MATH 137: Calculus 1 for Honours Mathematics
Solutions Manual

Fall 2024 Edition

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Contents

0 Pre-Calculus Review **4**
- 0.1 Review 4

1 Sequence Limits **10**
- 1.1 Absolute Values 10
- 1.2 Sequences and their limits 17
- 1.3 Arithmetic Rules for Limits 21
- 1.4 Squeeze Theorem 27
- 1.5 Recursive Sequences 32

2 Function Limits and Continuity **43**
- 2.1 Introduction to Function Limits 43
- 2.2 Sequential Characterization of Limits 46
- 2.3 One-Sided Limits 50
- 2.4 Fundamental Trigonometric Limit 54
- 2.5 Limits at Infinity and Horizontal Asymptotes 58
- 2.6 Fundamental Log Limit 62
- 2.7 Infinite Limits and Vertical Asymptotes 65
- 2.8 Continuity 70
- 2.9 Types of Discontinuities 75
- 2.10 Intermediate Value Theorem 79

3 Derivatives **82**
- 3.1 Average and Instantaneous Velocity 82
- 3.2 Definition of the Derivative 86
- 3.3 Derivative of Some Common Functions 90
- 3.4 Derivative Rules 94
- 3.5 Linear Approximation 97
- 3.6 Newton's Method 100
- 3.7 Derivatives of Inverse Functions 105
- 3.8 Implicit and Logarithmic Differentiation 108

4 Applications of the Derivative **111**
- 4.1 Related Rates 111
- 4.2 Extrema 115
- 4.3 Mean Value Theorem 118
- 4.4 Antiderivatives 122
- 4.5 The First Derivative and the Shape of a Function 124
- 4.6 The Second Derivative and the Shape of a Function 127
- 4.7 Classifying Critical Points 133

4.8 L'Hôpital's Rule 140
4.9 Curve Sketching 143
4.10 Optimization 147

Chapter 0

Pre-Calculus Review

0.1 Review

Chapter 0 Problems

0.1 Determine whether $f(x) = \dfrac{x^3}{(x^2+1)^2}$ is even, odd, or neither.

We have that

$$f(-x) = \frac{8(-x)^3}{((-x)^2+1)^2} = \frac{-8x^3}{(x^2+1)^2} = -f(x).$$

Since $f(-x) = -f(x)$, this is an odd function.

0.2 (a) If $f(x)$ is even and $g(x)$ is odd, show that $h(x) = f(x)g(x)$ is odd.

Since f is even, we have $f(-x) = f(x)$; and since $g(x)$ is odd, we have $g(-x) = -g(x)$. Thus,

$$h(-x) = f(-x)g(-x) = f(x) \cdot -g(x) = -f(x)g(x) = -h(x).$$

Thus, $h(x)$ is an odd function.

(b) If $f(x)g(x)$ is odd, is it necessarily true that one of $f(x)$ or $g(x)$ is even while the other is odd? Explain.

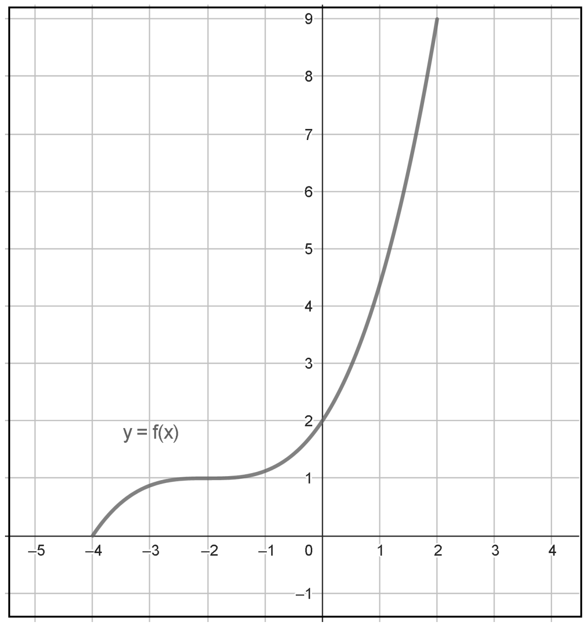
No, this is false. For instance, consider the functions

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

These functions are neither even nor odd, yet $f(x)g(x) = x$ is an odd function.

An even easier counterexample would be to let f be the constant function $f(x) = 0$ and let $g(x)$ be any function that isn't odd, say $g(x) = 1$. The product of these functions is $f(x)g(x) = 0$, which is indeed odd!

0.3 Consider the function f whose graph is shown below.



(a) Explain how you know that f is invertible.

This function is invertible by the horizontal line test: every horizontal line intersects the curve no more than once.

(b) What is $f^{-1}(1)$?

If $x = f^{-1}(1)$, then $f(x) = 1$. We see from the graph that this this occurs at $x = -2$, hence $f^{-1}(1) = -2$.

(c) What is $f^{-1}(f^{-1}(2))$?

Note that $f^{-1}(2) = 0$, since, from the graph, we have $f(0) = 2$. Furthermore, we see from the graph that $f^{-1}(0) = -4$, since $f(-4) = 0$. Thus,

$$f^{-1}\left(f^{-1}(2)\right) = f^{-1}(0) = -4.$$

0.4 Fully factor $x^3 - 6x^2 - 25x - 18$ given that -1 is a root.

Since -1 is a root, $x+1$ is a factor of the polynomial. We perform the long division as follows:

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & - & 7x & - & 18 \\
\hline
x + 1 & x^3 & - & 6x^2 & - & 25x & - & 18 \\
 & x^3 & + & x^2 & & & & \\
\cline{2-8}
 & & - & 7x^2 & - & 25x & - & 18 \\
 & & - & 7x^2 & - & 7x & & \\
\cline{3-8}
 & & & & - & 18x & - & 18 \\
 & & & & - & 18x & - & 18 \\
\cline{5-8}
 & & & & & & & 0
\end{array}
$$

As expected, the remainder is 0. Thus,

$$\frac{x^3 - 6x^2 - 25x - 18}{x+1} = x^2 - 7x - 18,$$

and therefore

$$x^3 - 6x^2 - 25x - 18 = (x+1)(x^2 - 7x - 18).$$

Since $x^2 - 7x - 18 = (x-9)(x+2)$, we have

$$x^3 - 6x^2 - 25x - 18 = (x+1)(x-9)(x+2).$$

0.5 Use polynomial long division to write

$$\frac{2x^3 - x^2 - 4}{x-3} = q(x) + \frac{r(x)}{x-3},$$

where $q(x)$ is a polynomial and $\deg(r(x)) < 1$ (i.e., $r(x)$ is constant).

The polynomial long division is as follows:

$$
\begin{array}{r|rrrrrrr}
 & & & 2x^2 & + & 5x & + & 15 \\
\hline
x - 3 & 2x^3 & - & x^2 & & & - & 4 \\
 & 2x^3 & - & 6x^2 & & & & \\
\cline{2-8}
 & & & 5x^2 & & & - & 4 \\
 & & & 5x^2 & - & 15x & & \\
\cline{3-8}
 & & & & & 15x & - & 4 \\
 & & & & & 15x & - & 45 \\
\cline{5-8}
 & & & & & & & 41
\end{array}
$$

We have a quotient of $q(x) = 2x^2 + 5x + 15$ and a remainder of $r(x) = 41$. Thus,

$$\frac{2x^3 - x^2 - 4}{x-3} = 2x^2 + 5x + 15 + \frac{41}{x-3}.$$

0.6 Throughout this question, you will make use of the sum and difference identities for sine and cosine.

(a) For all real numbers θ, prove that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Notice that

$$\begin{aligned}\sin(2\theta) &= \sin(\theta+\theta)\\ &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) && \text{(from the sum of angles identity)}\\ &= 2\sin(\theta)\cos(\theta),\end{aligned}$$

as desired.

(b) For all real numbers θ, prove that

$$\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}.$$

That is, you must prove each expression on the right-hand side is equal to $\cos(2\theta)$.

We proceed similarly to (a). We have

$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) && \text{(by the sum of angles identity)}\\ &= \cos^2(\theta) - \sin^2(\theta).\end{aligned}$$

For the remaining identities, we start with $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ from above. Recognizing that $\sin^2(\theta) = 1 - \cos^2(\theta)$, we get

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= \cos^2(\theta) - \big(1 - \cos^2(\theta)\big)\\ &= 2\cos^2(\theta) - 1.\end{aligned}$$

Similarly, recognizing that $\cos^2(\theta) = 1 - \sin^2\theta)$, we get

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= \big(1 - \sin^2(\theta)\big) - \sin^2(\theta)\\ &= 1 - 2\sin^2(\theta).\end{aligned}$$

(c) Compute the exact value of $\sin\left(\frac{\pi}{12}\right)$ and $\cos\left(\frac{\pi}{12}\right)$. It may help to recall that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, and $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

Observe that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. Thus,

$$\begin{aligned}
\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}.
\end{aligned}$$

0.7 Find all x that satisfy the given equation.

(a) $2^x + 2^{x+1} = 48$

Using exponent laws, we find that

$$\begin{aligned}
2^x + 2^{x+1} = 48 \quad &\Longrightarrow \quad 2^x + 2\cdot 2^x = 48 \\
&\Longrightarrow \quad 3\cdot 2^x = 48 \\
&\Longrightarrow \quad 2^x = 16 \\
&\Longrightarrow \quad 2^x = 2^4 \\
&\Longrightarrow \quad x = 4.
\end{aligned}$$

(b) $\log_5(x+4) + \log_5(x) = 1$

We have

$$\begin{aligned}
\log_5(x+4) + \log_5(x) = 1 \quad &\Longrightarrow \quad \log_5(x(x+4)) = 1 \\
&\Longrightarrow \quad x^2 + 4x = 5 \\
&\Longrightarrow \quad x^2 + 4x - 5 = 0 \\
&\Longrightarrow \quad (x+5)(x-1) = 0.
\end{aligned}$$

We see that $x = -5$ and $x = 1$ are possible solutions but neither $\log_5(x+4)$ nor $\log_5(x)$ are defined for $x = -5$. Thus, the only solution is $x = 1$.

(c) $5 = \ln(3 + 2\ln x)$

We have

$$\begin{aligned} 5 = \ln(3 + 2\ln x) \quad &\Longrightarrow \quad e^5 = 3 + 2\ln x \\ &\Longrightarrow \quad e^5 - 3 = 2\ln x \\ &\Longrightarrow \quad \frac{1}{2}(e^5 - 3) = \ln(x) \\ &\Longrightarrow \quad x = e^{(e^5-3)/2}. \end{aligned}$$

0.8 If $\ln(a) = 2$, $\ln(b) = 3$, and $\ln\left(\frac{ac}{b^2}\right) = -2$, what is the value of c?

Using properties of logarithms, we have

$$\begin{aligned} & \ln\left(\frac{ac}{b^2}\right) &= -2 \\ \Longrightarrow \quad & \ln(ac) - \ln\left(b^2\right) &= -2 \\ \Longrightarrow \quad & \ln(a) + \ln(c) - 2\ln(b) &= -2 \\ \Longrightarrow \quad & 2 + \ln(c) - 2 \cdot 3 &= -2 \\ \Longrightarrow \quad & \ln(c) &= 2, \end{aligned}$$

from which it follows that $c = e^2$.

Chapter 1

Sequence Limits

1.1 Absolute Values

Section 1.1 Problems

1.1.1. Solve the following equations.

(a) $|x-2|=5$

(b) $|x-4|=|3x+2|$

(a) We consider two cases.

- <u>Case 1: $x \geq 2$</u>

  In this case, $|x-2|=x-2$, and the equation becomes

  $$x-2=5.$$

  This has solution $x=7$.

- <u>Case 2: $x < 2$</u>

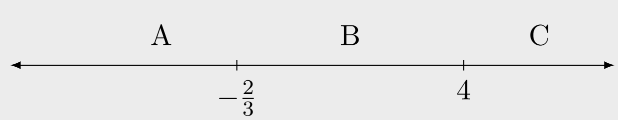
  In this case, $|x-2|=-(x-2)$, and the equation becomes

  $$-(x-2)=5.$$

  Equivalently, $-x+2=5$, so $x=-3$.

We conclude that there are two solutions: $x=-3$ or $x=7$.

(b) We consider three cases, one for each interval below:



- Case 1: $x \in A$ (i.e., $x \leq -\frac{2}{3}$)

  In this case we have that $|x-4| = -(x-4)$ and $|3x+2| = -(3x+2)$ so that our equation becomes

$$\begin{aligned} -(x-4) &= -(3x+2) \\ x-4 &= 3x+2 \\ -6 &= 2x \\ x &= -3. \end{aligned}$$

  Since $-3 \in A$ we have a valid solution: $x = -3$.

- Case 2: $x \in B$ (i.e., $-\frac{2}{3} < x \leq 4$)

  In this case we have that $|x-4| = -(x-4)$ and $|3x+2| = 3x+2$ so that our equation becomes

$$\begin{aligned} -(x-4) &= 3x+2 \\ 4-x &= 3x+2 \\ 2 &= 4x \\ x &= \frac{1}{2}. \end{aligned}$$

  Since $\frac{1}{2} \in B$ we have a valid solution: $x = \frac{1}{2}$.

- Case 3: $x \in C$ (i.e., $x > 4$)

  In this case we have that $|x-4| = x-4$ and $|3x+2| = 3x+2$ so that our equation becomes

$$\begin{aligned} x-4 &= 3x+2 \\ -6 &= 2x \\ x &= -3. \end{aligned}$$

  However, $-3 \notin C$. So, based on this case alone, we wouldn't be able to to use $x = -3$. It is, however, a solution from Case 1, so $x = -3$ *is* included in our final set of values.

Thus, there are two solutions: $x = -3, \frac{1}{2}$.

1.1.2. For real numbers a and b, write the expression

$$\frac{1}{2}(a+b+|a-b|)$$

as a piecewise function. You should consider 2 cases. What is this expression essentially calculating?

What about the expression
$$\frac{1}{2}(a+b-|a-b|)?$$

We'll first focus on the expression $\frac{1}{2}(a+b+|a-b|)$. We will consider the following two cases.

- **Case 1**: Suppose that $a \geq b$. In this case, $a-b \geq 0$. Thus, $|a-b| = a-b$ and the expression becomes
$$\frac{1}{2}(a+b+(a-b)) = a.$$
- **Case 2**: Suppose that $b > a$. In this case, $a-b < 0$. Thus, $|a-b| = -(a-b)$ and the expression becomes
$$\frac{1}{2}(a+b-(a-b)) = b.$$

In conclusion,
$$\frac{1}{2}(a+b+|a-b|) = \begin{cases} a & \text{if } a \geq b, \\ b & \text{if } b > a. \end{cases}$$
That is, this expression calculates the *maximum* of a and b.

A similar argument shows that
$$\frac{1}{2}(a+b-|a-b|)$$
calculates the *minimum* of a and b.

1.1.3. Let a and b be real numbers such that $|a^2b - ab^2| = 7$, $|a-b| = 4$, and $|a| - |b| = 3$. Determine $|a|$.

Using properties of absolute values, we have
$$7 = |a^2b - ab^2| = |ab(a-b)| = |a| \cdot |b| \cdot |a-b|.$$
Since we are given that $|a-b| = 4$ and $|a| - |b| = 3$ (hence $|b| = |a| - 3$), it follows that
$$7 = |a| \cdot (|a| - 3) \cdot 4 = 4|a|^2 - 12|a|.$$
Equivalently,
$$4|a|^2 - 12|a| - 7 = 0.$$
This is a quadratic equation involving $|a|$ – letting $x = |a|$, you can write the equation as $4x^2 - 12x - 7 = 0$ if you find this easier to think about. Solving this equation using the quadratic formula, we find that
$$|a| = \frac{12 \pm \sqrt{144 - 4(4)(-7)}}{2(4)} = \frac{12 \pm \sqrt{256}}{8} = \frac{12 \pm 16}{8} = -\frac{1}{2}, \frac{7}{2}.$$

Of course, since the absolute value of any number must be non-negative, only $|a| = \frac{7}{2}$ is possible.

1.1.4. Solve the following inequalities.

(a) $1 < |x + 1|$
(b) $|x + 3| + |1 - 2x| \leq 5$
(c) $|x - 4||x + 2| > 7$

(a) We consider two cases.

- Case 1: $x < -1$ This implies that $|x+1| = -(x+1)$ and so our inequality becomes

$$\begin{aligned} 1 &< -(x + 1) \\ 1 &< -x - 1 \\ x &< -2. \end{aligned}$$

Since we assumed $x < -1$, the solution $x < -2$ is consistent with this, so it is a valid range.

- Case 1: $x \geq -1$
In this case we get that $|x + 1| = x + 1$ and so our inequality becomes

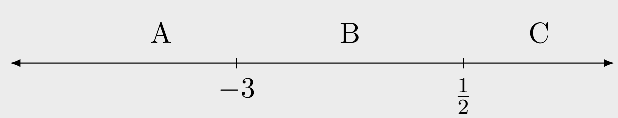
$$\begin{aligned} 1 &< x + 1 \\ 0 &< x. \end{aligned}$$

Since we assumed $x \geq -1$, the solution $x > 0$ is consistent with this, and so it is a valid range.

Putting these together we get an overall solution of

$$x > 0 \quad \text{or} \quad x < -2.$$

(b) We consider three cases, one for each interval below:



- Case 1: $x \in A$ (i.e., $x \leq -3$) In this case we have that $|x + 3| = -(x + 3)$ and $|1 - 2x| = 1 - 2x$, so our inequality becomes

$$\begin{aligned} |x + 3| + |1 - 2x| &\leq 5 \\ -(x + 3) + 1 - 2x &\leq 5 \\ -3x &\leq 7 \\ x &\geq -\frac{7}{3}. \end{aligned}$$

However, we assumed $x \in A$ which means $x \leq -3$. So, there are no solutions here.

