

CH 5 - Applications of Derivatives

Luke Lu • 2025-11-10

Related Rates

Tip — Steps for Related Rates Questions

1. Draw diagram
2. Identify **changing** quantities
3. Find **constant** quantities (if possible)
4. Derive equations relating the quantities that are changing
5. **Implicitly differentiate** the key equations
6. Solve for the desired rate of change, substituting in known quantities.
7. **Concluding statement** (and also check units)

Example:

1. Laindon is taking a hot air balloon ride. A giant fan is blowing hot air into the balloon in a rate of $200 \frac{\text{m}^3}{\text{min}}$. Assuming that at any given point in time the balloon sphere, find the rate at which the radius of the balloon is changing when the diameter is 12 m.

ANS:

1. Picture: The problem is trivial so the graph is omitted
2. Changing variable: Volume(m^3), Radius(m), time(t)
3. Constant quantities: $\frac{dV}{dt} = 200 \frac{\text{m}^3}{\text{min}}$
4. Key Equation: $V = \frac{4}{3}\pi r^3(t)$
5. Implicit Differentiation: $\frac{dV}{dt} = 4\pi r^2(t) \cdot \frac{dr}{dt}$
6. $\frac{dr}{dt} \Big|_{r=6} = \frac{1}{4\pi(6)^2} \cdot 200 = \frac{200}{144\pi} = \frac{25}{18\pi} \frac{\text{m}}{\text{min}}$
7. Concluding statement: When the diameter of the balloon is 12m, the rate of change of the radius is expanding by $\frac{25}{18\pi} \frac{\text{m}}{\text{min}}$
2. The construction workers building M4 accidentally left a 20 foot ladder propped up against a concrete wall that is 80 feet in height. The base of the ladder begins to slide away from the wall at a rate of 2ft/sec, and the top begins to move down as a result. When the base of the ladder is 14 ft from the wall, how fast is the top of the ladder sliding down the wall?

ANS:

1. Picture is omitted and left as an exercise for the reader
2. Changing variable: Distance from wall of base of ladder (m), Height where ladder touches the wall (m)
3. Constant quantities : $\frac{dx}{dt} = 2$
4. Key Equation: $x^2 + y^2 = 20^2$

5. Implicit Differentiation: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$
6. $\frac{dy}{dt} = -\frac{14}{\sqrt{400-14^2}} \cdot 2 = -\frac{14}{\sqrt{51}} \frac{\text{ft}}{\text{sec}}$
7. Concluding statement: When the base of ladder is 14cm, the top of the ladder is falling at a speed of $\frac{14}{\sqrt{51}} \frac{\text{ft}}{\text{sec}}$

Extrema

Info – Extrema

Let $f(x)$ be a function defined on an interval I , and let $c \in I$. We say f has

1. A **global minimum** on I at $x = c$ if $f(c) \leq f(x) \forall x \in I$
2. A **global maximum** on I at $x = c$ if $f(c) \geq f(x) \forall x \in I$
3. A **global extremum** on I at $x = c$ if f has either a global minimum or global maximum.
 - Every point on a constant function is both a global minimum and global maximum
 - Every global extremum can be a local extremum in some interval

Examples:

1. Find all global extrema of $f(x) = x^2$ on $[0, 1]$
 - The global minimum be $x = 0$ because $f(0) \leq f(x) \forall x \in [0, 1]$
 - The global maximum DNE as the end point is missing. That is infinitely numbers lie on the interval $[0, 1]$
2. Find all global extrema of $f(x) = \frac{1}{x}$ on $[-1, 1]$
 - The global extrema DNE as $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$.

Info – Extreme Value Theorem (Existence Thm)

Assume that $f(x)$ is continuous on the closed interval $[a, b]$. Then **there exist** two numbers $c_1, c_2 \in [a, b]$ s.t. $f(c_1) \leq f(x) \leq f(c_2) \forall x \in [a, b]$.

