CH 3 - Proving Mathematical Statements

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Definitions

1. **Proposition** — a statement to be proved true
2. **Theorem** — a significant proposition
3. **Lemma** — a subsidiary proposition
4. **Corollary** — a proposition that follows almost immediately from a theorem

Proving Universally Quantified Statements

1. Choose a representative object $x \in S$ (let x be arbitrary in S)
2. Show the open sentence is true for this x using facts about S

Example

Prove $\forall x, y \in \mathbb{R}, x^4 + x^2 y + y^2 \geq 5x^2 y - 3y^2$

Discovery

If $x^4 + x^2 y + y^2 \geq 5x^2 y - 3y^2$ => $x^4 - 4x^2 y + 4y^2 \geq 0$ => $\left(x^2 - 2y\right)^2 \geq 0$

This is a discovery, not a proof

Proof

Let $x, y \in \mathbb{R}$ be arbitrary

Then $\left(x^2 - 2y\right)^2 \geq 0$

So $x^4 - 4x^2 y + 4y^2 \geq 0$

Hence $x^4 + x^2 y + y^2 - 5x^2 y + 3y^2 \geq 0$

$\forall x, y \in \mathbb{R}, x^4 + x^2 y + y^2 \geq 5x^2 y - 3y^2$

Disprove Universally Quantified Statement

To disprove $\forall x \in S, P(x)$, find $x \in S$ with $\neg, P(x)$

Example

Disprove $\forall x \in \mathbb{R}, x^2 = 5$

Proof

Let $x = 0$

Then $x^2 = 0 \neq 5$

$\exists x \in \mathbb{R}$ with $x^2 \neq 5$, so $\forall x \in \mathbb{R}, x^2 = 5$ is false

Prove Existentially Quantified Statement

Find a specific $x \in S$ that makes the sentence true

Example 1

Prove $\exists m \in \mathbb{Z}$ s.t. $\frac{m-7}{2m+4} = 5$

Proof

$m - 7 = 5(2m + 4)$ => $m - 7 = 10m + 10$ => $-27 = 9m$ => $m = -3$

Let $m = -3$ and note $2m + 4 = -2 \neq 0$

Then $\frac{m-7}{2m+4} = \frac{-3-7}{2(-3)+4} = \frac{-10}{-6+4} = \frac{-10}{-2} = 5$

$\exists m \in \mathbb{Z}$ with $\frac{m-7}{2m+4} = 5$

Example 2

Prove there exists a perfect square k s.t. $k^2 - \frac{31}{2}k = 8$

Proof

Let $k = 16 = 4^2$

Then $k^2 - \frac{31}{2}k = 256 - 248 = 8$

There exists a perfect square k with $k^2 - \frac{31}{2}k = 8$

Disprove Existentially Quantified Statement

To disprove $\exists x \in S, P(x)$, prove $\forall x \in S, \neg, P(x)$

Example

Disprove $\exists x \in \mathbb{R}$ s.t. $\cos(2x) + \sin(2x) = 3$

Proof

For all $x \in \mathbb{R}$, we have $-1 \leq \cos(2x) \leq 1$ and $-1 \leq \sin(2x) \leq 1$

So $-2 \leq \cos(2x) + \sin(2x) \leq 2$

Thus $\cos(2x) + \sin(2x) \neq 3$ since $3 \notin [-2, 2]$

$\forall x \in \mathbb{R}, \cos(2x) + \sin(2x) \neq 3$ i.e. $\neg, (\exists x \in \mathbb{R}, \cos(2x) + \sin(2x) = 3)$

Prove/Disprove Nested Quantified Statement

Consider.

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$
2. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1$

1. True

   Let $x \in \mathbb{R}$ and set $y = \sqrt[3]{x^3 - 1}$

   Then $x^3 - y^3 = x^3 - \left(\sqrt[3]{x^3 - 1}\right)^3 = x^3 - (x^3 - 1) = 1$

   $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$

2. False

The negation is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ with $x^3 - y^3 \neq 1$

Let $x \in \mathbb{R}$ and choose $y = x$

Then $x^3 - y^3 = x^3 - x^3 = 0 \neq 1$

$\neg\left(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1\right)$