

CH 1 — Integration

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Definite Integrals

Info — Riemann Sums

Given $f(x)$ that is defined over $[a, b]$ with $a < b$, the area under function $f(x)$ can be found by

1. Left-Endpoint Riemann Sum

$$L_n = \sum_{i=0}^{n-1} f(x_i^*) \Delta x$$

- Underestimates Increasing Functions

2. Right-Endpoint Riemann Sum

$$R_n = \sum_{i=1}^n f(x_i^*) \Delta x$$

- Overestimates Increasing Functions

where

- $\Delta x = \frac{b-a}{n}$ under regular partition
- $x_i^* = a + i\Delta x = a + i\frac{b-a}{n}$

$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n$ for $f(x)$ on interval $[a, b]$

Regular Partition means interval $[a, b]$ is equally partitioned into n rectangles with identical width

Example:

Estimate area under the curve for $f(x) = x^2$ on $x \in [0, 1]$

$$R_n = \sum_{i=1}^n \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^n \frac{1}{n} \left(\frac{i}{n}\right)^2 = \frac{1}{n^3} \sum_{i=1}^n i^2 = \frac{(n+1)(2n+1)}{6n}$$

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n} = \frac{1}{3}$$

Info — Definite Integral

$f(x)$ defined on $x \in [a, b]$ with regular partition with n subintervals

The definite integral of $f(x)$ on $[a, b]$ is defined

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(a + i\Delta x) \Delta x$$

A function is integrable on $x \in [a, b]$ provided that the limit of Riemann Sum exists and has the same value regardless of the choice of x_i^*

Info – Integrability Theorem for Continuous Functions

Integrability: $\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n$

1. If f is continuous of $[a, b]$ then f is integrable on $[a, b]$
2. f is bounded on $[a, b]$ and has a **finite** number of discontinuities, then f is integrable on $[a, b]$

That is continuity implies integrability and the other way is false

Examples:

1. $f(x) = x^2$
2. $f(x) = \begin{cases} 2 & \text{if } x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}$, note that $f(x)$ is discontinuous
3. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ on $[0, 1]$
 - x_i^* is rational $\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^* + i) \Delta x = 1$
 - x_i^* is irrational $\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^* + i) \Delta x = 0$

Thus not integrable

For geometric interpretation, Riemann Sums and Definite Integrals measures the “signed” area where there is no more than 1 inflection point

- A positive result of w implies the area under the curve above x -axis is w
- A negative result of w implies the area under the curve under x -axis is w

Info – Parity of Functions and Definite Integrals

Let $f(x)$ be bounded and integrable on $[-a, a]$

1. If $f(x)$ is odd function, then

$$\int_{-a}^a f(x) dx = 0$$

2. If $f(x)$ is even function where $\int_0^a f(x) dx = w$

$$\int_{-a}^a f(x) dx = 2w$$

Examples:

1. $\int_1^3 x^2 - 3x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(1 + \frac{2i}{n}) \cdot \frac{2}{n} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^3 - 3\left(1 + \frac{2i}{n}\right) \right] \cdot \frac{2}{n}$
 $= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(-\frac{4i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n} \right) = -\frac{10}{3}$
2. $\int_0^5 x^3 - 2 dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(i \frac{5}{n}\right) \frac{5}{n} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{5}{n}\right) \left(\left(i \frac{5}{n}\right)^3 - 2 \right) \right] = \frac{583}{4}$

Info — Basic Property of Definite Integral

Let $f(x), g(x)$ be integrable on $[a, b]$

1. For any $c \in \mathbb{R}$, the function $cf(x)$ is integrable and

$$\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$$

2. The function $f + g$ is integrable and

$$\int_a^b (f + g)(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$$

3. If $m, M \in \mathbb{R}$ and $m \leq f(x) \leq M \forall x \in [a, b]$, then

$$m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$$

4. If $f(x) \geq 0 \forall x$, then

$$\int_a^b f(x) \, dx \geq 0$$

5. If $f(x) \leq g(x) \forall x \in [a, b]$, then

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$$

6. The function $|f|$ is integrable on $[a, b]$ and

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx$$

7. Bound flipping

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

- 8.

