Contents

Definitions

1) **Proposition** is a true statement has to be proved to be true
2) **Theorem** is a significant proposition
3) **Lemma** is a subsidiary proposition
4) **Corollary** is a proposition that follows almost immediately from a theorem

Proving Universally Quantified Statements

1) Choose a representative mathematical object $x \in S$ (e.g. Let the object be an arbitrary in S)
2) Show the open sentence must be true for representative x, using known facts about the elements of S Example: Proof that $\forall x, y \in \mathbb{R}$, $x^4+x^2y+y^2 \geq 5x^2y-3y^2$ Discovery: if $x^4+x^2y+y^2 \geq 5x^2y-3y^2 \implies x^4+x^2y+y^2-5x^2y+3y^2 \geq 0$ then $(x^2-y)^2 \geq 0$ However this is not a proof, it assumes the statement to be true already **Actual Proof:** Let x and y be arbitrary real numbers. Since $x, y \in \mathbb{R}$, then (x^2-2y) is a real number. Which means that $(x^2-2y)^2 \geq 0$, as all real numbers squared will be non-negative. Thus, $x^4-4x^2y+4y^2 \geq 0$. Then, $x^4+x^2y-5x^2y+y^2+3y^2 \geq 0$. So, $x^4+x^2y+y^2 \geq 5x^2y-3y^2$. Therefore, we have proved that $\forall x, y \in \mathbb{R}$, $x^4+x^2y+y^2 \geq 5x^2y-3y^2$ #### Disprove Universally Quantified Statement Find an element $x \in S$ for which the open sentence is false; that is proving an existentially quantified statement. Example: Proof that $\forall x \in \mathbb{R}, x^2 = 5$ Proof: Let $x = 0$ Thus $x^2 = 0$, which is not 5. Therefore $\exists x \in \mathbb{R}, x^2 \neq 5$, namely $x = 0$. So it is false $\forall x \in \mathbb{R}, x^2 = 5$

Prove Existentially Quantified Statement

Find an element $x \in S$ for which the open sentence is true.

Example 1: Prove that $\exists \in \mathbb{Z}$ s.t. $\frac{m-7}{2m+4} = 5$ Discovery: $m-7 = 5(2m+4) \iff m-7 = 10m+10 \iff -27 = 9m \iff m = -3$ **Actual Proof**: Let $m = -3$, clearly $-3 \in \mathbb{Z}$ and $\frac{m-7}{2m+4} = \frac{-3-7}{2(-3)+4} = \frac{-10}{-6+4} = \frac{-10}{-2} = 5$ Therefore, we have shown that $m \in \mathbb{Z}, \frac{m-7}{2m+4} = 5$ Example 2: Prove that there exists a perfect square k s.t. $k^2 - \frac{31}{2}k = 8$ Proof: Let $k = 16$, clearly, $k = 16 = 4^2$ Also, $k^2 - \frac{31}{2}k = 16^2 - \frac{31}{2}(16) = 16\left(16 - \frac{31}{2}\right) = \frac{16}{2} * 1 = 8$ Therefore, we have shown that there exists a perfect square k s.t. $k^2 - \frac{31}{2}k = 8$

Disprove Existentially Quantified Statement Example: Disprove that $\exists x \in \mathbb{R}, \cos(2x) + \sin(2x) = 3$ we need to prove $\forall x \in \mathbb{R}, \cos(2x) + \sin(2x) \neq 3$ Let $x \in \mathbb{R}$ Notice that $-1 \leq \cos 2x \leq 1$ and $-1 \leq \sin 2x \leq 1$ So we can add the inequalities to get: $-2 \leq \cos 2x + \sin 2x \leq 2$ Clearly, $\cos(2x) + \sin(2x) \neq 3$, since $3 \notin [-2, 2]$ $\therefore \forall x \in \mathbb{R}, \cos(2x) + \sin(2x) \neq 3$ i.e. $\neg(\exists x \in \mathbb{R}, \cos(2x) + \sin(2x) = 3)$

Prove/Disprove Nested Quantified Statement

Example: Consider the two statements: 1) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$ 2) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$

Prove or Disprove: 1) This is true Let $x \in \mathbb{R}$, let $y = (x^3-1)^{\frac{1}{3}}$ $x^3 - y^3 = x^3 - ((x^3-1)^{\frac{1}{3}})^3 = x^3 - (x^3-1) = x^3 - x^3 + 1 = 1$ $\therefore \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$ 2) This is false We will prove the negation, that is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$ Let $x \in \mathbb{R}$, let $y = x$ Then $x^3 - y^3 = x^3 - x^3 = 0 \neq 1$ We have shown that $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$
$\therefore \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$