

## CH 5 - Applications of Derivatives

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### Related Rates

💡 **Tip** — Steps for Related Rates Questions

1. Draw diagram
2. Identify **changing** quantities
3. Find **constant** quantities (if possible)
4. Derive equations relating the quantities that are changing
5. **Implicitly differentiate** the key equations
6. Solve for the desired rate of change, substituting in known quantities.
7. **Concluding statement** (and also check units)

Example:

1. Laindon is taking a hot air balloon ride. A giant fan is blowing hot air into the balloon in a rate of  $200 \frac{\text{m}^3}{\text{min}}$ . Assuming that at any given point in time the balloon sphere, find the rate at which the radius of the balloon is changing when the diameter is 12 m.

ANS:

1. Picture: The problem is trivial so the graph is omitted
  2. Changing variable: Volume( $\text{m}^3$ ), Radius(m), time(t)
  3. Constant quantities:  $\frac{dV}{dt} = 200 \frac{\text{m}^3}{\text{min}}$
  4. Key Equation:  $V = \frac{4}{3}\pi r^3(t)$
  5. Implicit Differentiation:  $\frac{dV}{dt} = 4\pi r^2(t) \cdot \frac{dr}{dt}$
  6.  $\left. \frac{dr}{dt} \right|_{r=6} = \frac{1}{4\pi(6)^2} \cdot 200 = \frac{200}{144\pi} = \frac{25}{18\pi} \frac{\text{m}}{\text{min}}$
  7. Concluding statement: When the diameter of the balloon is 12m, the rate of change of the radius is expanding by  $\frac{25}{18\pi} \frac{\text{m}}{\text{min}}$
2. The construction workers building M4 accidentally left a 20 foot ladder propped up against a concrete wall that is 80 feet in height. The base of the ladder begins to slide away from the wall at a rate of 2ft/sec, and the top begins to move down as a result. When the base of the ladder is 14 ft from the wall, how fast is the top of the ladder sliding down the wall?

ANS:

1. Picture is omitted and left as an exercise for the reader
2. Changing variable: Distance from wall of base of ladder (m), Height where ladder touches the wall (m)
3. Constant quantities :  $\frac{dx}{dt} = 2$
4. Key Equation:  $x^2 + y^2 = 20^2$

5. Implicit Differentiation:  $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$
6.  $\frac{dy}{dt} = -\frac{14}{\sqrt{400-14^2}} \cdot 2 = -\frac{14}{\sqrt{51}} \frac{\text{ft}}{\text{sec}}$
7. Concluding statement: When the base of ladder is 14cm, the top of the ladder is falling at a speed of  $\frac{14}{\sqrt{51}} \frac{\text{ft}}{\text{sec}}$

## Extrema

### Info – Extrema

Let  $f(x)$  be a function defined on an interval  $I$ , and let  $c \in I$ . We say  $f$  has

1. A **global minimum** on  $I$  at  $x = c$  if  $f(c) \leq f(x) \forall x \in I$
  2. A **global maximum** on  $I$  at  $x = c$  if  $f(c) \geq f(x) \forall x \in I$
  3. A **global extremum** on  $I$  at  $x = c$  if  $f$  has either a global minimum or global maximum.
- Every point on a constant function is both a global minimum and global maximum
  - Every global extremum can be a local extremum in some interval

Examples:

1. Find all global extrema of  $f(x) = x^2$  on  $[0, 1]$ 
  - The global minimum be  $x = 0$  because  $f(0) \leq f(x) \forall x \in [0, 1]$
  - The global maximum DNE as the end point is missing. That is infinitely numbers lie on the interval  $[0, 1]$
2. Find all global extrema of  $f(x) = \frac{1}{x}$  on  $[-1, 1]$ 
  - The global extrema DNE as  $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$  and  $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$ .

### Info – Extreme Value Theorem (Existence Thm)

Assume that  $f(x)$  is continuous on the closed interval  $[a, b]$ . Then **there exist** two numbers  $c_1, c_2 \in [a, b]$  s.t.  $f(c_1) \leq f(x) \leq f(c_2) \forall x \in [a, b]$ .

