CH 10 - Complex Numbers

Luke Lu • 2025-11-21

Standard Form

🐳 **Info — Definition of Complex Numbers**

A **complex number** z in **standard form** is an expression of the form $z = x + yi$ where $x, y \in \mathbb{R}$.

The real number x is called the **real part** of z, and is written $\mathfrak{R}(x)$.

The real number y is called the **imaginary part** of z, and is written $\mathfrak{I}(z)$.

The set of complex numbers is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$$

The complex number $z = x + yi$ and $w = u + vi$ are equal $(z = w)$ if and only if $x = u, y = v$

A complex number z is said to be purely real if $y = 0$ (i.e. $1 = 1 + 0i$)

A complex number z is said to be purely imaginary if $x = 0$ (i.e. $i = 0 + 1i$)

0 is purely real and purely imaginary (i.e. $0 = 0 + 0i$)

🐳 **Info — Complex Arithmetics**

Let $z = a + bi$ and $w = c + di$ be complex numbers. Then the

Addition is defined as

$$z + w = (a + c) + (b + d)i$$

Multiplication is defined as

$$zw = (ac - bd) + (ad + bc)i$$

Examples:

Let $z = 2 + 3i, w = -1 + 7i$

1. $z + w = (2 - 1) + (3 + 7)i = 1 + 10i$
2. $zw = (-2 - 21) + (14 - 3)i = -23 + 11i$
3. $i^2 = ii = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + (0 + 0)i = -1$

$\therefore i^2 = -1$

From (3), we can derive a easier way of multiplication

💡 **Tip** — Multiplication trick

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Properties of Complex Arithmetics

1. $z + 0 = 0 + z = z$
2. $z0 = 0z = 0$
3. $z + (-1)z = (-1)z + z = 0$
4. $z1 = 1z = z$

🐳 **Info — Multiplicative Inverse**

For all complex numbers z, the multiplicative inverse of z exists if and only if $z \neq 0$. Moreover, for $a + bi \neq 0$, the multiplicative inverse is unique given by

$$z^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i = \frac{a - bi}{a^2+b^2}$$

Examples:

1. $\frac{(1-2i)-(3+4i)}{(5-6i)} = [(1-2i)-(3+4i)] \cdot (5-6i)^{-1} = (-2-6i) \cdot \frac{1}{25+36} \cdot (5+6i)$

$= \frac{1}{61} \cdot (-10 - 30i - 12i + 36) = \frac{26-42i}{61} = \frac{26}{61} - \frac{42}{61} i$

2. $i^{2025} = \left(i^4\right)^{506} \cdot i = (-1)^{506} \cdot i = i$

💡 **Tip — i^n Solutions**

$$i^n = \begin{cases} 1, n \equiv 0 \pmod 4 \\ i, n \equiv 1 \pmod 4 \\ -1, n \equiv 2 \pmod 4 \\ -i, n \equiv 3 \pmod 4 \end{cases}$$

🐳 **Info — Properties of Complex Arithmetics**

In complex arithmetic, the following properties are vlaid for $u, v, z \in \mathbb{C}$

1. Associativity of addition: $(u+v)+z = u+(v+z)$
2. Communtativity of addition: $u + v = v + u$
3. Additive identity: $0 = 0 + 0i$ has the property that $z + 0 = z$
4. Additive inverses: If $z = a + bi$, then there exists an additive inverse of z, written $-z$, with the property that $z + (-z) = 0$. The additive inverse of $z = a + bi$ is $-z = -a - bi$
5. Associativity of multiplication: $(uv)z = u(vz)$
6. Communtativity of multiplication: $uv = vu$
7. Multiplicative identity: $1 = 1 + 0i$ has property that $z1 = z$
8. Multiplicative inverses: If $z = a + bi \neq 0$, then there exists a multiplicative inverse of z, written z^{-1}, with the property that $zz^{-1} = 1$. The multiplicative inverse of $z = a + bi \neq 0$ is $z^{-1} = \frac{a-bi}{a^2+b^2}$
9. Distributivity: $z(u+v) = zu + zv$

Example:

Proof of PCA Part 5: $(uv)z = u(vz) \ \forall u, v, z \in \mathbb{C}$

Let $u = a + bi, v = c + di, z = x + yi$ where $a, b, c, d, x, y \in \mathbb{R}$ so that $u, v, z \in \mathbb{C}$

$(uv)z = [(a + bi)(c + di)](x + yi) = (ac - bd + (ad + bc)i)(x + yi) =$

$= ((ac - bd)x - (ad + bc)y) + ((ac - bd)y + (ad + bc)x)i$

$= (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$u(vz) = (a + bi)[(c + di)(x + yi)] = (a + bi)(cx - dy) + (cy + dx)i$

