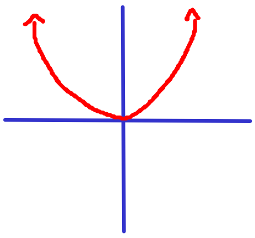
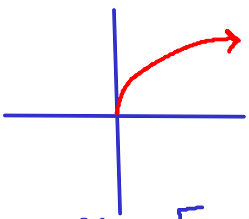
The Function Zoo



domain: $\mathbb{R}$

range: $[0, \infty)$

$f(x) = x^2$

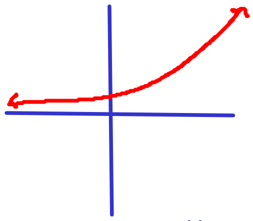


domain: $[0, \infty)$

range: $[0, \infty)$

$f(x) = \sqrt{x}$

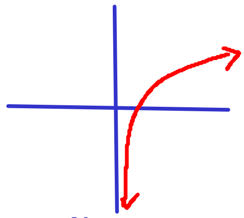
square and root



domain: $\mathbb{R}$

range: $(0, \infty)$

$f(x) = e^x$

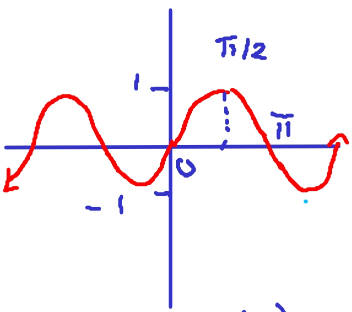


domain: $(0, \infty)$

range: $\mathbb{R}$

$f(x) = \ln(x)$

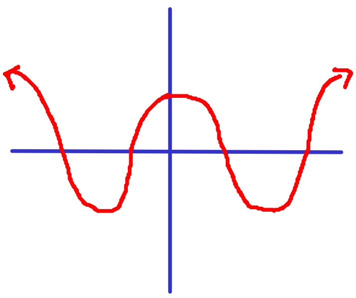
exp and log



domain: $\mathbb{R}$

range: $[-1, 1]$

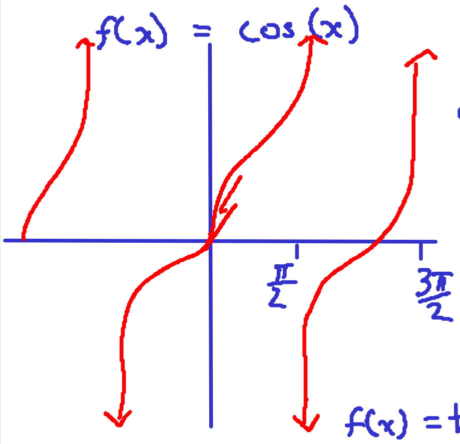
$f(x) = \sin(x)$



domain: $\mathbb{R}$

range: $[-1, 1]$

$f(x) = \cos(x)$

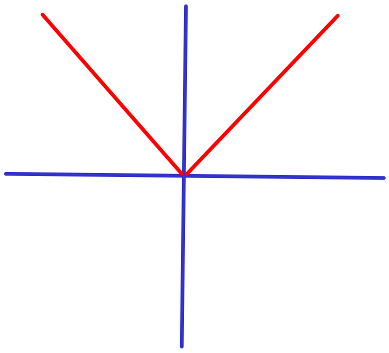


domain: $\mathbb{R} \setminus \{\frac{2k+1}{2}\pi : k \in \mathbb{Z}\}$

range: $\mathbb{R}$

$f(x) = \tan(x)$

trig functions



domain: $\mathbb{R}$

range: $[0, \infty)$

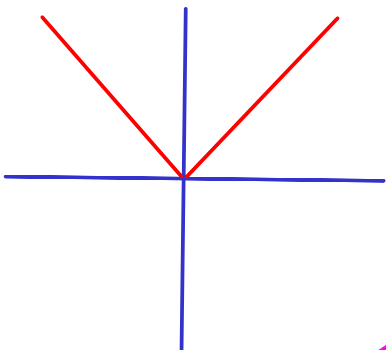
absolute value

Lecture 3: Absolute Values and the Triangle Inequality
Section 1.1 of the lecture notes