- Case 2: $x \in B$ (i.e., $-3 < x \leq \frac{1}{2}$)

  In this case we have that $|x+3| = x+3$ and $|1-2x| = 1-2x$ and so our inequality becomes

$$\begin{aligned} |x+3| + |1-2x| &\leq 5 \\ x+3+1-2x &\leq 5 \\ -x &\leq 1 \\ x &\geq -1. \end{aligned}$$

  Since we assumed $x \in B$, this means that all x in $[-1, \frac{1}{2}]$ are solutions.

- Case 3: $x \in C$ (i.e., $x > \frac{1}{2}$)

  In this case we have that $|x+3| = x+3$ and $|1-2x| = -(1-2x)$ and so our inequality becomes

$$\begin{aligned} |x+3| + |1-2x| &\leq 5 \\ x+3-(1-2x) &\leq 5 \\ x+3-1+2x &\leq 5 \\ 3x &\leq 3 \\ x &\leq 1. \end{aligned}$$

  Since we assumed $x \in C$, this means that all $x \in (\frac{1}{2}, 1]$ are solutions.

Combining the solutions from all three cases, we get an overall solution of

$$x \in [-1, 1].$$

(c) We consider three cases, one for each interval below:

- Case 1: $x \in A$ (i.e., $x \leq -2$)

  Here we get that $|x-4| = -(x-4)$ and $|x+2| = -(x+2)$ and so the inequality becomes

$$\begin{aligned} |x-4||x+2| &> 7 \\ -(x-4) \cdot -(x+2) &> 7 \\ (x-4)(x+2) &> 7 \\ x^2 - 2x - 15 &> 0. \end{aligned}$$

We are now investigating when the parabola $x^2-2x-15$ is positive. Note that it is a parabola opening upward so it will be positive on either side of its roots. The roots of the parabola are

$$x = \frac{2 \pm \sqrt{4+60}}{2} = 1 \pm 4 = -3, 5$$

meaning that the inequality $x^2 - 2x - 15 > 0$ is true when $x < -3$ or $x > 5$. However, in this case we are assuming $x \in A$ meaning $x \leq -2$ so this particular case only gives us the solutions $x < -3$.

- Case 2: $x \in B$ (i.e., $-2 < x \leq 4$)

  Here we get that $|x-4| = -(x-4)$ and $|x+2| = x+2$ and so the inequality becomes

$$\begin{aligned} |x-4||x+2| &> 7 \\ [-(x-4)][(x+2)] &> 7 \\ -(x-4)(x+2) &> 7 \\ x^2-2x-8 &< -7 \\ x^2-2x-1 &< 0. \end{aligned}$$

  We are now investigating when an upward opening parabola is negative. This occurs between the two roots which are

$$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2},$$

  and so $x^2 - 2x - 1 < 0$ when $1 - \sqrt{2} < x < 1 + \sqrt{2}$.

  Note that, the values $1-\sqrt{2}$ and $1+\sqrt{2}$ (approximately -0.4 and 2.4) both fall within the interval B and so there is no "cutting-off" that needs to take place. Thus, from this case we get that $1-\sqrt{2} < x < 1+\sqrt{2}$ is a valid solution interval.

- Case 3: $x \in C$ (i.e., $x > 4$)

  Here we get that $|x-4| = x-4$ and $|x+2| = x+2$ and so the inequality becomes

$$\begin{aligned} |x-4||x+2| &> 7 \\ (x-4)(x+2) &> 7 \\ x^2-2x-15 &> 0. \end{aligned}$$

  As we already established in the first case, is true when $x < -3$ or $x > 5$. Since we are assuming $x \in C$, then only $x > 5$ can be used here.

Putting all of this together gives an overall solution set of

$$x < -3 \quad \text{or} \quad 1-\sqrt{2} < x < 1+\sqrt{2} \quad \text{or} \quad x > 5.$$

1.1.5. Let x and y be real numbers.

(a) Prove that $|x| - |y| \leq |x - y|$.

(b) Prove that $||x| - |y|| \leq |x - y|$.

Hint: Consider using the triangle inequality.

(a) **Proof:** Using the "trick" that $x = x - y + y$, we can say

$$\begin{aligned} |x| &= |x - y + y| \\ &\leq |x - y| + |y| \quad \text{by the triangle inequality} \end{aligned}$$

Subtracting $|y|$ from both sides, we deduce that $|x| - |y| \leq |x - y|$, as desired. □

(b) **Proof:** From (a), we know

$$|x| - |y| \leq |x - y|.$$

But since x and y are arbitrary, applying (a) with x and y swapped, it follows that

$$|y| - |x| \leq |y - x| = |-(x - y)| = |-1||x - y| = |x - y|.$$

That is, $-(|x| - |y|) \leq |x - y|$, or equivalently, $-|x - y| \leq |x| - |y|$. Therefore, when combined with (a), we have

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

which is equivalent to the inequality

$$||x| - |y|| \leq |x - y|.$$

□

1.2 Sequences and their limits

Section 1.2 Problems

1.2.1. Use the formal definition of the limit to show that

$$\lim_{n\to\infty} \frac{n}{n+9} = 1.$$

Proof: Let $\varepsilon > 0$ be given. Let $N = \frac{9}{\varepsilon} - 9$. Then for $n > N$, we have

$$\left|\frac{n}{n+9} - 1\right| = \left|\frac{n-(n+9)}{n+9}\right| = \frac{9}{n+9} < \frac{9}{N+9} = \frac{9}{\left(\frac{9}{\varepsilon} - 9\right) + 9} = \varepsilon.$$

Consequently, $\lim_{n\to\infty} \frac{n}{n+9} = 1$, as required. $\square$

1.2.2. Use the formal definition of the limit to show that

$$\lim_{n\to\infty} \frac{n+(-1)^n}{n} = 1.$$

Proof: Let $\varepsilon > 0$ be given. Let $N = \frac{1}{\varepsilon}$. For all $n > N$, we have

$$\begin{aligned}\left|\frac{n+(-1)^n}{n} - 1\right| &= \left|\frac{(n+(-1)^n)-n}{n}\right| \\ &= \frac{|(-1)^n|}{n} \\ &= \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = 1$, as required. $\square$

1.2.3. Use the formal definition of the limit to show that

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}.$$

Proof: Let $\varepsilon > 0$ be given. Let $N = \sqrt{\frac{2}{\varepsilon}}$. For $n > N$, we have

$$\begin{aligned}\left|\frac{n^2+1}{2n^2+3} - \frac{1}{2}\right| &= \left|\frac{2\left(n^2+1\right) - \left(2n^2+3\right)}{2n^2+3}\right| \\ &= \frac{|-1|}{2n^2+3}\end{aligned}$$

$$= \frac{1}{2n^2+3} < \frac{1}{2n^2} < \frac{1}{2N^2} = \frac{1}{2 \cdot \frac{2}{\varepsilon}} = \varepsilon.$$

Thus, $\lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}$. □

1.2.4. Use the formal definition of the limit to show that

$$\lim_{n\to\infty} \frac{\sin(n)}{n^2+6} = 0.$$

Proof: Let $\varepsilon > 0$ be given. We will assume without loss of generality that $\varepsilon < \frac{1}{6}$, so $\frac{1}{\varepsilon} - 6 > 0$. Let $N = \sqrt{\frac{1}{\varepsilon} - 6}$. Then for $n > N$, we have

$$\left|\frac{\sin(n)}{n^2+6} - 0\right| = \frac{|\sin(n)|}{n^2+6} \leq \frac{1}{n^2+6} < \frac{1}{N^2+6} = \frac{1}{\left(\frac{1}{\varepsilon} - 6\right) + 6} = \varepsilon.$$

Thus, $\lim_{n\to\infty} \frac{\sin(n)}{n^2+6} = 0$. □

1.2.5. Let $a, b, c, d \in \mathbb{R}$ with $c > 0$. Use the formal definition of the limit to prove that

$$\lim_{n\to\infty} \frac{an+b}{cn+d} = \frac{a}{c}.$$

Proof: Let $\varepsilon > 0$ be given. Let N be any number larger than $\frac{|bc-ad|}{c^2\varepsilon} - \frac{d}{c}$. Note that this implies that $N > \frac{-d}{c}$, which we will soon see is important.

For all $n > N$, we have

$$\begin{aligned}
\left|\frac{an+b}{cn+d} - \frac{a}{c}\right| &= \left|\frac{c(an+b) - a(cn+d)}{c(cn+d)}\right| \\
&= \left|\frac{bc-ad}{c(cn+d)}\right| \\
&= \frac{|bc-ad|}{c(cn+d)} && \left(cn+d > 0 \text{ since } n > N > \tfrac{-d}{c}\right) \\
&< \frac{|bc-ad|}{c(cN+d)} \\
&< \frac{|bc-ad|}{c\left(c\left(\frac{|bc-ad|}{c^2\varepsilon} - \frac{d}{c}\right) + d\right)} \\
&= \frac{|bc-ad|}{c\left(\left(\frac{|bc-ad|}{c\varepsilon} - d\right) + d\right)}
\end{aligned}$$

$$= \frac{|bc - ad|}{c\left(\frac{|bc-ad|}{c\varepsilon}\right)}$$
$$= \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{an+b}{cn+d} = \frac{a}{c}$, as desired. □

1.2.6. Consider the function

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

Prove that $\lim_{n\to\infty} f(n)$ does not exist.

Hint: Assume that $\lim_{n\to\infty} f(n)$ exists and derive a contradiction.

Proof: Assume for contradiction that

$$L = \lim_{n\to\infty} f(n)$$

exists. Then, according to the formal definition of the limit with $\varepsilon = \frac{1}{4}$, there exists a cutoff N such that for all $n > N$, $|f(n) - L| < \frac{1}{4}$.

Since there are an infinite number of perfect squares, let n_0 be a perfect square with $n_0 > N$. Likewise, since there are an infinite number of natural numbers that are not perfect squares, let n_1 be any such number with $n_1 > N$ and $n_1 \geq 2$ (for reasons that will soon become apparent).

Since $n_0 > N$, we have $|f(n_0) - L| < \frac{1}{4}$; and since $n_1 > N$, we have $|f(n_1) - L| < \frac{1}{4}$. Thus,

$$\begin{aligned} |f(n_0) - f(n_1)| &= |\,(f(n_0) - L) + (f(n_1) - L)\,| \\ &\leq |f(n_0) - L| + |f(n_1) - L| \\ &< \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

But this is a contradiction, since

$$\begin{aligned} |f(n_0) - f(n_1)| = \left|1 - \frac{1}{n}\right| &= 1 - \frac{1}{n} \\ &\geq 1 - \frac{1}{2} && \text{(since } n_1 \geq 2\text{)} \\ &= \frac{1}{2}. \end{aligned}$$

We conclude that $\lim_{n\to\infty} f(n)$ does not exist. □

1.2.7. Use the formal definition of the limit to show that

$$\lim_{n\to\infty} \frac{n^2}{3n+1} = \infty.$$

Proof: Let $m > 0$ be given, and let $N = 6m$. Then for all $n > N$, we have

$$\begin{aligned} a_n = \frac{n^2}{3n+1} &> \frac{n^2}{3n+3n} && \text{(since } 3n > 1\text{)} \\ &= \frac{n^2}{6n} = \frac{n}{6} > \frac{N}{6} = \frac{6m}{6} = m. \end{aligned}$$

This proves that $\lim_{n\to\infty} \frac{n^2}{3n+1} = \infty$, as required. □

1.3 Arithmetic Rules for Limits

Section 1.3 Problems

1.3.1. Use the formal definition of the limit to prove the following.

$$\text{If } c \in \mathbb{R} \text{ and } \lim_{n\to\infty} a_n = a \text{ where } a \in \mathbb{R}, \text{ then } \lim_{n\to\infty} ca_n = ca.$$

Hint: Consider the $c = 0$ and $c \neq 0$ cases separately.

Proof: We will consider two possibilities.

Case 1: $c = 0$. Let $\varepsilon > 0$ be given, and let $N = 1$. Then for $n > N$, we have

$$|ca_n - ca| = |0 - 0| = 0 < \varepsilon.$$

This proves that $\lim_{n\to\infty} ca_n = ca$.

Case 2: $c \neq 0$. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = a$, let N be a cutoff such that $|a_n - a| < \frac{\varepsilon}{|c|}$ for all $n > N$. For $n > N$, we have

$$|ca_n - ca| = |c||a_n - a| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

This proves that $\lim_{n\to\infty} ca_n = ca$, as required. □

1.3.2. Use the formal definition of the limit to prove the following.

$$\text{If } \lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b \text{ where } a, b \in \mathbb{R}, \text{ then } \lim_{n\to\infty} (a_n - b_n) = a - b.$$

Proof: Let $\varepsilon > 0$ be given. Let N_1 be a cutoff such that for all $n > N_1$, $|a_n - a| < \frac{\varepsilon}{2}$. Let N_2 be a cutoff such that for all $n > N_2$, $|b_n - b| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then for $n > N$, we have

$$\begin{aligned} |\,(a_n - b_n) - (a - b)\,| &= |\,(a_n - a) + (b_n - b)\,| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

1.3.3. Use the formal definition of the limit to prove the following.

$$\text{If } \lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} a_n = a \text{ for some } a \in \mathbb{R}, \text{ then } \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Proof: Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = a$, let N_1 be a cutoff such that for all $n > N_1$, $|a_n - a| < 1$. Since $\lim\limits_{n\to\infty} b_n = \infty$, let N_2 be a cutoff such that for all $n > N_2$, $b_n > \frac{1+|a|}{\varepsilon}$. Let $N = \max\{N_1, N_2\}$. For $n > N$, we have

$$\begin{aligned}
\left|\frac{a_n}{b_n} - 0\right| = \left|\frac{a_n}{b_n}\right| &= \frac{|(a_n - a) + a|}{b_n} && (|b_n| = b_n \text{ since } b_n > \tfrac{1+|a|}{\varepsilon} > 0)\\
&\le \frac{|a_n - a| + |a|}{b_n} && \text{(by the triangle inequality)}\\
&< \frac{1 + |a|}{b_n} && (\text{since } n > N_1)\\
&< \frac{1+|a|}{\left(\frac{1+|a|}{\varepsilon}\right)} && (\text{since } n > N_2)\\
&= \varepsilon.
\end{aligned}$$

This proves that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, as required. □

1.3.4. Let $\{a_n\}$ be a sequence.

(a) Use the formal definition of a limit to prove the following.

$$\text{If } \lim_{n\to\infty} a_n = L, \text{ then } \lim_{n\to\infty} |a_n| = |L|.$$

Hint: It may help to recall that $||a| - |b|| \le |a - b|$ for all $a, b \in \mathbb{R}$, a fact proved in Exercise 1.5.1(b).

Proof: Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = L$, let N be a cutoff such that for all $n > N$, $|a_n - L| < \varepsilon$. Then, for all $n > N$, it follows that

$$||a_n| - |L|| \le |a_n - L| < \varepsilon.$$

This proves that $\lim\limits_{n\to\infty} |a_n| = |L|$, as required. □

(b) Consider the statement

$$\text{If } \lim_{n\to\infty} |a_n| = |L|, \text{ then } \lim_{n\to\infty} a_n = L \text{ or } \lim_{n\to\infty} a_n = -L.$$

Is this statement true or false? If it is true, prove it. If it is false, provide a counterexample.

This statement is false. For instance, consider the sequence $\{a_n\}$ given by $a_n = (-1)^n$ for $n \ge 1$. We have $|a_n| = |(-1)^n| = 1$ for all n, hence

$$\lim_{n\to\infty} |a_n| = |1|.$$

But note that $\lim\limits_{n\to\infty} a_n$ is neither 1 nor -1. In fact, $\lim\limits_{n\to\infty} a_n$ does not exist at all!

1.3.5. Let $a_n \geq 0$ for all n. Use the formal definition of the limit to prove the following:

$$\text{If } \lim_{n\to\infty} a_n = L, \text{ then } \lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}.$$

Hint: Consider the $L = 0$ and $L \neq 0$ cases separately.

Proof: We separately consider two possibilities.

Case 1: $L = 0$. Assume that $\lim_{n\to\infty} a_n = 0$ and let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = 0$, let N be a cutoff such that for all $n > N$, we have $a_n < \varepsilon^2$. Then for $n > N$, it follows that

$$|\sqrt{a_n} - \sqrt{0}| = \sqrt{a_n} < \sqrt{\varepsilon^2} = \varepsilon.$$

This proves that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{0} = 0$, as required.

Case 2: $L \neq 0$. Assume that $\lim_{n\to\infty} a_n = L$ and let $\varepsilon > 0$ be given. Since we have assumed that $\lim_{n\to\infty} a_n = L$, let N be a cutoff such that for all $n > N$, $|a_n - L| < \varepsilon\sqrt{L}$. For $n > N$, we then have

$$\begin{aligned}
|\sqrt{a_n} - \sqrt{L}| &= \left| \left(\sqrt{a_n} - \sqrt{L}\right) \cdot \frac{\sqrt{a_n} + \sqrt{L}}{\sqrt{a_n} + \sqrt{L}} \right| \\
&= \left| \frac{a_n - L}{\sqrt{a_n} + \sqrt{L}} \right| \\
&= \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \\
&\leq \frac{|a_n - L|}{\sqrt{L}} && \text{(since } \sqrt{a_n} + \sqrt{L} \geq \sqrt{L}\text{)} \\
&< \frac{\varepsilon\sqrt{L}}{\sqrt{L}} \\
&= \varepsilon.
\end{aligned}$$

This proves that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$, as required. □

1.3.6. (**Challenge Problem!**) There are many ways to find the average of two positive numbers a and b. The familiar arithmetic average is $A = (a + b)/2$. The perhaps-less-familiar geometric average is $G = \sqrt{ab}$. Both take a single step to evaluate.

