CH 3 — Function Limits and Continuity

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Definitions

If $f : \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}, \lim_{x \to a} f(x) = L$ if
for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

Examples:

1)Prove using the $\varepsilon - \delta$ definition that $\lim_{x \to 0} f(x)$ DNE where

$$f(x) = \begin{cases} -2 \text{ if } x < 0 \\ 3 \text{ if } x > 0 \end{cases}$$

Domain: $\mathbb{R} \setminus \{0\}$

Take $\varepsilon = 1$. Consider some $\delta > 0$. Whitin $(0 - \delta, 0 + \delta)$
We have both $(-\delta, 0)$ where $f(x) = -2$ and $(0, \delta)$ where $f(x) = 3$. If this δ exists for $\varepsilon = 1$ then the limit L would need to be distance 1 or both −2 and 3, where is impossible.

$$\therefore \lim_{x \to 0} f(x) = \text{DNE}$$

2) $\lim_{x \to 7} 8x - 3 = 53$

Let $\varepsilon > 0$ be arbitrary.
We want find δ s.t. if $0 < |x - 7| < \delta$ then $|8x - 3 - 53| < \varepsilon \to \delta = \frac{\varepsilon}{8}$
Pick $\delta = \frac{\varepsilon}{8}$.
Then if $0 < |x - 7| < \frac{\varepsilon}{8}, |(8x - 3) - 53| = |8x - 56| = 8|x - 7| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$

3) $\lim_{x \to 1} x^2 + 3x + 4 = 8$

We want for any $\varepsilon > 0$ and $\delta > 0 : |x - 1| < \delta$, then $|f(x) - L| < \varepsilon$

$\leftrightarrow |x^2 + 3x - 4| < \varepsilon \leftrightarrow |(x + 4)(x - 1)| < \varepsilon \leftrightarrow |x + 4| - |x - 1| < \varepsilon$

I can always make δ smaller if I need to.

take $\delta < 1$, then $|x - 1| < 1 \Longrightarrow 0 < x < 2$
$|x + 4| < 6 \to |x + 4\|x - 1| < 6\delta$, but $6\delta < \varepsilon \leftrightarrow \delta < \frac{\varepsilon}{4}$. Say $\frac{\varepsilon}{7} < \frac{\varepsilon}{6}$ for all epsilon.
Take $\delta < \min\left(1, \frac{\varepsilon}{6}\right)$

Proof

Let $\varepsilon > 0$ be given. Take $\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{7}\right)$. Then, if $|x - 1| < \delta, |x^2 + 3x + 4 - 8| = |x^2 + 3x - 4| = |(x + 4)(x - 1)| = |(x + 4)(x - 1)| < 6 \cdot \frac{\varepsilon}{7} < \varepsilon$

$\square$

Info — **Sequential Characterization of Limits Theorem**
Let $a \in \mathbb{R}$. let the function $f(x)$ be defined on an open interval containing a, expect possibly at $x = a$ itself. Then the following are equivailent:

1. $\lim_{x \to a} f(x) = L$
2. For all sequences $\{x_n\}$ satisfying $\lim_{n \to \infty} x_n = a$ and $x_n \neq a, \forall n \in \mathbb{N}$, we have that $\lim_{n \to \infty} f(x_n) = L$

💡 **Tip** — Usage of Sequential Characterization of Limits

1. Find a sequence $\{x_n\}$ with $x_n \to a$
2. Find two sequences $\{x_n\}, \{y_n\}$ with $x_n, y_n \to a$ and $x_n, y_n \neq a, \forall n \in \mathbb{N}$ but which $\{f(x_n)\}, \{f(y_n)\}$ converge to different values

Proof

$\Longrightarrow$: $\lim_{x \to a} f(x) = L$ means $\forall \varepsilon > 0, \exists \delta > 0 : |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
Let $\{x_n\}$ be s.t. $x_n \to a$(meaning that $\forall \varepsilon > 0, \exists N \in \mathbb{R} : \forall n > \mathbb{N}, |x_n - a| < \varepsilon_2$) and $x_n \neq a$ for any n.

In particular, let ε for $x_n \to a$ be δ. Then $\forall n > N$, $|x_n - a| < \delta$, and so $|f(x)_n| < \varepsilon_1$. Then $\forall n > N, |x_n - a| < \delta$ and so $|f(x_n) - L| < \varepsilon_1$. So by definition, $\lim_{n \to \infty} f(x_n) = L$

Side Question: We saw the limit of a sequence is unique. Is the same true for limits of functions?

