

## CH 2 — Differential Equations

Luke Lu • 2026-02-09

### Introduction to Differential Equations

#### Info — Differential Equations

A differential equation (DE), is an equation involving an unknown function and its derivatives. The term ordinary differential equation (ODE) refers to a differential equation involving single-variable functions, whereas the term partial differential equation (PDE) refers to a differential equation involving multivariable functions (i.e., functions with multiple inputs).

An ODE is expressed

$$F(x, y, y', y'', \dots, y^n) = 0$$

for some  $n \in \mathbb{N}$

The order of a DE is the order of the **highest derivative** that appears in the equation.

A function  $y = \varphi(x)$  is a solution to the differential equation  $F(x, y, y', y'', \dots, y^n) = 0$  if

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), \dots, \varphi^n(x)) = 0$$

The graph of a solution to a DE is called a **solution curve**

The complete collection of solutions to a DE, including any arbitrary constants, is called its general solution. A particular solution to a DE is one in which all arbitrary constants have been specified.

A differential equation together with one or more initial conditions is known as an initial value problem (IVP)

Examples:

1. First order differential equation  $\frac{dy}{dx} = x + y$
2. Second order differential equation  $\frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = \ln x + y$
3. Solve  $\frac{dy}{dx} = \sin x \implies y = -\cos x + C, c \in \mathbb{R}$
4.  $\frac{dy}{dx} = \sin x, y(0) = 2 \implies -\cos x + C \implies -\cos x + 3$
5.  $y' = x + y$

$$y = -1 - x + 2e^x \implies y' = -1 + 2e^x \implies (x + y) = x + (-1 - x + 2e^x) = -1 + 2e^x = y'$$

6.  $y = -1 - x - 5e^x$

$$y' = -1 - 5e^x \implies x + (-1 - x - 5e^x) = -1 - 5e^x = y'$$

7.  $y = -5 - x + 2e^x$

$$y' = -1 + 2e^x \implies x + (-5 - x + 2e^x) = -5 + 2e^x \neq y'$$

Thus  $y = -1 - x + Ce^x$ , for  $C \in \mathbb{R}$  is always a solution

8. Determine all real numbers  $k$  s.t.  $x(t) = \sin(kt)$  is a solution to the second-order differential equation  $\frac{d^2y}{dx^2} = -2x$

$$\begin{cases} x'(t) = k \cos(kt) \\ x''(t) = -k^2 \sin(kt) \end{cases} \implies -k^2 \sin kt = 0 \sin kt$$

$$(k^2 - 2) \sin(kt) = 0$$

$$k = \pm\sqrt{2}, k = 0$$

## Direction Fields



### Info – Direction Field

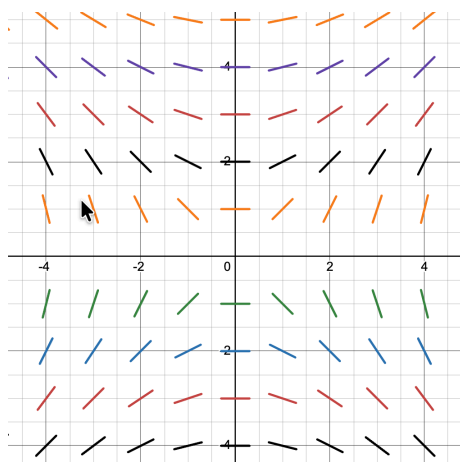
A direction field for the differential equation  $y' = F(x)$  displays short line segments of slope  $F(x, y)$  at various points in the Cartesian plane



**Tip** – Direction field plotter for DE: <https://www.desmos.com/calculator/p7vd3cdmei>

Examples:

1.  $\frac{dy}{dx} = \frac{x}{y}$

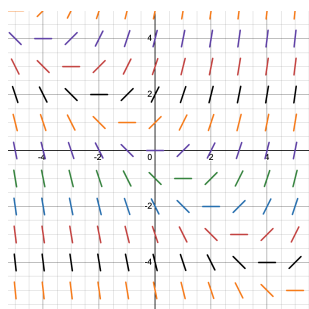


We can obtain this when plugging-in numbers.