In other words, there is a global minimum at  $x = c_1$  and a global maximum at  $x_{c_2}$

### Info – Local Extrema

Let  $f$  be a function. We say that  $f$  has

1. a **local minimum** at  $x = c$  if there exists an open interval  $(a, b)$  containing  $c$  such that  $f(c) \leq f(x) \forall x \in (a, b)$
2. a **local maximum** at  $x = c$  if there exists an open interval  $(a, b)$  containing  $c$  such that  $f(c) \geq f(x) \forall x \in (a, b)$
3. a **local extremum** at  $x = c$  if there is either a local minimum or a local maximum

### Warning – Local Extrema

If  $c$  is an endpoint of the domain of  $f$ ,  $c$  can never be a local extremum, even if it is a global extremum

### Info – Fermat's Theorem

If there is a local extremum for  $f(x)$  at  $x = c$  and  $f'(c)$  exists, hence  $f'(c) = 0$ . That is we cannot put an open interval around the point.

Examples:

1. Does the converse of Fermat's Theorem hold? That is if  $f'(0) = 0$ , then is a local extremum at  $x = c$ .

This is false. Let  $f(x) = x^3$ ,  $f'(x) = 3x^2$ ,  $f'(0) = 0$  but is not a local extremum on any interval containing  $x = 0$

2. Why is it worth mentioning  $f'(c)$  has to exist?

It is important because it is like saying  $f(x)$  is differentiable at  $x = c$ . If not, let  $f(x) = |x|$ .  $f(x)$  is continuous. It has a local minimum at  $x = 0$  but  $f'(0)$  DNE as it is not differentiable.

### Info – Critical Points

We say that a function  $f$  has a **critical point** at  $x = c$  if  $f'(c) = 0$  or  $f'(c) = \text{DNE}$  for  $c \in$  the domain of  $f$ . These are our candidates for local extrema.

### Tip – Closed Interval Method

Let  $f(x)$  be continuous function on  $[a, b]$ .

1. Calculate  $f(a)$  and  $f(b)$
2. Find  $f'(x)$
3. Find all the critical points of  $f$  on  $[a, b]$
4. Calculate  $f(c)$

Example:

$$f(x) = \frac{1}{3}x^3 - 3\sqrt[3]{x} \text{ on } [-8, 1]$$

$$f(-8) = -\frac{512}{3} - 3(-2) = -\frac{496}{3}$$

$$f(1) = \frac{1}{3} - 3 = -\frac{8}{3}$$

$$f'(x) = x^2 - x^{-\frac{2}{3}}$$

$$f'(c) = 0 \implies c^2 - c^{-\frac{2}{3}} = 0 \implies c^{\frac{8}{3}} = 1 \implies c = -1, 1. f'(c) = \text{DNE} \implies c = 0$$

$$f(0) = 0$$

$$f(-1) = -\frac{1}{3} + 3 = \frac{8}{3} \text{ Global maximum at } x = -1, \text{ global minimum at } f(-8)$$

### Info – Rolle's Theorem (Existence Thm)

If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $f(a) = f(b) = k \in \mathbb{R}$ , then there exists  $c \in (a, b)$  s.t.  $f'(c) = 0$

### Proof

If  $f(x) = k \forall x \in [a, b]$ , any value of  $c$  works.

Otherwise,  $\exists x_0 \in [a, b]$  s.t.  $f(x_0) \neq k$ . Since  $f$  is continuous on  $[a, b]$ , it attains a maximum/minimum on  $[a, b]$ .

Since  $f(x_0) \neq k \implies f(x_0) > k \iff f(a), f(b)$  are not maximum, or  $f(x_0) < k \iff f(a), f(b)$  are not minimum. So one of maximum or minimum is in  $(a, b)$ , thus differentiable at some  $c$ .

By Fermat's Theorem,  $f'(c) = 0$  or  $f'(c) = \text{DNE}$ . But  $f$  is differentiable on  $(a, b) \implies f'(c)$  exists.

$\therefore f'(c) = 0$

### Info – Mean Value Theorem (Existence Thm)

If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then  $\exists c \in (a, b)$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

### Proof

Let  $h(x) = f(x) - \left[ f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$

$$h(a) = f(a) - \left[ f(a) + \frac{f(b)-f(a)}{b-a} \cdot (a-a) \right] = 0$$

$$h(b) = f(b) - \left[ f(a) + \frac{f(b)-f(a)}{b-a} \cdot (b-a) \right] = 0$$

Since  $h(b) = h(a) \xrightarrow{\text{Rolle's Theorem}} \exists c \in (a, b)$  s.t.  $h'(c) = 0$

That is  $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \implies f'(x) = \frac{f(b)-f(a)}{b-a}$

Finally,  $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \iff f'(c) = \frac{f(b)-f(a)}{b-a}$

Tip: the construction of  $\left[ f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$  is the linear approximation of  $f(x)$  near  $a$