In other words, there is a global minimum at $x = c_1$ and a global maximum at $x = c_2$

Info – Local Extrema

Let f be a function. We say that f has

1. a **local minimum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \leq f(x) \forall x \in (a, b)$
2. a **local maximum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \geq f(x) \forall x \in (a, b)$
3. a **local extremum** at $x = c$ if there is either a local minimum or a local maximum

Warning – Local Extrema

If c is an endpoint of the domain of f , c can never be a local extremum, even if it is a global extremum

Info – Fermat's Theorem

If there is a local extremum for $f(x)$ at $x = c$ and $f'(c)$ exists, hence $f'(c) = 0$. That is we cannot put an open interval around the point.

Examples:

1. Does the converse of Fermat's Theorem hold? That is if $f'(0) = 0$, then is a local extremum at $x = c$.

This is false. Let $f(x) = x^3$, $f'(x) = 3x^2$, $f'(0) = 0$ but is not a local extremum on any interval containing $x = 0$

2. Why is it worth mentioning $f'(c)$ has to exist?

It is important because it is like saying $f(x)$ is differentiable at $x = c$. If not, let $f(x) = |x|$. $f(x)$ is continuous. It has a local minimum at $x = 0$ but $f'(0)$ DNE as it is not differentiable.

Info – Critical Points

We say that a function f has a **critical point** at $x = c$ if $f'(c) = 0$ or $f'(c)$ = DNE for $c \in$ the domain of f . These are our candidates for local extrema.

Tip – Closed Interval Method

Let $f(x)$ be continuous function on $[a, b]$.

1. Calculate $f(a)$ and $f(b)$
2. Find $f'(x)$
3. Find all the critical points of f on $[a, b]$
4. Calculate $f(c)$

Example:

$$f(x) = \frac{1}{3}x^3 - 3\sqrt[3]{x} \text{ on } [-8, 1]$$

$$f(-8) = -\frac{512}{3} - 3(-2) = -\frac{496}{3}$$

$$f(1) = \frac{1}{3} - 3 = -\frac{8}{3}$$

$$f'(x) = x^2 - x^{-\frac{2}{3}}$$

$$f'(c) = 0 \implies c^2 - c^{-\frac{2}{3}} = 0 \implies c^{\frac{8}{3}} = 1 \implies c = -1, 1. f'(c) = \text{DNE} \implies c = 0$$

$$f(0) = 0$$

$$f(-1) = -\frac{1}{3} + 3 = \frac{8}{3} \text{ Global maximum at } x = -1, \text{ global minimum at } f(-8)$$

Info – Rolle's Theorem (Existence Thm)

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = k \in \mathbb{R}$, then there exists $c \in (a, b)$ s.t. $f'(c) = 0$

Proof

If $f(x) = k \forall x \in [a, b]$, any value of c works.

Otherwise, $\exists x_0 \in [a, b]$ s.t. $f(x_0) \neq k$. Since f is continuous on $[a, b]$, it attains a maximum/minimum on $[a, b]$.

Since $f(x_0) \neq k \implies f(x_0) > k \leftrightarrow f(a), f(b)$ are not maximum, or $f(x_0) < k \leftrightarrow f(a), f(b)$ are not minimum. So one of maximum or minimum is in (a, b) , thus differentiable at some c .

By Fermat's Theorem, $f'(c) = 0$ or $f'(c) = \text{DNE}$. But f is differentiable on $(a, b) \implies f'(c)$ exists.

$$\therefore f'(c) = 0$$

Info – Mean Value Theorem (Existence Thm)

If f is continuous on $[a, b]$ and differentiable on (a, b) , then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

$$\text{Let } h(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$$

$$h(a) = f(a) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (a-a) \right] = 0$$

$$h(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (b-a) \right] = 0$$

Since $h(b) = h(a) \stackrel{\text{Rolle's Theorem}}{\implies} \exists c \in (a, b)$ s.t. $h'(c) = 0$

$$\text{That is } h'(x) = f'(x) \frac{f(b)-f(a)}{b-a} \implies f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$\text{Finally, } h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \leftrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Tip: the construction of $\left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$ is the linear approximation of $f(x)$ near a