We can, however, use the arithmetic and geometric averages to create an iterative combination called the *arithmetic-geometric mean*, written as $AGM(a, b)$, and defined as the limit of the coupled, recursively-defined sequences $\{a_k\}$ and $\{b_k\}$ constructed via the following rules:

Initialize:

$$a_0 = a$$

$$b_0 = b$$

Iterate: $(k = 0, 1, 2, \ldots)$

$$a_{k+1} = \frac{a_k + b_k}{2}$$
$$b_{k+1} = \sqrt{a_k b_k}$$

Then a_k and b_k converge to the same limit,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = AGM(a, b).$$

Although recursively-defined sequences are notoriously difficult to analyze (see Section 1.5), we can prove several useful properties of the arithmetic-geometric mean using the tools from this section.

(a) Use the AGM-algorithm definition and the limit rules to show that

$$AGM(a, b) = AGM\left(\frac{a+b}{2}, \sqrt{ab}\right) \tag{1.1}$$

and that for any positive real number λ,

$$AGM(\lambda a, \lambda b) = \lambda AGM(a, b) \tag{1.2}$$

(Eq. 1.1) Given the sequences generated from the AGM-algorithm, $\{a_1, a_2, a_3, \ldots\}$ and $\{b_1, b_2, b_3, \ldots\}$, if you initiate the algorithm with any pair (a_n, b_n) you simply start the iteration at step $k = n$, so you will reach the same limit $AGM(a, b)$. That is,

$$AGM(a_1, b_1) = AGM(a_2, b_2) = AGM(a_3, b_3) = \ldots = AGM(a, b)$$

Eq. 1.1 is $AGM(a_1, b_1) = AGM(\frac{a+b}{2}, \sqrt{ab}) = AGM(a, b)$.

(Eq. 1.2) Define two new variables $c = \lambda a$ and $d = \lambda b$. From the AGM-algorithm, multiplying a_{k+1} and b_{k+1} by λ, it follows that

$$c_{k+1} = \lambda a_{k+1} = \frac{\lambda a_k + \lambda b_k}{2} = \frac{c_k + d_k}{2}, \quad d_{k+1} = \lambda b_{k+1} = \sqrt{\lambda^2 a_k b_k} = \sqrt{c_k d_k}$$

Then, from the limit rules,

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} \lambda a_k = \lambda \lim_{k\to\infty} a_k = \lambda AGM(a, b).$$

(b) Use part 6a to show that

$$AGM(1, b) = \frac{1+b}{2}\, AGM\left(1, \frac{2\sqrt{b}}{1+b}\right)$$

Run the equations from part 6a backwards: Using $\lambda = \frac{1+b}{2}$, from Eq. 1.2,

$$\frac{1+b}{2} AGM\left(1, \frac{2\sqrt{b}}{1+b}\right) = AGM\left(\frac{1+b}{2}, \sqrt{b}\right)$$

From Eq. 1.1 with $a = 1$,

$$AGM\left(\frac{1+b}{2}, \sqrt{b}\right) = AGM(1, b)$$

(c) The $AGM(a, b)$ algorithm converges incredibly quickly – the number of correct decimal places approximately *doubles* with each iteration. Find the first three terms of the sequences $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ converging to $AGM(1, 1/\sqrt{2})$, and display your results to 16 decimal places.

k	a_k	b_k
0	1.00000 00000 00000 0	0.70710 67811 86547 5
1	0.**8**5355 33905 93273 7	0.8**4**089 64152 53714 5
2	0.847**2**2 49029 23494 2	0.847**2**0 12667 46891 5
3	0.84721 308**4**8 35192 9	0.84721 3084**7** 52765 4
∞	0.84721 30847 93979 1	0.84721 30847 93979 1

(d) The arithmetic-geometric mean $AGM(a, b)$ forms the basis of one of the most powerful algorithms to calculate π, and up until 2009, it was the method-of-choice (last used by Daisuke Takahashi and Yasumasa Kanada to compute 2.58 trillion digits). Denoting the half-distance between the two sequences by $c_{k+1} = \frac{1}{2}(a_k - b_k) = -(a_{k+1} - a_k)$, the iterative scheme is based on the formula by Gauss,

$$\pi = \frac{2AGM^2\left(1, \frac{1}{\sqrt{2}}\right)}{\frac{1}{2} - \sum_{n=1}^{\infty} 2^n c_n^2} \tag{1.3}$$

If we write the partial sum of the series in the denominator of Eq. 1.3 as $s_k = \frac{1}{2} - \sum_{n=1}^{k} 2^n c_n^2$, then the algorithm runs as follows. Initialize:

$$a_0 = 1$$
$$b_0 = \frac{1}{\sqrt{2}}$$
$$s_0 = \frac{1}{2}$$

Iterate: $(k = 0, 1, 2, \ldots, K-1)$

$$a_{k+1} = (a_k + b_k)/2$$
$$b_{k+1} = \sqrt{a_k b_k}$$
$$c_{k+1}^2 = (a_{k+1} - a_k)^2$$
$$s_{k+1} = s_k - 2^{k+1} c_{k+1}^2$$

Finally, using $AGM(1, 1/\sqrt{2}) \approx a_{K+1}$,

$$\pi \approx \pi_K = \frac{(a_K + b_K)^2}{2s_K}$$

Run the algorithm to $K = 3$, and display your result to 16 decimal places. For reference,

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ldots$$

K	π_K	Correct digits
0	2.91421 35623 73094 9	0
1	3.1**4**057 92505 22168 6	3
2	3.14159 2**6**462 13542 8	8
3	3.14159 26535 8979**3** 1	16*
∞	3.14159 26535 89793 2	

$\star$ This is actually due to the limitation of double precision in MATLAB – with higher numerical precision, one would find 19 correct digits after three iterations. It takes only 36 iterations of this scheme to calculate π to 206.1 billion decimal places. For more information on high-performance algorithms for calculating π, see 'π Unleashed' by Jörg Arndt and Christoph Haenel (Springer, 2001).

1.4 Squeeze Theorem

Section 1.4 Problems

1.4.1. Evaluate the following limits.

(a) $\lim_{n\to\infty} \frac{(-1)^n \cos(n)}{n}$

Note that $-1 \le \cos(n) \le 1$ for all n. Thus, since $(-1)^n$ is either 1 or -1 depending on whether n is even or odd, it follows that

$$-1 \le (-1)^n \cos(n) \le 1$$

for all n. Hence, dividing every term in the inequality by n,

$$\frac{-1}{n} \le \frac{(-1)^n \cos(n)}{n} \le \frac{1}{n}$$

for all n. Since $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} \frac{(-1)^n \cos(n)}{n} = 0.$$

(b) $\lim_{n\to\infty} \frac{n + \sin(n)\cos(n)}{n}$

Since $-1 \le \sin(n) \le 1$ and $-1 \le \cos(n) \le 1$ for all n, it follows that

$$-1 \le \sin(n)\cos(n) \le 1$$

for all n. Adding n to all terms in the inequality, we find that

$$n - 1 \le n + \sin(n)\cos(n) \le n + 1$$

for all n. Dividing by n, it follows that

$$\frac{n-1}{n} \le \frac{n + \sin(n)\cos(n)}{n} \le \frac{n+1}{n}.$$

Since $\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1$ and $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$, we conclude by the Squeeze Theorem that

$$\lim_{n\to\infty} \frac{n + \sin(n)\cos(n)}{n} = 1.$$

(c) $\lim_{n\to\infty} \frac{|n\sin(n) - 1|}{n^2 + 1}$

We will attempt to compare the numerator, $|n\sin(n)+1|$ to something simpler and easier to analyze. Since we are dealing with the a sum within an absolute value, perhaps the triangle inequality could help. Indeed, we note that

$$0 \leq |n\sin(n)+1| \leq |n\sin(n)| + |1| = |n|\underbrace{|\sin(n)|}_{\leq 1}+1 \leq n+1.$$

Dividing through by n^2+1, it follows that

$$0 \leq \frac{|n\sin(n)+1|}{n^2+1} \leq \frac{n+1}{n^2+1}.$$

Finally, since

$$\lim_{n\to\infty}\frac{n+1}{n^2+1} = \frac{n\left(1+\frac{1}{n}\right)}{n\left(n+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{n+\frac{1}{n}} = 0,$$

(and of course $\lim\limits_{n\to\infty} 0 = 0$), it follows from the Squeeze Theorem that

$$\lim_{n\to\infty}\frac{|n\sin(n)+1|}{n^2+1} = 0.$$

1.4.2. If $\{a_n\}$ and $\{b_n\}$ are sequences with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$, evaluate

$$\lim_{n\to\infty}\left(a_n\sin(n) + b_n\cos(n)\right).$$

Since $-1 \leq \sin(n) \leq 1$ and $-1 \leq \cos(n) \leq 1$ for all n, we have

$$-a_n - b_n \leq a_n\sin(n) + b_n\cos(n) \leq a_n + b_n$$

for all n. Since $\lim\limits_{n\to\infty}(-a_n - b_n) = 0$ and $\lim\limits_{n\to\infty}(a_n+b_n) = 0$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty}\left(a_n\sin(n) + b_n\cos(n)\right) = 0.$$

1.4.3. Evaluate $\lim\limits_{n\to\infty}\dfrac{n!}{n^n}$.

Hint: Write $n! = 1\cdot 2\cdot 3\cdots n$ and $n^n = n\cdot n\cdot n\cdots n$ and use the fact that $0 < \frac{n!}{n^n}$ for all $n \in \mathbb{N}$.

Following the hint, we note that

$$0 < \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} = \left(\frac{1}{n}\right)\underbrace{\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)}_{\leq 1\text{, since each factor is at most }1} \leq \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$ (and of course $\lim_{n\to\infty} 0 = 0$), it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

1.4.4. Use the Squeeze Theorem to show that $\lim_{n\to\infty} (2^n + 3^n)^{1/n} = 3$.

Notice that

$$(2^n + 3^n)^{1/n} \geq (0 + 3^n)^{1/n} = 3.$$

Moreover,

$$(2^n + 3^n)^{1/n} \leq (3^n + 3^n)^{1/n} = (2 \cdot 3^n)^{1/n} = 2^{1/n} \cdot 3.$$

Having established that

$$3 \leq (2^n + 3^n)^{1/n} \leq 2^{1/n}3,$$

we note that $\lim_{n\to\infty} 2^{1/n} \cdot 3 = 2^0 \cdot 3 = 3$ (and of course $\lim_{n\to\infty} 3 = 3$). Hence, by the Squeeze Theorem,

$$\lim_{n\to\infty} (2^n + 3^n)^{1/n} = 3,$$

as required.

1.4.5. Use the Squeeze Theorem to prove that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Notice that

$$-|a_n| \leq a_n \leq |a_n$$

for all n. (Indeed, notice that if $a_n \geq 0$, then $a_n = |a_n|$; while if $a_n < 0$, then $a_n = -|a_n|$.) Since

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} -|a_n| = 0,$$

it follows from the Squeeze Theorem that $\lim_{n\to\infty} a_n = 0$, as required.

1.4.6. Let $\{a_n\}$ be a sequence and M a real number such that $|a_n - n| \leq M$ for all $n \in \mathbb{N}$. Evaluate

$$\lim_{n\to\infty} \frac{a_n}{n}.$$

The inequality $|a_n - n| \leq M$ can be written as

$$-M \leq a_n - n \leq M.$$

Hence, adding n to all parts of the inequality, we find that

$$n - M \leq a_n \leq n + M.$$

Dividing by n, it follows that

$$\frac{n-M}{n} \leq \frac{a_n}{n} \leq \frac{n+M}{n}$$

for all n. Thus, since

$$\lim_{n\to\infty} \frac{n+M}{n} = \lim_{n\to\infty} \left(1 + \frac{M}{n}\right) = 1 + 0 = 1$$

and

$$\lim_{n\to\infty} \frac{n-M}{n} = \lim_{n\to\infty} \left(1 - \frac{M}{n}\right) = 1 - 0 = 1,$$

it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} \frac{a_n}{n} = 1.$$

1.4.7. Let $A_1 \geq A_2 \geq \cdots \geq A_k \geq 0$ where $k \in \mathbb{N}$ and define $S_n = (A_1^n + A_2^n + \ldots A_k^n)^{1/n}$. In this problem, we will use the squeeze theorem to evaluate the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (A_1^n + A_2^n + \ldots + A_k^n)^{1/n}$$

(a) Consider $(A_1, A_2, A_3) = (7, 5, 2)$ and use a calculator to compute $S_n = (A_1^n + A_2^n + A_3^n)^{1/n}$ for $n = 5$ and $n = 50$ to 3 decimal places. What does this suggests for a lower bound on S_n and what does this suggest for the limit of S_n?

For $n = 5$, we have

$$S_5 = (7^5 + 5^5 + 2^5)^{1/5} \approx 7.245$$

and for $n = 50$, we have

$$S_{50} = (7^{50} + 5^{50} + 2^{50})^{1/50} \approx 7.000$$

These computations suggest that $7 \leq S_n$ and that $S_n \to 7$ as $n \to \infty$.

(b) Consider $(A_1, A_2, A_3) = (2.1, 2, 1.9)$ and use a calculator to compute $S_n = (A_1^n + A_2^n + A_3^n)^{1/50}$ for $n = 5$ and $n = 50$ to 3 decimal places. What does this suggests for a lower bound on S_n and what does this suggest for the limit of S_n?

For $n = 5$, we have

$$S_5 = (2.1^5 + 2^5 + 1.9^5)^{1/5} \approx 2.500$$

and for $n = 50$, we have

$$S_{50} = (2.1^{50} + 2^{50} + 1.9^{50})^{1/50} \approx 2.104$$

These computations again suggest that $2.1 \leq S_n$ and that $S_n \to 2.1$ as $n \to \infty$

(c) Based on the computations above, bound S_n above and below by appropriate bounds and compute the limit using the squeeze theorem.

From parts (a) and (b), we have that $A_1^n \leq A_1^n + A_2^n + \ldots A_k^n$. (Notice that we can also infer this since each $A_k \geq 0$ so adding more of these positive terms to A_1^n makes the summation larger.) Further, since $A_1 \geq A_2 \geq \cdots \geq A_k \geq 0$, then we can bound this summation above by

$$A_1^n \leq A_1^n + A_2^n + \ldots A_k^n \leq kA_1^n$$
$$\Rightarrow A_1 \leq (A_1^n + A_2^n + \ldots A_k^n)^{1/n} \leq k^{1/n} A_1$$

(The upper bound can be rationalized since A_1^n is the largest term, so replacing A_k^n with A_1^n makes the summation larger.)

Now, since we've bounded $A_1 \leq S_n \leq k^{1/n} A_1$, then we can compute the limits of the bounding sequences to find that $\lim\limits_{n\to\infty} A_1 = A_1 = \lim\limits_{n\to\infty} k^{1/n} A_1$, so by the squeeze theorem, we have that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (A_1^n + A_2^n + \ldots + A_k^n)^{1/n} = A_1.$$

1.5 Recursive Sequences

Section 1.5 Problems

1.5.1. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = \dfrac{7+a_n}{6}$ for $n \geq 1$.

(a) Prove by induction that $\{a_n\}$ is increasing.

Base case: $a_1 = 1$, $a_2 = \dfrac{8}{6} = \dfrac{4}{3} > 1$, which means that $a_1 < a_2$ as required.

Inductive Step: Suppose $a_k < a_{k+1}$ for some $k \geq 1$. Then we get

$$\begin{aligned} a_k < a_{k+1} \quad &\Longrightarrow \quad 7 + a_k < 7 + a_{k+1} \\ &\Longrightarrow \quad \frac{7+a_k}{6} < \frac{7+a_{k+1}}{6} \\ &\Longrightarrow \quad a_{k+1} < a_{k+2} \end{aligned}$$

as desired. Therefore, by mathematical induction, we have $a_n < a_{n+1}$ for all $n \geq 1$.

(b) Prove by induction that $\{a_n\}$ is bounded above by 2.

Base case: $a_1 = 1 \leq 2$.

Inductive Step: Suppose $a_k \leq 2$ for some $k \geq 1$. Then we get

$$\begin{aligned} a_k \leq 2 \quad &\Longrightarrow \quad 7 + a_k \leq 9 \\ &\Longrightarrow \quad \frac{7+a_k}{6} \leq \frac{9}{6} \leq 2 \\ &\Longrightarrow \quad a_{k+1} \leq 2. \end{aligned}$$

Therefore, by mathematical induction, we have $a_n \leq 2$ for all $n \geq 1$.

(c) Prove that $\{a_n\}$ is convergent and find $\lim\limits_{n\to\infty} a_n$.

Since the sequence is bounded above and increasing, it converges by the Monotone Convergence Theorem. Therefore, the sequence has a limit, call it L. We get

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{7+a_n}{6} = \frac{7+L}{6}.$$

Hence,

$$\begin{aligned} L = \frac{7+L}{6} \quad &\Longrightarrow \quad 6L = 7 + L \\ &\Longrightarrow \quad 5L = 7 \\ &\Longrightarrow \quad L = \frac{7}{5}. \end{aligned}$$

1.5.2. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = \dfrac{1}{3 - a_n}$ for $n \geq 1$.

(a) Prove that $\{a_n\}$ is monotonic.

Consider some terms of the sequence:

$$a_1 = 1,\ a_2 = 0.5,\ a_3 = 0.4,\ a_4 \approx 0.384,\ a_5 \approx 0.382$$

Based on the above, we suspect that $\{a_n\}$ is decreasing.

Claim: $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Base Case: For $n = 1$, we have $a_1 = 1$ and $a_2 = 0.5$. Thus, $a_1 > a_2$, so the claim holds for $n = 1$.

