CH 1 — Integration

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Definite Integrals

> ### 🐳 Info — Riemann Sums
>
> Given $f(x)$ that is defined over $[a, b]$ with $a < b$, the area under function $f(x)$ can be found by
>
> 1. **Left-Endpoint Riemann Sum**
>
> $$L_n = \sum_{i=0}^{n-1} f(x_i^*)\Delta x$$
>
> - Underestimates Increasing Funcitons
>
> 2. **Right-Endpoint Riemann Sum**
>
> $$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$
>
> - Overestimates Increasing Functions
>
> where
>
> - $\Delta x = \frac{b-a}{n}$ under regular partition
> - $x_i^* = a + i\Delta x = a + i\frac{b-a}{n}$
>
> $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$ for $f(x)$ on interval $[a, b]$
>
> Regular Parition means interval $[a, b]$ is equally partitionined into n rectangles with identical width

Example:

Estimate area under the curve for $f(x) = x^2$ on $x \in [0, 1]$

$R_n = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6n}$

$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n} = \frac{1}{3}$

> ### 🐳 Info — Definite Integral
>
> $f(x)$ defined on $x \in [a, b]$ with regular partition with n subintervals
>
> The definite integral of $f(x)$ on $[a, b]$ is defined
>
> $$\int_a^b f(x)\,\mathrm{d}x = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$
>
> A function is integrable on $x \in [a, b]$ provided that the limit of Riemann Sum exists and has the same value regardless of the choice of x_i^*

> 🐳 **Info − Integrability Theorem for Continuous Functions**
>
> Integrability: $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$
>
> 1. If f is continuous of $[a, b]$ then f is integrable on $[a, b]$
>
> 2. f is bounded on $[a, b]$ and has a **finite** number of discontinuities, then f is integrable on $[a, b]$
>
> That is continuity implies integrability and the other way is false

Examples:

1. $f(x) = x^2$

2. $f(x) = \begin{cases} 2 \text{ if } x \le 3 \\ 1 \text{ if } x > 3 \end{cases}$, note that $f(x)$ is discontinuous

3. $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ on $[0, 1]$

   - x_i^* is rational

     $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 1$
   - x_i^* is irrational $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 0$

   Thus not integrable

For geometric interpretation, Riemann Sums and Definite Integrals measures the "signed" area where there is no more than 1 inflection point

- A positive result of w implies the area under the curve above x-axis is w

- A negative result of w implies the area under the curve under x-axis is w

> 🐳 **Info − Parity of Functions and Definite Integrals**
>
> Let $f(x)$ be bounded and integrable on $[-a, a]$
>
> 1. If $f(x)$ is odd function, then
>
> $$\int_{-a}^{a} f(x)\,dx = 0$$
>
> 2. If $f(x)$ is even function where $\int_0^a f(x)\,dx = w$
>
> $$\int_{-a}^{a} f(x)\,dx = 2w$$

Examples:

1. $\int_1^3 x^2 - 3x\,dx = \lim_{n\to\infty} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n} = \text{im}_{n\to\infty} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^3 - 3\left(1 + \frac{2i}{n}\right)\right] \cdot \frac{2}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^n \left(-\frac{4i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right) = -\frac{10}{3}$

2. $\int_0^5 x^3 - 2\,dx = \lim_{n\to\infty} \sum_{i=1}^n f\left(i\frac{5}{n}\right)\frac{5}{n} = \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{5}{n}\right)\left(\left(i\frac{5}{n}\right)^3 - 2\right)\right] = \frac{583}{4}$

🐳 Info — Basic Property of Definite Integral

Let $f(x), g(x)$ be integrable on $[a, b]$

1. For any $c \in \mathbb{R}$, the function $cf(x)$ is integrable and

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

2. The function $f + g$ is integrable and

$$\int_a^b (f+g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

3. If $m, M \in \mathbb{R}$ and $m \le f(x) \le M \forall x \in [a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$$

4. If $f(x) \ge 0 \forall x$, then

$$\int_a^b f(x)\,\mathrm{d}x \ge 0$$

5. If $f(x) \le g(x) \forall x \in [a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$$

6. The function $|f|$ is integrable on $[a, b]$ and

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \le \int_a^b |f(x)|\,\mathrm{d}x$$

7. Bound fliping

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

8.
$$\int_a^a f(x)\,\mathrm{d}x = 0$$

🐳 Info — Separation of Domain of Definite Integral

If $f(x)$ is also integrable on an interval containing a, b, c, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

> 🐋 **Info — Average Value of Function**
>
> Let f be a function that is continuous on an interval $[a, b]$ with $a < b$. The **average calue of f on $[a, b]$** is defined as
>
> $$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Examples:

1. Determine the average value of $f(x) = 1 - x^2$ on $[-1, 1]$

   $f_{\text{avg}} = \frac{1}{1-(-1)} \int_{-1}^1 f(x)\, dx = \int_0^1 f(x)\, dx = \lim_{x \to \infty} \sum_{i=1}^n \frac{1 - \left(\frac{i}{n}\right)^2}{n}$

   $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n 1 - \frac{i^2}{n^2} = \lim_{n \to \infty} \frac{1}{n}\left(n - \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right) = 1 - \frac{1}{3} = \frac{2}{3}$

2. Suppose that f, g are integrable on $[-1, 1]$, $\int_1^{-1} f(t)\, dt = 5$, and g is an evern function with $\int_0^1 g(t)\, dt = 2$.