## Antiderivative

### Info – Antiderivative

Given a function  $f(x)$ , an **antiderivative** is a function  $F(x)$  s.t.  $F'(x) = f(x)$ . If  $F'(x) = f(x)$  for all  $x \in I$  for some interval  $I$ , then  $F(x)$  is an antiderivative of  $f(x)$  on  $I$

e.g.  $\frac{d}{dx} - \ln(\cos x) = -\frac{1}{\cos} x \sin x = -\frac{-\sin x}{\cos x} = \tan x$

Note: one function can have infinitely many antiderivatives, that is why we insist an **antiderivative** of  $f(x)$

### Info – Constant Function Theorem

Suppose that  $f'(x) = 0 \forall x \in I$  for some interval  $I$ . Then  $\exists \alpha \in \mathbb{R}$  s.t.  $f(x) = \alpha \forall x \in I$

### Proof

Let  $x_1 < x_2 \in I$ .

Since  $f$  is differentiable on  $I$ , it is continuous on  $[x_1, x_2]$  and differentiable on  $(x_1, x_2)$ .

By MVT,  $\exists c \in (x_1, x_2)$  s.t.  $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ . But  $f'(c) = 0$  since  $f'(x) = 0$  on  $I$ .

Thus,  $0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \wedge x_2 - x_1 \neq 0 \implies f(x_2) - f(x_1) = 0 \iff f(x_2) = f(x_1)$ .

Since  $x_1, x_2$  are arbitrary, therefore  $f$  is constant on  $I$

### Info – Antiderivative Theorem

Suppose that  $F'(x) = G'(x) \forall x \in I$  for some interval  $I$ . Then there exists  $\alpha \in \mathbb{R}$  s.t.  $F(x) = G(x) + \alpha \forall x \in I$

### Proof

Let  $h(x) = F(x) - G(x)$ . Then  $h'(x) = F'(x) - G'(x) = 0$  on  $I$ .

By the CFT,  $h(x) = \alpha$  for some  $\alpha \in \mathbb{R}$ , so  $F(x) - G(x) = \alpha \implies F(x) = G(x) + \alpha \quad (\forall x \in I)$

## First Derivatives

### Info – Definition of Increasing/Decreasing

Let  $I$  be an interval and  $x_1, x_2 \in I$ , then  $f(x)$

- **increasing** on  $I$  if  $f(x_1) \leq f(x_2) \quad \forall x_1 < x_2$
- **decreasing** on  $I$  if  $f(x_1) \geq f(x_2) \quad \forall x_1 < x_2$
- **strictly increasing** on  $I$  if  $f(x_1) < f(x_2) \quad \forall x_1 < x_2$
- **strictly decreasing** on  $I$  if  $f(x_1) > f(x_2) \quad \forall x_1 < x_2$

Note: a constant function is both increasing and decreasing but not strictly

### Info – Increasing/Decreasing Function Theorem

Let  $I$  be an interval

1. If  $f'(x) \geq 0 \forall x \in I$ , then  $f(x)$  is increasing on  $I$
2. If  $f'(x) > 0 \forall x \in I$ , then  $f(x)$  is strictly increasing on  $I$
3. If  $f'(x) \leq 0 \forall x \in I$ , then  $f(x)$  is decreasing on  $I$
4. If  $f'(x) < 0 \forall x \in I$ , then  $f(x)$  is strictly decreasing on  $I$

### Proof

Let  $x_1 < x_2 \in I$ .  $f'(x) > 0$  on  $I$ , so it exists on  $I \implies f$  is differentiable on  $(x_1, x_2)$ , continuous on  $[x_1, x_2]$ .

By MVT,  $\exists c \in (x_1, x_2)$  s.t.  $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Given  $f'(x) > 0 \forall x \in I$  and  $x_1 < x_2$  and  $f(x_1) < f(x_2)$

Since  $x_1, x_2$  are arbitrary, thus  $f$  is strictly increasing on  $I$

Proof for increasing, strictly decreasing and decreasing is similar thus be omitted.

Question: If  $f$  is strictly increasing on  $I \implies f'(x) > 0 \forall x \in I$ ?

ANS: No, counterexample  $f(x) = x^3$

Question: If  $f$  is strictly decreasing on  $I \implies f'(x) < 0 \forall x \in I$ ?

ANS: No, counterexample  $f(x) = -\sqrt[3]{x}$



### Info – Bounded Derivative Theorem

Let  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

Suppose that  $m \leq f'(x) \leq M \forall x \in (a, b)$ . Then

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

$$\forall x \in [a, b]$$

### Proof

Let  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

It also applies to  $[a, x_1]$ .