Inductive Step: Suppose $a_k \geq a_{k+1}$ for some $k \in \mathbb{N}$. We get

$$\begin{aligned} -a_k \leq -a_{k+1} \quad &\Longrightarrow \quad 3 - a_k \leq 3 - a_{k+1} \\ &\Longrightarrow \quad \frac{1}{3 - a_k} \geq \frac{1}{3 - a_{k+1}} \\ &\Longrightarrow \quad a_{k+1} \geq a_{k+2}, \end{aligned}$$

so the claim holds for $n = k + 1$. Therefore, by induction, the claim is true for all $n \in \mathbb{N}$, and so the sequence is decreasing.

(b) Prove that $\{a_n\}$ is bounded.

Consider some terms of the sequence:

$$a_1 = 1,\ a_2 = 0.5,\ a_3 = 0.4,\ a_4 \approx 0.384,\ a_5 \approx 0.382$$

Based on the above, we suspect that $\{a_n\}$ is bounded between 0 and 1. Since the sequence begins at $a_1 = 1$ and we previously showed that the sequence is decreasing, it follows that $a_n \leq 1$ for all n. Thus, we need only prove that $a_n \geq 0$ for all n.

Claim: $a_n \geq 0$ for all $n \in \mathbb{N}$.

Base Case: $a_1 = 1 \geq 0$, so the claim holds for $n = 1$.

Inductive Step: Suppose $a_k \geq 0$ for some $k \in \mathbb{N}$. We get

$$\begin{aligned} a_k \geq 0 \quad &\Longrightarrow \quad -a_k \leq 0 \\ &\Longrightarrow \quad 3 - a_k \leq 3 \\ &\Longrightarrow \quad \frac{1}{3 - a_k} \geq \frac{1}{3} \qquad \text{(Note: } 3 - a_k > 0\text{, since } a_k \leq 1.\text{)} \\ &\Longrightarrow \quad a_{k+1} \geq \frac{1}{3} \geq 0 \end{aligned}$$

and so the claim holds for $n = k + 1$. Therefore, by induction, the claim is true for all $n \in \mathbb{N}$, and so the sequence is bounded below by 0.

(c) Prove that $\{a_n\}$ is convergent and find $\lim\limits_{n\to\infty} a_n$.

We have proven that the sequence is bounded and decreasing, so by the Monotone Convergence Theorem, it converges to a limit L. We have

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n} = \frac{1}{3-L},$$

so $L = \frac{1}{3-L}$, and hence $L^2 - 3L + 1 = 0$. We get

$$L = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Because $a_1 = 1$ and the sequence is decreasing, the limit has to be less or equal to 1. Thus, $\lim\limits_{n\to\infty} a_n = \dfrac{3-\sqrt{5}}{2}$.

1.5.3. Consider the sequence $\{a_n\}$ defined by $a_1 = 19$ and $a_{n+1} = \dfrac{\sqrt{9a_n - 2}}{3}$ for $n \in \mathbb{N}$. Prove that $\{a_n\}$ converges and find its limit.

Hint: As usual, your main tool will be the Monotone Convergence Theorem. However, in this example, it will be difficult to identify the limit of the sequence without knowing sharp bounds for a_n. It may therefore help to figure out what the possible limits are – something you would normally do at the *end* of such a problem – and then use this information to guess (and prove!) bounds for the sequence accordingly.

As suggested by the hint, we will work backwards in order to identify sharp bounds for this sequence. Assuming for a moment that the limit, L, does exist, we have

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{\sqrt{9a_n - 2}}{3} = \frac{\sqrt{9L-2}}{3}.$$

Solving for L, we find that

$$\begin{aligned} L = \frac{\sqrt{9L-2}}{3} \quad &\implies \quad 3L = \sqrt{9L-2} \\ &\implies \quad 9L^2 - 9L + 2 = 0 \\ &\implies \quad L = \frac{9 \pm \sqrt{(-9)^2 - 4(9)(2)}}{18} \\ &\implies \quad L = \frac{1}{3}, \frac{2}{3}. \end{aligned}$$

Since the sequence starts at $a_1 = 19$ and has possible limits of $\frac{1}{3}$ or $\frac{2}{3}$, we guess that the sequence is decreasing. We will also guess that the sequence is bounded below by $\frac{2}{3}$, as this would allow us to infer that $\frac{2}{3}$ is the limit.

In this problem, we will combine our arguments for boundedness and monotonicity into a single inductive proof. Feel free to prove these statements separately (as in the previous solutions) if you prefer.

Claim: For all $n \in \mathbb{N}$, $\frac{2}{3} \leq a_{n+1} \leq a_n$.

Base Case: We have $a_1 = 19$ and

$$a_2 = \frac{\sqrt{9a_1 - 2}}{3} = \frac{\sqrt{9(19) - 2}}{3} = \frac{\sqrt{169}}{3} = \frac{13}{3}.$$

Indeed, it is true that $\frac{2}{3} \leq \frac{13}{3} \leq 19$, hence $\frac{2}{3} \leq a_2 \leq a_1$.

Inductive Step: Assume $\frac{2}{3} \leq a_{k+1} \leq a_k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
\frac{2}{3} \leq a_{k+1} \leq a_k &\implies 6 \leq 9a_{k+1} \leq 9a_k && \text{(multiply by 9)}\\
&\implies 4 \leq 9a_{k+1} - 2 \leq 9a_k - 2 && \text{(subtract 2)}\\
&\implies 2 \leq \sqrt{9a_{k+1} - 2} \leq \sqrt{9a_k - 2} && \text{(square root)}\\
&\implies \frac{2}{3} \leq \frac{\sqrt{9a_{k+1} - 2}}{3} \leq \frac{\sqrt{9a_k - 2}}{3} && \text{(divide by 3)}\\
&\implies \frac{2}{3} \leq a_{k+2} \leq a_{k+1},
\end{aligned}$$

as desired.

This completes the proof of our claim. By the Monotone Convergence Theorem, since $\{a_n\}$ is decreasing and bounded below, it must converge to a limit L, which we know from earlier is either $\frac{1}{3}$ or $\frac{2}{3}$. Since $\frac{2}{3} \leq a_n$ for all n, the limit cannot be $\frac{1}{3}$. Thus,

$$\lim_{n\to\infty} a_n = \frac{2}{3}.$$

1.5.4. The Monotone Convergence Theorem states that if a sequence is both bounded and monotone, then the sequence must converge. In this exercise, we investigate the converse of this statement – which turns out to be false.

(a) Demonstrate by example that there exist convergent sequences that are not monotonic. Can you find an example of a convergent sequence such that no tail of the sequence is monotonic?

Consider the sequence

$$\left\{\frac{(-1)^{n+1}}{n}\right\}_{n=1}^{\infty} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \dots\right\}.$$

The sequence is not monotonic – in fact, no tail of the sequence is monotonic – and yet the sequence converges to 0. Try proving this using the Squeeze Theorem.

(b) However, prove that if $\{a_n\}$ converges to a limit L, then $\{a_n\}$ must be bounded. That is, prove that there exist numbers m and M such that $m \leq a_n \leq M$ for all n.

Proof: If $\lim_{n\to\infty} a_n = L$, then by Theorem 1.2.16, the interval $(L-1, L+1)$ contains a tail of $\{a_n\}$. That is, there exists a cutoff N such that

$$L - 1 < a_n < L + 1$$

for all $n > N$.

Let M be any number larger than $L+1$ and also larger than $a_1, a_2, \ldots a_N$. Such a number exists, since $L+1, a_1, a_2, \ldots, a_N$ is a finite list. Likewise, let m be any number smaller than $L-1$ and $a_1, a_2, \ldots, a_N$.

Based on how we have defined m and M, and in light of the fact that $L-1 < a_n < L+1$ for all $n > N$, it follows that

$$m \leq a_n \leq M$$

for all n. That is, $\{a_n\}$ is bounded. □

1.5.5. The *Fibonacci sequence* is given by $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ where each subsequent term is given by the sum of the preceding two terms, $a_{n+2} = a_{n+1} + a_n$ $(n \geq 1)$, starting with $a_1 = a_2 = 1$. It is clear that $\lim_{n\to\infty} a_n = \infty$. However, the limit of the ratio of consecutvie Fibonacci numbers, a_{n+1}/a_n, *does* exist!

In the following two problems, we will explore some methods to determine this limit.

(a) Show that the terms in the Fibonacci sequence are related as

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}}, \ n \geq 1.$$

The sequence is generated by the recursion $a_{n+2} = a_{n+1} + a_n$; dividing through by a_{n+1} gives,

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}}, \ n \geq 1.$$

(b) Assume that the sequence $b_n = a_{n+1}/a_n$ converges and find the value of the limit $\lim_{n\to\infty} b_n = \varphi$. Start by constructing a recursive relationship for b_{n+1}.

Create a new sequence $\{b_n\}$ with terms given by $b_n = a_{n+1}/a_n$. In terms of b_n, the recursion is given as $b_{n+1} = 1 + 1/b_n$. To determine the limit, we use limit rules to write

$$\lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} 1 + \frac{1}{b_n} \Rightarrow \varphi = 1 + \frac{1}{\varphi},$$

or $\varphi^2 - \varphi - 1 = 0$. From the quadratic formula, the roots are,

$$\varphi = \frac{1 \pm \sqrt{5}}{2} \approx -0.618, 1.618.$$

All the terms in the sequence are positive, so the limit, if it exists, must be $\varphi = (1+\sqrt{5})/2$.

(c) (**Challenge Problem!**) Prove that the sequence $\{b_n = a_{n+1}/a_n\}$ indeed converges by proving that the sequence is a *contraction*.

(i) First, show that

$$b_{n+1} = 1 + \frac{b_{n-1}}{1 + b_{n-1}} \qquad (n \geq 2),$$

and explain why this implies that $1 < b_n < 2$ for $n \geq 3$.

(ii) Second, using the recursion formula for b_n in part (a), and the limit φ from part (b), show that

$$|b_{n+1} - \varphi| = \left| \frac{1}{b_n} - \frac{1}{\varphi} \right| < \frac{1}{\varphi} |b_n - \varphi|,$$

for $n \geq 3$.

(iii) Finally, show that

$$|b_{n+1} - \varphi| < \frac{1}{\varphi^{n-2}} |b_3 - \varphi|,$$

and explain why we can conclude that $b_n \to \varphi$ as $n \to \infty$.

(i) From the recursion formula defining the sequence,

$$b_{n+1} = 1 + \frac{1}{b_n}, \quad \text{and} \quad b_n = 1 + \frac{1}{b_{n-1}} = \frac{b_{n-1} + 1}{b_{n-1}},$$

so that,

$$b_{n+1} = 1 + \frac{b_{n-1}}{1 + b_{n-1}} \qquad (n \geq 2).$$

All of the terms in the sequence are positive, so $b_{n+1} > 1$ (setting $b_{n-1} = 0$), and the maximum value for $b_{n+1} < 2$ (letting $n \to \infty$ implies that $\frac{b_{n-1}}{1+b_{n-1}} \to 1$). Altogether, $1 < b_{n+1} < 2$ for $n \geq 2$.

(ii) From part (b), $\varphi = 1 + 1/\varphi$; subtracting from the recursion formula for b_{n+1}, and taking the absolute value,

$$|b_{n+1} - \varphi| = \left| 1 + \frac{1}{b_n} - 1 - \frac{1}{\varphi} \right| = \left| \frac{1}{b_n} - \frac{1}{\varphi} \right| = \left| \frac{\varphi - b_n}{\varphi b_n} \right| = \frac{|b_n - \varphi|}{\varphi b_n} < \frac{1}{\varphi} |b_n - \varphi|,$$

because, from part (i), $b_n > 1$ for $n \geq 3$.

(iii) If $|b_{n+1} - \varphi| < \frac{1}{\varphi} |b_n - \varphi|$, then applying the same argument to $|b_n - \varphi|$ results in the cascade

$$|b_{n+1} - \varphi| < \frac{1}{\varphi} |b_n - \varphi| < \frac{1}{\varphi^2} |b_{n-1} - \varphi| < \cdots < \frac{1}{\varphi^{n-2}} |b_3 - \varphi|.$$

We know from part (b) that the limit, if it exists, must be $\varphi = (1 + \sqrt{5})/2 > 1$, and so for any given tolerance ϵ, we can always find an index N such that

$$|b_{n+1} - \varphi| < \frac{1}{\varphi^{n-2}} |b_3 - \varphi| < \epsilon \quad \text{for all } n > N,$$

that is, we can always find N to make $1/\varphi^{n-2}$ as small as we want whenever $n > N$.

Therefore, by the definition of convergence, $\lim\limits_{n \to \infty} b_n = \varphi$.

1.5.6. (**Challenge Problem!**) As in the previous problem, the *Fibonacci sequence* is given by $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ where each subsequent term is given by the sum of the preceding two terms,

$$a_{n+2} = a_{n+1} + a_n \ (n \geq 0), \tag{1.4}$$

starting with $a_0 = 0$ and $a_1 = 1$.

Recursively-defined sequences are notoriously difficult to analyze because we cannot simply use limit rules to determine convergence. The goal of this question is to convert the recursive sequence defined by Eq. 1.4 into an *explicit* sequence.

(a) Assume that the terms in the Fibonacci sequence are given explicitly by $a_n = c\lambda^n$ where c and λ are constants. Substitute this expression into Eq. (1.4), and solve for λ (you should find two values, λ_1 and λ_2).

Substituting $a_n = c\lambda^n$ into the equation defining the sequence,

$$c\lambda^{n+2} = c\lambda^{n+1} + c\lambda^n, \quad \text{or (dividing through by } c\lambda^n\text{)}, \quad \lambda^2 = \lambda + 1,$$

which has two solutions: $\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$.

(b) The full expression for a_n is written as the linear combination, $a_n = c_1\lambda_1^n + c_2\lambda_2^n$. Use the initial terms in the sequence, $a_0 = 0$ and $a_1 = 1$, to determine c_1 and c_2. Then, use a calculator to compute a few terms of a_n using this expression and compare with the Fibonacci sequence.

If $a_n = c_1\lambda_1^n + c_2\lambda_2^n$, then from the initial terms in the sequence,

$$a_0 = c_1 + c_2 = 0, \quad \text{and} \quad a_1 = c_1\frac{1+\sqrt{5}}{2} + c_2\frac{1-\sqrt{5}}{2} = 1.$$

Solving this system of equations for c_1 and c_2,

$$c_1 = \frac{1}{\sqrt{5}}, \qquad c_2 = -\frac{1}{\sqrt{5}},$$

and so,

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

It is nothing short of a miracle that this expression evaluates to an integer for every integer choice of n, let alone that it evaluates to the members of the Fibonacci sequence!

(c) Using the explicit expression for a_n from part (b), determine the limit of the ratio of subsequent Fibonnaci terms, $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$.

From the explicit form for the members of the sequence found in part (b), the limit of the ratio can be determined using the limit rules,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}}$$
$$=\lim_{n\to\infty}\frac{1}{2}\frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}}.$$

This looks overwhelming, but if we factor out $(1+\sqrt{5})^n$ from the numerator and denominator, then,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2}\frac{(1+\sqrt{5})^n}{(1+\sqrt{5})^n}\frac{(1+\sqrt{5})-(1-\sqrt{5})\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^n}{1-\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^n}$$
$$=\lim_{n\to\infty}\frac{1}{2}\frac{(1+\sqrt{5})-(1-\sqrt{5})\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^n}{1-\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^n}.$$

The two expressions in square-brackets are less than 1, so as $n\to\infty$,

$$\lim_{n\to\infty}\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^n=\lim_{n\to\infty}\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^{n+1}=0$$

Consequently,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2}\frac{(1+\sqrt{5})-(1-\sqrt{5})\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^n}{1-\left[\frac{1-\sqrt{5}}{1+\sqrt{5}}\right]^n}$$
$$=\frac{1}{2}\frac{(1+\sqrt{5})-(1-\sqrt{5})\cdot 0}{1-0}=\frac{1+\sqrt{5}}{2}.$$

In fact, this will be the limit no matter how you initialize the sequence, i.e., for any (a,b), you will get the same limit for the ratio of subsequent terms starting from $a_0=a$ and $a_1=b$.

(d) Using induction, prove that for $n\geq 1$,

$$a_1^2+a_2^2+\cdots+a_n^2=a_na_{n+1}.$$

First, because $a_1=a_2=1$ and $a_3=2$,

$$a_1^2+a_2^2=1^2+1^2=2=a_2a_3 \quad ✓$$

Next, if we assume that $a_1^2+a_2^2+\cdots+a_n^2=a_na_{n+1}$, can we show that $a_1^2+a_2^2+\cdots+a_n^2+a_{n+1}^2=a_{n+1}a_{n+2}$? Start with,

$$a_1^2+a_2^2+\cdots+a_n^2=a_na_{n+1}$$

and add a_{n+1}^2 to both sides,

$$a_1^2+a_2^2+\cdots+a_n^2+a_{n+1}^2=a_na_{n+1}+a_{n+1}^2.$$

Then factor the right-hand side,

$$a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2 = a_{n+1}\left(a_n + a_{n+1}\right).$$

But the term in brackets is equal to a_{n+2} by the definition of the sequence, so

$$a_1^2 + a_2^2 + \cdots + a_n^2 + a_{n+1}^2 = a_{n+1}a_{n+2} \quad \checkmark$$

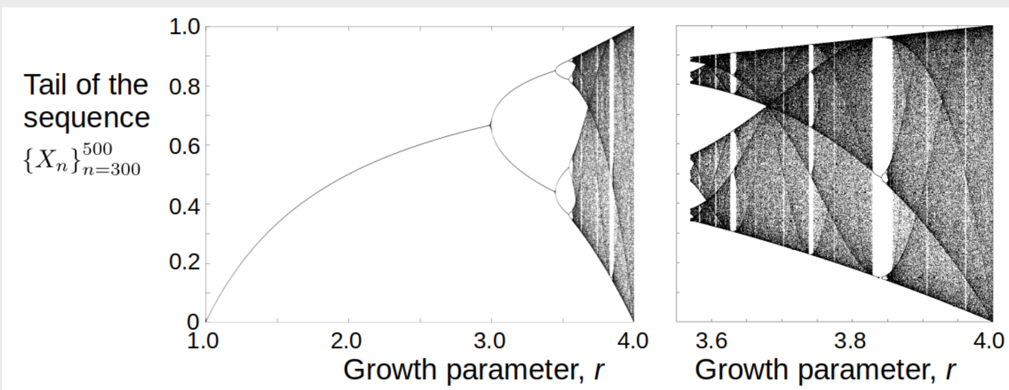
Amazing! It is not obvious that a sequence defined by recursive summation should impose such clean constraints on the sum of all squared terms in the sequence, but it only takes two lines to demonstrate that our intuition is way off!