ANS: NO, it is like saying $\lim_{x \to a} f(x) = L$ and $= M$ and $L \neq M$ Suppose true. By Sequential Characterization of Limits, $\forall \{x_n\} \to a$ but $x_n \neq a \forall n, f(x_n) \to L$ and $f(x_n) \to M$ but $L \neq M$ Since the limits of sequences are unique, thus there is a contradiction.

Examples:

Prove that $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist
We take sequences of peak points of $\cos\left(\frac{1}{x}\right)$, that is $-1, 1$. Then will converge to $-1, 1$ repeatedly, so by Sequential Characterization, $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ will not exist.

$\cos\left(\frac{1}{x}\right) = 1$ if $x = \frac{1}{2k\pi}, k \in \mathbb{Z}$, and $\cos\left(\frac{1}{x}\right) = -1$ if $x = \frac{1}{(2k+1)\pi}, k \in \mathbb{Z}$.

Let $x_n = \frac{1}{2}n\pi$ and $y_n = \frac{1}{2n+1}\pi$. Then $x_n, y_n \to 0, x_n, y_n \neq 0 \forall n$. It converges to both -1 and 1. By Sequential Characterization, the limit DNE.

Limit Laws

Info — Let f, g be functions with $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$ for some $L, M \in \mathbb{R}$ then:

1. For any $c \in \mathbb{R}$, if $f(x) = c$ for all n then $L = c$
2. For any $c \in \mathbb{R}$, if $\lim_{x \to a} cf(x) = cL$
3. $\lim_{x \to a}(f(x) + g(x)) = L + M$
4. $\lim_{x \to} f(x) \cdot g(x) = LM$
5. $\lim_{n \to \infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
6. If $\alpha > 0$ and $L > 0$, then $\lim_{x > a} f(x)^\alpha = L^\alpha$

Proof

We assume functions f, g are defined on a punctured neighborhood of a and $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$. In the quotient law we also assume $M \neq 0$.

1. Product law

Claim. $\lim_{x \to a}(f(x)g(x)) = LM$.

Proof. Let $\varepsilon > 0$. Then $|f(x)g(x) - LM| = |f(x)(g(x) - M) + M(f(x) - L)| \leq |f(x)| \cdot |g(x) - M| + |M| \cdot |f(x) - L|$.

Since $f(x) \to L$, choose $\delta_0 > 0$ with $|x - a| < \delta_0 \Rightarrow |f(x) - L| < 1$, hence $|f(x)| \leq |L| + 1$ there.

Choose $\delta_1, \delta_2 > 0$ so that $|x - a| < \delta_1 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2(|L| + 1)}$ and $|x - a| < \delta_2 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2(|M| + 1)}$.

Let $\delta = \min(\delta_0, \delta_1, \delta_2)$. For $0 < |x - a| < \delta$, $|f(x)g(x) - LM| \leq (|L| + 1) \cdot \frac{\varepsilon}{2(|L| + 1)} + |M| \cdot \frac{\varepsilon}{2(|M| + 1)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus $\lim_{x \to a}(fg) = LM$.

2. Quotient law (with $M \neq 0$)

Claim. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Proof. Let $\varepsilon > 0$. Because $g(x) \to M \neq 0$, there exists $\delta_0 > 0$ such that $|x - a| < \delta_0 \Rightarrow |g(x) - M| < |M \frac{|}{2}$, hence $|g(x)| \geq |M \frac{|}{2}$.

Now $| \frac{f(x)}{g(x)} - \frac{L}{M} | = | M f(x) - L g(x) \frac{|}{|M| \cdot |g(x)|} \leq \frac{|M| \cdot |f(x) - L| + |L| \cdot |g(x) - M|}{|M| \cdot |g(x)|} \leq \left(\frac{2}{|} M|\right) \cdot |f(x) - L| + \left(2\ |L \frac{|}{|} M|^2\right) \cdot |g(x) - M|$.

Choose $\delta_1, \delta_2 > 0$ with $|x - a| < \delta_1 \Rightarrow |f(x) - L| < \left(|M \frac{|}{4}\right) \cdot \varepsilon$ and $|x - a| < \delta_2 \Rightarrow |g(x) - M| < \left(|M \frac{|^2}{4(|L| + 1)}\right) \cdot \varepsilon$.