$= (a(cx - dy) - b(cy + dx)) + (a(cy + dx) + b(cx - dy))i$

$= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i$

Thus $(uv)z = u(vz) = (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$\square$

🐳 **Info** — Other Arithemtic of Complex Numbers

For $z \in \mathbb{C}$

1. $z^0 = 1$
2. $z^1 = z$
3. $z^{k+1} = z^k z \ \forall k \in \mathbb{N}$
4. $(z^n)^m = z^{nm}$ and $z^n z^m = z^{n+m} \ \forall n, m \in \mathbb{N} \cup \{0\}$

For $n \notin \mathbb{N} \cup \{0\}$ will be discussed in later lectures

Examples:

Find a real solution to $6z^3 + \left(1 + 3\sqrt{2}i\right)z^2 - \left(11 - 2\sqrt{2}\right)i - 6 = 0$

Let $z = x \in \mathbb{R}$, that is $z = x + 0i$

$\Longrightarrow 6x^3 + \left(1 + 3\sqrt{2}i\right)x^2 - \left(11 - \sqrt{2}i\right)x - 6 = 0$

$\Longrightarrow (6x^3 + x^2 - 11x - 6) + \left(3\sqrt{2}x^2 + 2\sqrt{2}x\right)i = 0 + 0i$

$\Longrightarrow (6x^3 + x^2 - 11x - 6) = 0$ and $3\sqrt{2}x^2 + 2\sqrt{2}x = 0$

$x = 0$ or $x = -\frac{2}{3}$ for the imaginary part.

However, $x = 0$ does not satisfy the real part.

$\therefore x = -\frac{2}{3} \Longrightarrow z = -\frac{2}{3} + 0i$

Conjugate and Modulus

🐳 **Info — Complex Conjugate**

The complex **conjugate** of a complext number $z = x + yi$ written $\overline{z}$ is the complex number

$$\overline{z} = x - yi$$

Info — Properties of Conjugate

For the complex conjugate, the following properties hold $\forall z, w \in \mathbb{C}$

1. $\overline{(\overline{z})} = z$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $z + \overline{z} = 2\mathfrak{R}(z)$ and $z - \overline{z} = 2\mathfrak{I}(z)i$
4. $\overline{zw} = \overline{z} \cdot \overline{w}$
5. If $z \neq 0$, $\overline{(z^{-1})} = (\overline{z})^{-1}$
6. If $w \neq 0$, $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

Proof

Part 3

$z + \overline{z} = 2\mathfrak{R}(z)$ and $z - \overline{z} = 2\mathfrak{I}(z)i$

Let $z = x + yi$

$z + \overline{z} = (x + yi) + (x - yi) = 2x + 0i = 2\mathfrak{R}(z)$

$z - \overline{z} = (x + yi) - (x - yi) = 0x + 2i = 2\mathfrak{I}(z)$

Part 4

$\overline{zw} = \overline{z} \cdot \overline{w}$

Let $z = x + yi, w = a + bi$

$\overline{zw} = \overline{(x + yi)(a + bi)} = \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb + ya)i$

$\overline{zw} = \overline{x + yi}\,\overline{a + bi} = (x - yi)(a - bi) = (xa - (-y)(-b)) + (x(-b) + (-y)a)i$

$= (xa - yb) - (xb + ya)i$

Thus $\overline{zw} = \overline{z} \cdot \overline{w}$

□

Examples:

1. Prove $z \in \mathbb{R} \iff z = \overline{z}$

   Let $z = x + yi \ \forall x, y \in \mathbb{R}$

   $\Longrightarrow$

   Suppose $z \in \mathbb{R}$, then $y = 0$, so that $z = x + 0i = x \in \mathbb{R}$.

T. hen $\overline{z} = x - 0i = x$

$\therefore z = \overline{z}$

$\Longleftarrow$

Suppose $z = \overline{z}$, then $x + yi = x - yi$

This implies $x = x, y = -y$. Thus $y = 0$

$\therefore z = x + 0i = x \in \mathbb{R}$

2. Prove that z is purely imaginary $\Longleftrightarrow z = -\overline{z}$
3. Solve $z^2 = i\overline{z}$

   Let $z = x + yi$. Then $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$

   and $i\overline{z} = i(x - yi) = ix - i^2 y = xi + y$

   Then the equation becomes $x^2 - y^2 + 2xyi = xi + y$

   $\Longrightarrow \begin{cases} x^2 - y^2 = y \\ 2xy = x \end{cases} \Longrightarrow x = 0 \land y = \frac{1}{2} \Longrightarrow \begin{cases} x^2 - \frac{1}{4} = \frac{1}{2} \\ -y^2 = 0 \end{cases} \Longrightarrow z = \left\{0, -i, \pm\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}i\right\}$

