CH 2 — Differential Equations

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Introduction to Differential Equations

🐳 **Info — Differential Equations**

A differential equation (DE), is an equation involving an unknown function and its derivatives. The term ordinary differential equation (ODE) refers to a differential equation involving single-variable functions, whereas the term partial differential equation (PDE) refers to a differential equation involving multivariable functions (i.e., functions with multiple inputs).

An ODE is expressed

$$F(x, y, y', y'', ..., y^n) = 0$$

for some $n \in \mathbb{N}$

The order of a DE is the order of the **highest derivative** that appears in the equation.

A function $y = \varphi(x)$ is a solution to the differential equation $F(x, y, y', y'', ..., y^n) = 0$ if

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), ..., \varphi^n(x)) = 0$$

The graph of a solution to a DE is called a **solution curve**

The complete collection of solutions to a DE, including any arbitrary constants, is called its general solution. A particular solution to a DE is one in which all arbitrary constants have been specified.

A differential equation together with one or more initial conditions is known as an initial value problem (IVP)

Examples:

1. First order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$
2. Second order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \ln x + y$
3. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \Longrightarrow y = -\cos x + C, c \in \mathbb{R}$
4. $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x, y(0) = 2 \Longrightarrow -\cos x + C \Longrightarrow -\cos x + 3$
5. $y' = x + y$

   $y = -1 - x + 2e^x \Longrightarrow y' = -1 + 2e^x \Longrightarrow (x + y) = x + (-1 - x + 2e^x) = -1 + 2e^x = y'$
6. $y = -1 - x - 5e^x$

   $y' = -1 - 5e^x \Longrightarrow x + (-1 - x - 5e^x) = -1 - 5e^x = y'$
7. $y = -5 - x + 2e^x$

   $y' = -1 + 2e^x \Longrightarrow x + (-5 - x + 2e^x) = -5 + 2e^x \neq y'$

Thus $y = -1 - x + Ce^x$, for $C \in \mathbb{R}$ is always a solution

8. Determine all real numbers k s.t. $x(t) = \sin(kt)$ is a solution to the second-order differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2x$

   $\begin{cases} x'(t) = k\cos(kt) \\ x''(t) = -k^2\sin(kt) \end{cases} \implies -k^2 \sin kt = 02 \sin kt$

   $$(k^2 - 2)\sin(kt) = 0$$

   $$k = \pm\sqrt{2}, k = 0$$

Direction Fields

Separable Differential Equations

Linear First-Order Differential Equations