CH 10 - Complex Numbers

Luke Lu • 2025-11-24

Standard Form

> 🐳 **Info — Definition of Complex Numbers**
>
> A **complex number** z in **standard form** is an expression of the form $z = x + yi$ where $x, y \in \mathbb{R}$.
>
> The real number x is called the **real part** of z, and is written $\mathfrak{R}(x)$.
>
> The real number y is called the **imaginary part** of z, and is written $\mathfrak{I}(z)$.
>
> The set of complex numbers is
>
> $$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$$
>
> The complex number $z = x + yi$ and $w = u + vi$ are equal $(z = w)$ if and only if $x = u, y = v$
>
> A complex number z is said to be purely real if $y = 0$ (i.e. $1 = 1 + 0i$)
>
> A complex number z is said to be purely imaginary if $x = 0$ (i.e. $i = 0 + 1i$)
>
> 0 is purely real and purely imaginary (i.e. $0 = 0 + 0i$)

> 🐳 **Info — Complex Arithmetics**
>
> Let $z = a + bi$ and $w = c + di$ be complex numbers. Then the
>
> Addition is defined as
>
> $$z + w = (a + c) + (b + d)i$$
>
> Multiplication is defined as
>
> $$zw = (ac - bd) + (ad + bc)i$$

Examples:

Let $z = 2 + 3i, w = -1 + 7i$

1. $z + w = (2 - 1) + (3 + 7)i = 1 + 10i$

2. $zw = (-2 - 21) + (14 - 3)i = -23 + 11i$

3. $i^2 = ii = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + (0 + 0)i = -1$

$\therefore i^2 = -1$

From (3), we can derive a easier way of multiplication

> 💡 **Tip** — Multiplication trick
>
> $$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Properties of Complex Arithmetics

1. $z + 0 = 0 + z = z$
2. $z0 = 0z = 0$
3. $z + (-1)z = (-1)z + z = 0$
4. $z1 = 1z = z$

> 🐳 **Info** — **Multiplicative Inverse**
>
> For all complex numbers z, the multiplicative inverse of z exists if and only if $z \neq 0$. Moreover, for $a + bi \neq 0$, the multiplicative inverse is unique given by
>
> $$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i = \frac{a - bi}{a^2 + b^2}$$

Examples:

1. $\frac{(1-2i)-(3+4i)}{(5-6i)} = [(1 - 2i) - (3 + 4i)] \cdot (5 - 6i)^{-1} = (-2 - 6i) \cdot \frac{1}{25+36} \cdot (5 + 6i)$

$= \frac{1}{61} \cdot (-10 - 30i - 12i + 36) = \frac{26-42i}{61} = \frac{26}{61} - \frac{42}{61}i$

2. $i^{2025} = \left(i^4\right)^{506} \cdot i = (-1)^{506} \cdot i = i$

> 💡 **Tip** — i^n **Solutions**
>
> $$i^n = \begin{cases} 1, n \equiv 0 \pmod 4 \\ i, n \equiv 1 \pmod 4 \\ -1, n \equiv 2 \pmod 4 \\ -i, n \equiv 3 \pmod 4 \end{cases}$$

> 🐳 **Info** — **Properties of Complex Arithmetics**
>
> In complex arithmetic, the following properties are vlaid for $u, v, z \in \mathbb{C}$
>
> 1. Associativity of addition: $(u + v) + z = u + (v + z)$
> 2. Communtativity of addition: $u + v = v + u$
> 3. Additive identity: $0 = 0 + 0i$ has the property that $z + 0 = z$
> 4. Additive inverses: If $z = a + bi$, then there exists an additive inverse of z, written $-z$, with the property that $z + (-z) = 0$. The additive inverse of $z = a + bi$ is $-z = -a - bi$
> 5. Associativity of multiplication: $(uv)z = u(vz)$
> 6. Communtativity of multiplication: $uv = vu$
> 7. Multiplicative identity: $1 = 1 + 0i$ has property that $z1 = z$
> 8. Multiplicative inverses: If $z = a + bi \neq 0$, then there exists a multiplicative inverse of z, written z^{-1}, with the property that $zz^{-1} = 1$. The multiplicative inverse of $z = a + bi \neq 0$ is $z^{-1} = \frac{a-bi}{a^2+b^2}$
> 9. Distributivity: $z(u + v) = zu + zv$

