

# CH 10 - Complex Numbers

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## Standard Form

### Info – Definition of Complex Numbers

A **complex number**  $z$  in **standard form** is an expression of the form  $z = x + yi$  where  $x, y \in \mathbb{R}$ .

The real number  $x$  is called the **real part** of  $z$ , and is written  $\Re(z)$ .

The real number  $y$  is called the **imaginary part** of  $z$ , and is written  $\Im(z)$ .

The set of complex numbers is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$$

The complex number  $z = x + yi$  and  $w = u + vi$  are equal ( $z = w$ ) if and only if  $x = u, y = v$

A complex number  $z$  is said to be purely real if  $y = 0$  (i.e.  $1 = 1 + 0i$ )

A complex number  $z$  is said to be purely imaginary if  $x = 0$  (i.e.  $i = 0 + 1i$ )

0 is purely real and purely imaginary (i.e.  $0 = 0 + 0i$ )

### Info – Complex Arithmetics

Let  $z = a + bi$  and  $w = c + di$  be complex numbers. Then the

Addition is defined as

$$z + w = (a + c) + (b + d)i$$

Multiplication is defined as

$$zw = (ac - bd) + (ad + bc)i$$

Examples:

Let  $z = 2 + 3i, w = -1 + 7i$

1.  $z + w = (2 - 1) + (3 + 7)i = 1 + 10i$
2.  $zw = (-2 - 21) + (14 - 3)i = -23 + 11i$
3.  $i^2 = ii = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + (0 + 0)i = -1$   
 $\therefore i^2 = -1$

From (3), we can derive a easier way of multiplication

### Tip – Multiplication trick

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

## Properties of Complex Arithmetics

1.  $z + 0 = 0 + z = z$
2.  $z0 = 0z = 0$
3.  $z + (-1)z = (-1)z + z = 0$
4.  $z1 = 1z = z$

### Info – Multiplicative Inverse

For all complex numbers  $z$ , the multiplicative inverse of  $z$  exists if and only if  $z \neq 0$ . Moreover, for  $a + bi \neq 0$ , the multiplicative inverse is unique given by

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i = \frac{a - bi}{a^2 + b^2}$$

## Examples:

1.  $\frac{(1-2i)-(3+4i)}{(5-6i)} = [(1-2i) - (3+4i)] \cdot (5-6i)^{-1} = (-2-6i) \cdot \frac{1}{25+36} \cdot (5+6i)$   
 $= \frac{1}{61} \cdot (-10-30i-12i+36) = \frac{26-42i}{61} = \frac{26}{61} - \frac{42}{61}i$
2.  $i^{2025} = (i^4)^{506} \cdot i = (-1)^{506} \cdot i = i$

### Tip – $i^n$ Solutions

$$i^n = \begin{cases} 1, & n \equiv 0 \pmod{4} \\ i, & n \equiv 1 \pmod{4} \\ -1, & n \equiv 2 \pmod{4} \\ -i, & n \equiv 3 \pmod{4} \end{cases}$$

### Info – Properties of Complex Arithmetics

In complex arithmetic, the following properties are valid for  $u, v, z \in \mathbb{C}$

1. Associativity of addition:  $(u + v) + z = u + (v + z)$
2. Commutativity of addition:  $u + v = v + u$
3. Additive identity:  $0 = 0 + 0i$  has the property that  $z + 0 = z$
4. Additive inverses: If  $z = a + bi$ , then there exists an additive inverse of  $z$ , written  $-z$ , with the property that  $z + (-z) = 0$ . The additive inverse of  $z = a + bi$  is  $-z = -a - bi$
5. Associativity of multiplication:  $(uv)z = u(vz)$
6. Commutativity of multiplication:  $uv = vu$
7. Multiplicative identity:  $1 = 1 + 0i$  has property that  $z1 = z$
8. Multiplicative inverses: If  $z = a + bi \neq 0$ , then there exists a multiplicative inverse of  $z$ , written  $z^{-1}$ , with the property that  $zz^{-1} = 1$ . The multiplicative inverse of  $z = a + bi \neq 0$  is  $z^{-1} = \frac{a - bi}{a^2 + b^2}$
9. Distributivity:  $z(u + v) = zu + zv$

