

## CH 3 — Function Limits and Continuity

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### Definitions

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function and  $a \in \mathbb{R}$ ,  $\lim_{x \rightarrow a} f(x) = L$  if for all  $\varepsilon > 0$  there exists  $\delta > 0$  s.t. if  $0 < |x - a| < \delta$ , then  $|f(x) - L| < \varepsilon$

Examples:

1) Prove using the  $\varepsilon - \delta$  definition that  $\lim_{x \rightarrow 0} f(x)$  DNE where

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 3 & \text{if } x > 0 \end{cases}$$

Domain:  $\mathbb{R} \setminus \{0\}$

Take  $\varepsilon = 1$ . Consider some  $\delta > 0$ . Within  $(0 - \delta, 0 + \delta)$

We have both  $(-\delta, 0)$  where  $f(x) = -2$  and  $(0, \delta)$  where  $f(x) = 3$ . If this  $\delta$  exists for  $\varepsilon = 1$  then the limit  $L$  would need to be distance 1 or both -2 and 3, where is impossible.

$$\therefore \lim_{x \rightarrow 0} f(x) = \text{DNE}$$

2)  $\lim_{x \rightarrow 7} 8x - 3 = 53$

Let  $\varepsilon > 0$  be arbitrary.

We want find  $\delta$  s.t. if  $0 < |x - 7| < \delta$  then  $|8x - 3 - 53| < \varepsilon \rightarrow \delta = \frac{\varepsilon}{8}$

Pick  $\delta = \frac{\varepsilon}{8}$ .

Then if  $0 < |x - 7| < \frac{\varepsilon}{8}$ ,  $|(8x - 3) - 53| = |8x - 56| = 8|x - 7| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$

3)  $\lim_{x \rightarrow 1} x^2 + 3x + 4 = 8$

We want for any  $\varepsilon > 0$  and  $\delta > 0$  :  $|x - 1| < \delta$ , then  $|f(x) - L| < \varepsilon$

$$\Leftrightarrow |x^2 + 3x - 4| < \varepsilon \Leftrightarrow |(x + 4)(x - 1)| < \varepsilon \Leftrightarrow |x + 4| - |x - 1| < \varepsilon$$

I can always make  $\delta$  smaller if I need to.

take  $\delta < 1$ , then  $|x - 1| < 1 \Rightarrow 0 < x < 2$

$|x + 4| < 6 \rightarrow |x + 4||x - 1| < 6\delta$ , but  $6\delta < \varepsilon \Leftrightarrow \delta < \frac{\varepsilon}{6}$ . Say  $\frac{\varepsilon}{7} < \frac{\varepsilon}{6}$  for all epsilon.

Take  $\delta < \min(1, \frac{\varepsilon}{6})$

### Proof

Let  $\varepsilon > 0$  be given. Take  $\delta = \min(\frac{1}{2}, \frac{\varepsilon}{7})$ . Then, if  $|x - 1| < \delta$ ,  $|x^2 + 3x + 4 - 8| = |x^2 + 3x - 4| = |(x + 4)(x - 1)| = |x + 4||x - 1| < 6 \cdot \frac{\varepsilon}{7} < \varepsilon$

□

### Info – Sequential Characterization of Limits Theorem

Let  $a \in \mathbb{R}$ . let the function  $f(x)$  be defined on an open interval containing  $a$ , except possibly at  $x = a$  itself. Then the following are equivalent:

1.  $\lim_{x \rightarrow a} f(x) = L$
2. For all sequences  $\{x_n\}$  satisfying  $\lim_{n \rightarrow \infty} x_n = a$  and  $x_n \neq a, \forall n \in \mathbb{N}$ , we have that  $\lim_{n \rightarrow \infty} f(x_n) = L$

### 💡 Tip – Usage of Sequential Characterization of Limits

1. Find a sequence  $\{x_n\}$  with  $x_n \rightarrow a$
2. Find two sequences  $\{x_n\}, \{y_n\}$  with  $x_n, y_n \rightarrow a$  and  $x_n, y_n \neq a, \forall n \in \mathbb{N}$  but which  $\{f(x_n)\}, \{f(y_n)\}$  converge to different values

### Proof

$\Rightarrow$  :  $\lim_{x \rightarrow a} f(x) = L$  means  $\forall \varepsilon > 0, \exists \delta > 0 : |x - a| < \delta$ , then  $|f(x) - L| < \varepsilon$ .