Case1:

For  $x_1 \in (a, b]$

By MVT,  $\exists c \in (a, x_1)$  s.t.  $f'(c) = \frac{f(x_1) - f(a)}{x_1 - a}$

Since  $m \leq f'(c) \leq M$  so  $m(x_1 - a) \leq f(x_1) - f(a) \leq M(x_1 - a)$ .

Then  $m(x_1 - a) + f(a) \leq f(x_1) \leq M(x_1 - a) + f(a)$

Case 2:

When  $x = a$ ,  $m(x - a) + f(a) = f(a)$  and similar applies to  $M(x - a)$ .

Resulting to  $f(a) \leq f(a) \leq f(a)$

Example:

Prove that  $\sqrt{51} \in [7 + \frac{1}{8}, 7 + \frac{1}{7}]$ .

Let  $f(x) = \sqrt{x}$  and let  $[a, b]$  be  $[49, 64]$

Since  $f(x)$  is continuous on  $[49, 64]$  and differentiable on  $[49, 64]$

By Bounded Derivative Theorem,  $f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$

where  $m \leq f'(x) \leq M$  on  $[49, 64] \forall x$ .

$$f'(x) = \frac{1}{2\sqrt{x}}, \frac{1}{\sqrt{64}} \leq \frac{1}{2\sqrt{x}} \leq \frac{1}{2\sqrt{49}} \implies 7 + \frac{1}{2} \cdot \frac{1}{8}(51 - 49) \leq \sqrt{51} \leq \sqrt{49} + \frac{1}{2} \cdot \frac{1}{7}(51 - 49) \\ \implies 7 + \frac{1}{8} \leq \sqrt{51} \leq 7 + \frac{1}{7}$$



### Info – Comparison via First Derivative Theorem

Assume  $f(x)$  and  $g(x)$  are continuous at  $x = a$  with  $f(a) = g(a)$ . Then

- $f$  and  $g$  are differentiable for  $x > a$  and  $f'(x) \leq g'(x) \forall x > a \implies f(x) \leq g(x) \forall x > a$
- $f$  and  $g$  are differentiable for  $x < a$  and  $f'(x) \leq g'(x) \forall x < a \implies f(x) \geq g(x) \forall x < a$

Example:

Show that  $x - \frac{1}{2}x^2 < \ln(1 + x) \forall x > 0$

Let  $f(x) = x - \frac{1}{2}x^2, g(x) = \ln(1 + x)$

At  $x = 0, f(0) = 0, g(0) = 0$ , that is  $f(0) = g(0)$

All functions are differentiable on  $(0, \infty)$

$$f'(x) = 1 - x, g'(x) = \frac{1}{1+x}.$$

Since  $x > 0, (1 - x)(1 + x) < 1 \implies 1 - x^2 < 1 \implies -x^2 < 0$

Thus by CFDT,  $x - \frac{1}{2}x^2 < \ln(1 + x) \forall x > 0$

## Second Derivatives



### Info – Concavity and Second Derivative Theorem

1. We say  $f$  is **concave up** on an interval  $I$  if for all  $a, b \in I$ , the secant line between  $(a, f(a))$  and  $(b, f(b))$  lies **above** the graph of  $f(x)$ .
2. We say  $f$  is **concave down** on an interval  $I$  if for all  $a, b \in I$ , the secant line between  $(a, f(a))$  and  $(b, f(b))$  lies **below** the graph of  $f(x)$ .
3. If  $f''(x) > 0 \forall x$  in an interval  $I$  then  $f(x)$  is **concave up** on  $I$
4. If  $f''(x) < 0 \forall x$  in an interval  $I$  then  $f(x)$  is **concave down** on  $I$

Example:

$f(x) = |x|$  is neither concave up nor concave down as the secant line lies on the graph if  $a$  and  $b$  are same side of the absolute function.

### **Info – Point of Inflection**

A point  $(c, f(x))$  is called a **point of inflection** of  $f(x)$  if  $f(x)$  is continuous at  $x = c$  and the concavity of  $f(x)$  changes at  $x = c$

If  $f''(x)$  is continuous at  $c$  and  $(c, f(c))$  is a point of inflection, then  $f''(x) = 0$  or DNE

However, the converse is false:  $f(x) = x^4$ ,  $f''(0) = 0$ , but the concavity does not change

## **Derivative Tests**

## **Optimization**

## **L'Hôpital Rule**