantified Statement

Example: Consider the two statements: 1)  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$  2)  $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$

Prove or Disprove: 1) This is true Let  $x \in \mathbb{R}$ , let  $y = (x^3 - 1)^{\frac{1}{3}}$   $x^3 - y^3 = x^3 - ((x^3 - 1)^{\frac{1}{3}})^3 = x^3 - (x^3 - 1) = x^3 - x^3 + 1 = 1 \therefore \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$  2) This is false We will prove the negation, that is  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$  Let  $x \in \mathbb{R}$ , let  $y = x$  Then  $x^3 - y^3 = x^3 - x^3 = 0 \neq 1$  We have shown that  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$   $\therefore \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$