1.5.7. (**Challenge Problem!**) The implicitly-defined sequence, with growth parameter $1 < r < 4$, and initial state $0 < X_0 < 1$,

$$X_{n+1} = rX_n(1 - X_n), \quad n = 0, 1, 2, 3, \ldots$$

was popularized by the mathematical biologist Robert May. Although it is a simple-looking sequence, the limiting-behaviour has an unusual dependence on the growth parameter r.

For values of $1 < r < 4$, initialize the sequence at $X_0 = \frac{1}{2}$, generate 500 terms and plot the two-hundred-term 'tail' of the sequence $\{X_n\}_{n=300}^{500}$ as a function of the parameter r. Pay careful attention to the parameter ranges $1 < r < 3$, $3 \leq r < 3.57$ and $3.57 \leq r < 4$.



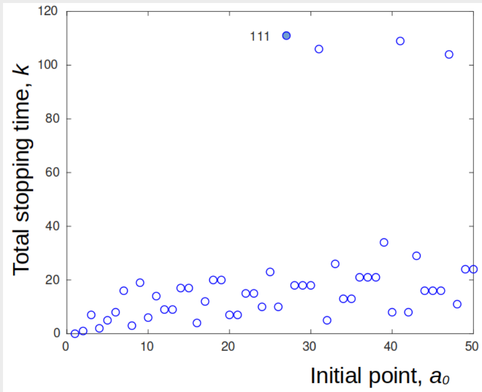
For $1 < r < 3$, the sequence converges to the limit $L = 1 - \frac{1}{r}$. In the range $3 \leq r < 3.45$, the tail oscillates between two values, and the period of that oscillation doubles at $r \approx 3.45$ and again at $r \approx 3.54$. At $r \approx 3.57$, the system becomes *chaotic*; for almost all initial conditions, the tail no longer exhibits finite-period oscillations (although there are isolated domains of r that exhibit non-chaotic behaviour).

1.5.8. (**Challenge Problem!**) The sequence

$$a_{n+1} = \begin{cases} a_n/2, & a_n \text{ is even} \\ 3a_n + 1, & a_n \text{ is odd} \end{cases}$$

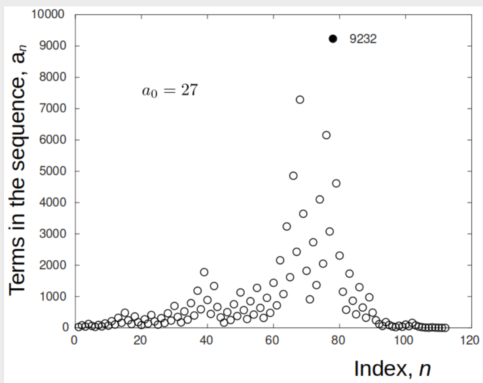
is called the *hail stone sequence* because, for many initial (positive integer) values a_0, the terms of the sequence make large excursions up and down, like hail stones in a storm cloud. Once the sequence reaches $a_k = 1$, it will oscillate forever as $\{4, 2, 1\}$. The index k at which the sequence first reaches $a_k = 1$ is called the *total stopping time of the sequence.*

(a) For initial points $a_0 = [1, 50]$, plot the total stopping time k as a function of a_0.



Most sequences terminate after less than 40 terms, but there are four exceptionally-long-lived sequences for $a_0 \in [1, 50]$. The largest stopping time ($k = 111$) corresponds to the initial point $a_0 = 27$.

(b) You should find that the largest stopping time in part (a) is $k = 111$ when $a_0 = 27$. Plot the sequence $\{a_n\}$ as a function of the index n starting at $a_0 = 27$. How large does the sequence climb?



The terms in the sequence increase and decrease many times, with the largest magnitude term $a_{77} = 9232$.

Collatz (1937) conjectured that every initial point a_0 has a finite total stopping time. As of 2020, Collatz's conjecture has been verified up to $a_0 = 2^{68}$, but its proof remains an infamous unsolved problem in mathematics. The great 20^{th} century mathematician, Paul Erdös (1913-1996), said that the search for a proof of the Collatz conjecture is 'hopeless, absolutely hopeless' and that 'mathematics may not be ripe for such problems.'

See ‘The ultimate challenge: the $3x+1$ problem’ edited by Jeffrey C. Lagarias (AMS, 2010). Particularly noteworthy is the contribution by John Conway (1937-2020) describing his FRACTRAN programming language.

Chapter 2

Function Limits and Continuity

2.1 Introduction to Function Limits

Section 2.1 Problems

2.1.1. Use the $\varepsilon-\delta$ definition of limits to show that $\lim\limits_{x\to 3}(4x+1)=13$.

Let $\epsilon>0$ be given. Let $\delta=\dfrac{\epsilon}{4}$. Then, if $0<|x-3|<\delta$, we get

$$|(4x+1)-13|=|4x-12|=4|x-3|<4\delta=4\cdot\frac{\varepsilon}{4}=\varepsilon$$

as desired.

2.1.2. Use the $\varepsilon-\delta$ definition of limits to show that $\lim\limits_{x\to -1}(1-9x)=10$.

Let $\epsilon>0$ be given. Let $\delta=\dfrac{\epsilon}{9}$. Then, if $0<|x+1|<\delta$, we get

$$|(1-9x)-10|=|-9x-9|=9|x+1|<9\delta=9\cdot\frac{\varepsilon}{9}=\varepsilon$$

as desired.

2.1.3. Use the $\varepsilon-\delta$ definition of limits to show that $\lim\limits_{x\to 2}\left(x^2-4x+4\right)=0$.

Let $\epsilon>0$ be given. Let $\delta=\sqrt{\epsilon}$. Then, if $0<|x-2|<\delta$, we get

$$|x^2-4x+4-0|=|x^2-4x+4|=|x-2|^2<\delta^2=\left(\sqrt{\varepsilon}\right)^2=\varepsilon$$

as desired.

2.1.4. Use the $\varepsilon - \delta$ definition of limits to show that $\lim_{x\to 4}\sqrt{x} = 2$.

Let $\varepsilon > 0$ be given. Let $\delta = 2\varepsilon$. Then, if $0 < |x-4| < \delta$, we get

$$\begin{aligned}
|\sqrt{x}-2| &= \left|\left(\sqrt{x}-2\right)\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}\right| \\
&= \left|\frac{x-4}{\sqrt{x}+2}\right| \\
&= \frac{|x-4|}{\sqrt{x}+2} \\
&\le \frac{|x-4|}{2} && \text{(since } \sqrt{x}+2 \ge 2\text{)} \\
&< \frac{\delta}{2} = \frac{2\varepsilon}{2} = \varepsilon,
\end{aligned}$$

as desired.

2.1.5. Use the $\varepsilon - \delta$ definition of limits to show that $\lim_{x\to 1} x^3 = 1$.

Hint: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

Let $\varepsilon > 0$ be given. Let $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$. If $0 < |x-1| < \delta$, observe that since $\delta < 1$, we have $|x-1| < 1$, and therefore $0 < x < 2$. Consequently,

$$\begin{aligned}
|x^3-1| &= |(x-1)(x^2+x+1)| && \text{(from the hint)} \\
&= |x-1||x^2+x+1| \\
&< \delta|x^2+x+1| \\
&\le \delta\left(|x|^2+|x|+1\right) && \text{(triangle inequality)} \\
&\le \delta\left(2^2+2+1\right) && \text{(since } 0 < x < 2\text{)} \\
&= 7\delta \\
&\le 7\cdot\frac{\varepsilon}{7} \\
&= \varepsilon,
\end{aligned}$$

as desired.

2.1.6. Use the $\varepsilon - \delta$ definition of limits to show that $\lim_{x\to 3}\frac{1}{x^2} = \frac{1}{9}$.

Let $\varepsilon > 0$ be given. Let $\delta = \min\left\{1, \frac{36\varepsilon}{7}\right\}$. If $0 < |x-3| < \delta$, observe that since $\delta < 1$, we have $|x-3| < 1$, and therefore $2 < x < 4$. Consequently,

$$\left|\frac{1}{x^2}-\frac{1}{9}\right| = \left|\frac{9-x^2}{9x^2}\right|$$

$$
\begin{aligned}
&= \frac{|9-x^2|}{9x^2} \\
&= \frac{|3-x||3+x|}{9x^2} \\
&< \frac{\delta|3+x|}{9x^2}.
\end{aligned}
$$

Since $2 < x < 4$, it follows from the triangle inequality that

$$|3+x| \leq |3| + |x| \leq 3 + 4 = 7.$$

Furthermore, $9x^2 \geq 9(2)^2 = 36$. Thus,

$$\left|\frac{1}{x^2} - \frac{1}{9}\right| < \frac{\delta|3+x|}{9x^2} \leq \delta \cdot \frac{7}{36} < \frac{36\varepsilon}{7} \cdot \frac{7}{36} = \varepsilon,$$

as desired.

2.1.7. Let $f(x)$ and $g(x)$ be functions such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Use the $\varepsilon - \delta$ definition of limits to show that $\lim_{x\to a} (f(x) + g(x)) = L + M$.

Note: We saw the same phenomenon when working with limits of sequences, and the proof there followed a similar structure. In the next section, you'll learn about the Sequential Characterization of Function Limits, which lets us transfer many familiar results from sequences to functions—without having to reprove everything from scratch using the ε–δ definition!

Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \frac{\varepsilon}{2}$. Similarly, since $\lim_{x\to a} f(x) = M$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|g(x) - M| < \frac{\varepsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, if $0 < |x-a| < \delta$, we have

$$
\begin{aligned}
|\,(f(x)+g(x)) - (L+M)\,| &= |\,(f(x)-L) + (g(x)-M)\,| \\
&\leq |f(x)-L| + |g(x)-M| \qquad \text{(triangle inequality)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}
$$

as desired.

2.2 Sequential Characterization of Limits

Section 2.2 Problems

2.2.1. Use the Sequential Characterization of Function Limits to evaluate

$$\lim_{x\to 4} \frac{\sqrt{x}}{1+x}.$$

Let $f(x) = \dfrac{\sqrt{x}}{1+x}$. To evaluate $\lim\limits_{n\to 4} f(x)$, let $\{x_n\}$ be a sequence with $x_n \neq 4$ for all n and $\lim\limits_{n\to\infty} x_n = 4$. We compute $\lim\limits_{n\to\infty} f(x_n)$, referencing specific parts from Theorem **??** so you can see which properties of sequences are being used:

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \frac{\sqrt{x_n}}{1+x_n} \\
&= \frac{\lim\limits_{n\to\infty} \sqrt{x_n}}{\lim\limits_{n\to\infty} (1+x_n)} && \text{(Property 5.)} \\
&= \frac{\sqrt{\lim\limits_{n\to\infty} x_n}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} x_n} && \text{(Properties 3. and 7.)} \\
&= \frac{\sqrt{4}}{1+4} = \frac{2}{5}.
\end{aligned}$$

We have shown that $\lim\limits_{n\to\infty} f(x_n) = \frac{2}{5}$ for any sequence $\{x_n\}$ with $x_n \neq 4$ for all n and $\lim\limits_{n\to\infty} x_n = 4$. Thus, by the Sequential Characterization of Function Limits,

$$\lim_{x\to 4} f(x) = \frac{2}{5}.$$

2.2.2. Consider the function

$$f(x) = \cos^2\left(\frac{1}{x}\right).$$

Find sequences $\{x_n\}$ and $\{y_n\}$ that both converge to 0, $x_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n)$. Clearly explain why this implies that $\lim\limits_{x\to 0} f(x)$ does not exist.

Pick $x_n = \dfrac{1}{2n\pi}$ and $y_n = \dfrac{1}{\frac{\pi}{2} + 2n\pi}$. Clearly, both $\{x_n\}$ and $\{y_n\}$ converge to 0, and $x_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$. We have

$$f(x_n) = \cos^2\left(\frac{1}{\frac{1}{2n\pi}}\right) = \cos^2(2n\pi) = 1,$$

hence $\lim_{n\to\infty} f(x_n) = 1$. Next,

$$f(y_n) = \cos^2\left(\frac{1}{\frac{1}{\frac{\pi}{2}+2n\pi}}\right) = \cos^2\left(\frac{\pi}{2} + 2n\pi\right) = 0,$$

hence $\lim_{n\to\infty} f(y_n) = 0$. Since $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, we conclude that $\lim_{x\to 0} f(x)$ does not exist. Indeed, if the limit did exist, by the Sequential characterization of Function Limits, the above calculations would suggest that $\lim_{x\to 0} f(x)$ is both 1 and 0, which is impossible as limits are unique.

2.2.3. Consider the function

$$f(x) = \frac{x^2 - 9}{|x-3|}.$$

Find sequences $\{x_n\}$ and $\{y_n\}$ that both converge to 3, $x_n \neq 3$ and $y_n \neq 3$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$. Clearly explain why this implies that $\lim_{x\to 3} f(x)$ does not exist.

Pick $x_n = 3 + \dfrac{1}{n}$ and $y_n = 3 - \dfrac{1}{n}$. Clearly, both x_n and y_n converge to 3, and $x_n \neq 3$, $y_n \neq 3$ for all $n \in \mathbb{N}$. Now, we compute:

$$f(x_n) = \frac{\left(3 + \frac{1}{n}\right)^2 - 9}{\left|3 + \frac{1}{n} - 3\right|} = \frac{9 + \frac{6}{n} + \frac{1}{n^2} - 9}{\left|\frac{1}{n}\right|} = n\left(\frac{6}{n} + \frac{1}{n^2}\right) = 6 + \frac{1}{n}.$$

Therefore, $\lim_{n\to\infty} f(x_n) = 6$. Next,

$$f(y_n) = \frac{\left(3 - \frac{1}{n}\right)^2 - 9}{\left|3 - \frac{1}{n} - 3\right|} = \frac{9 - \frac{6}{n} + \frac{1}{n^2} - 9}{\left|\frac{-1}{n}\right|} = n\left(-\frac{6}{n} + \frac{1}{n^2}\right) = -6 + \frac{1}{n}.$$

Therefore, $\lim_{n\to\infty} f(y_n) = -6$.

Since $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, we conclude that $\lim_{x\to 3} f(x)$ does not exist. Indeed, if the limit did exist, by the Sequential characterization of Function Limits, the above calculations would suggest that $\lim_{x\to 3} f(x)$ is both 6 and -6, which is impossible as limits are unique.

2.2.4. For $x \in \mathbb{R}$, define the *floor* of x, denoted $\lfloor x \rfloor$, to be the largest integer less than or equal to x. For instance,

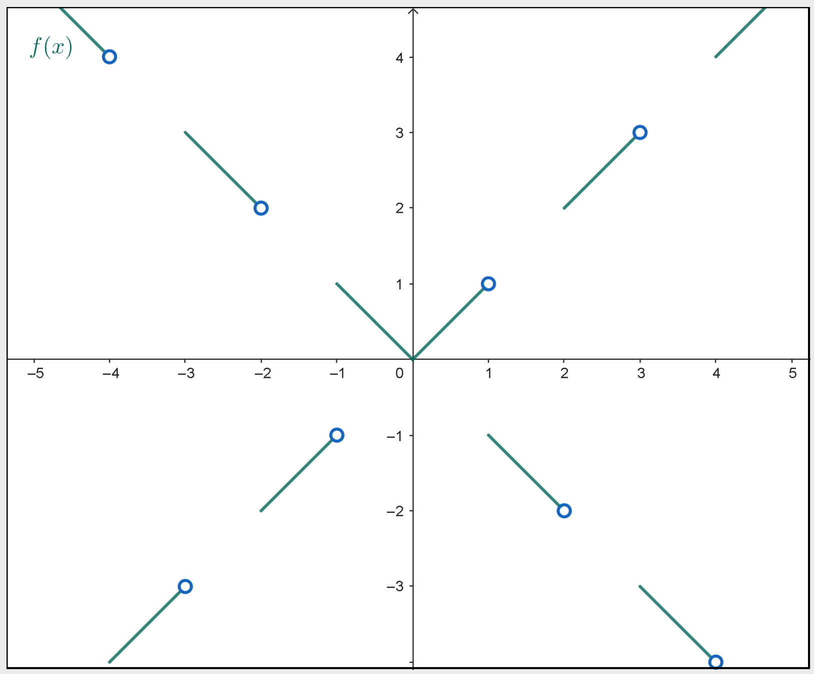
$$\lfloor 2 \rfloor = 2, \quad \lfloor 1.003 \rfloor = 1, \quad \lfloor \pi \rfloor = 3, \quad \lfloor -7.4 \rfloor = -8.$$

Consider the function

$$f(x) = \begin{cases} x & \text{if } \lfloor x \rfloor \text{ is even}, \\ -x & \text{if } \lfloor x \rfloor \text{ is odd}. \end{cases}$$

(a) Sketch the graph of $f(x)$.

The graph of f is shown below.



(b) Prove that $\lim_{x\to 2} f(x)$ does not exist.

