CH 2 — Differential Equations

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Introduction to Differential Equations

🐳 **Info — Differential Equations**

A differential equation (DE), is an equation involving an unknown function and its derivatives. The term ordinary differential equation (ODE) refers to a differential equation involving single-variable functions, whereas the term partial differential equation (PDE) refers to a differential equation involving multivariable functions (i.e., functions with multiple inputs).

An ODE is expressed

$$F(x, y, y', y'', ..., y^n) = 0$$

for some $n \in \mathbb{N}$

The order of a DE is the order of the **highest derivative** that appears in the equation.

A function $y = \varphi(x)$ is a solution to the differential equation $F(x, y, y', y'', ..., y^n) = 0$ if

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), ..., \varphi^n(x)) = 0$$

The graph of a solution to a DE is called a **solution curve**

The complete collection of solutions to a DE, including any arbitrary constants, is called its general solution. A particular solution to a DE is one in which all arbitrary constants have been specified.

A differential equation together with one or more initial conditions is known as an initial value problem (IVP)

Examples:

1. First order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$
2. Second order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \ln x + y$
3. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \Longrightarrow y = -\cos x + C, c \in \mathbb{R}$
4. $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x, y(0) = 2 \Longrightarrow -\cos x + C \Longrightarrow -\cos x + 3$
5. $y' = x + y$

   $y = -1 - x + 2e^x \Longrightarrow y' = -1 + 2e^x \Longrightarrow (x + y) = x + (-1 - x + 2e^x) = -1 + 2e^x = y'$
6. $y = -1 - x - 5e^x$

   $y' = -1 - 5e^x \Longrightarrow x + (-1 - x - 5e^x) = -1 - 5e^x = y'$
7. $y = -5 - x + 2e^x$

   $y' = -1 + 2e^x \Longrightarrow x + (-5 - x + 2e^x) = -5 + 2e^x \neq y'$

Thus $y = -1 - x + Ce^x$, for $C \in \mathbb{R}$ is always a solution

8. Determine all real numbers k s.t. $x(t) = \sin(kt)$ is a solution to the second-order differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2x$

   $\begin{cases} x'(t) = k\cos(kt) \\ x''(t) = -k^2\sin(kt) \end{cases} \implies -k^2 \sin kt = 02 \sin kt$

   $$(k^2 - 2)\sin(kt) = 0$$

   $$k = \pm\sqrt{2}, k = 0$$

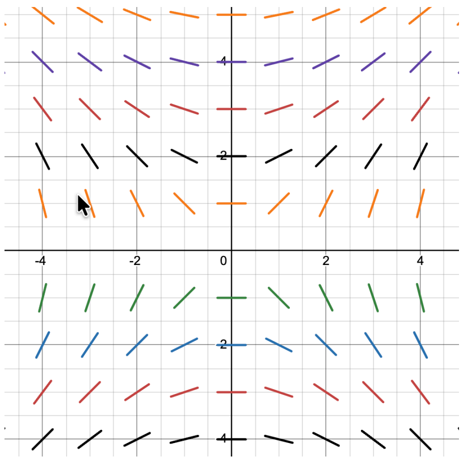
Direction Fields

> 🐳 **Info — Direction Field**
>
> A direction field for the different equation $y' = F(x)$ displays short line segments of slope $F(x, y)$ at various points in the Cartesian plane

> 💡 **Tip** — Direction field plotter for DE: https://www.desmos.com/calculator/p7vd3cdmei

Examples:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$

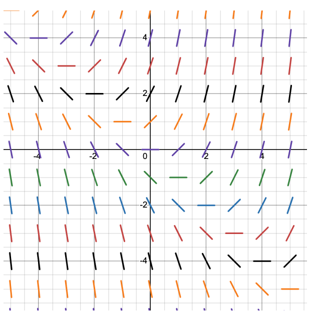


   We can obtain this when plugging-in numbers.

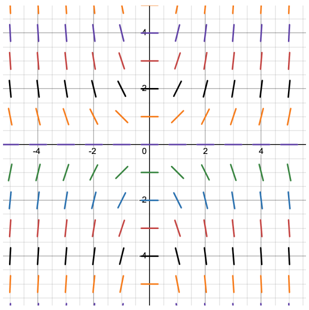
   At $(-2, 1)$, $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(-2,1)} = \left.\frac{x}{y}\right|_{(x,y)=(-2,1)} \implies -\frac{2}{1} = -2$