Modulus of Complex Numbers

🐳 **Info — Modulus of Complex Number**

The **modulus** of the complex number $z = x + yi$, written $|z|$, is the non-negative real number

$$|z| = \sqrt{x^2 + y^2}$$

🐳 **Info — Properties of Modulus**

For the modulus, the following properties $\forall z, w \in \mathbb{C}$:

1. $|z| = 0 \Longleftrightarrow z = 0$
2. $|\overline{z}| = |z|$
3. $\overline{z}z = |z|^2$
4. $|zw| = |z||w|$
5. If $z \neq 0$ then $|z^{-1}| = |z|^{-1}$

Side note: for $z \neq 0$, $z^{-1} = \frac{\overline{z}}{|z|^2}$

6. If $w \neq 0$, $\frac{z}{w} = z \cdot \frac{\overline{w}}{|w|^2}$
7. $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$
8. $\overline{\sum z_i} = \sum \overline{(z_i)}$
9. $\overline{\prod z_i} = \prod \overline{(z_i)}$
10. $|\prod z_i| = \prod(|z_i|)$

Proof

Part 3

$\overline{z}z = |z|^2$

Let $z = a + bi, \forall a, b \in \mathbb{R}$

then $\overline{z}z = (a - bi)(a + bi) = (a^2 - (-b)(b)) + (ab + (-b)a)i$

$= a^2 + b^2 + 0i = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2$

□

The Complex Plane and Polar Form

Complex Plane

💡 **Tip — Geometric Interpretation and Graphical Properties**

- z and $\overline{z}$ are reflection of each other over real axis
- Modulus is the distance from the point z to origin
- For addition, it is similar to vector addition that is the parallelogram rule, $z + w$
- For subtraction, consider $z + w - w = z$ and the rest is same for addition

🐳 **Info — Triangle Inequality**

For all $z, w \in \mathbb{C}$, we have

$$|z + w| \leq |z| + |w|$$

Note that $|z|$ is the modulus of z, not absolute value

Proof

Let $z = x + yi, w = u + vi$, where $x, y, u, v \in \mathbb{R}$

$|z + w| = |(x + u) + (y + v)i| = \sqrt{(x + u)^2 + (y + v)^2}$

$= \sqrt{(x - (-u))^2 + (y - (-v))^2}$

$(\text{The Euclidean distance formulat between } (x, y) \text{ and } (-u, -v))$

Consider thr triangle ABC constructed from points

$A : (0, 0);\ \ B(x, y) = (z = x + yi);\ \ C : (-u, -v) = (-w = -u - vi)$

Let $l_{AB} = \text{length of side AB}, l_{BC}$ and l_{AC} have the similar constructed

From geometric perspective, $l_{BC} \leq l_{AB} + l_{AC}$

Note that $l_{AB} = \sqrt{x^2 + y^2} = |z|, l_{AC} = \sqrt{(-u)^2 + (-v)^2} = |w|$

$l_{BC} = \sqrt{(x - (-u))^2 + (y - (-v))^2} = |z + w|$

Therefore $|z + w| \leq |z| + |w|$

□

Exercise:

Let $z \neq \pm i$. Prove that $\frac{z}{1+z^2}$ is real $\Longleftrightarrow z \in \mathbb{R}$ or $|z = 1|$

Polar Form

🐳 **Info — Polar Form**

A **polar form** of a complex number z is denoted

$$z = r(\cos\theta + i\sin\theta)$$

where $r \geq 0$, being the modulus of z and angle $\theta \in \mathbb{R}$ be an **argument** of z

- θ is not unique unless given restriction of $\theta \in [0, 2\pi)$

Notice that in standard form $z = x + yi$

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = |z| = \sqrt{x^2 + y^2}$
- $\theta = \arctan\left(\frac{y}{x}\right)$

Examples:

Convert polar form to standard form

1. $z = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$
2. $z = \cos\frac{15}{6}\pi + i\sin\frac{15}{6}\pi = 0 + 1i = i$

Convert from standard form to polar form

3. $z = \sqrt{6} + \sqrt{2}i$

   $r = \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2} = 2\sqrt{2}$

   $\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ is one possibility as the angle is not unique

   $z = 2\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
4. $z = -3\sqrt{2} + 3\sqrt{6}i = 6\sqrt{2}\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right)$