Example:

Proof of PCA Part 5:  $(uv)z = u(vz) \forall u, v, z \in \mathbb{C}$

Let  $u = a + bi, v = c + di, z = x + yi$  where  $a, b, c, d, x, y \in \mathbb{R}$  so that  $u, v, z \in \mathbb{C}$

$$(uv)z = [(a + bi)(c + di)](x + yi) = (ac - bd + (ad + bc)i)(x + yi) =$$

$$= ((ac - bd)x - (ad + bc)y) + ((ac - bd)y + (ad + bc)x)i$$

$$= (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$$

$$u(vz) = (a + bi)[(c + di)(x + yi)] = (a + bi)(cx - dy) + (cy + dx)i$$

$$= (a(cx - dy) - b(cy + dx)) + (a(cy + dx) + b(cx - dy))i$$

$$= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i$$

$$\text{Thus } (uv)z = u(vz) = (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$$

□

### Info – Other Arithmetic of Complex Numbers

For  $z \in \mathbb{C}$

1.  $z^0 = 1$

2.  $z^1 = z$

3.  $z^{k+1} = z^k z \forall k \in \mathbb{N}$

4.  $(z^n)^m = z^{nm}$  and  $z^n z^m = z^{n+m} \forall n, m \in \mathbb{N} \cup \{0\}$

For  $n \notin \mathbb{N} \cup \{0\}$  will be discussed in later lectures

Examples:

Find a real solution to  $6z^3 + (1 + 3\sqrt{2}i)z^2 - (11 - 2\sqrt{2})i - 6 = 0$

Let  $z = x \in \mathbb{R}$ , that is  $z = x + 0i$

$$\Rightarrow 6x^3 + (1 + 3\sqrt{2}i)x^2 - (11 - 2\sqrt{2})x - 6 = 0$$

$$\Rightarrow (6x^3 + x^2 - 11x - 6) + (3\sqrt{2}x^2 + 2\sqrt{2}x)i = 0 + 0i$$

$$\Rightarrow (6x^3 + x^2 - 11x - 6) = 0 \text{ and } 3\sqrt{2}x^2 + 2\sqrt{2}x = 0$$

$x = 0$  or  $x = -\frac{2}{3}$  for the imaginary part.

However,  $x = 0$  does not satisfy the real part.

$$\therefore x = -\frac{2}{3} \Rightarrow z = -\frac{2}{3} + 0i$$

## Conjugate and Modulus

### Info – Complex Conjugate

The complex **conjugate** of a complex number  $z = x + yi$  written  $\bar{z}$  is the complex number

$$\bar{z} = x - yi$$

## Info – Properties of Conjugate

For the complex conjugate, the following properties hold  $\forall z, w \in \mathbb{C}$

1.  $\overline{(\bar{z})} = z$
2.  $\overline{z + w} = \bar{z} + \bar{w}$
3.  $z + \bar{z} = 2\Re(z)$  and  $z - \bar{z} = 2\Im(z)i$
4.  $\overline{zw} = \bar{z} \cdot \bar{w}$
5. If  $z \neq 0$ ,  $\overline{(z^{-1})} = (\bar{z})^{-1}$
6. If  $w \neq 0$ ,  $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

### Proof

#### Part 3

$$z + \bar{z} = 2\Re(z) \text{ and } z - \bar{z} = 2\Im(z)i$$

Let  $z = x + yi$

$$z + \bar{z} = (x + yi) + (x - yi) = 2x + 0i = 2\Re(z)$$

$$z - \bar{z} = (x + yi) - (x - yi) = 0x + 2i = 2\Im(z)$$

#### Part 4

$$\overline{zw} = \bar{z} \cdot \bar{w}$$

Let  $z = x + yi, w = a + bi$

$$\overline{zw} = \overline{(x + yi)(a + bi)} = \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb + ya)i$$

$$\overline{zw} = \overline{x + yia + bi} = (x - yi)(a - bi) = (xa - (-y)(-b)) + (x(-b) + (-y)a)i$$

$$= (xa - yb) - (xb + ya)i$$

$$\text{Thus } \overline{zw} = \bar{z} \cdot \bar{w}$$

□

Examples:

1. Prove  $z \in \mathbb{R} \iff z = \bar{z}$

Let  $z = x + yi \quad \forall x, y \in \mathbb{R}$

$\implies$

Suppose  $z \in \mathbb{R}$ , then  $y = 0$ , so that  $z = x + 0i = x \in \mathbb{R}$ .