   Hence a slope of -2 at $(-2, 1)$ and other points have the same method

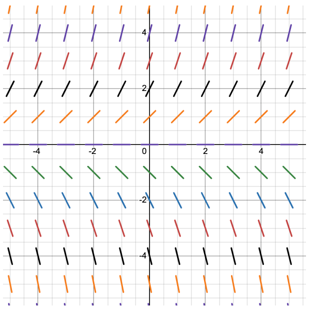
2. $y'x = x + y$



3. $y' = xy$



4. $y' = y$



Separable Differential Equations

🐳 **Info — Separable Differential Equation**

A first order differential equation is said to be separable if it is written in format

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(x)$$

Examples:

1. $y' = \sin x$
2. $y' = \frac{x}{y}$
3. $y' = 5y$
4. $y^2 y' = 2y + xy$
5. $y' = x + y$ is not separable but linear, see in Linear First Order DEs

💡 **Tip — Solving Separable DE** We have two cases

1. Determine any solution y with $h(y) = 0$
2. Find the solutions y where $h(y) \not\equiv 0$ by evaluating

$$\int \frac{1}{h(y)} \, \mathrm{d}y = \int g(x) \, \mathrm{d}x$$

   If possible, isolate y as a function of x in the resulting equation.

The general solution is the collection of all solutions obtained in Case 1 together with all solutions obtained in Case 2.

Examples:

1. $y' = y$

   Case 1: $y = 0$

   Case 2:

$$\begin{aligned} \int \frac{1}{y} \, \mathrm{d}y &= \int 1 \, \mathrm{d}x \\ \ln|y| &= x + C_1 \\ |y| &= e^{x+C_1} y = \pm e^{C_1} e^x = \pm C e^x, \ C \in \mathbb{R} \end{aligned}$$

   The general solution is $y = Ae^x, \ A \in \mathbb{R}$

2. $y' = \frac{x}{y}$

   Case 1: $h(y) = \frac{1}{y} \neq 0$

   Case 2:

$$\begin{aligned} \int y \, \mathrm{d}y &= \int x \, \mathrm{d}x \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ y^2 &= x^2 + 2C \\ y &= \pm\sqrt{x^2 + 2C} \end{aligned}$$

   The general solution $y = \pm\sqrt{x^2 + D}, \ D \in \mathbb{R}$

3. $y' = xy^2, y(0) = 3$

   Case 1: $h(y) = 0 \Longrightarrow y \equiv 0$

   Case 2:

$$\int \frac{1}{y^2} \, \mathrm{d}y = \int x \, \mathrm{d}x \Longrightarrow -\frac{1}{y} = \frac{x^2}{2} + C \Longrightarrow y = -2\frac{2}{x^2 + 2C}$$

   The general solution is $y \equiv 0$ or $y = \frac{-2}{x^2+D}, \ D \in \mathbb{R}$

   For $y(0) = 3 \Longrightarrow -\frac{2}{D} = 3 \Longrightarrow D = -\frac{2}{3}$ ==> the unique solution is $y = \frac{-2}{x^2 - \frac{2}{3}}$

4. $y' - x^2 - y^2 - 2xy + 1 = 0 \implies y' = (x+y)^2 - 1 \overset{v=x+y}{\implies} v' = v^2$

   Case 1: $h(v) = v^2 \equiv 0 \implies v \equiv 0$

   Case 2: $\int \frac{1}{v^2} \,\mathrm{d}v = x \implies -\frac{1}{v} = x + C \implies v = -\frac{1}{x+C}, C \in \mathbb{R}$

   The general solution $v \equiv 0$ or $v = -\frac{1}{x+C}, C \in \mathbb{R}$

   $y = -x$ or $y = -x - \frac{1}{x+C}, C \in \mathbb{R}$

Linear First-Order Differential Equations

🐳 **Info — First-Order Linear Differential Equation**

First order Linear DE has the form

$$A_0(x)y + A_1(x)y' = B(x) \quad A_1(x) \not\equiv 0$$

Such an equation can be written inthe form $y' + P(x)y = Q(x)$ called the standard form.

🐳 **Info — Integrating Factor**

Given a DE of the form $y' + P(x)y = Q(x)$,

$$\mu = e^{\int P(x) \,\mathrm{d}x}$$

is called the integrating factor for the DE.