Let  $\{x_n\}$  be s.t.  $x_n \rightarrow a$  (meaning that  $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, |x_n - a| < \varepsilon_2$ ) and  $x_n \neq a$  for any  $n$ .

In particular, let  $\varepsilon$  for  $x_n \rightarrow a$  be  $\delta$ . Then  $\forall n > N, |x_n - a| < \delta$ , and so  $|f(x_n) - L| < \varepsilon_1$ . Then  $\forall n > N, |x_n - a| < \delta$  and so  $|f(x_n) - L| < \varepsilon_1$ . So by definition,  $\lim_{n \rightarrow \infty} f(x_n) = L$

**Side Question:** We saw the limit of a sequence is unique. Is the same true for limits of functions?

**ANS:** NO, it is like saying  $\lim_{x \rightarrow a} f(x) = L$  and  $= M$  and  $L \neq M$  Suppose true. By Sequential Characterization of Limits,  $\forall \{x_n\} \rightarrow a$  but  $x_n \neq a \forall n, f(x_n) \rightarrow L$  and  $f(x_n) \rightarrow M$  but  $L \neq M$  Since the limits of sequences are unique, thus there is a contradiction.

Examples:

Prove that  $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$  does not exist

We take sequences of peak points of  $\cos\left(\frac{1}{x}\right)$ , that is  $-1, 1$ . Then will converge to  $-1, 1$  repeatedly, so by Sequential Characterization,  $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$  will not exist.

$$\cos\left(\frac{1}{x}\right) = 1 \text{ if } x = \frac{1}{2k\pi}, k \in \mathbb{Z}, \text{ and } \cos\left(\frac{1}{x}\right) = -1 \text{ if } x = \frac{1}{(2k+1)\pi}, k \in \mathbb{Z}.$$

Let  $x_n = \frac{1}{2n\pi}$  and  $y_n = \frac{1}{(2n+1)\pi}$ . Then  $x_n, y_n \rightarrow 0, x_n, y_n \neq 0 \forall n$ . It converges to both  $-1$  and  $1$ . By Sequential Characterization, the limit DNE.

## Limit Laws

**Info** – Let  $f, g$  be functions with  $\lim_{x \rightarrow a} f(x) = L, \lim_{x \rightarrow a} g(x) = M$  for some  $L, M \in \mathbb{R}$  then:

1. For any  $c \in \mathbb{R}$ , if  $f(x) = c$  for all  $x$  then  $L = c$
2. For any  $c \in \mathbb{R}$ , if  $\lim_{x \rightarrow a} cf(x) = cL$
3.  $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$
4.  $\lim_{x \rightarrow a} f(x) \cdot g(x) = LM$
5.  $\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{L}{M}$  if  $M \neq 0$
6. If  $\alpha > 0$  and  $L > 0$ , then  $\lim_{x \rightarrow a} f(x)^\alpha = L^\alpha$

## Proof

We assume functions  $f, g$  are defined on a punctured neighborhood of  $a$  and  $\lim_{x \rightarrow a} f(x) = L$ ,  $\lim_{x \rightarrow a} g(x) = M$ . In the quotient law we also assume  $M \neq 0$ .

### 1. Product law

**Claim.**  $\lim_{x \rightarrow a} (f(x)g(x)) = LM$ .

**Proof.** Let  $\varepsilon > 0$ . Then  $|f(x)g(x) - LM| = |f(x)(g(x) - M) + M(f(x) - L)| \leq |f(x)| \cdot |g(x) - M| + |M| \cdot |f(x) - L|$ .

Since  $f(x) \rightarrow L$ , choose  $\delta_0 > 0$  with  $|x - a| < \delta_0 \Rightarrow |f(x) - L| < 1$ , hence  $|f(x)| \leq |L| + 1$  there.