VIF (Very Important Function): $f(x) = |x|$, the **absolute value of** x, defined by

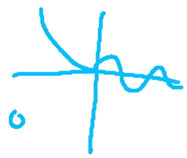
$$f(x) = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

piecewise function is defined in multiple branches



domain: $\mathbb{R}$
range: $[0, \infty)$

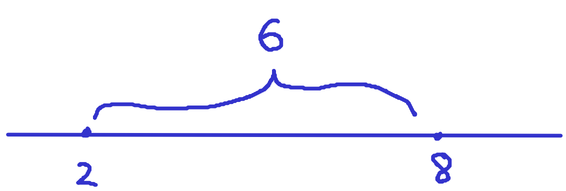
$$f(x) = \begin{cases} x^2 \text{ if } x < 0 \\ \sin(x) \text{ if } x > 0 \end{cases}$$



$|ab| = |a||b|$

in particular, if $b = -1$ $\quad |-a| = |a| \cdot |-1| = |a|$

The quantity $|a-b|$ represents the *distance* between a and b, one of the most crucial concepts in calculus!

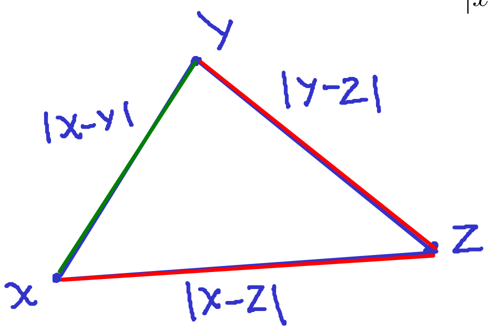


$|8-2| = 6$ ✓
$|2-8| = 6$ ✓

From this point onward, all material will be covered on quiz 2. The last thing you need for quiz 1 is the definition of absolute value!

Theorem (The Triangle Inequality): For all $x, y, z \in \mathbb{R}$, we have that

$$|x-y| \leq |x-z| + |z-y|$$



"the shortest distance between two points is a direct line"

swap x only

$|x-y| \leq |x-z| + |z-y|$

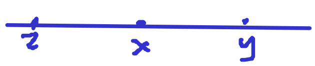
$|y-x| \leq |y-z| + |z-x|$

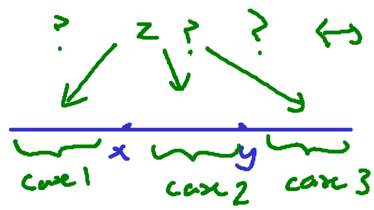
$\Leftrightarrow |x-y| \leq |z-y| + |x-z|$

$\Leftrightarrow \underbrace{|x-y| \leq |x-z| + |z-y|}_{\text{get same equation back}}$

Proof: Without loss of generality, suppose $x \leq y$

case 1: $z \leq x \leq y$



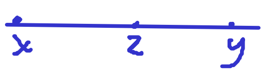


$|x-y| \leq |z-y|$
$\leq \underbrace{|x-z|}_{\geq 0} + |z-y|$ ✓

case 3: $x \leq y \leq z$

✓

case 2: $x \leq z \leq y$



$|x-y| = |x-z| + |z-y|$ ✓

so $|x-y| \leq |x-z| + |z-y|$ ✓

$|x-y| \leq |x-z|$

$|x-y| \leq |x-z| + \underbrace{|z-y|}_{\geq 0}$

Theorem (The Triangle Inequality II): For all $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$. ← more useful version

Proof: By Triangle Inequality I, $|x-y| \leq |x-z| + |z-y|$ for all $x, y, z \in \mathbb{R}$

in particular, it's true for $x = a$, $y = -b$, $z = 0$

subbing in, $|a+b| \leq |a-0| + |0-(-b)|$

$= |a| + |b|$ ✓

Question: If we replace the $+$ with a $-$ in the statement of The Triangle Inequality II, is it still true? In other words, is it true that $|a-b| \leq |a| - |b|$ for all $a, b \in \mathbb{R}$?

