CH 10 - Complex Numbers

Luke Lu • 2025-12-17

Standard Form

Info — Definition of Complex Numbers

A **complex number** z in **standard form** is an expression of the form $z = x + yi$ where $x, y \in \mathbb{R}$.

The real number x is called the **real part** of z, and is written $\Re(x)$.

The real number y is called the **imaginary part** of z, and is written $\Im(z)$.

The set of complex numbers is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$$

The complex number $z = x + yi$ and $w = u + vi$ are equal $(z = w)$ if and only if $x = u, y = v$

A complex number z is said to be purely real if $y = 0$ (i.e. $1 = 1 + 0i$)

A complex number z is said to be purely imaginary if $x = 0$ (i.e. $i = 0 + 1i$)

0 is purely real and purely imaginary (i.e. $0 = 0 + 0i$)

Info — Complex Arithmetics

Let $z = a + bi$ and $w = c + di$ be complex numbers. Then the

Addition is defined as

$$z + w = (a + c) + (b + d)i$$

Multiplication is defined as

$$zw = (ac - bd) + (ad + bc)i$$

Examples:

Let $z = 2 + 3i, w = -1 + 7i$

1. $z + w = (2 - 1) + (3 + 7)i = 1 + 10i$
2. $zw = (-2 - 21) + (14 - 3)i = -23 + 11i$
3. $i^2 = ii = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + (0 + 0)i = -1$

$\therefore i^2 = -1$

From (3), we can derive a easier way of multiplication

💡 **Tip** — Multiplication trick

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Properties of Complex Arithmetics

1. $z + 0 = 0 + z = z$
2. $z0 = 0z = 0$
3. $z + (-1)z = (-1)z + z = 0$
4. $z1 = 1z = z$

🐳 **Info — Multiplicative Inverse**

For all complex numbers z, the multiplicative inverse of z exists if and only if $z \neq 0$. Moreover, for $a + bi \neq 0$, the multiplicative inverse is unique given by

$$z^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = \frac{a-bi}{a^2+b^2}$$

Examples:

1. $\frac{(1-2i)-(3+4i)}{(5-6i)} = [(1-2i)-(3+4i)] \cdot (5-6i)^{-1} = (-2-6i) \cdot \frac{1}{25+36} \cdot (5+6i)$

$= \frac{1}{61} \cdot (-10 - 30i - 12i + 36) = \frac{26-42i}{61} = \frac{26}{61} - \frac{42}{61}i$

2. $i^{2025} = \left(i^4\right)^{506} \cdot i = (-1)^{506} \cdot i = i$

💡 **Tip — i^n Solutions**

$$i^n = \begin{cases} 1, n \equiv 0 \pmod 4 \\ i, n \equiv 1 \pmod 4 \\ -1, n \equiv 2 \pmod 4 \\ -i, n \equiv 3 \pmod 4 \end{cases}$$

🐳 **Info — Properties of Complex Arithmetics**

In complex arithmetic, the following properties are vlaid for $u, v, z \in \mathbb{C}$

1. Associativity of addition: $(u + v) + z = u + (v + z)$
2. Communtativity of addition: $u + v = v + u$
3. Additive identity: $0 = 0 + 0i$ has the property that $z + 0 = z$
4. Additive inverses: If $z = a + bi$, then there exists an additive inverse of z, written $-z$, with the property that $z + (-z) = 0$. The additive inverse of $z = a + bi$ is $-z = -a - bi$
5. Associativity of multiplication: $(uv)z = u(vz)$
6. Communtativity of multiplication: $uv = vu$
7. Multiplicative identity: $1 = 1 + 0i$ has property that $z1 = z$
8. Multiplicative inverses: If $z = a + bi \neq 0$, then there exists a multiplicative inverse of z, written z^{-1}, with the property that $zz^{-1} = 1$. The multiplicative inverse of $z = a + bi \neq 0$ is $z^{-1} = \frac{a-bi}{a^2+b^2}$
9. Distributivity: $z(u + v) = zu + zv$

Example:

Proof of PCA Part 5: $(uv)z = u(vz)\ \forall u, v, z \in \mathbb{C}$

Let $u = a + bi, v = c + di, z = x + yi$ where $a, b, c, d, x, y \in \mathbb{R}$ so that $u, v, z \in \mathbb{C}$