Let $\delta = \min(\delta_0, \delta_1, \delta_2)$. Then for $0 < |x - a| < \delta$, $| \frac{f(x)}{g(x)} - \frac{L}{M} | \leq \left(\frac{2}{|} M|\right) \cdot \left(|M \frac{|}{4}\right) \cdot \varepsilon + \left(2\ |L \frac{|}{|} M|^2\right) \cdot \left(|M \frac{|^2}{4(|L| + 1)}\right) \cdot \varepsilon \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Therefore $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Info – Limit of Polynomial Functions Let $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polyomial.

Then $\lim_{x \to a} p(x) = p(a)$

Proof

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} x^i = a^i$$

$$\lim_{x \to a} a_i x^i = a_i a^i$$

$$\lim_{x \to a} \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} a_i a^i$$

Info — Limit of Rational Functions

Let $f(x) = \frac{p(x)}{q(x)}$ when p, q be polynomial functions and $a \in \mathbb{R}$

1. If $q(a) \neq 0$ then $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$
2. If $\lim_{x \to a} q(a) = 0$ but then $\lim_{x \to a} p(x) \neq 0$ then $\lim_{x \to a} \frac{p(x)}{q(x)}$ is DNE.
   If $x \to a, x < 0$, then the limit diverges to $-\infty$.
   If $x \to a, x > 0$, then the limit diverges to ∞.
3. Otherwise, $p(a) = 0 = q(a)$, so both $p(x)$ and $q(x)$ have $(x - a)$ as a factor. Divide it out and then repeat the process.

Examples:

1. $\lim_{x \to -3} \frac{x^3 + 10x^2 + 13x - 24}{x^2 - 4x - 21}$

$\Rightarrow \overset{[\frac{0}{0}]}{=} \lim_{x \to -3} \frac{(x+3)(x-1)(x+8)}{(x+3)(x-7)} = \lim_{x \to -3} \frac{(x-1)(x+8)}{(x-7)} = \frac{(-3-1)(-3+8)}{(-3-7)} = \frac{-20}{-10} = 2$

Info — Squeeze Theorem(Functions):

If $g(x) \leq f(x) \leq h(x)$ be functions defined in an open interval I around a except possibly at a.

If $\forall a \in I \setminus \{a\}$ we have $g(x) < f(x) \leq h(x)$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$

💡 **Tip** — When to apply Squeeze Theorem
1. Trigonometeric functions with clear bounds and polynomial terms before
2. Exponential Functions with constants terms or by defining a certain inverval

2. Evaluate $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$$

Notice that x^2 are polynomial function that is defined in $x \in \mathbb{R}$.

$$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$$

By Squeeze Theorem, $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

One Sided Limits and the Fundamental Trig Limit

1. We say that L is the **right side limit** of f at a, and write $\lim_{x \to a^+} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x - a| < \delta$ and $x > a$ then $|f(x) - L| < \varepsilon$
2. We say that L is the **left side limit** of f at a, and write $\lim_{x \to a^-} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x - a| < \delta$ and $x < a$ then $|f(x) - L| < \varepsilon$

Info – Theorem
$\lim_{x \to a} f(x) = L \Longleftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Example:
Show that $\lim_{x \to 0} \sin(x) = 0, \lim_{x \to 0} \cos(x) = 1$, and $\lim_{x \to 0} \tan(x) = 0$

1. $\lim_{x \to 0} \sin(x)$:

Construct a unit circle with a triangle that lies in the first quadrant and an arbitrary point, say $P(x, y)$. Then $P(x, y) = P(\cos(x), \sin(y))$. The area of the triangle can be represented as $\frac{1}{2} \sin(x)$.

Contruct another unit circle and draw $P(x, y)$ at the same location as the previous triangle, however, contruct an sector. The area of this new sector is $\frac{1}{2} x$.

Notice that the area bounded by the sector is bigger than the triangle.

We then have $0 \leq \frac{1}{2} \sin(x) \leq \frac{1}{2} x \Longrightarrow 0 \leq \sin(x) \leq x$. Since $\lim_{x \to 0^+} 0 = \lim_{x \to 0^+} x = 0$, by Squeeze Theorem, $\lim_{x \to 0^+} \sin(x) = 0$

$\lim_{x \to 0^-} \sin(x) = 0$ can be achieved similarly to the prove of right side limit and will be omitted.