Example:

Proof of PCA Part 5: $(uv)z = u(vz)\ \forall u, v, z \in \mathbb{C}$

Let $u = a + bi, v = c + di, z = x + yi$ where $a, b, c, d, x, y \in \mathbb{R}$ so that $u, v, z \in \mathbb{C}$

$(uv)z = [(a + bi)(c + di)](x + yi) = (ac - bd + (ad + bc)i)(x + yi) =$

$= ((ac - bd)x - (ad + bc)y) + ((ac - bd)y + (ad + bc)x)i$

$= (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$u(vz) = (a + bi)[(c + di)(x + yi)] = (a + bi)(cx - dy) + (cy + dx)i$

$= (a(cx - dy) - b(cy + dx)) + (a(cy + dx) + b(cx - dy))i$

$= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i$

Thus $(uv)z = u(vz) = (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$\square$

> 🐳 **Info** — Other Arithemtic of Complex Numbers
>
> For $z \in \mathbb{C}$
>
> 1. $z^0 = 1$
> 2. $z^1 = z$
> 3. $z^{k+1} = z^k z\ \forall k \in \mathbb{N}$
> 4. $(z^n)^m = z^{nm}$ and $z^n z^m = z^{n+m}\ \forall n, m \in \mathbb{N} \cup \{0\}$
>
> For $n \notin \mathbb{N} \cup \{0\}$ will be discussed in later lectures

Examples:

Find a real solution to $6z^3 + \left(1 + 3\sqrt{2}i\right)z^2 - \left(11 - 2\sqrt{2}\right)i - 6 = 0$

Let $z = x \in \mathbb{R}$, that is $z = x + 0i$

$\implies 6x^3 + \left(1 + 3\sqrt{2}i\right)x^2 - \left(11 - \sqrt{2}i\right)x - 6 = 0$

$\implies (6x^3 + x^2 - 11x - 6) + \left(3\sqrt{2}x^2 + 2\sqrt{2}x\right)i = 0 + 0i$

$\implies (6x^3 + x^2 - 11x - 6) = 0$ and $3\sqrt{2}x^2 + 2\sqrt{2}x = 0$

$x = 0$ or $x = -\frac{2}{3}$ for the imaginary part.

However, $x = 0$ does not satisfy the real part.

$\therefore x = -\frac{2}{3} \implies z = -\frac{2}{3} + 0i$

Conjugate and Modulus

> 🐳 **Info** — Complex Conjugate
>
> The complex **conjugate** of a complext number $z = x + yi$ written $\overline{z}$ is the complex number
>
> $$\overline{z} = x - yi$$

> 🐳 **Info − Properties of Conjugate**
>
> For the complex conjugate, the following properties hold $\forall z, w \in \mathbb{C}$
>
> 1. $\overline{(\overline{z})} = z$
> 2. $\overline{z + w} = \overline{z} + \overline{w}$
> 3. $z + \overline{z} = 2\Re(z)$ and $z - \overline{z} = 2\Im(z)i$
> 4. $\overline{zw} = \overline{z} \cdot \overline{w}$
> 5. If $z \neq 0, \overline{(z^{-1})} = (\overline{z})^{-1}$
> 6. If $w \neq 0, \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

Proof

Part 3

$z + \overline{z} = 2\Re(z)$ and $z - \overline{z} = 2\Im(z)i$

Let $z = x + yi$

$z + \overline{z} = (x + yi) + (x - yi) = 2x + 0i = 2\Re(z)$

$z - \overline{z} = (x + yi) - (x - yi) = 0x + 2i = 2\Im(z)$

Part 4

$\overline{zw} = \overline{z} \cdot \overline{w}$

Let $z = x + yi, w = a + bi$

$\overline{zw} = \overline{(x + yi)(a + bi)} = \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb + ya)i$

$\overline{zw} = \overline{x + yi}\,\overline{a + bi} = (x - yi)(a - bi) = (xa - (-y)(-b)) + (x(-b) + (-y)a)i$

$= (xa - yb) - (xb + ya)i$

Thus $\overline{zw} = \overline{z} \cdot \overline{w}$

$\square$

Examples:

1. Prove $z \in \mathbb{R} \iff z = \overline{z}$

   Let $z = x + yi \;\; \forall x, y \in \mathbb{R}$

   $\implies$

   Suppose $z \in \mathbb{R}$, then $y = 0$, so that $z = x + 0i = x \in \mathbb{R}$.

