CH 1 — Vectors

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Introduction

> 🐳 **Info — Vector**
>
> The set $\mathbb{R}^n$ is defined as $\left\{ \vec{x} = \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} : x_1, ..., x_n \in \mathbb{R} \right\}$
>
> A **vector** is an element $\vec{x} = \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ of $\mathbb{R}^n$
>
> The row notation of $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix}$ is $\vec{v} = [v_1 \ v_2 \ v_3]^T$

Operations

> 🐳 **Info — Equality**
>
> We say that vectors $\vec{w} = \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$ are **equal** if $u_i = v_i \, \forall \ i = 1, 2, ..., n$.
>
> Denoted $\vec{w} = \vec{v}$

> 🐳 **Info — Addition and Properties**
>
> Let $\vec{w} = \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ ... \\ w_n \end{pmatrix} \in \mathbb{R}^n.$
>
> Then $\vec{w} + \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ ... \\ u_n + v_n \end{pmatrix}$
>
> 1. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$
> 2. $\vec{w} + \vec{v} + \vec{w} = \vec{w} + (\vec{v} + \vec{w})$
> 3. There is a zero **vector**, $\vec{0} = [0 \ 0 \ 0 \ ... \ 0]^T \in \mathbb{R}^n$
> 4. $\vec{v} + \vec{0} = \vec{v}$

🐳 Info − Additive Inverse

Let $\vec{w} = \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix} \in \mathbb{R}^n$. The additive inverse of $\vec{w}$ denoted $-\vec{w}$ is defined as

$$-\vec{w} = \begin{pmatrix} -u_1 \\ -u_2 \\ ... \\ -u_n \end{pmatrix}$$

$$\vec{w} - \vec{w} = \vec{w} + (-\vec{w}) = \vec{0}$$

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \begin{pmatrix} v_1 - u_1 \\ v_2 - u_2 \\ ... \\ v_n - u_n \end{pmatrix}$$

🐳 Info − Scalar Multiplication

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix}, \vec{w} \in \mathbb{R}^n, c, d \in \mathbb{R}$. Then the scalar product $c\vec{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ ... \\ cv_n \end{pmatrix}$

1. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
2. $c(\vec{w} + \vec{v}) = c\vec{w} + c\vec{v}$
3. $0\vec{w} = \vec{0}$
4. If $c\vec{v} = \vec{0}$ then $c = 0 \lor \vec{v} = 0$

🐳 Info − Standard Basis

In $\mathbb{R}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set $E = \{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$ is called the **standard basis for** $\mathbb{R}^n$

(i.e. $\mathbb{R}^3$ is $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$)

If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix} = v_1\vec{e_1} + v_2\vec{e_2} + ... + v_n\vec{e_n}$ then we call $v_1, v_2, ..., v_n$ the **components of** $\vec{v}$

Vectors in $\mathbb{C}^n$

> 🐳 **Info — Vectors in $\mathbb{C}^n$**
>
> The set $\mathbb{C}^n$ is defined as $\left\{ \vec{z} = \begin{pmatrix} z_1 \\ ... \\ z_n \end{pmatrix} : z_1, ..., z_n \in \mathbb{C} \right\}$
>
> The **vector** is an element $\vec{z} = \begin{pmatrix} z_1 \\ ... \\ z_n \end{pmatrix}$ of $\mathbb{C}^n$
>
> In $\mathbb{C}^n$, let $\vec{e}_i$ be the vector whose i^{th} component is 1 with all other components 0. The set $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ is called the **standard basis for** $\mathbb{C}^n$

Dot Product

> 🐳 **Info — Dot Product**
>
> Let $\vec{w} = \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ ... \\ w_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$. We defined their **dot product** by
>
> $$\vec{w} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + ... + u_n v_n$$
>
> 1. $\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$
>
> 2. $(\vec{w} + \vec{v}) \cdot \vec{w} = \vec{w} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
>
> 3. $(c\vec{w}) \cdot \vec{v} = c(\vec{w} \cdot \vec{v})$
>
> 4. $\vec{w} \cdot \vec{w} \geq 0$, with $\vec{w} \cdot \vec{w} = 0 \Longleftrightarrow \vec{w} = 0$