Choose  $\delta_1, \delta_2 > 0$  so that  $|x - a| < \delta_1 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2(|L|+1)}$  and  $|x - a| < \delta_2 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2(|M|+1)}$ .

Let  $\delta = \min(\delta_0, \delta_1, \delta_2)$ . For  $0 < |x - a| < \delta$ ,  $|f(x)g(x) - LM| \leq (|L| + 1) \cdot \frac{\varepsilon}{2(|L|+1)} + |M| \cdot \frac{\varepsilon}{2(|M|+1)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ .

Thus  $\lim_{x \rightarrow a} (fg) = LM$ .

### 2. Quotient law (with $M \neq 0$ )

**Claim.**  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$ .

**Proof.** Let  $\varepsilon > 0$ . Because  $g(x) \rightarrow M \neq 0$ , there exists  $\delta_0 > 0$  such that  $|x - a| < \delta_0 \Rightarrow |g(x) - M| < |M|/2$ , hence  $|g(x)| \geq |M|/2$ .

Now  $|\frac{f(x)}{g(x)} - \frac{L}{M}| = |\frac{Mf(x) - Lg(x)}{g(x)M}| \leq \frac{|M| \cdot |f(x) - L| + |L| \cdot |g(x) - M|}{|M| \cdot |g(x)|} \leq \left(\frac{2}{|M|}\right) \cdot |f(x) - L| + \left(2|L|\frac{1}{|M|^2}\right) \cdot |g(x) - M|$ .

Choose  $\delta_1, \delta_2 > 0$  with  $|x - a| < \delta_1 \Rightarrow |f(x) - L| < \left(|M|/4\right) \cdot \varepsilon$  and  $|x - a| < \delta_2 \Rightarrow |g(x) - M| < \left(|M|/4\right)^2 \cdot \varepsilon$ .

Let  $\delta = \min(\delta_0, \delta_1, \delta_2)$ . Then for  $0 < |x - a| < \delta$ ,  $|\frac{f(x)}{g(x)} - \frac{L}{M}| \leq \left(\frac{2}{|M|}\right) \cdot \left(|M|/4\right) \cdot \varepsilon + \left(2|L|\frac{1}{|M|^2}\right) \cdot \left(|M|/4\right)^2 \cdot \varepsilon \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ .

Therefore  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$ .

**Info – Limit of Polynomial Functions** Let  $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$  be a polynomial.

Then  $\lim_{x \rightarrow a} p(x) = p(a)$

## Proof

$$\lim_{x \rightarrow a} x = a$$

$$\lim_{x \rightarrow a} x^i = a^i$$

$$\lim_{x \rightarrow a} a_i x^i = a_i a^i$$

$$\lim_{x \rightarrow a} \sum_{i=0}^n a_i x^i = \sum_{i=0}^n a_i a^i$$

### Info – Limit of Rational Functions

Let  $f(x) = \frac{p(x)}{q(x)}$  when  $p, q$  be polynomial functions and  $a \in \mathbb{R}$

1. If  $q(a) \neq 0$  then  $\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$
2. If  $\lim_{x \rightarrow a} q(x) = 0$  but then  $\lim_{x \rightarrow a} p(x) \neq 0$  then  $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$  is DNE.  
If  $x \rightarrow a, x < 0$ , then the limit diverges to  $-\infty$ .  
If  $x \rightarrow a, x > 0$ , then the limit diverges to  $\infty$ .
3. Otherwise,  $p(a) = 0 = q(a)$ , so both  $p(x)$  and  $q(x)$  have  $(x - a)$  as a factor. Divide it out and then repeat the process.

Examples:

$$\begin{aligned} 1. \lim_{x \rightarrow -3} \frac{x^3 + 10x^2 + 13x - 24}{x^2 - 4x - 21} \\ \Rightarrow \lim_{x \rightarrow -3} \frac{(x+3)(x-1)(x+8)}{(x+3)(x-7)} = \lim_{x \rightarrow -3} \frac{(x-1)(x+8)}{(x-7)} = \frac{(-3-1)(-3+8)}{(-3-7)} = \frac{-20}{-10} = 2 \end{aligned}$$

### Info – Squeeze Theorem(Functions):

If  $g(x) \leq f(x) \leq h(x)$  be functions defined in an open interval  $I$  around  $a$  except possibly at  $a$ .