   $\int_{-1}^1 3f(x) - g(x)\, dx = 3 \int_{-1}^1 f(x)\, dx - \int_{-1}^1 g(x)\, dx = -3 \int_1^{-1} f(x)\, dx - 2 \int_0^1 g(x)\, dx = -19$

> 🐋 **Info — Fundamental Theorem of Calculus (FTC - 1)**
>
> Let $a \in \mathbb{R}$. If f is continuous on an open interval I containing a, then the funciton
>
> $$G(x) = \int_a^x f(t)\, dt$$
>
> is differentiable $\forall x \in I$ and $G'(x) = f(x)$. That is,
>
> $$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$
>
> **General Extended FTC 1**
>
> Let f be continuous, g, h be differentiable
>
> $$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Proof

Given $x \in I$, from the definition of the derivative, we have
$G'(x) = \lim_{h \to \infty} \frac{G(x+h) - G(x)}{h}$

$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h} = \lim_{h \to 0} \frac{\int_a^x f(t)\, dt + \int_x^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$.

For all $h \neq 0$, sufficiently close to 0, and $h > 0$ f is continuous on $[x, x+h]$.

$\forall h, \exists c = c(h)$ in $[x, x+h]$ s.t.

$$f(c(h)) = \frac{1}{h} \int_x^{x+h} f(t) \, dt$$

Since $x \le c(h) \le x + h$, by Squeeze Theorem, $\lim_{h \to 0} c_h = x$, thus

$$G'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt = \lim_{h \to 0} (c_h) = f(x)$$

$\square$

Examples

1. $G(x) = \int_0^x \frac{1}{1+t^2} \, dt$

   Since $f(t) = \frac{1}{1+t^2}$ is continuous on $\mathbb{R}$, by FTC 1

   $G'(x) = \frac{d}{dx} \int_0^x \frac{1}{1+t^2} \, dt = \frac{1}{1+t^2}$

2. $H(x) = \int_2^{e^x} \cos(t^2) \, dt$

   Since $f(t) = \cos(t^2)$ is continuous on $\mathbb{R}$, by FTC 1

   $H'(u) = \frac{d}{du} \int_2^u \cos(t^2) \, dt \cdot \frac{du}{dx} = \frac{du}{dx} \cos(u^2) \overset{u=e^x}{=} e^x \cos(e^{2x})$

3. Assume f is continuous and g, h differentiable

   $G(x) = \int_{g(x)}^{h(x)} f(t) \, dt = \int_{g(x)}^0 f(t) \, dt + \int_0^{h(x)} f(t) \, dt = - \int_0^{g(x)} f(t) \, dt + \int_0^{h(x)} f(t) \, dt$

   $G'(x) = -\frac{d}{dx} \int_0^{g(x)} f(t) \, dt + \frac{d}{dx} \int_0^{h(x)} f(t) \, dt \overset{\text{by FTC 1}}{=} -f'(g(x))g'(x) + f'(h(x))h'(x)$

> 🐳 **Info − Fundamental Theorem of Calculus (FTC - 2)**
>
> If f, F are continuous on $[a, b]$ and $F'(x) = f(x) \; \forall x \in (a, b)$, then
>
> $$\int_a^b f(x) \, dx = F(b) - F(a)$$

Proof

Let F be any antiderivative of f. Then $F(x)$ and the antiderivative $G(x) = \int_a^x f(t) \, dt$ have the relation that $G(x) = F(x) + C$

$$\int_a^b f(x) \, dx = \int_a^b f(t) \, dt - \int_a^a f(t) \, dt$$
$$= G(b) - G(a)$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a)$$

$\square$

Example

If $H(x) = \int_5^x x^2 \, dx$, $\int_1^2 = H(2) - H(1) = \int_5^2 x^2 \, dx - \int_5^1 x^2 \, dx = \int_1^2 x^2 \, dx$

