Classical Related Rates Problems

This note summarizes the most common and instructive examples of **related rates** problems encountered in calculus. Each scenario is described with a short setup, diagram description, and key relations.

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1. Ladder Sliding Down a Wall

Setup: A ladder of fixed length L leans against a vertical wall. The bottom slides away at a rate dx/dt . Find how fast the top slides down (dy/dt).

Relation: $x^2 + y^2 = L^2$

Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$.

Illustration:

```
     |
     |
     |
     |   \   ← ladder (L)
     |____
     x
```

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2. Shadow of a Person under a Lamp

Setup: A person of height h_1 walks away/toward a lamp of height h_2 . The shadow length is y , and the distance from lamp to person is x . The person walks at dx/dt .

Relation: $\dfrac{h_2}{x+y} = \dfrac{h_1}{y}$

Differentiate: $h_2 y'(x+y) - h_1(x+y)y' = h_1 y x'$

Simplify to: $\frac{dy}{dt} = \frac{h_1}{h_2 - h_1}\frac{dx}{dt}$.

Illustration:

```
Lamp (h₂)
  |
  |
  |
  |   \__ person (h₁)
  |_______\____ shadow (y)
      x
```

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3. Water Pouring into a Conical Tank

Setup: Water is poured into an inverted cone. Given dh/dt , find dV/dt or vice versa.

Relation: Volume of a cone $V = \frac{1}{3}\pi r^2 h$. For similar triangles, $r/h = R/H$, so $r = (R/H)h$.

Differentiate: Substitute for r and differentiate V with respect to t :

$$\frac{dV}{dt} = \frac{\pi R^2}{H^2} h^2 \frac{dh}{dt}.$$

Illustration:

```
       apex
        •
       /|
      / |
     /  |
    /___|___\  ← top radius R, height H
```

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4. Expanding Circle (Area & Radius)

Setup: The radius r of a circle increases at dr/dt . Find how fast the area A or circumference C increases.

Relations: - $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. - $C = 2\pi r \Rightarrow \frac{dC}{dt} = 2\pi \frac{dr}{dt}$.

Illustration:

```
   •──r──○
   A = πr²
```

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5. Two Cars Moving at Right Angles

Setup: Car A moves east at dx/dt , Car B north at dy/dt . Distance between them is z .

Relation: $z^2 = x^2 + y^2$

Differentiate: $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\Rightarrow \frac{dz}{dt} = \frac{xx'+yy'}{z}$.

Illustration:

```
     Car B ↑ y
           |
           |
           |  \  z
           |
  Car A → |_____\  x
```

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6. Rotating Beacon on a Shoreline

Setup: A beacon located a fixed distance d from the shore rotates, sweeping a beam along the shore. The angle is θ , and the distance along the shore is x .

Relation: $\tan\theta = x/d$

Differentiate: $\sec^2\theta\frac{d\theta}{dt} = \frac{1}{d}\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = d\sec^2\theta\frac{d\theta}{dt}$.

Illustration:

```
     beacon •—— d —— shore


                 \\ θ
                  \
                   \\ x (along shore)
```

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7. Draining Cylindrical Tank

Setup: A cylindrical tank drains so that h decreases at a known rate. Find dV/dt .

Relation: $V = \pi r^2 h$

$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$.

If r also changes, use product rule.

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8. Inflating a Sphere

Setup: A balloon's radius r increases at dr/dt . Find dV/dt .

Relation: $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Illustration:

```
    ○
   /|\  radius r
```

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9. Airplane Flying Over Radar Station

Setup: A plane flying horizontally at constant altitude h and horizontal speed dx/dt . The radar measures distance s from station to plane.

Relation: $s^2 = x^2 + h^2$

Differentiate: $2s\frac{ds}{dt} = 2x\frac{dx}{dt}$

$\Rightarrow \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$.

Illustration:

```
Plane →      h
        ________
       |      /  s
       |     /
```

```
|_____/    x
Radar
```

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10. Boats Pulling In or Out (Rope Problem)

Setup: A boat is pulled toward a dock by a rope passing through a pulley located h meters above the boat's bow. The rope is pulled at dr/dt , and the distance from the boat to dock is x .

Relation: $r^2 = x^2 + h^2$

Differentiate: $2r\frac{dr}{dt} = 2x\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{r}{x}\frac{dr}{dt}$.

Summary Table

Scenario	Key Equation	Goal
Ladder	$x^2 + y^2 = L^2$	dy/dt given dx/dt
Shadow	$\frac{h_2}{x+y} = \frac{h_1}{y}$	$dy/dt, d(x+y)/dt$
Cone	$V = \frac{1}{3}\pi r^2 h$	dV/dt or dh/dt
Circle	$A = \pi r^2$	dA/dt
Cars	$z^2 = x^2 + y^2$	dz/dt
Beacon	$\tan\theta = x/d$	dx/dt
Cylinder	$V = \pi r^2 h$	dV/dt
Sphere	$V = \frac{4}{3}\pi r^3$	dV/dt
Airplane	$s^2 = x^2 + h^2$	ds/dt
Boat	$r^2 = x^2 + h^2$	dx/dt