CH 4 - Mathematical Induction

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Notations

$$\sum_{i=m}^{n} x_i = x_m^2 + x_{m+1} = x_{m+2} + ... + x_{n-1} + x_n$$

$$\prod_{i=m}^{n} x_i = x_m \cdot x_{m+1} \cdot x_{m+2} \cdot ... \cdot x_{n-1} \cdot x_n$$

Properties

Constant multiplication

$$\sum_{i=m}^{k} cx_i = c \cdot \sum_{i=m}^{k} x_i$$

Addition/Subtraction

$$\sum_{i=m}^{k} x_i \pm \sum_{i=m}^{k} y_i = \sum_{i=m}^{k} x_i \pm \sum_{i=m}^{k} y_i$$

Index Shift

$$\sum_{i=m}^{k} x_i = \sum_{m \pm n}^{k \pm n} x_{i \mp n}$$

Breaking Sum

$$\sum_{i=m}^{k} x_i = \sum_{i=m}^{r} x_i + \sum_{i=r+1}^{k} x_i$$

Recurrence Relation

A sequence of values by giving one or more initial terms, together with an equation expressing each subsequent term in terms of earlier ones. (i.e. $s_1 = 1, s_n = s_{n-1} + n$ is the same as $\sum_{i=1}^{n} i$)

Proof by Induction

An **axiom** of a mathematical system is a statement that is assumed to be true. No proof is given. From axioms we derive proposition and theorems.

Info — Axiom
Principle of Mathematical Induction
Let $P(n)$ be an open sentence that dependes on $n \in \mathbb{N}$
If statements 1 and 2 are both true:
1. $P(1)$
2. $\forall k \in \mathbb{N}, \text{if } P(k), \text{then } P(k+1)$

Then statement 3 is true:
3. $\forall n \in \mathbb{N}, P(n)$

Examples

1. Let $P(n)$ be the open sentence $\sum_{i=1}^{n} i(i+1) = \frac{1}{3}n(n+1)(n+2)$

Prove that $P(n)$ is true $\forall n \in \mathbb{N}$

Proof

We will use the Principle of Mathematical Induction
Base case:
When $n = 1$
$\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{1} i(i+1) = 1(2) = 2$
$\frac{1}{3}n(n+1)(n+2) = \frac{1}{3}(1)(2)(3) = 2$
$\therefore P(1)$ is true.

Induction:
Inductive hypothesis:
Let $k \in \mathbb{N}$.
Assume that $P(k)$ is true.

Then we have $\sum_{i=1}^{k} i(i+1) = \frac{1}{3}k(k+1)(k+2)$
Now,

$$\sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k} i(i+1) + (k+1)((k+1)+1) \quad \text{(by Breaking Sum Property)}$$

$$= \frac{1}{3}(k)(k+1)(k+1+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+1+1)(k+1+2)$$

By indution we have shown that $P(n)$ is true $\forall n \in \mathbb{N}$

□

2. Prove that $n! > 2^n, \forall$ positive integers $n \geq 4$.

Proof

We will use the Principle of Mathematical Induction
We are trying to prove the open sentence $P(n) : n! \geq 2^n$ is true $\forall n \in \mathbb{Z}$ s.t. $n \geq 4$

Base case:
When $n = 4$

$$n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$2^n = 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

We have that $24 > 16$
$\therefore P(4)$ is true.

Induction:
Inductive hypothesis:
Let $k \in \mathbb{N}$ and that $k > 4$ is true
Assume that $P(k)$ is true
Then

$$
\begin{aligned}
(k+1)! = (k+1)k! &> (k+1)2^k \qquad \text{(By Inductive Hypothesis)} \\
&> (1+1)2^k = 2 \cdot 2^k = 2^{k+1}
\end{aligned}
$$

Thus, by induction, we have shown that $n! \geq 2^n$ is true $\forall n \in \mathbb{Z}$ s.t. $n \geq 4$

□

3. Use inducitOn to prove that $6 \mid (2n^3 + 3n^2 + n) \forall n \in \mathbb{N}$

Quiz 4 Cutoff

Proof by Strong Induction

Info —
Principle of Strong Induction
Let $P(n)$ be an open sentence that dependes on $n \in \mathbb{N}$
If statements 1 and 2 are both true:
1. $P(1)$
2. $\forall k \in \mathbb{N}$, if $P(1) \land P(2) \land ... \land P(k)$, then $P(k+1)$

Then statement 3 is true:
3. $\forall n \in \mathbb{N}, P(n)$

Example:

1. Consider the sequence defined recusively by

   $x_1 = 4, x_2 = 68$ and $x_m = 2x_{m-1} + 15x_{m-2} \forall m \geq 3$

Prove that $x_n = 2(-3)^n + 10(5)^{n-1} \forall n \in \mathbb{N}$

Proof

We proceed by strong induction.
Base cases:
When $n = 1, x_1 = 4$ and $2(-3)^1 + 10(5)^1 = -6 + 10 = 4$
So the claim holds for $n = 1$
When $n = 2, x_1 = 68$ and $2(-3)^2 + 10(5)^2 = 18 + 50 = 68$
So the claim holds for $n = 2$

Induction :
Assume the claim holds for $n = 1, n = 2, ...n = k$ for some $k \in \mathbb{N}$ with $k \geq 2$

Assume

$$x_k = 2(-3)^k + 10(5)^{k-2}$$

$$x_{k-1} = 2(-3)^{k-1} + 10(5)^{k-2}$$

$$x_{k-2} = 2(-3)^{k-2} + 10(5)^{k-2}$$

Then

$$x_{k+1} = 2x_k + 15x_{k-1}$$

$$= 2\big(2(-3)^{k-1} + 10(5)^{k-2}\big) + 15\big(2(-3)^{k-1} + 10(5)^{k-2}\big) \qquad \text{(by I.H)}$$

$$= 4(-3)^k + 20(5)^{k-1} + 5 \cdot 3 \cdot 2(-3)^{k-1} + 10 \cdot 5 \cdot 3 \cdot (5)^{k-2}$$

$$= 4(-3)^k + 5 \cdot 4 \cdot 5^{k-1} - 5 \cdot 2(-3)^k + 2 \cdot 3 \cdot 5^k$$

$$= (-3)^{k(4-10)} + 5^{k(4+6)}$$

$$= 2(-3)^{k+1} + 105^k$$

$\therefore x_{k+1} = 2(-3)^{k+1} + 105^k$ so the claim holds for $n = k+1$ $\therefore$
by strong induction the claim holds $\forall n \in \mathbb{N}$

$\square$

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