If  $\forall a \in I \setminus \{a\}$  we have  $g(x) < f(x) \leq h(x)$  and  $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$ , then  $\lim_{x \rightarrow a} f(x) = L$

### Tip – When to apply Squeeze Theorem

1. Trigonometric functions with clear bounds and polynomial terms before
2. Exponential Functions with constants terms or by defining a certain interval

2. Evaluate  $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right)$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$$

Notice that  $x^2$  are polynomial function that is defined in  $x \in \mathbb{R}$ .

$$\lim_{x \rightarrow 0} -x^2 = \lim_{x \rightarrow 0} x^2 = 0$$

By Squeeze Theorem,  $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

## One Sided Limits and the Fundamental Trig Limit

1. We say that  $L$  is the **right side limit** of  $f$  at  $a$ , and write  $\lim_{x \rightarrow a^+} f(x) = L$  if  $\forall \varepsilon > 0, \exists \delta > 0$  s.t. if  $|x - a| < \delta$  and  $x > a$  then  $|f(x) - L| < \varepsilon$
2. We say that  $L$  is the **left side limit** of  $f$  at  $a$ , and write  $\lim_{x \rightarrow a^-} f(x) = L$  if  $\forall \varepsilon > 0, \exists \delta > 0$  s.t. if  $|x - a| < \delta$  and  $x < a$  then  $|f(x) - L| < \varepsilon$

### Info – Theorem

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

Example:

Show that  $\lim_{x \rightarrow 0} \sin(x) = 0$ ,  $\lim_{x \rightarrow 0} \cos(x) = 1$ , and  $\lim_{x \rightarrow 0} \tan(x) = 0$

1.  $\lim_{x \rightarrow 0} \sin(x)$ :

Construct a unit circle with a triangle that lies in the first quadrant and an arbitrary point, say  $P(x, y)$ . Then  $P(x, y) = P(\cos(x), \sin(y))$ . The area of the triangle can be represented as  $\frac{1}{2} \sin(x)$ .

Construct another unit circle and draw  $P(x, y)$  at the same location as the previous triangle, however, construct an sector. The area of this new sector is  $\frac{1}{2}x$ .

Notice that the area bounded by the sector is bigger than the triangle.

We then have  $0 \leq \frac{1}{2} \sin(x) \leq \frac{1}{2}x \implies 0 \leq \sin(x) \leq x$ . Since  $\lim_{x \rightarrow 0^+} 0 = \lim_{x \rightarrow 0^+} x = 0$ , by Squeeze Theorem,  $\lim_{x \rightarrow 0^+} \sin(x) = 0$

$\lim_{x \rightarrow 0^-} \sin(x) = 0$  can be achieved similarly to the prove of right side limit and will be omitted.

Thus  $\lim_{x \rightarrow 0} \sin(x) = 0$

2.  $\lim_{x \rightarrow 0} \cos(x) = 1$  :