Consider the sequence $\{x_n\}$ given by

$$x_1 = 1.9, \quad x_2 = 1.99, \quad x_3 = 1.999, \quad x_4 = 1.9999, \quad \text{etc.},$$

Note that $\lim_{n\to\infty} x_n = 2$. Since $\lfloor x_n \rfloor = 1$ (odd) for all n, we have $f(x_n) = -x_n$ for all n, hence

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} -x_n = -2.$$

Next, consider the sequence $\{y_n\}$ given by

$$x_1 = 2.1, \quad x_2 = 2.01, \quad x_3 = 2.001, \quad x_4 = 2.0001, \quad \text{etc.},$$

Note that $\lim_{n\to\infty} y_n = 2$. Since $\lfloor y_n \rfloor = 2$ (even) for all n, we have $f(y_n) = y_n$ for all n, hence

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_n = 2.$$

Since $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, we conclude that $\lim_{x\to 2} f(x)$ does not exist. Indeed, if the limit did exist, by the Sequential Characterization of Function Limits, the above calculations would suggest that $\lim_{x\to 2} f(x)$ is both -2 and 2, which is impossible as limits are unique.

2.2.5. Prove the Squeeze Theorem for Functions using the Sequential Characterization of Function Limits and the Squeeze Theorem for Sequences.

Proof: Let $f(x)$, $g(x)$, and $h(x)$ be functions with

$$g(x) \leq f(x) \leq h(x)$$

and

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x).$$

Let $\{x_n\}$ be a sequence such that $x_n \neq a$ for all n and $\lim_{n\to\infty} x_n = a$. It follows from the Sequential Characterization of Function Limits that

$$\lim_{n\to\infty} g(x_n) = L = \lim_{n\to\infty} h(x_n);$$

and since

$$g(x_n) \leq f(x_n) \leq h(x_n),$$

we deduce from the Squeeze Theorem for Sequences that $\lim_{n\to\infty} f(x_n) = L$. Therefore, since $\lim_{n\to\infty} f(x_n) = L$ for this arbitrary sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = a$, it follows from the Sequential Characterization of Function Limits that $\lim_{x\to a} f(x) = L$, as required. $\square$

2.3 One-Sided Limits

Section 2.3 Problems

2.3.1. Evaluate $\displaystyle\lim_{x\to 2^-} \frac{x-2}{\sqrt{x^2-4x+4}}$.

Notice that
$$\sqrt{x^2-4x+4} = \sqrt{(x-2)^2} = |x-2|.$$
As x approaches 2 from the left, $x-2$ is negative. This means that $|x-2| = -(x-2)$, hence
$$\lim_{x\to 2^-} \frac{x-2}{\sqrt{x^2-4x+4}} = \lim_{x\to 2^-} \frac{x-2}{|x-2|} = \lim_{x\to 2^-} \frac{x-2}{-(x-2)} = \lim_{x\to 2^-} -1 = -1.$$

2.3.2. Evaluate $\displaystyle\lim_{x\to 0^+} \sqrt{x}\cos\left(\frac{\pi}{x}\right)$.

Note that $-1 \le \cos\left(\frac{\pi}{x}\right) \le 1$ for all $x \neq 0$. Thus, multiplying through by $\sqrt{x}$, we have
$$-\sqrt{x} \le \sqrt{x}\cos\left(\frac{\pi}{x}\right) \le \sqrt{x}$$
for all $x > 0$. Since $\displaystyle\lim_{x\to 0^+} -\sqrt{x} = \lim_{x\to 0^+} \sqrt{x} = 0$, it follows from the Squeeze Theorem that
$$\lim_{x\to 0^+} \sqrt{x}\cos\left(\frac{\pi}{x}\right) = 0.$$

2.3.3. Evaluate $\displaystyle\lim_{x\to 4} \frac{x^2-8x+16}{|4-x|}$.

We consider the one-sided limits as x approaches 4.

<u>Left-sided limit:</u>
$$\begin{aligned}\lim_{x\to 4^-} \frac{x^2-8x+16}{|4-x|} &= \lim_{x\to 4^-} \frac{(x-4)(x-4)}{4-x} && \text{(since } 4-x>0\text{)}\\ &= \lim_{x\to 4^-} -(x-4)\\ &= 0.\end{aligned}$$

<u>Right-sided limit:</u>
$$\begin{aligned}\lim_{x\to 4^+} \frac{x^2-8x+16}{|4-x|} &= \lim_{x\to 4^+} \frac{(x-4)(x-4)}{-(4-x)} && \text{(since } 4-x<0\text{)}\\ &= \lim_{x\to 4^+} (x-4)\end{aligned}$$

$$= 0.$$

Since $\lim\limits_{x\to 4^-} \dfrac{x^2-8x+16}{|4-x|} = \lim\limits_{x\to 4^+} \dfrac{x^2-8x+16}{|4-x|} = 0$, we conclude that

$$\lim_{x\to 4} \frac{x^2-8x+16}{|4-x|} = 0.$$

2.3.4. Evaluate $\lim\limits_{x\to 1} \dfrac{|x-1|+|x+1|-2}{x-1}$.

We consider the left- and right-sided limits separately, noting that

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1, \\ -(x-1) & \text{if } x < 1. \end{cases}$$

Regardless of which sided limit we are considering, note that around $x = 1$, we have $|x+1| = x+1$ as $x+1$ is positive.

Left-sided limit:

$$\begin{aligned} \lim_{x\to 1^-} \frac{|x-1|+|x+1|-2}{x-1} &= \lim_{x\to 1^-} \frac{-(x-1)+(x+1)-2}{x-1} \\ &= \lim_{x\to 1^-} \frac{0}{x-1} \\ &= \lim_{x\to 1^-} 0 \\ &= 0 \end{aligned}$$

Right-sided limit:

$$\begin{aligned} \lim_{x\to 1^+} \frac{|x-1|+|x+1|-2}{x-1} &= \lim_{x\to 1^+} \frac{(x-1)+(x+1)-2}{x-1} \\ &= \lim_{x\to 1^+} \frac{2(x-1)}{x-1} \\ &= \lim_{x\to 1^+} 2 \\ &= 2 \end{aligned}$$

As the sided limits differ, $\lim\limits_{x\to 1} \dfrac{|x-1|+|x+1|-2}{x-1}$ does not exist.

2.3.5. Let a be a real number and consider the function defined by

$$f(x) = \begin{cases} \dfrac{x^2-1}{x^2+(a-1)x+a} & \text{if } 0 < x < 1, \\[2ex] \dfrac{\sqrt{x}-1}{x-1} & \text{if } x > 1. \end{cases}$$

Determine the value of a such that $\lim_{x\to 1} f(x)$ exists.

In order for $\lim_{x\to 1} f(x)$ to exist, it must be the case that the one-sided limits exist and are equal. We compute these limits separately.

Right-Sided Limit:

$$\begin{aligned}
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x-1} \\
&= \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \\
&= \lim_{x\to 1^+} \frac{x-1}{(x-1)\left(\sqrt{x}+1\right)} \\
&= \lim_{x\to 1^+} \frac{1}{\sqrt{x}+1} \\
&= \frac{1}{2}.
\end{aligned}$$

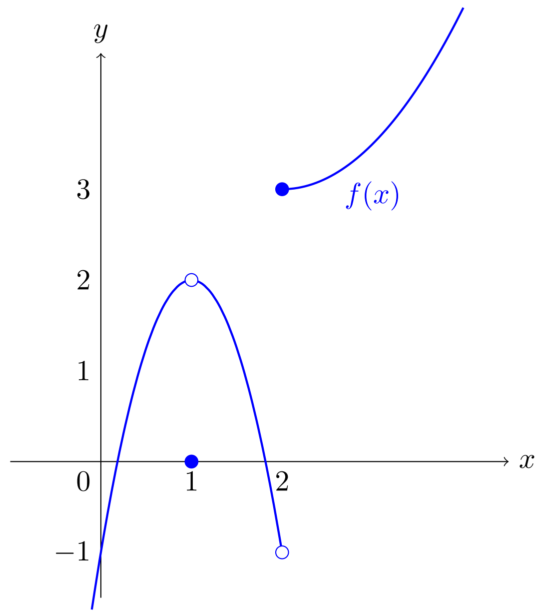
Left-Sided Limit:

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{x^2-1}{x^2+(a-1)x+a} \\
&= \lim_{x\to 1^-} \frac{(x-1)(x+1)}{(x-1)(x+a)} \\
&= \lim_{x\to 1^-} \frac{x+1}{x+a} \\
&= \frac{2}{2+a}.
\end{aligned}$$

(In final step in the evaluation of the left-sided limit above, we have implicitly assumed that $a \neq -1$, otherwise the limit has the form “2/0” and the limit does not exist.)

Setting the sided limits equal to each other, we have $\frac{1}{2} = \frac{2}{2+a}$. Solving for a, we get $2+a=4$, hence $a=2$.

2.3.6. Consider the graph of a function $y = f(x)$ shown below. Determine $\lim_{x\to 1} f(f(x))$. Explain your reasoning.



We can see from the graph that as x approaches 1, whether from the left or from the right, $f(x)$ approaches 2 *from below*. Thus,

$$\lim_{x\to 1} f(\underbrace{f(x)}_{\to 2^-}) = \lim_{x\to 2^-} f(x) = -1.$$

2.4 Fundamental Trigonometric Limit

Section 2.4 Problems

2.4.1. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x}$

We multiply and divide by 3 to make the expression look more like the Fundamental Trig Limit:

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot 3 = 1 \cdot 3 = 3.$$

(b) $\displaystyle\lim_{x\to 0} \frac{x^2 + 2x}{x\sin(x) + 2\sin(x)}$

This may look intimidating, but there are some common terms to be factored:

$$\lim_{x\to 0} \frac{x^2 + 2x}{x\sin(x) + 2\sin(x)} = \lim_{x\to 0} \frac{x(x+2)}{(x+2)\sin(x)} = \lim_{x\to 0} \frac{x}{\sin(x)} = \lim_{x\to 0} \frac{1}{\frac{\sin(x)}{x}} = \frac{1}{1} = 1.$$

(c) $\displaystyle\lim_{x\to 0} \frac{2x}{\sin(x) + \sin(5x)}$

We have

$$\begin{aligned}
\lim_{x\to 0} \frac{2x}{\sin(x) + \sin(5x)} &= \lim_{x\to 0} \frac{2}{\frac{\sin(x)}{x} + \frac{\sin(5x)}{x}} \\
&= \lim_{x\to 0} \frac{2}{\frac{\sin(x)}{x} + \frac{\sin(5x)}{5x} \cdot 5} \\
&= \frac{2}{1 + 1 \cdot 5} \\
&= \frac{1}{3}.
\end{aligned}$$

2.4.2. Prove the following for all real numbers a and b with $a, b \neq 0$.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$

Appealing to the result from (a), we have

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)} &= \lim_{x\to 0}\frac{\sin(ax)}{ax}\cdot\frac{bx}{\sin(bx)}\cdot\frac{ax}{bx}\\ &= \lim_{x\to 0}\frac{\sin(ax)}{ax}\cdot\frac{1}{\frac{\sin(bx)}{bx}}\cdot\frac{a}{b}\\ &= 1\cdot\frac{1}{1}\cdot\frac{a}{b}\\ &= \frac{a}{b}.\end{aligned}$$

(b) $\lim\limits_{x\to 0}\dfrac{\tan(ax)}{\tan(bx)} = \dfrac{a}{b}$

From (a), we have

$$\lim_{x\to 0}\frac{\tan(ax)}{\tan(bx)} = \lim_{x\to 0}\frac{\frac{\sin(ax)}{\cos(ax)}}{\frac{\sin(bx)}{\cos(bx)}} = \lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}\cdot\frac{\cos(bx)}{\cos(ax)} = \frac{a}{b}\cdot 1 = \frac{a}{b}.$$

(c) $\lim\limits_{x\to 0}\dfrac{\tan(ax)}{\sin(bx)} = \dfrac{a}{b}$

Appealing to the result from (a), we have

$$\lim_{x\to 0}\frac{\tan(ax)}{\sin(bx)} = \lim_{x\to 0}\frac{\frac{\sin(ax)}{\cos(ax)}}{\sin(bx)} = \lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}\cdot\frac{1}{\cos(bx)} = \frac{a}{b}\cdot 1 = \frac{a}{b}.$$

2.4.3. Evaluate $\lim\limits_{x\to 0}\dfrac{x^3\sin\left(\frac{1}{x}\right)}{\sin(x)}$.

We'll begin by factoring out terms that resemble our Fundamental Trig Limit:

$$\lim_{x\to 0}\frac{x}{\sin(x)}\cdot x^2\sin\left(\frac{1}{x}\right).$$

Now, we know that $\lim\limits_{x\to 0}\dfrac{x}{\sin(x)} = \lim\limits_{x\to 0}\dfrac{1}{\frac{\sin(x)}{x}} = 1$. To understand the behaviour of $x^2\sin\left(\frac{1}{x}\right)$ as x approaches 0, note that for all $x\neq 0$,

$$-1\le\sin\left(\frac{1}{x}\right)\le 1.$$

Thus, multiplying through by x^2, it follows that

$$-x^2\le x^2\sin\left(\frac{1}{x}\right)\le x^2$$

for all $x \neq 0$. Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, we conclude from the Squeeze Theorem that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Putting everything together, we find that

$$\lim_{x\to 0} \frac{x^3 \sin\left(\frac{1}{x}\right)}{\sin(x)} = \lim_{x\to 0} \frac{x}{\sin(x)} \cdot x^2 \sin\left(\frac{1}{x}\right) = 1 \cdot 0 = 0.$$

2.4.4. Evaluate $\lim_{x\to 0} \frac{\sin(\sin(x))}{x}$.

We rewrite the limit by multiplying and dividing by $\sin(x)$:

$$\lim_{x\to 0} \frac{\sin(\sin(x))}{x} = \lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} \cdot \frac{\sin(x)}{x}.$$

Of course, we know that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. To understand the behaviour of $\frac{\sin(\sin(x))}{\sin(x)}$, it might help to assign a new variable to $\sin(x)$, say u. As x approaches 0, $u = \sin(x)$ approaches 0 too. So,

$$\lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1.$$

Thus, putting everything together, we find that

$$\lim_{x\to 0} \frac{\sin(\sin(x))}{x} = \lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} \cdot \frac{\sin(x)}{x} = 1 \cdot 1 = 1.$$

2.4.5. Evaluate $\lim_{x\to 0} \frac{\cos(x) - 1}{x}$.

Hint: Rewrite the limit by multiplying and dividing by $\cos(x) + 1$. This limit will be useful when calculating the derivative of $\sin(x)$ in Chapter 3.

Following the hint, we have

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x) - 1}{x} &= \lim_{x\to 0} \frac{\cos(x) - 1}{x} \cdot \frac{\cos(x) + 1}{\cos(x) + 1} \\
&= \lim_{x\to 0} \frac{\cos^2(x) - 1}{x\left(\cos(x) + 1\right)} \\
&= \lim_{x\to 0} \frac{-\sin^2(x)}{x\left(\cos(x) + 1\right)} \\
&= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{-\sin(x)}{\cos(x) + 1} \\
&= 1 \cdot \frac{0}{2}
\end{aligned}$$

$$= 0.$$

2.5 Limits at Infinity and Horizontal Asymptotes

Section 2.5 Problems

2.5.1. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{2x^4-3x+4}{x^4+x+8}$

We have

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^4-3x+4}{x^4+x+8} &= \lim_{x\to\infty} \frac{x^4\left(2-\frac{3}{x^3}+\frac{4}{x^4}\right)}{x^4\left(1+\frac{1}{x^3}+\frac{8}{x^4}\right)} \\
&= \lim_{x\to\infty} \frac{2-\frac{3}{x^3}+\frac{4}{x^4}}{1+\frac{1}{x^3}+\frac{8}{x^4}} \\
&= \frac{2-0+0}{1+0+0} \\
&= 2.
\end{aligned}$$

(b) $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}-\sqrt{x+1}}$

The denominator has the form "$\infty - \infty$", which we cannot evaluate directly. We must first manipulate the expression by multiplying and dividing by the conjugate of the denominator:

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{\sqrt{x}-\sqrt{x+1}} &= \lim_{x\to\infty} \frac{1}{\sqrt{x}-\sqrt{x+1}} \cdot \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x}+\sqrt{x+1}}{x-(x+1)} \\
&= \lim_{x\to\infty} \frac{\left(\sqrt{x}+\sqrt{x+1}\right)}{-1} \\
&= -\infty.
\end{aligned}$$

(c) $\displaystyle\lim_{x\to-\infty} \frac{\cos(x+1)}{x+2}$

Note that $-1 \le \cos(x+1) \le 1$ for all x. Thus, dividing by $x+2$ (and remembering to reverse the inequalities since $x+2$ is negative as $x \to -\infty$), we have

$$\frac{-1}{x+2} \ge \frac{\cos(x+1)}{x+2} \ge \frac{1}{x+2}.$$

Since $\displaystyle\lim_{x\to-\infty} \frac{-1}{x+2} = 0$ and $\displaystyle\lim_{x\to-\infty} \frac{1}{x+2} = 0$, it follows from the Squeeze Theorem that

$$\lim_{x\to-\infty} \frac{\cos(x+1)}{x+2} = 0.$$

2.5.2. The following approach allows one to convert limits to $-\infty$ into limits to $+\infty$ (which some students may find easier to think about):

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x).$$

Use this approach to compute

$$\lim_{x\to-\infty} \frac{1+x+(-x)^{3/2}}{1+2(-x)^{3/2}}.$$

We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{1+x+(-x)^{3/2}}{1+2(-x)^{3/2}} &= \lim_{x\to\infty} \frac{1-x+x^{3/2}}{1+2x^{3/2}} \\
&= \lim_{x\to\infty} \frac{x^{3/2}\left(\frac{1}{x^{3/2}} - \frac{1}{x^{1/2}} + 1\right)}{x^{3/2}\left(\frac{1}{x^{3/2}} + 2\right)} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x^{3/2}} - \frac{1}{x^{1/2}} + 1}{\frac{1}{x^{3/2}} + 2} \\
&= \frac{0-0+1}{0+2} \\
&= \frac{1}{2}.
\end{aligned}$$

2.5.3. How many horizontal asymptotes can a function have? Explain your reasoning.

A function can have at most 2 horizontal asymptotes: one corresponding to the limit as x approaches ∞ and another corresponding to the limit as x approaches $-\infty$.