Then  $\bar{z} = x - 0i = x$

$\therefore z = \bar{z}$

$\iff$

Suppose  $z = \bar{z}$ , then  $x + yi = x - yi$

This implies  $x = x, y = -y$ . Thus  $y = 0$

$\therefore z = x + 0i = x \in \mathbb{R}$

2. Prove that  $z$  is purely imaginary  $\Leftrightarrow z = -\bar{z}$

3. Solve  $z^2 = i\bar{z}$

Let  $z = x + yi$ . Then  $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$

and  $i\bar{z} = i(x - yi) = ix - i^2y = xi + y$

Then the equation becomes  $x^2 - y^2 + 2xyi = xi + y$

$$\Rightarrow \begin{cases} x^2 - y^2 = y \\ 2xy = x \end{cases} \Rightarrow x = 0 \wedge y = \frac{1}{2} \Rightarrow \begin{cases} x^2 - \frac{1}{4} = \frac{1}{2} \\ -y^2 = 0 \end{cases} \Rightarrow z = \left\{ 0, -i, \pm \left( \frac{\sqrt{3}}{2} \right) + \frac{1}{2}i \right\}$$

## Modulus of Complex Numbers

### Info – Modulus of Complex Number

The **modulus** of the complex number  $z = x + yi$ , written  $|z|$ , is the non-negative real number

$$|z| = \sqrt{x^2 + y^2}$$

### Info – Properties of Modulus

For the modulus, the following properties  $\forall z, w \in \mathbb{C}$ :

1.  $|z| = 0 \Leftrightarrow z = 0$

2.  $|\bar{z}| = |z|$

3.  $\bar{z}z = |z|^2$

4.  $|zw| = |z||w|$

5. If  $z \neq 0$  then  $|z^{-1}| = |z|^{-1}$

Side note: for  $z \neq 0$ ,  $z^{-1} = \frac{\bar{z}}{|z|^2}$

6. If  $w \neq 0$ ,  $\frac{z}{w} = z \cdot \frac{\bar{w}}{|w|^2}$

7.  $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$

8.  $\overline{\sum z_i} = \sum (\bar{z}_i)$

9.  $\overline{\prod z_i} = \prod (\bar{z}_i)$

10.  $|\prod z_i| = \prod (|z_i|)$

## Proof

Part 3

$$\bar{z}z = |z|^2$$

Let  $z = a + bi$ ,  $\forall a, b \in \mathbb{R}$

$$\text{then } \bar{z}z = (a - bi)(a + bi) = (a^2 - (-b)(b)) + (ab + (-b)a)i$$

$$= a^2 + b^2 + 0i = (\sqrt{a^2 + b^2})^2 = |z|^2$$

□

## The Complex Plane and Polar Form

### Complex Plane

#### 💡 Tip – Geometric Interpretation and Graphical Properties

- $z$  and  $\bar{z}$  are reflection of each other over real axis
- Modulus is the distance from the point  $z$  to origin
- For addition, it is similar to vector addition that is the parallelogram rule,  $z + w$
- For subtraction, consider  $z + w - w = z$  and the rest is same for addition

#### 🌐 Info – Triangle Inequality

For all  $z, w \in \mathbb{C}$ , we have

$$|z + w| \leq |z| + |w|$$

Note that  $|z|$  is the modulus of  $z$ , not absolute value

### Proof

Let  $z = x + yi, w = u + vi$ , where  $x, y, u, v \in \mathbb{R}$

$$\begin{aligned} |z + w| &= |(x + u) + (y + v)i| = \sqrt{(x + u)^2 + (y + v)^2} \\ &= \sqrt{(x - (-u))^2 + (y - (-v))^2} \end{aligned}$$

(The Euclidean distance formula between  $(x, y)$  and  $(-u, -v)$ )

Consider the triangle ABC constructed from points

$$A : (0, 0); B(x, y) = (z = x + yi); C : (-u, -v) = (-w = -u - vi)$$

Let  $l_{AB}$  = length of side AB,  $l_{BC}$  and  $l_{AC}$  have the similar constructed