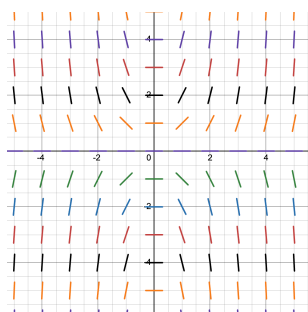
$$\text{At } (-2, 1), \left. \frac{dy}{dx} \right|_{(x,y)=(-2,1)} = \left. \frac{x}{y} \right|_{(x,y)=(-2,1)} \implies -\frac{2}{1} = -2$$

Hence a slope of  $-2$  at  $(-2, 1)$  and other points have the same method

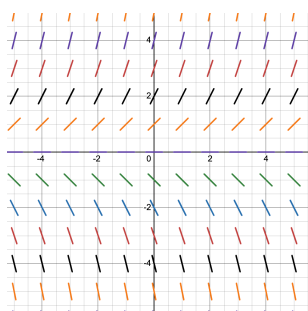
2.  $y'x = x + y$



3.  $y' = xy$



4.  $y' = y$



## Separable Differential Equations



### Info — Separable Differential Equation

A first order differential equation is said to be separable if it is written in format

$$\frac{dy}{dx} = g(x)h(y)$$

Examples:

1.  $y' = \sin x$

2.  $y' = \frac{x}{y}$

3.  $y' = 5y$

4.  $y^2 y' = 2y + xy$

5.  $y' = x + y$  is not separable but linear, see in Linear First Order DEs

 **Tip – Solving Separable DE** We have two cases

1. Determine any solution  $y$  with  $h(y) = 0$
2. Find the solutions  $y$  where  $h(y) \neq 0$  by evaluating

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

If possible, isolate  $y$  as a function of  $x$  in the resulting equation.

The general solution is the collection of all solutions obtained in Case 1 together with all solutions obtained in Case 2.

Examples:

1.  $y' = y$

Case 1:  $y = 0$

Case 2:

$$\begin{aligned}\int \frac{1}{y} dy &= \int 1 dx \\ \ln|y| &= x + C_1 \\ |y| &= e^{x+C_1} y = \pm e^{C_1} e^x = \pm C e^x, \quad C \in \mathbb{R}\end{aligned}$$

The general solution is  $y = Ae^x$ ,  $A \in \mathbb{R}$

2.  $y' = \frac{x}{y}$

Case 1:  $h(y) = \frac{1}{y} \neq 0$

Case 2:

$$\begin{aligned}\int y dy &= \int x dx \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ y^2 &= x^2 + 2C \\ y &= \pm \sqrt{x^2 + 2C}\end{aligned}$$

The general solution  $y = \pm \sqrt{x^2 + D}$ ,  $D \in \mathbb{R}$

3.  $y' = xy^2, y(0) = 3$

Case 1:  $h(y) = 0 \implies y \equiv 0$

Case 2:

$$\int \frac{1}{y^2} dy = \int x dx \implies -\frac{1}{y} = \frac{x^2}{2} + C \implies y = -2 \frac{2}{x^2 + 2C}$$

The general solution is  $y \equiv 0$  or  $y = \frac{-2}{x^2 + D}$ ,  $D \in \mathbb{R}$

For  $y(0) = 3 \implies -\frac{2}{D} = 3 \implies D = -\frac{2}{3} \implies$  the unique solution is  $y = \frac{-2}{x^2 - \frac{2}{3}}$

$$4. \quad y' - x^2 - y^2 - 2xy + 1 = 0 \implies y' = (x + y)^2 - 1 \xrightarrow{v=x+y} v' = v^2$$

$$\text{Case 1: } h(v) = v^2 \equiv 0 \implies v \equiv 0$$

$$\text{Case 2: } \int \frac{1}{v^2} dv = x \implies -\frac{1}{v} = x + C \implies v = -\frac{1}{x+C}, C \in \mathbb{R}$$

$$\text{The general solution } v \equiv 0 \text{ or } v = -\frac{1}{x+C}, C \in \mathbb{R}$$

$$y = -x \text{ or } y = -x - \frac{1}{x+C}, C \in \mathbb{R}$$

## Linear First-Order Differential Equations



### Info – First-Order Linear Differential Equation

First order Linear DE has the form

$$A_0(x)y + A_1(x)y' = B(x) \quad A_1(x) \neq 0$$

Such an equation can be written in the form  $y' + P(x)y = Q(x)$  called the standard form.