$$\int_a^a f(x) \, dx = 0$$

Info — Separation of Domain of Definite Integral

If $f(x)$ is also integrable on an interval containing a, b, c , then

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

Info — Average Value of Function

Let f be a function that is continuous on an interval $[a, b]$ with $a < b$. The **average value of f on $[a, b]$** is defined as

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

Examples:

1. Determine the average value of $f(x) = 1 - x^2$ on $[-1, 1]$

$$f_{\text{avg}} = \frac{1}{1-(-1)} \int_{-1}^1 f(x) \, dx = \int_0^1 f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1 - (\frac{i}{n})^2}{n}$$
$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 1 - \frac{i^2}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \left(n - \frac{1}{n^2} \left(\frac{n(n+1)(2n+1)}{6} \right) \right) = 1 - \frac{1}{3} = \frac{2}{3}$$

2. Suppose that f, g are integrable on $[-1, 1]$, $\int_1^{-1} f(t) \, dt = 5$, and g is an even function with $\int_0^1 g(t) \, dt = 2$.

$$\int_{-1}^1 3f(x) - g(x) \, dx = 3 \int_{-1}^1 f(x) \, dx - \int_{-1}^1 g(x) \, dx = -3 \int_1^{-1} f(x) \, dx - 2 \int_0^1 g(x) \, dx = -19$$

Info — Fundamental Theorem of Calculus (FTC - 1)

Let $a \in \mathbb{R}$. If f is continuous on an open interval I containing a , then the function

$$G(x) = \int_a^x f(t) \, dt$$

is differentiable $\forall x \in I$ and $G'(x) = f(x)$. That is,

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

General Extended FTC 1

Let f be continuous, g, h be differentiable

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = f'(h(x))h'(x) - f'(g(x))g'(x)$$

Proof

Given $x \in I$, from the definition of the derivative, we have

$$G'(x) = \lim_{h \rightarrow 0} \frac{G(x+h) - G(x)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h} = \lim_{h \rightarrow 0} \frac{\int_a^x f(t) \, dt + \int_x^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt.$$

For all $h \neq 0$, sufficiently close to 0, and $h > 0$ f is continuous on $[x, x+h]$.

$\forall h, \exists c = c(h)$ in $[x, x+h]$ s.t.

$$f(c(h)) = \frac{1}{h} \int_x^{x+h} f(t) dt$$

Since $x \leq c(h) \leq x + h$, by Squeeze Theorem, $\lim_{h \rightarrow 0} c_h = x$, thus

$$G'(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt = \lim_{h \rightarrow 0} f(c_h) = f(x)$$

□

Examples

$$1. G(x) = \int_0^x \frac{1}{1+t^2} dt$$

Since $f(t) = \frac{1}{1+t^2}$ is continuous on \mathbb{R} , by FTC 1

$$G'(x) = \frac{d}{dx} \int_0^x \frac{1}{1+t^2} dt = \frac{1}{1+x^2}$$

$$2. H(x) = \int_2^{e^x} \cos(t^2) dt$$

Since $f(t) = \cos(t^2)$ is continuous on \mathbb{R} , by FTC 1

$$H'(u) = \frac{d}{du} \int_2^u \cos(t^2) dt \cdot \frac{du}{dx} = \frac{du}{dx} \cos(u^2) \stackrel{u=e^x}{=} e^x \cos(e^{2x})$$

3. Assume f is continuous and g, h differentiable

$$G(x) = \int_{g(x)}^{h(x)} f(t) dt = \int_{g(x)}^0 f(t) dt + \int_0^{h(x)} f(t) dt = - \int_0^{g(x)} f(t) dt + \int_0^{h(x)} f(t) dt$$

$$G'(x) = - \frac{d}{dx} \int_0^{g(x)} f(t) dt + \frac{d}{dx} \int_0^{h(x)} f(t) dt \stackrel{\text{by FTC 1}}{=} -f'(g(x))g'(x) + f'(h(x))h'(x)$$

Info – Fundamental Theorem of Calculus (FTC - 2)

If f, F are continuous on $[a, b]$ and $F'(x) = f(x) \forall x \in (a, b)$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Proof

Let F be any antiderivative of f . Then $F(x)$ and the antiderivative $G(x) = \int_a^x f(t) dt$ have the relation that $G(x) = F(x) + C$

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b f(t) dt - \int_a^a f(t) dt \\ &= G(b) - G(a) \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) \end{aligned}$$