Thus $\lim_{x \to 0} \sin(x) = 0$

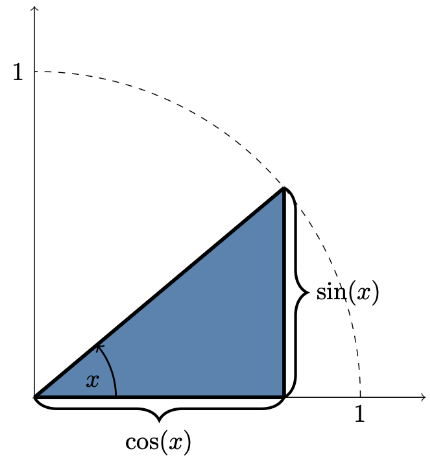
2. $\lim_{x \to 0} \cos(x) = 1$:

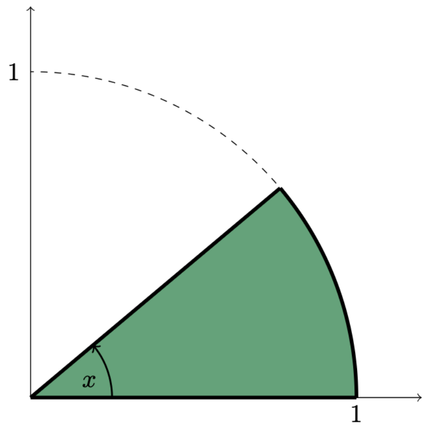
$\lim_{x \to 0} \cos(x) = \lim_{x \to 0} \sqrt{1 - \sin^2(x)} = 1$

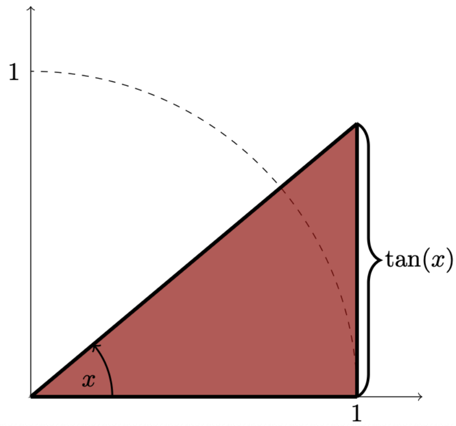
3. $\lim_{x \to 0} \tan(x) = \lim_{x \to 0} \frac{\sin(x)}{\cos(x)} = 1$

⚠️ **Warning – The Fundamental Trig Limit**:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$







We have that $\frac{1}{2}\cos(x)\sin(x) \leq \frac{1}{2}x \leq \frac{1}{2}\tan(x) \Longrightarrow \cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}$.

By Squeeze Theorem, $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$.

Since $\sin(x)$ is a even function, then $\frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x}$ so $\lim_{x\to 0^-} \frac{\sin(x)}{x} = 1$

$$\therefore \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Examples:

1. $\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(s)} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{\cos(x)} = 1 \cdot 1 = 1$
2. $\lim_{x\to 0} \frac{\sin(72x)}{\tan(9x)} = \lim_{x\to 0} \frac{\sin(72x)}{72x} \cdot \frac{9x}{\tan(9x)} \cdot 8 = \lim_{x\to 0} \frac{\sin(72x)}{72x} \cdot \lim_{x\to 0} \frac{9x}{\tan(9x)} \cdot 8 = 1 \cdot 1 \cdot 8$
3. $\lim_{x\to 0} \frac{\sin(x^2-1)}{\sin(x-1)} = \lim_{x\to 0} \frac{\sin(x^2-1)}{\sin(x-1)} \cdot \frac{x^2-1}{x^2-1} \cdot \frac{x-1}{x-1} = \lim_{x\to 0} \frac{\sin(x^2-1)}{x^2-1} \cdot \lim_{x\to 0} \frac{x-1}{\sin(x-1)} \cdot$
   $\lim_{x\to 0}(x+1) = 1 \cdot 1 \cdot 1 = 1$

Horizontal Asymptotes and the Fundamental Log Limit

Info – Limit at $\pm\infty$
Let $L \in \mathbb{R}$. We say that $\lim_{x\to\infty}$ if $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ s.t. if $x > N$, then $|f(x) - L| < \varepsilon$.