> 🐳 **Info — Basic Integraion Rules**
>
> $$\int x^r \, \mathrm{d}x = \frac{x^{r+1}}{r+1} + C \ \forall r \neq -1$$
>
> $$\int x^{-1} \, \mathrm{d}x = \ln|x| + C$$
>
> $$\int \sin x \, \mathrm{d}x = -\cos x + C$$
>
> $$\int \cos x \, \mathrm{d}x = \sin x + C$$
>
> $$\int \sec^2 x \, \mathrm{d}x = \tan x + C$$
>
> $$\int -\csc^2 x \, \mathrm{d}x = \cot x + C$$
>
> $$\int \sec x \tan x \, \mathrm{d}x = \sec x + C$$
>
> $$\int \csc x \cot x \, \mathrm{d}x = -\csc x + C$$
>
> $$\int e^x \, \mathrm{d}x = e^x + C$$
>
> $$\int -\frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x = \arccos x + C$$
>
> $$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x = \arcsin x + C$$
>
> $$\int \frac{1}{1 + x^2} \, \mathrm{d}x = \arctan x + C$$
>
> $$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + C$$
>
> $$\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C$$
>
> $$\int \csc x \, \mathrm{d}x = -\ln|\csc x + \cot x| + C$$

Examples:

1. $\int e^{5x} \, \mathrm{d}x = \frac{e^{5x}}{5} + C$

2. $\int \frac{t}{t+1} \, \mathrm{d}t = \int 1 - \frac{1}{t+1} \, \mathrm{d}t = t - \ln|t + 1| + C$

Examples:

1.
$$\int_0^4 2x^2 - x \,\mathrm{d}x = \frac{2}{3}x^3 - \frac{x^2}{2}\bigg|_0^4 = \frac{2}{3}(4)^2 - \frac{4^3}{2} - 0 = \frac{128}{3} - 8 = \frac{104}{3}$$

2.
$$\int_1^3 \frac{x + |x - 2|}{x} \,\mathrm{d}x = \int_1^2 \frac{2}{x} \,\mathrm{d}x + \int_2^3 \frac{2x - 2}{x} \,\mathrm{d}x$$

$$= 2\ln|x|\bigg|_1^2 + \int_2^3 2 - \frac{2}{x} \,\mathrm{d}x$$

$$= 2\ln 2 + 2x - 2\ln|x|\bigg|_2^3$$

$$= 2\ln 2 + 6 - 2\ln 3 - 4 + 2\ln 2$$

$$= 4\ln 2 + 2 - 2\ln 3$$

Substitution Rule / U-Substitution

> 🐳 **Info − U-Substitution**
>
> Let f, g be functions s.t. $g'(x)$ is continuous on $[a, b]$ and f is continuous on range of g
>
> $$\int f(g(x))g(x) \,\mathrm{d}x = \int f(u) \,\mathrm{d}u\bigg|_{u=g(x)}$$

Example:

1. $\int 2x\sqrt{1 + x^2} \,\mathrm{d}x$

   Let $u = 1 + x^2 \implies \mathrm{d}u = 2x \,\mathrm{d}x \implies \mathrm{d}x = \frac{\mathrm{d}u}{2x} \overset{u=1+x^2}{\implies} \int u^{\frac{1}{2}} \,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} \overset{u=1+x^2}{\implies} +C$

   $= \frac{2}{3}(1 + x^2)^{\frac{3}{2}} + C$

2. $\int x^2 e^{x^3} \,\mathrm{d}x$

   Let $u = x^3 \implies \mathrm{d}u = 3x^2 \,\mathrm{d}x \implies \mathrm{d}x = \frac{\mathrm{d}u}{3x^2} \overset{u=x^3}{\implies} \int \frac{1}{3}e^u \,\mathrm{d}u = \frac{1}{3}e^u \overset{u=x^3}{\implies} +C = \frac{1}{3}e^{x^3} + C$

3. $\int \frac{\cos(\ln x)}{x} \,\mathrm{d}x$

   Let $u = \ln x \implies \mathrm{d}u = \frac{1}{x} \,\mathrm{d}x \implies \mathrm{d}x = x \,\mathrm{d}u \overset{u=\ln x}{\implies} \int \cos u \,\mathrm{d}u = \sin u \overset{u=\ln x}{\implies} +C = \sin(\ln x) + C$

4. $\int \frac{x}{3\sqrt[3]{x+2}} \,\mathrm{d}x$

   Let $u = x + 2 \implies \mathrm{d}u = \mathrm{d}x$

   $\overset{u=x+2}{\implies} \int \frac{u-2}{\sqrt[3]{u}} \,\mathrm{d}u = \int u^{\frac{2}{3}} \,\mathrm{d}u - 2 \int u^{-\frac{1}{3}} \,\mathrm{d}u = \frac{3}{5}u^{\frac{5}{3}} + 3u^{\frac{2}{3}} \overset{u=x+2}{\implies} +C = (x + 2)^{\frac{5}{3}} + 3(x + 2)^{\frac{2}{3}} + C$

5. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x} \,\mathrm{d}x$

   Let $u = 1 + \sin x \implies \mathrm{d}u = \cos x \,\mathrm{d}x \implies \mathrm{d}x = \frac{\mathrm{d}u}{\cos x} \implies \int_1^2 \frac{1}{u} \,\mathrm{d}u = \ln u\bigg|_1^2 = \ln 2$

6. $\int_0^{\frac{\pi}{3}} \tan x \,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \,\mathrm{d}x$

   Let $u = \cos x \implies \mathrm{d}u = -\sin x \,\mathrm{d}x \implies \int_1^{\frac{1}{3}} -\frac{1}{u} \,\mathrm{d}u = \ln u\bigg|_{\frac{1}{3}}^1 = -\ln\frac{1}{3}$

Trigonometry Substitution

Example:

1. $\int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$

   Let $u = \arcsin x \implies x = \sin u, \forall u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies \mathrm{d}x = \cos u\,\mathrm{d}u \implies$

   $\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u} \cos u\,\mathrm{d}u = \int_0^{\frac{\pi}{2}} \cos^2 u\,\mathrm{d}u = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos(2u))\,\mathrm{d}u \frac{1}{2}(u + \sin(2u))\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

2. $\int \sqrt{1 - x^2}\,\mathrm{d}x$

   Let $u = \arcsin x \implies x = \sin u\ \forall u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies \mathrm{d}x = \cos u\,\mathrm{d}u \implies$

   $\int \sqrt{1 - \sin^2 u} \cos u\,\mathrm{d}u = \int \cos^2 u\,\mathrm{d}u = \int \frac{1 + \cos 2x}{2}\,\mathrm{d}u \overset{u = \arcsin x}{\implies} \frac{1}{2}\left(\arcsin x + \sqrt{1 - x^2}\right) + C$

3. $\int \frac{1}{\sqrt{x^2 + 9}}$

   Let $x = 3\tan\theta\ \forall\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \implies \mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta \implies \int \frac{1}{\sqrt{9\tan^\theta + 9}} 3\sec^2\theta\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta$

   $\int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\,\mathrm{d}\theta = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,\mathrm{d}\theta \overset{u = \sec\theta + \tan\theta}{\implies} \int \frac{1}{u}\,\mathrm{d}u = \ln|\sec\theta + \tan\theta| + C$

   $\overset{\theta = \arctan\frac{x}{3}}{\implies} \ln|\sec\arctan\frac{x}{3} + \tan\arctan\frac{x}{3}| + C = \ln\left(\frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3}\right) + C$

4. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x$

Let $x = \sec\theta \;\; \forall \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, 3\frac{\pi}{2}\right)$

$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}\sec\theta\tan\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{3}} \tan^2\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{3}} \sec^2\theta - 1\,\mathrm{d}\theta = \left.\tan\theta - \theta\right|_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}$

5. $\int \frac{1}{(5-4x-x^2)^{\frac{5}{2}}}\,\mathrm{d}x = \int \frac{1}{(1-(2-x)^2)^{\frac{5}{2}}}\,\mathrm{d}x$

Let $2 - x = \sin\theta,\;\; \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Longrightarrow \mathrm{d}x = -\cos\theta\,\mathrm{d}\theta$

$= \int \frac{-\cos\theta}{(1-(\sin\theta)^2)^{\frac{5}{2}}}\,\mathrm{d}\theta = \int \frac{-\cos\theta}{\cos^5\theta}\,\mathrm{d}\theta = \int -\sec^4\theta\,\mathrm{d}\theta = -\int \left(\sec^2\theta\right)^2\,\mathrm{d}\theta = -\frac{2}{7}\sec^{\frac{7}{2}}\theta + C =$

$-\frac{2}{7}\left(\frac{1}{\sqrt{5-4x-x^2}}\right)^{\frac{7}{2}} + C$

Integration By Parts

> 🐋 **Info — Integration by Parts**
>
> If f and g are differentiable functions of x, then
>
> $$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$
>
> The rule of choosing parts being $f(x)$ is
>
> 1. Logarithmic
>
> 2. Inverse trigonometric
>
> 3. Algebraic
>
> 4. Trigonometric
>
> 5. Exponential

Examples

1. $\int xe^x\,\mathrm{d}x$

   Let $\begin{cases} u=x \\ \mathrm{d}v=e^x\,\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}u=\mathrm{d}x \\ v=e^x \end{cases} \Longrightarrow xe^x - \int e^x = xe^x - e^x + C$

2. $\int x^2 \sin x\,\mathrm{d}x$

   Let $\begin{cases} u=x^2 \\ \mathrm{d}v=\sin x\,\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}u=2x\,\mathrm{d}x \\ v=-\cos x \end{cases} \Longrightarrow -x^2\cos x + 2\int x\cos x\,\mathrm{d}x = -x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C$

3.