□

Example

$$\text{If } H(x) = \int_5^x x^2 dx, \int_1^2 = H(2) - H(1) = \int_5^2 x^2 dx - \int_5^1 x^2 dx = \int_1^2 x^2 dx$$

Info – Basic Integraion Rules

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad \forall r \neq -1$$

$$\int x^{-1} \, dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int -\csc^2 x \, dx = \cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

Examples:

1. $\int e^{5x} \, dx = \frac{e^{5x}}{5} + C$

2. $\int \frac{t}{t+1} \, dt = \int 1 - \frac{1}{t+1} \, dt = t - \ln|t+1| + C$

Examples:

$$1. \int_0^4 2x^2 - x \, dx = \left. \frac{2}{3}x^3 - \frac{x^2}{2} \right|_0^4 = \frac{2}{3}(4)^2 - \frac{4^2}{2} - 0 = \frac{128}{3} - 8 = \frac{104}{3}$$

$$\begin{aligned} 2. \int_1^3 \frac{x + |x-2|}{x} \, dx &= \int_1^2 \frac{2}{x} \, dx + \int_2^3 \frac{2x-2}{x} \, dx \\ &= 2 \ln|x| \Big|_1^2 + \int_2^3 2 - \frac{2}{x} \, dx \\ &= 2 \ln 2 + 2x - 2 \ln|x| \Big|_2^3 \\ &= 2 \ln 2 + 6 - 2 \ln 3 - 4 + 2 \ln 2 \\ &= 4 \ln 2 + 2 - 2 \ln 3 \end{aligned}$$

Substitution Rule / U-Substitution



Info – U-Substitution

Let f, g be functions s.t. $g'(x)$ is continuous on $a, b]$ and f is continuous on range of g

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du \Big|_{u=g(x)}$$

Example:

$$1. \int 2x\sqrt{1+x^2} \, dx$$

$$\begin{aligned} \text{Let } u = 1 + x^2 &\Rightarrow du = 2x \, dx \Rightarrow dx = \frac{du}{2x} \xRightarrow{u=1+x^2} \int u^{\frac{1}{2}} \, du = \frac{2}{3}u^{\frac{3}{2}} \xRightarrow{u=1+x^2} +C \\ &= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C \end{aligned}$$

$$2. \int x^2 e^{x^3} \, dx$$

$$\text{Let } u = x^3 \Rightarrow du = 3x^2 \, dx \Rightarrow dx = \frac{du}{3x^2} \xRightarrow{u=x^3} \int \frac{1}{3} e^u \, du = \frac{1}{3} e^u \xRightarrow{u=x^3} +C = \frac{1}{3} e^{x^3} + C$$

$$3. \int \frac{\cos(\ln x)}{x} \, dx$$

$$\text{Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx \Rightarrow dx = x \, du \xRightarrow{u=\ln x} \int \cos u \, du = \sin u \xRightarrow{u=\ln x} +C = \sin(\ln x) + C$$

$$4. \int \frac{x}{3\sqrt[3]{x+2}} \, dx$$

$$\text{Let } u = x + 2 \Rightarrow du = dx$$

$$\xRightarrow{u=x+2} \int \frac{u-2}{3\sqrt[3]{u}} \, du = \int \frac{u^{\frac{2}{3}}}{3} \, du - 2 \int \frac{1}{\sqrt[3]{u}} \, du = \frac{3}{5}u^{\frac{5}{3}} + 3u^{\frac{2}{3}} \xRightarrow{u=x+2} +C = (x+2)^{\frac{5}{3}} + 3(x+2)^{\frac{2}{3}} + C$$

$$5. \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} \, dx$$

$$\text{Let } u = 1 + \sin x \Rightarrow du = \cos x \, dx \Rightarrow dx = \frac{du}{\cos x} \Rightarrow \int_1^2 \frac{1}{u} \, du = \ln u \Big|_1^2 = \ln 2$$

$$6. \int_0^{\frac{\pi}{3}} \tan x \, dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx$$

$$\text{Let } u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow \int_1^{\frac{1}{2}} -\frac{1}{u} \, du = -\ln u \Big|_{\frac{1}{2}}^1 = -\ln \frac{1}{2}$$