NO!

$a = 10, \quad b = -9$

$|10 - (-9)| = |19| = 19$

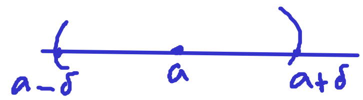
$|10| - |-9| = 10 - 9 = 1$

19 is not ≤ 1

There are three main types of inequalities that we will see again and again in this course:

- $|x - a| < \delta$

"x is within distance δ of a"



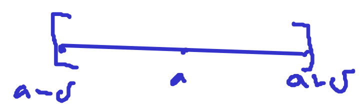
$x \in (a-\delta, a+\delta)$

$|x-3| < 5$

$(-2, 8)$

- $|x - a| \leq \delta$

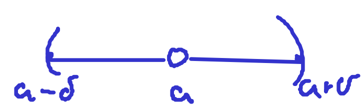
" . . . "



$x \in [a-\delta, a+\delta]$

- $0 < |x - a| < \delta$

" . . ., but $x \neq a$"



$x \in (a-\delta, a) \cup (a, a+\delta)$

Let's get used to working with inequalities by solving some problems!

Example: Solve $|-2x+6| < 5$.

$$|-2(x-3)| < 5$$

$$\iff |-2||x-3| < 5$$

$$\iff 2|x-3| < 5$$

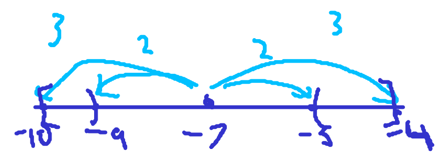
$$\iff |x-3| < \frac{5}{2} \qquad |x-a| < \delta \qquad x \in (a-\delta, a+\delta)$$

$$x \in \left(3-\frac{5}{2},\ 3+\frac{5}{2}\right) = \left(\frac{1}{2}, \frac{11}{2}\right) \checkmark$$

Example: Solve $2 < |x+7| \leq 3$.

$$\iff 2 < |x-(-7)| \leq 3$$

"the distance from x to -7 is at least 2, and at most 3, including 3 but not 2"



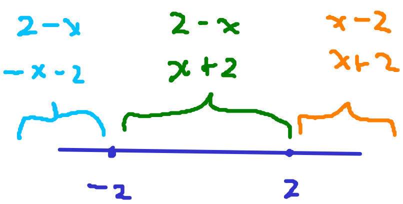
$$x \in [-10, -9) \cup (-5, -4]$$

$x = 2$ has the inequality not defined (can't divide by 0)

Example: Solve $|x+2|/|x-2| > 5$.

$$|x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ 2-x & \text{if } x < 2 \end{cases}$$

$$|x+2| = \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$



case 1: $x < -2$

$$\frac{-x-2}{2-x} > 5 \iff -x-2 > 10-5x$$

$$4x > 12$$

$$x > 3 \quad \text{impossible}$$

case 2: $-2 \leq x < 2$

$$\frac{x+2}{2-x} > 5 \iff x+2 > 10-5x$$

$$\iff 6x > 8$$

$$\iff x > \frac{4}{3}$$

$$x \in \left(\frac{4}{3}, 2\right)$$

case 3: $x > 2$

$$\frac{x+2}{x-2} > 5$$

$$x+2 > 5x-10$$

$$12 > 4x$$

$$3 > x \qquad x \in (2, 3)$$

$$\therefore x \in \left(\frac{4}{3}, 2\right) \cup (2, 3)$$