### Info – Integrating Factor

Given a DE of the form  $y' + P(x)y = Q(x)$ ,

$$\mu = e^{\int P(x) dx}$$

is called the integrating factor for the DE.



### Tip – Solving First-Order Linear DE

Given  $A_1(x)y' + A_0(x)y = B(x)$

1. Divide by  $A_1(x)$  to rewrite the DE in standard form
2. Multiply the equation by the integrating factor.
3. Rewrite LHS as  $(\mu(x)y)'$
4. Integrate in respect to  $x$
5. Isolate  $y$

Example:

$$1. \quad y' + \frac{3}{x}y = 1$$

$$\mu(x) = e^{3 \ln(x)} \implies x^3 y' + 3x^2 y = x^3$$

$$\frac{d}{dx} x^3 y = x^3$$

$$\int \frac{d}{dx} x^3 y dx = \int x^3 dx$$

$$y = \frac{1}{4}x + \frac{C}{x^3}$$

$$2. \quad y' = x + y \implies y' - y = x$$

$$\mu(x) = e^{-x} \implies e^{-x} y' - e^{-x} y = e^{-x} x$$

$$\frac{d}{dx} e^{-x} y = x e^{-x}$$

$$y = -x - 1 + C e^x, C \in \mathbb{R}$$

$$3. \quad y' + 2xy = 1$$

$$\stackrel{\mu(x)=e^{x^2}}{\implies} e^{x^2} y' + 2xye^{x^2} = e^{x^2}$$


$$\frac{d}{dx} e^{x^2} y = e^{x^2}$$

$$y = e^{-x^2} \int_0^1 e^{t^2} dt + Ce^{-x^2}, C \in \mathbb{R}$$

Q5 CUTOFF

## Applications

### Mixing Problem

 **Tip – Formula - Mixing Problem**

$$\frac{dA}{dt} = (\text{rate of substance in}) - (\text{rate of substance out})$$

Q1:

Suppose that a tank has 1000L of salt water at an initial concentration of 0.1kg/L. Salt water at a concentration of 0.3kg/L flows into the tank at a rate of 10 L/min. The solution is kept mixed and drain out at a rate of 10 L/min.

Determine the amount of salt, in kg, in the tank at time  $t$

Let  $A(t)$  represent the amount of salt(kg) in the tank at time  $t$ (min)

$$\frac{dA}{dt} = \frac{ds_i}{dt} - \frac{ds_o}{dt} \text{ in kg/min}$$

$$\frac{ds_i}{dt} = 0.3 \cdot 10 = 3 \text{ kg/min}$$

$$\frac{ds_o}{dt} = \frac{A(t)}{1000} \cdot 10 = \frac{A}{100} \text{ kg/min}$$

$$A(0) = 100 \text{ kg}$$

$$\frac{dA}{dt} = 3 - \frac{A}{100} = -\frac{A-300}{100} \implies \int \frac{1}{A-300} dA = \int -\frac{1}{100} dt \implies \ln|A-300| = -\frac{t}{100} + C$$

$$A = 300 \pm e^{-\frac{t}{100} + C}$$

$$\text{General solution } A = 300 + Be^{-\frac{t}{100}}, B \in \mathbb{R}$$

$$B = -200 \text{ since } A(0) = 100$$

$$\text{Particular solution } A = 300 - 200e^{-\frac{t}{100}}$$

### Newton Heat

### Population Growth

MIDTERM CUTOFF