$(uv)z = [(a + bi)(c + di)](x + yi) = (ac - bd + (ad + bc)i)(x + yi) =$

$= ((ac - bd)x - (ad + bc)y) + ((ac - bd)y + (ad + bc)x)i$

$= (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$u(vz) = (a + bi)[(c + di)(x + yi)] = (a + bi)(cx - dy) + (cy + dx)i$

$= (a(cx - dy) - b(cy + dx)) + (a(cy + dx) + b(cx - dy))i$

$= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i$

Thus $(uv)z = u(vz) = (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$\square$

🐳 **Info** — Other Arithmetic of Complex Numbers

For $z \in \mathbb{C}$

1. $z^0 = 1$
2. $z^1 = z$
3. $z^{k+1} = z^k z\ \forall k \in \mathbb{N}$
4. $(z^n)^m = z^{nm}$ and $z^n z^m = z^{n+m}\ \forall n, m \in \mathbb{N} \cup \{0\}$

For $n \notin \mathbb{N} \cup \{0\}$ will be discussed in later lectures

Examples:

Find a real solution to $6z^3 + \left(1 + 3\sqrt{2}i\right)z^2 - \left(11 - 2\sqrt{2}\right)i - 6 = 0$

Let $z = x \in \mathbb{R}$, that is $z = x + 0i$

$\implies 6x^3 + \left(1 + 3\sqrt{2}i\right)x^2 - \left(11 - \sqrt{2}i\right)x - 6 = 0$

$\implies (6x^3 + x^2 - 11x - 6) + \left(3\sqrt{2}x^2 + 2\sqrt{2}x\right)i = 0 + 0i$

$\implies (6x^3 + x^2 - 11x - 6) = 0$ and $3\sqrt{2}x^2 + 2\sqrt{2}x = 0$

$x = 0$ or $x = -\frac{2}{3}$ for the imaginary part.

However, $x = 0$ does not satisfy the real part.

$\therefore x = -\frac{2}{3} \implies z = -\frac{2}{3} + 0i$

Conjugate and Modulus

🐳 **Info — Complex Conjugate**

The complex **conjugate** of a complext number $z = x + yi$ written $\overline{z}$ is the complex number

$$\overline{z} = x - yi$$

Info — Properties of Conjugate

For the complex conjugate, the following properties hold $\forall z, w \in \mathbb{C}$

1. $\overline{(\overline{z})} = z$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $z + \overline{z} = 2\mathfrak{R}(z)$ and $z - \overline{z} = 2\mathfrak{I}(z)i$
4. $\overline{zw} = \overline{z} \cdot \overline{w}$
5. If $z \neq 0$, $\overline{(z^{-1})} = (\overline{z})^{-1}$
6. If $w \neq 0$, $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

Proof

Part 3

$z + \overline{z} = 2\mathfrak{R}(z)$ and $z - \overline{z} = 2\mathfrak{I}(z)i$

Let $z = x + yi$

$z + \overline{z} = (x + yi) + (x - yi) = 2x + 0i = 2\mathfrak{R}(z)$

$z - \overline{z} = (x + yi) - (x - yi) = 0x + 2i = 2\mathfrak{I}(z)$

Part 4

$\overline{zw} = \overline{z} \cdot \overline{w}$

Let $z = x + yi, w = a + bi$

$\overline{zw} = \overline{(x + yi)(a + bi)} = \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb + ya)i$

$\overline{zw} = \overline{x + yi}\,\overline{a + bi} = (x - yi)(a - bi) = (xa - (-y)(-b)) + (x(-b) + (-y)a)i$

$= (xa - yb) - (xb + ya)i$

Thus $\overline{zw} = \overline{z} \cdot \overline{w}$

$\square$

Examples:

1. Prove $z \in \mathbb{R} \iff z = \overline{z}$

   Let $z = x + yi \ \forall x, y \in \mathbb{R}$

   $\Longrightarrow$

   Suppose $z \in \mathbb{R}$, then $y = 0$, so that $z = x + 0i = x \in \mathbb{R}$.