T. hen $\overline{z} = x - 0i = x$

$\therefore z = \overline{z}$

$\impliedby$

Suppose $z = \overline{z}$, then $x + yi = x - yi$

This implies $x = x, y = -y$. Thus $y = 0$

$\therefore z = x + 0i = x \in \mathbb{R}$

2. Prove that z is purely imaginary $\iff z = -\overline{z}$

3. Solve $z^2 = i\overline{z}$

   Let $z = x + yi$. Then $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$

   and $i\overline{z} = i(x - yi) = ix - i^2 y = xi + y$

   Then the equation becomes $x^2 - y^2 + 2xyi = xi + y$

   $\implies \begin{cases} x^2 - y^2 = y \\ 2xy = x \end{cases} \implies x = 0 \wedge y = \frac{1}{2} \implies \begin{cases} x^2 - \frac{1}{4} = \frac{1}{2} \\ -y^2 = 0 \end{cases} \implies z = \left\{ 0, -i, \pm\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}i \right\}$

Modulus of Complex Numbers

> 🐳 **Info — Modulus of Complex Number**
>
> The **modulus** of the complex number $z = x + yi$, written $|z|$, is the non-negative real number
>
> $$|z| = \sqrt{x^2 + y^2}$$

> 🐳 **Info — Properties of Modulus**
>
> For the modulus, the following properties $\forall z, w \in \mathbb{C}$:
>
> 1. $|z| = 0 \iff z = 0$
>
> 2. $|\overline{z}| = |z|$
>
> 3. $\overline{z}z = |z|^2$
>
> 4. $|zw| = |z\|w|$
>
> 5. If $z \neq 0$ then $|z^{-1}| = |z|^{-1}$
>
> Side note: for $z \neq 0$, $z^{-1} = \frac{\overline{z}}{|z|^2}$
>
> 6. If $w \neq 0$, $\frac{z}{w} = z \cdot \frac{\overline{w}}{|w|^2}$
>
> 7. $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$
>
> 8. $\overline{\sum z_i} = \sum \overline{(z_i)}$
>
> 9. $\overline{\prod z_i} = \prod \overline{(z_i)}$
>
> 10. $|\prod z_i| = \prod(|z_i|)$

Proof

Part 3

$\overline{z}z = |z|^2$

Let $z = a + bi, \forall a, b \in \mathbb{R}$

then $\overline{z}z = (a - bi)(a + bi) = (a^2 - (-b)(b)) + (ab + (-b)a)i$

$= a^2 + b^2 + 0i = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2$

□

The Complex Plane and Polar Form

Complex Plane

> 💡 **Tip — Geometric Interpretation and Graphical Properties**
>
> - z and $\bar{z}$ are reflection of each other over real axis
>
> - Modulus is the distance from the point z to origin
>
> - For addition, it is similar to vector addition that is the parallelogram rule, $z + w$
>
> - For subtraction, consider $z + w - w = z$ and the rest is same for addition

> 🐳 **Info — Triangle Inequality**
>
> For all $z, w \in \mathbb{C}$, we have
>
> $$|z + w| \leq |z| + |w|$$
>
> Note that $|z|$ is the modulus of z, not absolute value

Proof

Let $z = x + yi, w = u + vi$, where $x, y, u, v \in \mathbb{R}$

$|z + w| = |(x + u) + (y + v)i| = \sqrt{(x + u)^2 + (y + v)^2}$
$= \sqrt{(x - (-u))^2 + (y - (-v))^2}$

(The Euclidean distance formulat between (x, y) and $(-u, -v)$)