💡 **Tip — Solving First-Order Linear DE**

Given $A_1(x)y' + A_0(x)y = B(x)$

1. Divide by $A_1(x)$ to rewrite the DE in standard form
2. Multiply the equation by the integrating factor.
3. Rewrite LHS as $(\mu(x)y)'$
4. Integrate in respext to x
5. Isolate y

Example:

1. $$y' + \frac{3}{x}y = 1$$

$$\overset{\mu(x)=e^{3\ln(x)}}{\implies} x^3y' + 3x^2y = x^3$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^3y = x^3$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}x^3y \,\mathrm{d}x = \int x^3 \,\mathrm{d}x$$

$$y = \frac{1}{4}x + \frac{C}{x^3}$$

2. $y' = x + y \implies y' - y = x$

$$\overset{\mu(x)=e^{-x}}{\implies} e^{-x}y' - e^{-x}y = e^{-x}x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{-x}y = xe^{-x}$$

$$y = -x - 1 + Ce^x, C \in \mathbb{R}$$

3. $$y' + 2xy = 1$$

$$\overset{\mu(x)=e^{x^2}}{\Longrightarrow} \quad e^{x^2} y' + 2xye^{x^2} = e^{x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{x^2} y = e^{x^2}$$

$$y = e^{-x^2} \int_0^1 e^{t^2} \,\mathrm{d}t + Ce^{-x^2}, C \in \mathbb{R}$$

Q5 CUTOFF

Applications

Mixing Problem

> 💡 **Tip — Formula - Mixing Problem**
>
> $$\frac{\mathrm{d}A}{\mathrm{d}t} = (\text{rate of substance in}) - (\text{rate of substance out})$$

Q1:

Suppose that a tank has 1000L of salt water at an initial concentration of 0.1kg/L. Salt water at a concentration of 0.3kg/L flows into tht tank at a rate of 10 L/min. The solution is kept mixed and drain out at a rate of 10 L/min.

Determine the amount of salt, in kg, in thte tank at time t

Let $A(t)$ represent the amount of salt(kg) in the tank at time t(min)

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}s_i}{\mathrm{d}t} - \frac{\mathrm{d}s_o}{\mathrm{d}t}$ in kg/min

$\frac{\mathrm{d}s_i}{\mathrm{d}t} = 0.3 \cdot 10 = 3\text{kg/min}$

$\frac{\mathrm{d}s_0}{\mathrm{d}t} = \frac{A(t)}{1000} \cdot 10 = \frac{A}{100}\text{kg/min}$

$A(0) = 100\text{kg}$

$\frac{\mathrm{d}A}{\mathrm{d}t} = 3 - \frac{A}{100} = -\frac{A-300}{100} \Longrightarrow \int \frac{1}{A-300} \,\mathrm{d}A = \int -\frac{1}{100} \,\mathrm{d}t \Longrightarrow \ln|A - 300| = -\frac{t}{100} + C$

$A = 300 \pm e^{-\frac{t}{100}+C}$

General solution $A = 300 + Be^{-\frac{t}{100}}, B \in \mathbb{R}$

$B = -200$ since $A(0) = 100$

Particular solution $A = 300 - 200e^{-\frac{t}{100}}$

Newton's Law of Heating/Cooling

💡 **Tip – Formula Newton's Law**

$T(t)$ represents the temperature of an object at time t, and T_s is the constant temperature of tis surroundings, then theres exists a constant $k > 0$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_s)$$

The general solution to this DE is given

$$T(t) = T_s + Ae^{-kt}, \ A \in \mathbb{R}$$

If $T(0) > T_s$, the object will cool.

An equality suggests no change.

$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_s)$

1. Constant solution: $T \equiv T_s$
2. $\int \frac{1}{T-T_s} \, \mathrm{d}T = \int -k \, \mathrm{d}t \implies \ln|T - T_s| = -kt + C$

   $= T = \pm e^{-kt+C} + T_s = \pm e^{-kt} e^C + T_s = Ae^{-kt} + T_s, \ A \in \mathbb{R}$

General Solution:

$$T_s + Ae^{-kt}, \ A \in \mathbb{R}$$

Q2:

A cup of coffee has a temperature of 98˚C in a room of 20˚C. After 1 minute, the temperature of the coffee is 96˚C. How long will it reach 80˚C?

$T_s = 20, \exists k > 0$ s.t. $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - 20)$ holds.

Initial conditions, $T(0) = 98, T(1) = 96$

The general solution will be $T = 20 + Ae^{-kt}$

With initial condition, we obtain $T = 20 + 78e^{\ln \frac{38}{39} t}$

$80 = 20 + 78e^{\ln\left(\frac{38}{39}\right)t} \implies t = \frac{\ln\left(\frac{10}{13}\right)}{\ln\left(\frac{38}{39}\right)} \approx 10.1$ min

Population Growth

💡 **Tip – Exponential Growth/Decay**

The general solution to the exponential growth DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is given

$$P(t) = Ae^{kt}, \ A = P(0)$$

- $k > 0$ for growth
- $k < 0$ for decay

💡 **Tip — Logistic Growth**

- If P is significantly smaller than M, $\frac{dP}{dt} \approx kP$
- If $P \approx M$, $\frac{dP}{dt} \approx 0$
- If $P > M$, $\frac{dP}{dt} < 0$

Formula

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

General Solution:

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