   Then $\overline{z} = x - 0i = x$

   $\therefore z = \overline{z}$

   $\Longleftarrow$

   Suppose $z = \overline{z}$, then $x + yi = x - yi$

   This implies $x = x, y = -y$. Thus $y = 0$

   $\therefore z = x + 0i = x \in \mathbb{R}$

2. Prove that z is purely imaginary $\iff z = -\overline{z}$
3. Solve $z^2 = i\overline{z}$

   Let $z = x + yi$. Then $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$

   and $i\overline{z} = i(x - yi) = ix - i^2y = xi + y$

   Then the equation becomes $x^2 - y^2 + 2xyi = xi + y$

   $\implies \begin{cases} x^2 - y^2 = y \\ 2xy = x \end{cases} \implies x = 0 \land y = \frac{1}{2} \implies \begin{cases} x^2 - \frac{1}{4} = \frac{1}{2} \\ -y^2 = 0 \end{cases} \implies z = \left\{0, -i, \pm\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}i\right\}$

Modulus of Complex Numbers

Info — Modulus of Complex Number

The **modulus** of the complex number $z = x + yi$, written $|z|$, is the non-negative real number

$$|z| = \sqrt{x^2 + y^2}$$

Info — Properties of Modulus

For the modulus, the following properties $\forall z, w \in \mathbb{C}$:

1. $|z| = 0 \iff z = 0$
2. $|\overline{z}| = |z|$
3. $\overline{z}z = |z|^2$
4. $|zw| = |z||w|$
5. If $z \neq 0$ then $|z^{-1}| = |z|^{-1}$

Side note: for $z \neq 0$, $z^{-1} = \frac{\overline{z}}{|z|^2}$

6. If $w \neq 0$, $\frac{z}{w} = z \cdot \frac{\overline{w}}{|w|^2}$
7. $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$
8. $\overline{\sum z_i} = \sum \overline{(z_i)}$
9. $\overline{\prod z_i} = \prod \overline{(z_i)}$
10. $|\prod z_i| = \prod(|z_i|)$

Proof

Part 3

$\overline{z}z = |z|^2$

Let $z = a + bi, \forall a, b \in \mathbb{R}$

then $\overline{z}z = (a - bi)(a + bi) = (a^2 - (-b)(b)) + (ab + (-b)a)i$

$= a^2 + b^2 + 0i = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2$

$\square$

The Complex Plane and Polar Form

Complex Plane

💡 **Tip — Geometric Interpretation and Graphical Properties**

- z and $\overline{z}$ are reflection of each other over real axis
- Modulus is the distance from the point z to origin
- For addition, it is similar to vector addition that is the parallelogram rule, $z + w$
- For subtraction, consider $z + w - w = z$ and the rest is same for addition

🐳 **Info — Triangle Inequality**

For all $z, w \in \mathbb{C}$, we have

$$|z + w| \leq |z| + |w|$$

Note that $|z|$ is the modulus of z, not absolute value

Proof

Let $z = x + yi, w = u + vi$, where $x, y, u, v \in \mathbb{R}$

$|z + w| = |(x + u) + (y + v)i| = \sqrt{(x + u)^2 + (y + v)^2}$
$= \sqrt{(x - (-u))^2 + (y - (-v))^2}$

$(\text{The Euclidean distance formulat between } (x, y) \text{ and } (-u, -v))$

Consider thr triangle ABC constructed from points

$A : (0, 0);\ \ B(x, y) = (z = x + yi);\ \ C : (-u, -v) = (-w = -u - vi)$

Let $l_{AB} = \text{length of side AB}, l_{BC}$ and l_{AC} have the similar constructed

From geometric perspective, $l_{BC} \leq l_{AB} + l_{AC}$

Note that $l_{AB} = \sqrt{x^2 + y^2} = |z|, l_{AC} = \sqrt{(-u)^2 + (-v)^2} = |w|$

$l_{BC} = \sqrt{(x - (-u))^2 + (y - (-v))^2} = |z + w|$

Therefore $|z + w| \leq |z| + |w|$

$\square$

Exercise:

Let $z \neq \pm i$. Prove that $\frac{z}{1+z^2}$ is real $\Longleftrightarrow z \in \mathbb{R}$ or $|z = 1|$

Polar Form

🐳 Info — Polar Form

A **polar form** of a complex number z is denoted

$$z = r(\cos\theta + i\sin\theta)$$

where $r \geq 0$, being the modulus of z and angle $\theta \in \mathbb{R}$ be an **argument** of z

- θ is not unique unless given restriction of $\theta \in [0, 2\pi)$

Notice that in standard form $z = x + yi$

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = |z| = \sqrt{x^2 + y^2}$
- $\theta = \arctan\left(\frac{y}{x}\right)$
- $\overline{z} = r(\cos(-\theta) + i\sin(-\theta))$

Examples:

Convert polar form to standard form

1. $z = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$
2. $z = \cos\frac{15}{6}\pi + i\sin\frac{15}{6}\pi = 0 + 1i = i$

Convert from standard form to polar form

3. $z = \sqrt{6} + \sqrt{2}i$

   $r = \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2} = 2\sqrt{2}$

   $\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ is one possibility as the angle is not unique