Info – $f(ax)$

Let $a \in \mathbb{R}, a \neq 0$ If $\int f(x) dx = F(x) + C$, then

$$\int f(ax) dx = \frac{1}{a} F(ax) + C$$

Trigonometry Substitution

Info – Trig-Sub

Integral contains	Substitution	Domain for θ	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \tan \theta$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\sec^2 \theta - 1 = \tan^2 \theta$

Tip – Half-Angle

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

Example:

1. $\int_0^1 \sqrt{1-x^2} dx$

Let $u = \arcsin x \Rightarrow x = \sin u, \forall u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow dx = \cos u du \Rightarrow$

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u} \cos u du = \int_0^{\frac{\pi}{2}} \cos^2 u du = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos(2u)) du = \frac{1}{2} \left(u + \sin(2u) \right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

2. $\int \sqrt{1-x^2} dx$

Let $u = \arcsin x \Rightarrow x = \sin u \forall u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow dx = \cos u du \Rightarrow$

$$\int \sqrt{1 - \sin^2 u} \cos u du = \int \cos^2 u du = \int \frac{1 + \cos 2u}{2} du \xrightarrow{u = \arcsin x} \frac{1}{2} \left(\arcsin x + \sqrt{1-x^2} \right) + C$$

3. $\int \frac{1}{\sqrt{x^2+9}} dx$

Let $x = 3 \tan \theta \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow dx = 3 \sec^2 \theta d\theta \Rightarrow \int \frac{1}{\sqrt{9 \tan^2 \theta + 9}} 3 \sec^2 \theta d\theta = \int \sec \theta d\theta$

$$\int \sec \theta \cdot \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta = \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \xrightarrow{u = \sec \theta + \tan \theta} \int \frac{1}{u} du = \ln |\sec \theta + \tan \theta| + C$$

$$\xrightarrow{\theta = \arctan \frac{x}{3}} \ln \left| \sec \arctan \frac{x}{3} + \tan \arctan \frac{x}{3} \right| + C = \ln \left(\frac{\sqrt{x^2+9}}{3} + \frac{x}{3} \right) + C$$

$$4. \int_1^2 \frac{\sqrt{x^2-1}}{x} dx$$

$$\text{Let } x = \sec \theta \quad \forall \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, 3\frac{\pi}{2}\right)$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta} \sec \theta \tan \theta d\theta = \int_0^{\frac{\pi}{3}} \tan^2 \theta d\theta = \int_0^{\frac{\pi}{3}} \sec^2 \theta - 1 d\theta = \tan \theta - \theta \Big|_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}$$

$$5. \int \frac{1}{(5-4x-x^2)^{\frac{5}{2}}} dx = \int \frac{1}{(1-(2-x)^2)^{\frac{5}{2}}} dx$$

$$\text{Let } 2-x = \sin \theta, \quad \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies dx = -\cos \theta d\theta$$

$$= \int \frac{-\cos \theta}{(1-(\sin \theta)^2)^{\frac{5}{2}}} d\theta = \int \frac{-\cos \theta}{\cos^5 \theta} d\theta = \int -\sec^4 \theta d\theta = -\int (\sec^2 \theta)^2 d\theta = -\frac{2}{7} \sec^{\frac{7}{2}} \theta + C =$$

$$-\frac{2}{7} \left(\frac{1}{\sqrt{5-4x-x^2}} \right)^{\frac{7}{2}} + C$$

Integration By Parts



Info – Integration by Parts

If f and g are differentiable functions of x , then

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

The rule of choosing parts being $f(x)$ is

1. Logarithmic
2. Inverse trigonometric
3. Algebraic
4. Trigonometric
5. Exponential

Examples

$$1. \int x e^x dx$$

$$\text{Let } \begin{cases} u=x \\ dv=e^x dx \end{cases} \implies \begin{cases} du=dx \\ v=e^x \end{cases} \implies x e^x - \int e^x = x e^x - e^x + C$$

$$2. \int x^2 \sin x dx$$

$$\text{Let } \begin{cases} u=x^2 \\ dv=\sin x dx \end{cases} \implies \begin{cases} du=2x dx \\ v=-\cos x \end{cases} \implies -x^2 \cos x + 2 \int x \cos x dx = -x^2 \cos x + 2(x \sin x + \cos x) + C = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

3.