2.5.4. Find all horizontal asymptotes of the function

$$f(x) = \frac{\sqrt{x^2+1}}{x+3}.$$

We separately compute the limits as x approaches ∞ and $-\infty$. We have

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+3} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)}$$

$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} \\
&= \lim_{x\to\infty} \frac{x\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} && (|x| = x \text{ since } x > 0) \\
&= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{3}{x}} \\
&= \frac{\sqrt{1+0}}{1+0} \\
&= 1.
\end{aligned}
$$

Thus, $y = 1$ is a horizontal asymptote as $x \to \infty$.

We also have

$$
\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+3} &= \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} \\
&= \lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} \\
&= \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{3}{x}\right)} && (|x| = -x \text{ since } x < 0) \\
&= \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{1}{x^2}}}{1+\frac{3}{x}} \\
&= \frac{-\sqrt{1+0}}{1+0} \\
&= -1.
\end{aligned}
$$

Thus, $y = -1$ is a horizontal asymptote as $x \to -\infty$.

2.5.5. Find all horizontal asymptotes for the function

$$f(x) = \frac{e^{2x}+1}{e^x+1}.$$

We separately compute the limits as x approaches ∞ and $-\infty$. We have

$$
\begin{aligned}
\lim_{x\to\infty} \frac{e^{2x}+1}{e^x+1} &= \lim_{x\to\infty} \frac{e^{2x}\left(1+\frac{1}{e^{2x}}\right)}{e^x\left(1+\frac{1}{e^x}\right)} \\
&= \lim_{x\to\infty} \underbrace{e^x}_{\to\infty} \cdot \underbrace{\frac{1+\frac{1}{e^{2x}}}{1+\frac{1}{e^x}}}_{\to\frac{1+0}{1+0}=1}
\end{aligned}
$$

$$= \infty.$$

Thus, there is no horizontal asymptote as $x \to \infty$.

As $x \to -\infty$, we have

$$\lim_{x \to -\infty} \frac{e^{2x} + 1}{e^x + 1} = \frac{0 + 1}{0 + 1} = 1.$$

Thus, $y = 1$ is a horizontal asymptote as $x \to -\infty$.

2.6 Fundamental Log Limit

Section 2.6 Problems

2.6.1. Use the fact that $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{e^x}{x} = \infty$ to evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{x^3}$

We have

$$\lim_{x\to\infty} \frac{\ln(x)}{x^3} = \lim_{x\to\infty} \frac{\ln\left(\left(x^3\right)^{1/3}\right)}{x^3} = \lim_{x\to\infty} \frac{\frac{1}{3}\ln(x^3)}{x^3} = \frac{1}{3}\cdot 0 = 0.$$

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x+\sqrt{x}}$

Note that for large values of x,

$$0 \le \frac{\ln(x)}{x+\sqrt{x}} \le \frac{\ln(x)}{x}.$$

Thus, since $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$ (and clearly $\lim_{x\to\infty} 0 = 0$), it follows from the Squeeze Theorem that

$$\lim_{x\to\infty} \frac{\ln(x)}{x+\sqrt{x}} = 0.$$

(c) $\lim_{x\to\infty} \frac{\ln(x)}{e^x}$

We have

$$\lim_{x\to\infty} \frac{\ln(x)}{e^x} = \lim_{x\to\infty} \frac{\ln(x)}{x}\cdot\frac{x}{e^x} = \lim_{x\to\infty} \frac{\ln(x)}{x}\cdot\frac{1}{\frac{e^x}{x}} = 0\cdot 0 = 0.$$

2.6.2. Use the fact that $\lim_{x\to\infty} \frac{\ln(x)}{x^p} = 0$ and $\lim_{x\to\infty} \frac{e^x}{x^p} = \infty$ for all $p > 0$ to evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{1+e^x+e^{x^2}}{1+x+x^2}$

Dividing the numerator and denominator by x^2, we get

$$\lim_{x\to\infty} \frac{1+e^x+e^{x^2}}{1+x+x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}+\frac{e^x}{x^2}+\frac{e^{x^2}}{x^2}}{\frac{1}{x^2}+\frac{1}{x}+1}.$$

The denominator approaches $0+0+1=1$, while the numerator approaches ∞. Thus,

$$\lim_{x\to\infty} \frac{1+e^x+e^{x^2}}{1+x+x^2} = \infty.$$

(b) $\displaystyle\lim_{x\to\infty} \frac{x+1}{x+\sqrt{x}\,\ln(x)+1}$

Dividing the numerator and denominator by x, we get

$$\lim_{x\to\infty} \frac{x+1}{x+\sqrt{x}\,\ln(x)+1} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1+\frac{\ln(x)}{\sqrt{x}}+\frac{1}{x}} = \frac{1+0}{1+0+0} = 1.$$

(c) $\displaystyle\lim_{x\to\infty} e^{-x}\left(1-x\sqrt{e^x}\right)$

We have

$$\begin{aligned}
\lim_{x\to\infty} e^{-x}\left(1-x\sqrt{e^x}\right) &= \lim_{x\to\infty}\left(e^{-x}-xe^{-x}e^{x/2}\right)\\
&= \lim_{x\to\infty}\left(e^{-x}-\frac{x}{e^{x/2}}\right)\\
&= \lim_{x\to\infty}\left(e^{-x}-\frac{1}{\frac{e^{x/2}}{x}}\right)\\
&= \lim_{x\to\infty}\left(e^{-x}-\frac{1}{\left(\frac{e^x}{x^2}\right)^{1/2}}\right)\\
&= 0-0\\
&= 0.
\end{aligned}$$

2.6.3. Let $a>0$, $a\neq 1$. Given $p>0$, use the fact that $\displaystyle\lim_{x\to\infty}\frac{\ln(x)}{x^p}=0$ to prove that

$$\lim_{x\to\infty} \frac{\log_a(x)}{x^p} = 0.$$

From the logarithm change of base formula, we have

$$\lim_{x\to\infty} \frac{\log_a(x)}{x^p} = \lim_{x\to\infty} \frac{\frac{\ln(x)}{\ln(a)}}{x^p} = \frac{1}{\ln(a)}\lim_{x\to\infty}\frac{\ln(x)}{x^p} = \frac{1}{\ln(a)}\cdot 0 = 0.$$

2.6.4. Let $a > 1$. Given $p > 0$, use the fact that $\lim_{x\to\infty} \frac{e^x}{x^p} = \infty$ to prove that

$$\lim_{x\to\infty} \frac{a^x}{x^p} = \infty$$

We note that $a^x = \left(e^{\ln(a)}\right)^x = (e^x)^{\ln(a)}$. Hence,

$$\lim_{x\to\infty} \frac{a^x}{x} = \lim_{x\to\infty} \frac{(e^x)^{\ln(a)}}{x} = \lim_{x\to\infty} \left(\frac{e^x}{x^{\frac{1}{\ln(a)}}}\right)^{\ln(a)} = 0^{\ln(a)} = 0.$$

2.7 Infinite Limits and Vertical Asymptotes

Section 2.7 Problems

2.7.1. Evaluate $\displaystyle\lim_{x\to 1}\frac{3-x}{(x-1)^2}$.

As x approaches 1 – whether from the left or the right – the denominator, $(x-1)^2$, approaches 0 from the positive direction (since the quantity is squared). Since $3-x\to 3-1=2$, we have

$$\lim_{x\to 1}\frac{3-x}{(x-1)^2}=\frac{2}{0^+}=\infty.$$

2.7.2. Evaluate $\displaystyle\lim_{x\to 3^+}\frac{x^2+10}{x^2-7x+12}$.

We have

$$\begin{aligned}\lim_{x\to 3^+}\frac{x^2+10}{x^2-7x+12}&=\lim_{x\to 3^+}\frac{x^2+10}{(x-4)(x-3)}\\&=\frac{(3)^2+10}{(3-4)(3^+-3)}\\&=\frac{19}{-1\cdot 0^+}\\&=-\infty.\end{aligned}$$

2.7.3. Determine all vertical asymptotes of the function $f(x)=\dfrac{\sqrt{x+2}}{x^2+4x+4}$.

Vertical asymptotes may occur when

$$x^2+4x+4=(x+2)^2=0,$$

hence when $x=-2$.

To know whether $x=-2$ is indeed a vertical asymptote, we need to calculate the limit of $f(x)$ as $x\to -2^+$. Notice that $f(x)$ is not defined when $x<-2$, so we only consider the limit from the right. We have

$$\begin{aligned}\lim_{x\to -2^+}\frac{\sqrt{x+2}}{x^2+4x+4}&=\lim_{x\to -2^+}\frac{\sqrt{x+2}}{(x+2)^2}\\&=\lim_{x\to -2^+}\frac{1}{(x+2)^{3/2}}\\&=\frac{1}{(0^+)^{3/2}}\\&=\infty\end{aligned}$$

Thus, $x = -2$ is a vertical asymptote of $f(x)$.

2.7.4. Determine all vertical asymptotes of the function $f(x) = \dfrac{\sin(x)}{2x^2 - \pi x}$.

Vertical asymptotes may occur when

$$2x^2 - \pi x = x(2x - \pi) = 0,$$

hence when $x = 0$ or $x = \frac{\pi}{2}$. To know whether $x = 0$ and $x = \frac{\pi}{2}$ are indeed asymptotes, we need to calculate the limits of $f(x)$ as x approaches these values.

As $x \to 0$, we have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x)}{2x^2 - \pi x} &= \lim_{x\to 0} \frac{\sin(x)}{x(2x - \pi)} \\
&= \lim_{x\to 0} \frac{\sin(x)}{x} \frac{1}{2x - \pi} \\
&= 1 \cdot \frac{1}{2(0) - \pi} && \text{(by the Fundamental Trig Limit)} \\
&= \frac{-1}{\pi}.
\end{aligned}$$

Thus, $x = 0$ is *not* a vertical asymptote of $f(x)$.

As $x \to \frac{\pi}{2}$, the denominator, $2x^2 - \pi x = x(2x - \pi)$, approaches 0, while the numerator, $\sin(x)$, approaches $\sin\left(\frac{\pi}{2}\right) = 1$. Thus, $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin(x)}{2x^2 - \pi x}$ will be $\pm\infty$ depending on whether x approaches $\frac{\pi}{2}$ from the left or from the right. This means that $x = \frac{\pi}{2}$ is a vertical asymptote of $f(x)$.

2.7.5. A line $y = mx + b$ is said to be a **slant asymptote** (also known as an **oblique asymptote**) of a function f if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{x\to-\infty} [f(x) - (mx + b)] = 0.$$

Graphically, if $y = mx + b$ is a slant asymptote of f, then the graph of f becomes very close to the line $y = mx + b$ as $x \to \infty$ or $x \to -\infty$.

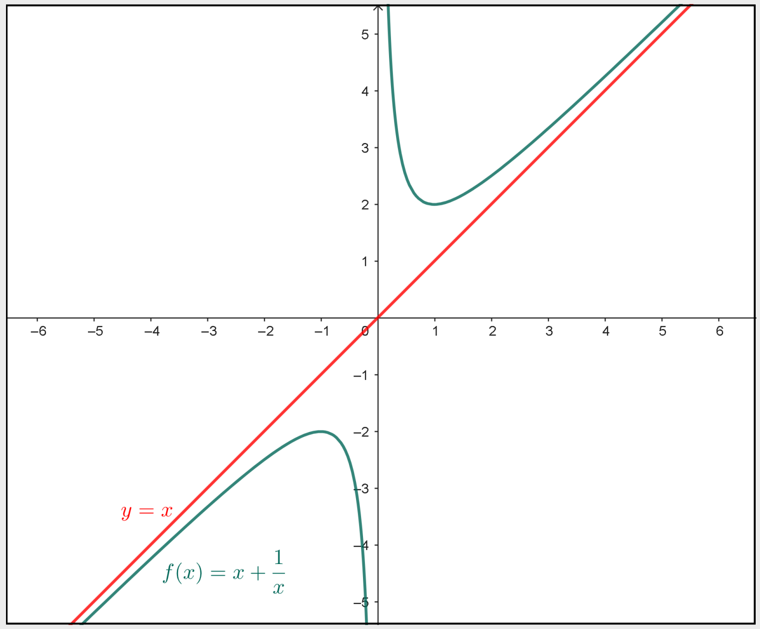
Find the slant asymptote for each function below. Try graphing these functions using software to see its behavior along the slant asymptote.

(a) $f(x) = x + \dfrac{1}{x}$

Notice that

$$\lim_{x\to\pm\infty}[f(x)-x]=\lim_{x\to\pm\infty}\frac{1}{x}=0.$$

Thus, $y=x$ is a slant asymptote of $f(x)$. We can see this in the graph of f, which is shown below.



(b) $f(x)=\dfrac{2x^2+10\sin(x)}{x}$

Observe that

$$f(x)=\frac{2x^2+10\sin(x)}{x}=2x+\frac{10\sin(x)}{x},$$

hence

$$f(x)-2x=\frac{10\sin(x)}{x}.$$

Since $-1\le\sin(x)\le 1$ for all x, we have

$$\frac{-10}{x}\le\frac{10\sin(x)}{x}\le\frac{10}{x}$$

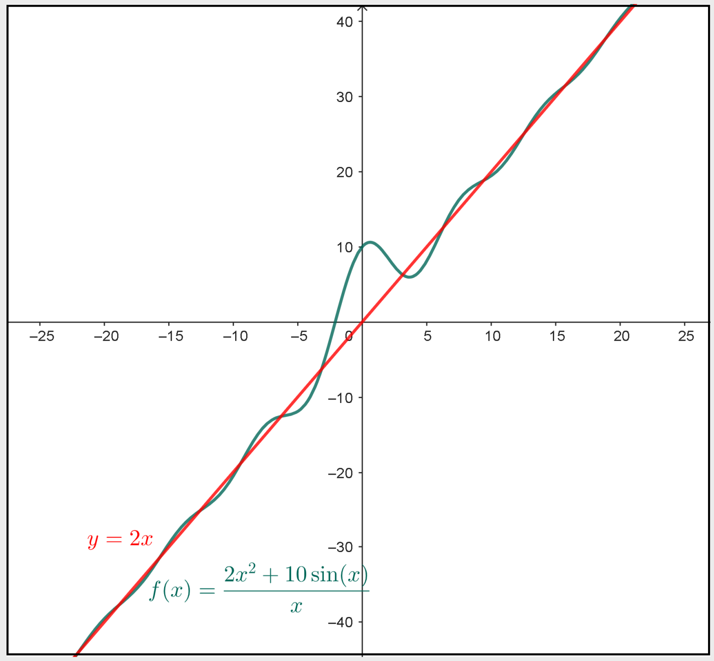
for $x>0$. Furthermore, since

$$\lim_{x\to\infty}\frac{-10}{x}=\lim_{x\to\infty}\frac{10}{x}=0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to\infty}[f(x)-2x]=\lim_{x\to\infty}\frac{10\sin(x)}{x}=0.$$

Thus, $y=2x$ is a slant asymptote for $f(x)$ as $x\to\infty$ (and a similar argument shows that $y=2x$ is also a slant asymptote as $x\to-\infty$). This is seen in the graph of f below.