   $z = 2\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
4. $z = -3\sqrt{2} + 3\sqrt{6}i = 6\sqrt{2}\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right)$

🐳 Info — Polar Multiplication for $\mathbb{C}$

For all complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then their product is

$$z_1 z_2 = r_1 r_2(\cos\theta(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Proof

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \in \mathbb{C}$

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta + 2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$= r_1 r_2(\cos\theta(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

□

Examples:

1. Compute $(i+1)i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \overset{\text{By PMC}}{=} \sqrt{2}\left(\cos\frac{3\pi}{4} i\sin\frac{3\pi}{4}\right)$
   $= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -1 + i$
2. Find $\left(\sqrt{6} + \sqrt{2}i\right)\left(-3\sqrt{2} + 3\sqrt{6}i\right)$
   $= 2\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \cdot 6\sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 24\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 24\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$
   $= -12\sqrt{3} + 12i$

De Moivre's Theorem

Info — De Moivre's Theorem (DMT)

$\forall \theta, n \in \mathbb{R}$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Info — Corollary of DMT

For all complex number $z = r(\cos\theta + i\sin\theta)$ and integer n, except $|z| = r = 0$ and $n < 0$ we have

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Examples:

1. $\left(\sqrt{3} - i\right)^{10} = \left[2\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)\right]^{10} = 1024\left(\cos -\frac{10\pi}{6} + i\sin -\frac{10\pi}{6}\right) = 512 + 512\sqrt{3}i$
2. Prove that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ (Hint: Consider $(\cos\theta + i\sin\theta)^4$ expand normally and apply DMT then compare real parts)
3. Prove that $\forall n \in \mathbb{Z}$ if $w \in \mathbb{C}, |w| = 1$ and θ is an argument of w, then $-\frac{i}{2}(w^n - w^{-n} = \sin(n\theta))$

Info — Additional Information

For notation, we write $\operatorname{cis}(\theta) = \cos\theta + i\sin\theta$ notation wise

Also, $\operatorname{cis}(\theta) = e^{i\theta}$ that is having the similar properties of exponentials

$e^{i\pi} = \operatorname{cis}(\pi) = -1 \implies e^{i\pi} + 1 = 0$ which is the Euler's Formula

Examples:

1. How many square roots does 64 has $\in \mathbb{C}$?

   ANS: Write $z = \operatorname{cis}(\theta), 64 = 64(1 + 0i) = 64(\cos 0 + i\sin 0)$.

   Then $z^2 = r^2 \operatorname{cis}(2\theta) \implies 64 \operatorname{cis}(0)$ thus $r^2 = 64$ and $2\theta = 0 + 2\pi k,\ \forall k \in \mathbb{Z}$.

   $r = 8$ since $r \geq 0$ being the modulus. $\theta = \pi k \implies$ only two unique positions in circle

   $\therefore r = 8, \theta = 0, \pi$ we get two solution. that is $z_1 = 8 \operatorname{cis}(0) = 8, z_2 = 8 \operatorname{cis}(\pi) = -8$

2. How many cube roots does 64 has $\in \mathbb{C}$

   ANS: $z^3 = 64 \Longrightarrow r^3 \text{ cis}(3\theta), r^3 = 64, 3\theta = 0 + 2\pi k, \ \forall k \in \mathbb{Z}$

   $\Longrightarrow r = 4, \theta = \frac{2\pi}{3}k = 0, \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}, \pm 2\pi$

   Three unique positions: $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

   $\therefore z_1 = 4 \text{ cis}(0) = 4, z_2 = 4 \text{ cis}\left(\frac{2\pi}{3}\right) = -2 + 2\sqrt{3}i, z_3 = 4 \text{ cis}\left(\frac{4\pi}{3}\right) = -2 - 2\sqrt{3}i$

Info — Complex n-th Root Theorem

For a complex number a and positive integer n, the complex numbers z that satisfy the equation

$$z^n = a$$

are called the **complex nth root** of a

For all complex numbers $a = r(\cos\theta + i\sin\theta)$ and natural numbers n, the complex n-th root of a are given by

$$\sqrt[n]{r}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \quad k = 0, 1, 2, 3, ..., n - 1$$

Examples:

1. Find all $z \in \mathbb{C}$ that satisfy $z^4 = -27\overline{z}$

   Let $z = r \text{ cis}(\theta)$ then $\overline{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta)) = r \text{ cis}(-\theta)$