Consider thr triangle ABC constructed from points

$A : (0, 0); \ B(x, y) = (z = x + yi); \ C : (-u, -v) = (-w = -u - vi)$

Let l_{AB} = length of side AB, l_{BC} and l_{AC} have the similar constructed

From geometric perspective, $l_{BC} \leq l_{AB} + l_{AC}$

Note that $l_{AB} = \sqrt{x^2 + y^2} = |z|, l_{AC} = \sqrt{(-u)^2 + (-v)^2} = |w|$

$l_{BC} = \sqrt{(x - (-u))^2 + (y - (-v))^2} = |z + w|$

Therefore $|z + w| \leq |z| + |w|$

□

Exercise:

Let $z \neq \pm i$. Prove that $\frac{z}{1+z^2}$ is real $\iff z \in \mathbb{R}$ or $|z = 1|$

Polar Form

> 🐳 **Info − Polar Form**
>
> A **polar form** of a complex number z is denoted
>
> $$z = r(\cos\theta + i\sin\theta)$$
>
> where $r \geq 0$, being the modulus of z and angle $\theta \in \mathbb{R}$ be an **argument** of z
>
> - θ is not unique unless given restriction of $\theta \in [0, 2\pi)$
>
> Notice that in standard form $z = x + yi$
>
> - $x = r\cos\theta$
> - $y = r\sin\theta$
> - $r = |z| = \sqrt{x^2 + y^2}$
> - $\theta = \arctan\left(\frac{y}{x}\right)$

Examples:

Convert polar form to standard form

1. $z = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

2. $z = \cos\frac{15}{6}\pi + i\sin\frac{15}{6}\pi = 0 + 1i = i$

Convert from standard form to polar form

3. $z = \sqrt{6} + \sqrt{2}i$

   $r = \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2} = 2\sqrt{2}$

   $\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ is one possibility as the angle is not unique

   $z = 2\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

4. $z = -3\sqrt{2} + 3\sqrt{6}i = 6\sqrt{2}\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right)$

> 🐳 **Info − Polar Multiplication for $\mathbb{C}$**
>
> For all complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then their product is
>
> $$z_1 z_2 = r_1 r_2(\cos\theta(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Proof

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \in \mathbb{C}$

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta + 2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$= r_1 r_2(\cos\theta(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$\square$

Examples:

1. Compute $(i + 1)i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \overset{\text{By } \mathrm{PM}\mathbb{C}}{=} \sqrt{2}\left(\cos\frac{3\pi}{4}i\sin\frac{3\pi}{4}\right)$

   $= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -1 + i$

2. Find $\left(\sqrt{6} + \sqrt{2}i\right)\left(-3\sqrt{2} + 3\sqrt{6}i\right)$

   $= 2\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \cdot 6\sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 24\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 24\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

   $= -12\sqrt{3} + 12i$

De Moivre's Theorem

> 🐳 **Info** — **De Moivre's Theorem (DMT)**
>
> $\forall \theta, n \in \mathbb{R},$
>
> $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

> 🐳 **Info** — **Corollary of DMT**
>
> For all complex number $z = r(\cos\theta + i\sin\theta)$ and integer n, except $|z| = r = 0$ and $n < 0$ we have
>
> $$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Examples:

1. $\left(\sqrt{3} - i\right)^{10} = \left[2\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)\right]^{10} = 1024\left(\cos -\frac{10\pi}{6} + i\sin -\frac{10\pi}{6}\right) = 512 + 512\sqrt{3}i$

2. Prove that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ (Hint: Consider $(\cos\theta + i\sin\theta)^4$ expand normally and apply DMT then compare real parts)

3. Prove that $\forall n \in \mathbb{Z}$ if $w \in \mathbb{C}$, $|w| = 1$ and θ is an argument of w, then $-\frac{i}{2}(w^n - w^{-n} = \sin(n\theta))$

> 🐳 **Info** — **Additional Information**
>
> For notation, we write $\mathrm{cis}(\theta) = \cos\theta + i\sin\theta$ notation wise
>
> Also, $\mathrm{cis}(\theta) = e^{i\theta}$ that is having the similar properties of exponentials
>
> $e^{i\pi} = \mathrm{cis}(\pi) = -1 \implies e^{i\pi} + 1 = 0$ which is the Euler's Formula