From geometric perspective,  $l_{BC} \leq l_{AB} + l_{AC}$

$$\begin{aligned} \text{Note that } l_{AB} &= \sqrt{x^2 + y^2} = |z|, l_{AC} = \sqrt{(-u)^2 + (-v)^2} = |w| \\ l_{BC} &= \sqrt{(x - (-u))^2 + (y - (-v))^2} = |z + w| \end{aligned}$$

Therefore  $|z + w| \leq |z| + |w|$

□

Exercise:

Let  $z \neq \pm i$ . Prove that  $\frac{z}{1+z^2}$  is real  $\iff z \in \mathbb{R}$  or  $|z| = 1$

## Polar Form

### Info – Polar Form

A **polar form** of a complex number  $z$  is denoted

$$z = r(\cos \theta + i \sin \theta)$$

where  $r \geq 0$ , being the modulus of  $z$  and angle  $\theta \in \mathbb{R}$  be an **argument** of  $z$

- $\theta$  is not unique unless given restriction of  $\theta \in [0, 2\pi)$

Notice that in standard form  $z = x + yi$

- $x = r \cos \theta$
- $y = r \sin \theta$
- $r = |z| = \sqrt{x^2 + y^2}$
- $\theta = \arctan\left(\frac{y}{x}\right)$

Examples:

Convert polar form to standard form

$$1. z = 3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

$$2. z = \cos \frac{15}{6}\pi + i \sin \frac{15}{6}\pi = 0 + 1i = i$$

Convert from standard form to polar form

$$3. z = \sqrt{6} + \sqrt{2}i$$

$$r = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = 2\sqrt{2}$$

$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$  is one possibility as the angle is not unique

$$z = 2\sqrt{2}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$$

$$4. z = -3\sqrt{2} + 3\sqrt{6}i = 6\sqrt{2}\left(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi\right)$$

### Info – Polar Multiplication for $\mathbb{C}$

For all complex numbers  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ ,  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ , then their product is

$$z_1 z_2 = r_1 r_2(\cos \theta(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

## Proof

Let  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ ,  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \in \mathbb{C}$

$$\begin{aligned} z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + 2) + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) \\ &= r_1 r_2(\cos \theta(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \end{aligned}$$

□

Examples:

1. Compute  $(i+1)i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \cdot (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$  By PMC  $= \sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$   
 $= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -1 + i$
2. Find  $(\sqrt{6} + \sqrt{2}i)(-3\sqrt{2} + 3\sqrt{6}i)$   
 $= 2\sqrt{2}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \cdot 6\sqrt{2}(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) = 24(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}) = 24\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$   
 $= -12\sqrt{3} + 12i$

## De Moivre's Theorem

### Info – De Moivre's Theorem (DMT)

$\forall \theta, n \in \mathbb{R}$ ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

### Info – Corollary of DMT

For all complex number  $z = r(\cos \theta + i \sin \theta)$  and integer  $n$ , except  $|z| = r = 0$  and  $n < 0$  we have

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

Examples:

1.  $(\sqrt{3} - i)^{10} = [2(\cos -\frac{\pi}{6} + i \sin -\frac{\pi}{6})]^{10} = 1024(\cos -\frac{10\pi}{6} + i \sin -\frac{10\pi}{6}) = 512 + 512\sqrt{3}i$
2. Prove that  $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$  (Hint: Consider  $(\cos \theta + i \sin \theta)^4$  expand normally and apply DMT then compare real parts)
3. Prove that  $\forall n \in \mathbb{Z}$  if  $w \in \mathbb{C}, |w| = 1$  and  $\theta$  is an argument of  $w$ , then  $-\frac{i}{2}(w^n - w^{-n}) = \sin(n\theta)$

### Info – Additional Information

For notation, we write  $\text{cis}(\theta) = \cos \theta + i \sin \theta$  notation wise

Also,  $\text{cis}(\theta) = e^{i\theta}$  that is having the similar properties of exponentials

$e^{i\pi} = \text{cis}(\pi) = -1 \implies e^{i\pi} + 1 = 0$  which is the Euler's Formula