   $r^4 \text{ cis}(4\theta) = -27 \cdot r \text{ cis}(-\theta)$

   $r = 0 \Longrightarrow z_1 = 0 \text{ cis}0 = 0$

   for $z \neq 0, r \neq 0 \Longrightarrow r^3 \text{ cis}(4\theta) = -27 \text{ cis}(-\theta) \Longrightarrow r^3 \text{ cis}(5\theta) = 27 \text{ cis}(0) = 27 \text{ cis}(\pi)$

   $r = 3, 5\theta = \pi + 2\pi k, \ \forall k \in \mathbb{Z}$

   $\theta = \frac{\pi + 2\pi k}{5} = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$ are unique poisitions on the circle

   $z_2 = 3 \text{ cis}\left(\frac{\pi}{5}\right), z_3 = 3 \text{ cis}\left(\frac{3\pi}{5}\right), z_4 = 3 \text{ cis}(\pi), z_5 3 \text{ cis}\left(\frac{7\pi}{5}\right), z_6 = 3 \text{ cis}\left(\frac{9\pi}{5}\right)$

2. $z^8 = 1$

   ANS: $z \in \left\{\text{cis}\left(\frac{k\pi}{4}\right), \forall k \in \{0, 1, 2, 3, 4, 5, 6, 7\}\right\}$

Info — Quadratic Formula for $\mathbb{C}$

For all complex numbers a, b, c with $a \neq 0$, the solution to $az^2 + bz + c = 0$ are given by

$$z = \frac{-b \pm w}{2a}$$

where w is a solution to $w^2 = b^2 - 4ac$

Examples:

1. $z^2 - 2z + 6 - 12i = 0$
   ANS: $z = 3 + 3i, -1 - 3i$

🐳 **Info — Polynomials**

An expression in form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

Where $a_i \in \mathbb{R}(\text{or } \mathbb{C})\ \forall i \in \mathbb{N} \cup \{0\}$

We denote $\mathbb{R}[x]$ or $\mathbb{C}[x]$ be a Polynomials.

The polynomial where x^n has coefficient non-zero, then we say the degree of polynomial is n.

Zero polynomial has all coefficients equal to zero with degree undefined.

A constant polynomial is either the zero polynomial or polynomial of degree zero.

Two polynomials are equal if and only if all coefficients are equal

🐳 **Info — Arithmetic**

Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$

- Addition

$$f(x) + g(x) = \sum_{k=0}^{\max(m,n)} (a_k + b_k) x^k$$

- Multiplication

$$f(x) g(x) = \sum_{l=0}^{m+n} (a_l b_{m+n-l}) x^l$$

Example:

Consider $f(x) = ix^3 + 4x^2 - ix + 6$ and $g(x) = x - 3i$, prove that $g(x)$ is a factor of $f(x)$ and find the complex polynomial $h(x)$ s.t. $f(x) = g(x)h(x)$

ANS: $h(x) = \frac{ix^3+4x^2-ix+6}{x-3i} = ix^2 + x + 2i$

Roots of Polynomial / Factoring Polynomial

🐳 **Info — Factoring Theorem**

For all polynomial $f(x) \in \mathbb{R}[x](\text{or } \mathbb{C}[x])$ and all $c \in \mathbb{R}(\text{or } \mathbb{C})$, the linear polynomial $x - c$ is a factor of polynomial $f(x)$ if and only if $f(c) = 0$ (equivalently, c is a root of the polynomial $f(x)$)

Examples:

1. Determine if $x = 1$ is a root of $f(x) = x^3 - 2x^2 + 4x - 5$

   $f(1) = -2 \neq 0 \Longrightarrow$ not a root

2. Proof that there does not exist a real linear factor of $f(x) = x^8 + x^3 + 1$

   Consider interval $x \in [0, \infty), (-\infty, -1], (-1, 0)$ separately

- $[0, \infty) : f(0) = 1 > 0$ then $f(x) \geq 1 \ \forall x \in [0, \infty)$
- $(-\infty, -1] : f(x) = x^3(x^5 + 1) + 1 \Longrightarrow (\leq -1)(\leq 0) + 1 \Longrightarrow (\geq 0) + 1 \Longrightarrow \geq 0$
- $(-1, 0) :$
  Notice that $-1 < x^3 < 1 \ \forall x \in (-1, 0)$ and $x^8 > 0 \ \forall x \in (-1, 0)$.
  So that $x^3 + 1 > 0 \ \forall x \in (-1, 0)$ thus $f(x) \geq 0 \ \forall x \in (-1, 0)$