> 🐳 **Info — Vector Unit Basics**
>
> Let $\vec{v}, \vec{w} \in \mathbb{R}^n$
>
> 1. The **length** of vector $\vec{w}$ is $\|\vec{w}\| = \sqrt{\vec{w} \cdot \vec{w}}$
>
> 2. If $c \in \mathbb{R}, \vec{w} \in \mathbb{R}^n$, then $\|\vec{w}\| = |c| \, \|\vec{w}\|$
>
> 3. $\vec{v}$ is a **unit vector** if $\|\vec{v}\| = 1$
>
> 4. **Normalization** is when some $\vec{v}$ is a non-zero vector,
>
> $$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$
>
>   in the direction of $\vec{v}$ by scaling $\vec{v}$
>
> 5. With $\vec{v}, \vec{v}$ non-zero vectors. The angle $\theta, 0 \leq \theta \leq \pi$ between $\vec{v}$ is such that
>
> $$\vec{w} \cdot \vec{v} = \|\vec{w}\| \|\vec{v}\| \cos \theta \text{ that is } \theta = \arccos\left(\frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\| \|\vec{v}\|} \right)$$
>
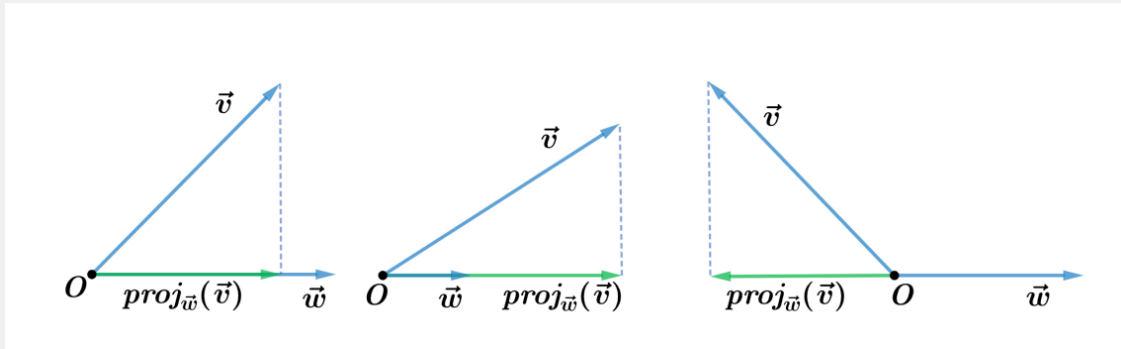> 6. $\vec{w}, \vec{v}$ are **orthogonal/perpendicular** if $\vec{w} \cdot \vec{v} = 0$

Projection

> 🐋 **Info − Projection**
>
> Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq 0$.
>
> 1. The **projection** of $\vec{v}$ onto $\vec{w}$ is defined by
>
> $$\texttt{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$
>
> We also refer to this as the **projection of $\vec{v}$ in the $\vec{w}$ direction**
>
> Illustration of $\texttt{proj}_{\vec{w}}(\vec{v})$:
>
>
>
> 2. We refer to the quantity
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> $$\|\vec{v}\| \cos \theta = \vec{v} \cdot \hat{w}$$
>
> as the **component** (or scalar component) **of $\vec{v}$ along $\vec{w}$**
>
> 3. The **perpendicular** of $\vec{v}$ onto $\vec{w}$ is defined by $\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \texttt{proj}_{\vec{w}}(\vec{v})$
>
> 4. The projection and the perpendicular of a vector $\vec{v}$ onto $\vec{w}$ are orthogonal; that is
>
> $$\text{perp}_{\vec{w}}(\vec{v}) \cdot \texttt{proj}_{\vec{w}}(\vec{v}) = 0$$

Standard Inner Product in $\mathbb{C}^n$

> 🐋 **Info − Standard inner product**
>
> The **standard inner product** of $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix} \in \mathbb{C}^n$ is
>
> $$\texttt{ip}$$