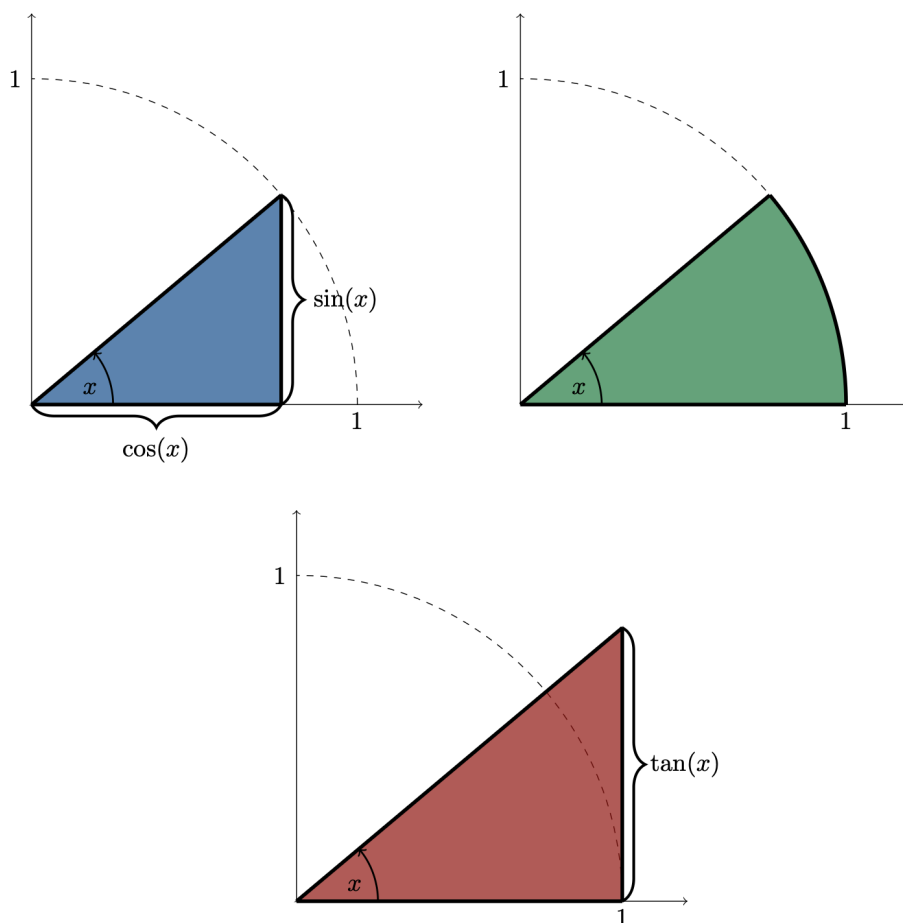
$$\lim_{x \rightarrow 0} \cos(x) = \lim_{x \rightarrow 0} \sqrt{1 - \sin^2(x)} = 1$$

3.  $\lim_{x \rightarrow 0} \tan(x) = \lim_{x \rightarrow 0} \frac{\sin(x)}{\cos(x)} = 0$

### ⚠ Warning –

The Fundamental Trig Limit:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$



We have that  $\frac{1}{2} \cos(x) \sin(x) \leq \frac{1}{2} x \leq \frac{1}{2} \tan(x) \implies \cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}$ .

By Squeeze Theorem,  $\lim_{x \rightarrow 0^+} \frac{\sin(x)}{x} = 1$ .

Since  $\sin(x)$  is an even function, then  $\frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x}$  so  $\lim_{x \rightarrow 0^-} \frac{\sin(x)}{x} = 1$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

Examples:

1.  $\lim_{x \rightarrow 0} \frac{\tan(x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos(x)} = 1 \cdot 1 = 1$
2.  $\lim_{x \rightarrow 0} \frac{\sin(72x)}{\tan(9x)} = \lim_{x \rightarrow 0} \frac{\sin(72x)}{72x} \cdot \frac{9x}{\tan(9x)} \cdot 8 = \lim_{x \rightarrow 0} \frac{\sin(72x)}{72x} \cdot \lim_{x \rightarrow 0} \frac{9x}{\tan(9x)} \cdot 8 = 1 \cdot 1 \cdot 8$
3.  $\lim_{x \rightarrow 0} \frac{\sin(x^2-1)}{\sin(x-1)} = \lim_{x \rightarrow 0} \frac{\sin(x^2-1)}{\sin(x-1)} \cdot \frac{x^2-1}{x^2-1} \cdot \frac{x-1}{x-1} = \lim_{x \rightarrow 0} \frac{\sin(x^2-1)}{x^2-1} \cdot \lim_{x \rightarrow 0} \frac{x-1}{\sin(x-1)} \cdot$   
 $\lim_{x \rightarrow 0} (x+1) = 1 \cdot 1 \cdot 1 = 1$