> 🐳 **Info — FUndamental Theorem of Algebra**
> For all complex polynomial $f(z)$ with degree greater than 1, there exists a $z_0 \in \mathbb{C}$ s.t. $f(z_0) = 0$

> 🐳 **Info — Complex Polynomial of Degree n have n roots**
> For all integers $n \geq 1$ and all complex polynomials $f(z)$ of degree n, there exist complex numbers $c \neq 0$ and $c_1, c_2, c_3, \ldots, c_n$ s.t.
>
> $$f(z) = c(z - c_1)(z - c_2)\ldots(z - c_n)$$
>
> Moreover, the roots of $f(z)$ are $c_1, c_2, c_3, \ldots, c_n$
>
> For all integers $n \geq$ and all real or complex polynomials $f(x)$ degree n, the polynoial f(x) has at most n roots

Examples:

1. - $x^2 + 1$ is irreducible in $\mathbb{R}[x]$,
   - Can be reduced in $\mathbb{C}[x] : (x - i)(x + i)$
2. - $x^4 + 2x^2 + 1$ is reducible but has no real roots
   - $x^4 + 2x^2 + 1$ is reducible and has real roots: $(x - i)^2(x + i)^2$

> 🐳 **Info — Conjugate Roots Theorem**
> For all olynomials $f(x)$ with real coefficient, if $c \in \mathbb{C}$ is a root of $f(x)$, then $\overline{c} \in \mathbb{C}$ is a root of $f(x)$

Proof

Let $f(x)$ be a polynomail with real coefficients.

Suppose $c \in \mathbb{C}$ is a root of $f(x)$.

We write $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$.

Since c is a root of $f(x) \Longrightarrow f(c) = 0$

$a_n c^n + a_{n-1} c^{n-1} + \ldots + a_2 c^2 + a_1 c + a_0 = 0$

$\overline{a_n c^n + a_{n-1} c^{n-1} + \ldots + a_2 c^2 + a_1 c + a_0} = \overline{0}$

$\overline{a_n c^n} + \overline{a_{n-1} c^{n-1}} + \ldots + \overline{a_2 c^2} + \overline{a_1 c} + \overline{a_0} = 0$

$a_n \overline{c}^n + a_{n-1} \overline{c}^{n-1} + \ldots + a_2 \overline{c}^2 + a_1 \overline{c} + a_0 = 0$

Therefore $\overline{c}$ is a root of $f(x)$

Example:

Express $x^4 - 5x^3 + 16x^2 - 9x - 13$ as a product of irreducible factors given $2 - 3i$ is a root

By CJRT, $2 + 3i$ is also a root, so $(x - 2 - 3i)$, $(x - 2 + 3i)$ are factors of this polynomial

$(x - 2 - 3i)(x - 2 + 3i) = x^2 - 4x + 13$ is a factor of the polynomial

$\frac{x^4-5x^3+16x^2-9x-13}{x^2-4x+13} = x^2 - x - 1 = \left(x - \left(\frac{1+\sqrt{5}}{2}\right)\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$

$(x - 2 - 3i)(x - 2 + 3i)\left(x - \left(\frac{1+\sqrt{5}}{2}\right)\right)\left(x - \frac{1-\sqrt{5}}{2}\right) \in \mathbb{C}[x]$

$= (x^2 - 4x + 13)\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right) \in \mathbb{R}[x]$

🐳 **Info** — Lemma

For all complex number c, $(x - c)(x - \overline{c}) \in \mathbb{R}[x]$

🐳 **Info — Real Factors or Real Polynomials**

For all real polynomial $f(x)$ of positive degree, $f(x)$ can be written as a product of real linear and real quadratic factors

Example:

One root of the polynomial $f(x) = 3x^5 - 14x^4 + 14x^3 + 32x^2 - 69x + 30$ is $2 + i$). Write $f(x)$ as a product of irreducible polynomials in $\mathbb{R}[x]$ and $\mathbb{C}[x]$

Given $2 + i$ is a root $\Longrightarrow (x - (2 + i))(x - (2 - i)) = x^2 - 4x + 5$ is a factor.

$f(x) = (x^2 - 4x + 5)(3x^3 - 2x^2 - 9x + 6) = (x^2 - 4x + 5)(3x - 2)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) \in \mathbb{R}[x]$

$f(x) = (x - 2 - i)(x - 2 + i)(3x - 2)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) \in \mathbb{C}[x]$