Similarly, $\lim_{x\to -\infty}$ if $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ s.t. if $x < N$, then $|f(x) - L| < \varepsilon$.

Info — Horizontal Asymptotes
If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ for some $L \in \mathbb{R}$ then we way $y = L$ is a **Horizontal Asymptote** of f

Note: you can cross horizontal asymptotes multiple times

Info — Divergence of Limits

1. We say that $\lim_{x\to\infty} f(x) = \infty$ if, $\forall M > 0, \exists N \in \mathbb{R}$ s.t. if $x > N$ we have $f(x) > M$.
2. We say that $\lim_{x\to-\infty} f(x) = \infty$ if, $\forall M > 0, \exists N \in \mathbb{R}$ s.t. if $x < N$ we have $f(x) > M$.
3. We say that $\lim_{x\to\infty} f(x) = -\infty$ if, $\forall M < 0, \exists N \in \mathbb{R}$ s.t. if $x > N$ we have $f(x) < M$.
4. We say that $\lim_{x\to-\infty} f(x) = -\infty$ if, $\forall M < 0, \exists N \in \mathbb{R}$ s.t. if $x < N$ we have $f(x) < M$.

Info — Squeeze Theorem at $\pm\infty$
If $g(x) \leq f(x) \leq h(x) \forall x \geq N$ for some $N \in \mathbb{R}$, and if $\lim_{x\to\infty} g(x) = L = \lim_{x\to\infty} h(x)$, then $\lim_{x\to\infty} f(x) = L$

⚠️ **Warning — The Fundamental Log Limit**

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$$

Proof

$0 \leq \frac{\ln(x)}{x}$ true whenever $x \geq 1$. Since $x \to \infty$, assume $x \geq 1$.

$\frac{\ln(x)}{x} = \frac{\ln(\sqrt{x}\cdot\sqrt{x})}{\sqrt{x}\cdot\sqrt{x}} = \frac{2\ln(\sqrt{x})}{\sqrt{x}\cdot\sqrt{x}} = \frac{2}{\sqrt{x}} \cdot \frac{\ln(\sqrt{x})}{\sqrt{x}} \leq 1 \leq \frac{2}{\sqrt{x}}$ (since $\ln(z) \leq z, \forall z$ arbitrarily large)

$0 \leq \frac{\ln(x)}{x} \leq \frac{2}{\sqrt{x}}$. By Squeeze Theorem $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$

Examples:

1. Show that $\lim_{x\to\infty} \frac{\ln(x)}{x^p} = 0, \forall p > 0$

$\lim_{x\to\infty} \frac{\ln(x)}{x^p} = \lim_{x\to\infty} \frac{\ln(x^p)}{x^p} \cdot \frac{1}{p}$ Let $u = x^p, x \to \infty, u \to \infty$

$\lim_{u\to\infty} \frac{\ln(u)}{u} \cdot \frac{1}{p} = 0 \cdot \frac{1}{p} = 0$

2. Show that $\lim_{x\to\infty} \frac{\ln(x^p)}{x} = 0, \forall p > 0$

$\lim_{x\to\infty} \frac{\ln(x^p)}{x} = \lim_{x\to\infty} p \cdot \lim_{x\to\infty} \frac{\ln(x)}{x} = p \cdot 0 = 0$

3. Show that $\lim_{x\to\infty} \frac{x^p}{e^x} = 0, \forall p > 0$

$\lim_{x\to\infty} \frac{x^p}{e^x}$ Let $x = \ln u \Leftrightarrow u = e^x, x \to \infty, u \to \infty$
$\lim_{u\to\infty} \frac{\ln(u)^p}{u} = \lim_{u\to\infty} \left(\frac{\ln(u)}{u^{\frac{1}{p}}}\right)^p = 0^p = 0$

4. Show that $\lim_{x\to 0^+} \frac{x^p}{\ln(x)}, \forall p > 0$

$\lim_{x\to 0^+} \frac{x^p}{\ln(x)}$ (Let $u = \frac{1}{x}$)
$\lim_{u\to\infty} \frac{1}{u^p} \cdot \ln\left(\frac{1}{u}\right)$