(c) $f(x) = \dfrac{-x^3 + x^2 - x + 31}{x^2 + 1}$

Hint: Polynomial long division

Following the hint, we will begin with polynomial long division:

$$
\begin{array}{cccc|cccccccc}
 & & & & & & & & - & x & + & 1 \\ \cline{5-12}
x^2 & + & 1 & & - & x^3 & + & x^2 & - & x & + & 31 \\
 & & & & - & x^3 & & & - & x & & \\ \cline{5-12}
 & & & & & & & x^2 & & & + & 31 \\
 & & & & & & & x^2 & & & + & 1 \\ \cline{8-12}
 & & & & & & & & & & & 30
\end{array}
$$

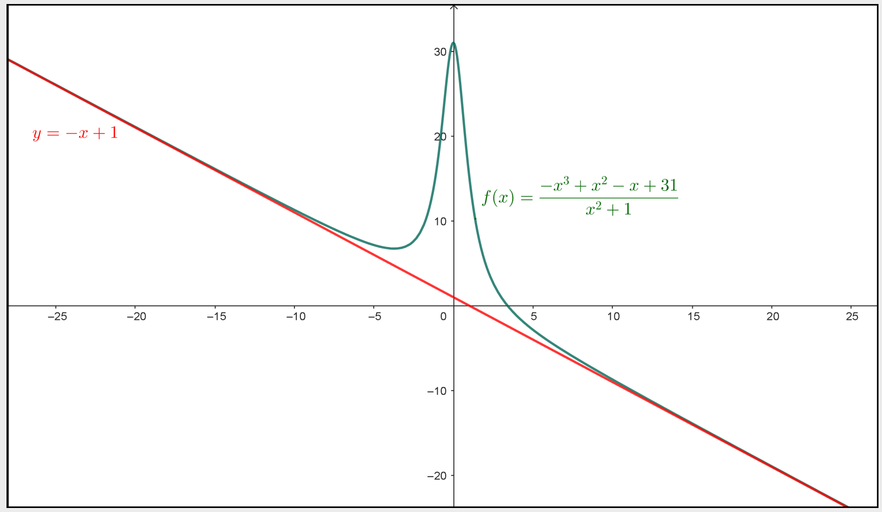
We therefore have

$$f(x) = \frac{-x^3 + x^2 - x + 31}{x^2 + 1} = -x + 1 + \frac{30}{x^2 + 1}.$$

From this, we observe that

$$\lim_{x \to \pm\infty} [f(x) - (-x + 1)] = \lim_{x \to \pm\infty} \frac{30}{x^2 + 1} = 0,$$

hence $y = -x+1$ is a slant asymptote of $f(x)$. This is visible in the graph of f below:


$y = -x + 1$
$f(x) = \frac{-x^3 + x^2 - x + 31}{x^2 + 1}$
30
20
10
−10
−20
−25
−20
−15
−10
−5
0
5
10
15
20
25

2.8 Continuity

Section 2.8 Problems

2.8.1. Evaluate the following limits, making appropriate reference to the continuity of familiar functions.

(a) $\lim_{x\to 0} \sin\left(\frac{\sin(x)}{x}\right)$

Recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Thus, since $\sin(x)$ is continuous at $x = 1$, we have

$$\lim_{x\to 0} \sin\left(\frac{\sin(x)}{x}\right) = \sin\left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) = \sin(1).$$

(b) $\lim_{x\to 1} e^{\frac{x-1}{\sqrt{x}-1}}$

We first note that

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} &= \lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \\ &= \lim_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{x-1} \\ &= \lim_{x\to 1} \left(\sqrt{x}+1\right) \\ &= \sqrt{1}+1 \\ &= 2.\end{aligned}$$

Since e^x is continuous at $x = 2$, it follows that

$$\lim_{x\to 1} e^{\frac{x-1}{\sqrt{x}-1}} = e^{\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}} = e^2.$$

(c) $\lim_{x\to\infty} (\ln(3x+1) - \ln(x+1))$

We first apply properties of logarithms to get

$$\lim_{x\to\infty} (\ln(3x+1) - \ln(x+1)) = \lim_{x\to\infty} \ln\left(\frac{3x+1}{x+1}\right).$$

Since

$$\lim_{x\to\infty} \frac{3x+1}{x+1} = \lim_{x\to\infty} \frac{x\left(3+\frac{1}{x}\right)}{x\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{1+\frac{1}{x}} = \frac{3+0}{1+0} = 3$$

and $\ln(x)$ is continuous at $x = 3$, it follows that

$$\begin{aligned}\lim_{x\to\infty}(\ln(3x+1)-\ln(x+1)) &= \lim_{x\to\infty}\ln\left(\frac{3x+1}{x+1}\right)\\ &= \ln\left(\lim_{x\to\infty}\frac{3x+1}{x+1}\right)\\ &= \ln(3).\end{aligned}$$

2.8.2. Let $a > 0$, $a \neq 1$.

(a) Use the fact that $\ln(x)$ is continuous to prove that $\log_a(x)$ is continuous on $(0, \infty)$,

Note that $\log_a(x) = \frac{\ln(x)}{\ln(a)} = c\ln(x)$ where $c = \frac{1}{\ln(a)}$. Since $\ln(x)$ is continuous on $(0, \infty)$ and $\log_a(x)$ is merely a multiple of $\ln(x)$, $\log_a(x)$ must also be continuous on $(0, \infty)$.

(b) Prove that a^x is continuous on $(-\infty, \infty)$.

From Theorem 2.8.10, if a function $f(x)$ is continuous at $x = a$ with $f(a) = b$, then $f^{-1}(y)$ is continuous at $y = b$. Thus, since we argued in (a) that $f(x) = \log_a(x)$ is continuous on $(0, \infty)$, $f^{-1}(x) = a^x$ is continuous on $(-\infty, \infty)$.

2.8.3. Use the $\varepsilon - \delta$ definition of continuity to prove that $f(x) = 2x^2 + 9$ is continuous at $x = 2$.

Let $f(x) = 2x^2 + 9$. To prove that f is continuous at $x = 2$, we must show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x-2| < \delta \quad \implies \quad |f(x) - f(2)| < \varepsilon.$$

First, compute $f(2)$:

$$f(2) = 2(2)^2 + 9 = 8 + 9 = 17.$$

So we want:

$$|f(x) - 17| = |2x^2 + 9 - 17| = |2x^2 - 8| = 2|x^2 - 4| = 2|x+2||x-2| < \varepsilon.$$

Without loss of generality, we can assume that $|x - 2| < 1$. As such,

$$|x-2| < 1 \quad \iff \quad 2-1 < x < 2+1 \quad \iff \quad 1 < x < 3.$$

For $1 < x < 3$, we have $6 < |2(x + 2)| < 10$, hence

$$|2(x-2)(x+2)| < 10|x-2|.$$

As a result,

$$10|x-2|<\epsilon \quad \Longleftrightarrow \quad |x-2|<\frac{\epsilon}{10}.$$

We are now ready to provide our proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \frac{\epsilon}{10}\}$. Then from the above estimates, we see that if $|x-2|<\delta$, then

$$|(2x^2+9)-17| = |2(x-2)(x+2)| < 10|x-2| < 10 \cdot \frac{\epsilon}{10} = \epsilon.$$

It follows that $\lim\limits_{x\to 2}(2x^2+9) = 17$, hence $f(x) = 2x^2+9$ is continuous at $x=2$.

2.8.4. Let f be a function defined as

$$f(x) = \begin{cases} \dfrac{x^2-4}{x^2+x-6}\cos(x^2) & \text{if } x \neq -3, 2, \\ 0 & \text{if } x = -3, 2. \end{cases}$$

Find the intervals where f is continuous. Justify your answer.

By the Arithmetic Rules for Continuous Functions and Continuity of Composition, the function

$$\frac{x^2-4}{x^2+x-6}\cos(x^2)$$

is continuous everywhere except when $x^2+x-6=0$. Note that

$$x^2+x-6 = (x+3)(x-2) = 0 \quad \Longleftrightarrow \quad x=-3 \text{ or } x=2.$$

Thus, f is continuous everywhere except possibly at $x=-3, 2$.

To determine whether f is continuous at $x=-3$ or $x=2$, we will compute the limits as x approaches these values.

First, note that as x approaches -3, we have $x^2-4 \to 5$, $x^2+x-6 \to 0$ and $\cos(x^2) \to \cos 9$. Thus, $\lim\limits_{x\to -3} \dfrac{x^2-4}{x^2+x-6}\cos(x^2)$ does not exist, hence we conclude that f is not continuous at $x=-3$.

Next, let's examine the limit as x approaches 2. We have

$$\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}\cos(x^2) = \lim_{x\to 2}\frac{x+2}{x+3}\cos(x^2) = \frac{4}{5}\cos(4).$$

Even though this limit exists, it is not equal to $f(2)=0$, and therefore f is not continuous at $x=2$.

Combining the above calculations, we see that f is continuous on

$$(-\infty,-3)\cup(-3,2)\cup(2,\infty).$$

2.8.5. Find all values of a and b such that

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < -1, \\ ax - 2b & \text{if } -1 \leq x < 1, \\ x^2 - bx + a & \text{if } x \geq 1, \end{cases}$$

is continuous for all $x \in \mathbb{R}$. Justify your answer.

Clearly f is continuous if $x \neq \pm 1$ because polynomials are continuous, so we only need to find out what happens at $x = \pm 1$.

Continuity at $x = -1$: We require that

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1). \tag{$*$}$$

Computing these one-sided limits, we have

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x^2 - 1) = 0,$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (ax - 2b) = -a - 2b.$$

Finding a and b that satisfy $(*)$ amounts to solving the equations

$$0 = -a - 2b = -a - 2b.$$

These expression will be equal whenever $-a - 2b = 0$, hence when $a = -2b$.

Continuity at $x = 1$: We require that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1). \tag{$**$}$$

Computing these one-sided limits, we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax - 2b) = a - 2b,$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - bx + a) = 1 - b + a$$

Satisfying $(**)$ amounts to finding a and b such that

$$a - 2b = 1 - b - a = a - 2b.$$

From the equation $a - 2b = 1 - b + a$, we find that $b = -1$. Since we also require that $a = -2b$, it follows that $a = 2$.

Thus, $a = 2$ and $b = -1$ are the values that make $f(x)$ continuous on $\mathbb{R}$.

2.8.6. Show that if a function is continuous at $x = 0$ and satisfies:

(a) $f(x + y) = f(x) + f(y)$ then it is continuous everywhere;

We need to show that for any $a \in \mathbb{R}$ we have that

$$\lim_{x \to a} f(x) = f(a).$$

We can do this by using the fact that this is equivalent to showing

$$\lim_{h \to 0} f(a+h) = f(a).$$

So let $a \in \mathbb{R}$ be given. We proceed by computing $\lim\limits_{h \to 0} f(a+h)$ directly:

$$\begin{aligned}
\lim_{h \to 0} f(a+h) &= \lim_{h \to 0} f(a) + f(h) \\
&= \lim_{h \to 0} f(a) + \lim_{h \to 0} f(h) \\
&= f(a) + \lim_{h \to 0} f(h) && \text{(since } f(a) \text{ does not depend on } h\text{)} \\
&= f(a) + f(0) && \text{(since we know } f \text{ is continuous at } 0\text{)} \\
&= f(a+0) && \text{(since } f(x) + f(y) = f(x+y)\text{)} \\
&= f(a).
\end{aligned}$$

(b) $f(x+y) = f(x)f(y)$ then it is continuous everywhere.

Again we are trying to show that $\lim\limits_{x \to a} f(x) = f(a) = \lim\limits_{h \to 0} f(a+h)$.

$$\begin{aligned}
\lim_{h \to 0} f(a+h) &= \lim_{h \to 0} f(a)f(h) \\
&= \left(\lim_{h \to 0} f(a)\right)\left(\lim_{h \to 0} f(h)\right) \\
&= f(a) \cdot \lim_{h \to 0} f(h) && \text{(since } f(a) \text{ does not depend on } h\text{)} \\
&= f(a) \cdot f(0) && \text{(since } f \text{ is continuous at } 0\text{)} \\
&= f(a+0) && \text{(since } f(x) \cdot f(y) = f(x+y)\text{)} \\
&= f(a).
\end{aligned}$$

2.9 Types of Discontinuities

Section 2.9 Problems

2.9.1. Consider the function

$$f(x) = \frac{x^2 - x - 2}{x^2 - 5x + 6}.$$

Identify any points at which f is discontinuous, then classify its discontinuities.

The function is not defined at points where $x^2 - 5x + 6 = (x-2)(x-3) = 0$ – that is, at $x = 2$ and $x = 3$. Thus, being undefined at $x = 2$ and $x = 3$, f is discontinuous at $x = 2$ and $x = 3$.

To classify the discontinuity at $x = 2$, we need to examine the behavior of f as x approaches 2. We have

$$\begin{aligned}
\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 5x + 6} \\
&= \lim_{x\to 2} \frac{(x-2)(x+1)}{(x-2)(x-3)} \\
&= \lim_{x\to 2} \frac{x+1}{x-3} \\
&= \frac{2+1}{2-3} \\
&= -3.
\end{aligned}$$

Since $\lim_{x\to 2} f(x)$ exists but $f(2)$ is undefined, this is a removable discontinuity.

To classify the discontinuity at $x = 3$, we again perform a limit calculation. Notice that as $x \to 3^+$, the denominator, $x^2 - 5x + 6 = (x-2)(x-3)$ approaches 0 from the positive direction, while the numerator, $x^2 - x - 2$, approaches $3^2 - 3 - 2 = 4$. Thus,

$$\lim_{x\to 3^+} f(x) = \infty.$$

Thus, there is an infinite discontinuity at $x = 3$.

2.9.2. Consider the function

$$f(x) = \frac{|x|}{x^2}.$$

Is f continuous at $x = 0$? If not, classify the type of discontinuity.

The function is not continuous, as $f(0)$ is undefined. To determine the type of discontinuity, we compute a limit. Examining the behavior of $f(x)$ as $x \to 0^+$,we find that

$$\lim_{x\to 0^+} \frac{|x|}{x^2} = \lim_{x\to 0^+} \frac{x}{x^2} \qquad (|x| = x \text{ since } x > 0)$$

$$= \lim_{x \to 0^+} \frac{1}{x}$$
$$= \infty.$$

Thus, there is an infinite discontinuity at $x = 0$.

2.9.3. Consider the function
$$f(x) = \frac{x|1-x|}{x-1}.$$
Is f continuous at $x = 1$? If not, classify the type of discontinuity.

The function is not continuous at $x = 1$, since $f(1)$ is not defined. Computing the sided limits as $x \to 1$, we find that

$$\begin{aligned}
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} \frac{x|1-x|}{x-1} \\
&= \lim_{x \to 1^-} \frac{x(1-x)}{x-1} && \text{(as } 1-x > 0 \text{ when } x < 1\text{)} \\
&= \lim_{x \to 1^-} -x \\
&= -1,
\end{aligned}$$

while

$$\begin{aligned}
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} \frac{x|1-x|}{x-1} \\
&= \lim_{x \to 1^+} \frac{x(-(1-x))}{x-1} && \text{(as } 1-x < 0 \text{ when } x > 1\text{)} \\
&= \lim_{x \to 1^+} x \\
&= 1.
\end{aligned}$$

Since $\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$ exist but these one-sided limits are not equal, there is a jump discontinuity at $x = 1$.

2.9.4. Consider the function
$$f(x) = \begin{cases} \sin^2(x) \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$
Is f continuous at $x = 0$? If not, classify the type of discontinuity.

For all $x \neq 0$,
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Hence, multiplying through by $\sin^2(x)$ (which is non-negative for all x), we find that for $x \neq 0$,

$$-\sin^2(x) \leq \sin^2(x)\sin\left(\frac{1}{x}\right) \leq \sin^2(x).$$

Since $\lim_{x\to 0} -\sin^2(x) = 0$ and $\lim_{x\to 0}\sin^2(x) = 0$, it follows from the Squeeze Theorem that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\sin^2(x)\sin\left(\frac{1}{x}\right) = 0.$$

While $\lim_{x\to 0} f(x)$ exists, it is not equal to $f(0) = 1$. Thus, f has a removable discontinuity at $x = 0$.

2.9.5. Consider the function

$$f(x) = \begin{cases} \cos(x)\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous at $x = 0$? If not, classify the type of discontinuity.

Notice that $f(x)$ is bounded around $x = 0$ (in fact, everywhere), since

$$\left|\cos(x)\sin\left(\frac{1}{x}\right)\right| = |\cos(x)|\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \cdot 1 = 1.$$

Furthermore, since $\sin\left(\frac{1}{x}\right)$ oscillates infinitely often around $x = 0$ and $\cos(x) \approx \cos(0) = 1$ around $x = 0$, the product $\cos(x)\sin\left(\frac{1}{x}\right)$ is approximately equal to $1 \cdot \sin\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right)$ and hence will also oscillate infinitely often around $x = 0$. Thus, f has an oscillatory discontinuity at $x = 0$.

2.9.6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a jump discontinuity at $x = 0$ with

$$\lim_{x\to 0^-} f(x) = -1, \quad \lim_{x\to 0^+} f(x) = 1, \quad \text{and} \quad f(0) = 0.$$

Determine whether the function $\cos(f(x))$ is continuous at $x = 0$, and if not, classify the discontinuity.

Since $\cos(x)$ is continuous at $x = 0$, we have

$$\lim_{x\to 0^-}\cos(f(x)) = \cos\left(\lim_{x\to 0^-} f(x)\right) = \cos(-1),$$

while

$$\lim_{x\to 0^+}\cos(f(x)) = \cos\left(\lim_{x\to 0^+} f(x)\right) = \cos(1).$$

Observe that $\cos(-1) = \cos(1)$ since $\cos(x)$ is an even function, hence

$$\lim_{x\to 0^-}\cos(f(x)) = \lim_{x\to 0^+}\cos(f(x)) = \cos(1).$$

Since the value of this limit is different from $\cos(f(0)) = \cos(0) = 1$, we conclude that $\cos(f(x))$ has a removable discontinuity at $x = 0$.

2.10 Intermediate Value Theorem

Section 2.10 Problems

2.10.1. Consider the function $f(x) = x^3 - 4x - 1$.

(a) Prove that f has at least one root in the interval $(0, 4)$.

Since f is a polynomial, it is continuous everywhere. Furthermore, since $f(0) = -1 < 0$ and $f(4) = 48 > 0$, by the Intermediate Value Theorem, there exists $x \in [0, 4]$ such that $f(x) = 0$. That is, f has a root in $(0, 4)$.

(b) Find an interval of length $\frac{1}{2}$ that contains a root of f.

From (a), we know that there is a root in $(0, 4)$. Checking the value of f at the midpoint of the interval, $x = 2$, we find that $f(2) = -1 < 0$. Hence, since $f(2) < 0$ and $f(4) > 0$, by the Intermediate Value Theorem, f must have a root in $(2, 4)$.

Repeating the above argument in $[2, 4]$, we check the value of f at the midpoint of the interval, $x = 3$. Since $f(2) < 0$ and $f(3) = 14 > 0$, by the Intermediate Value Theorem, f has a root in $(2, 3)$.

Lastly, we check the value of f at the midpoint of $[2, 3]$, $x = \frac{5}{2}$. Since $f(2) < 0$ and $f\left(\frac{5}{2}\right) = \frac{37}{8} > 0$, by the Intermediate Value Theorem, f has a root in $\left(2, \frac{5}{2}\right)$. Since $\left(2, \frac{5}{2}\right)$ has length $\frac{1}{2}$, this interval is sufficient.

2.10.2. Prove that $f(x) = x^3 - 12x + 10$ has at least two roots in $(0, 3)$.

Since f is a polynomial, it is continuous everywhere. Moreover, computing the values of f at a few points in $(0, 3)$, we find that

$$f(0) = 10, \quad f(1) = -1, \quad \text{and} \quad f(3) = 1.$$

Since $f(0) > 0$ and $f(1) < 0$, by the Intermediate Value Theorem, f must have a root in $(0, 1)$. Similarly, since $f(1) < 0$ and $f(3) > 0$, by the Intermediate Value Theorem, f must have a root in $(1, 3)$.

Since $(0, 1)$ and $(1, 3)$ do not overlap, the two roots we have discovered must be distinct. Hence, f has at least two roots in $(0, 3)$, as required.

2.10.3. Prove that there exists a real number c such that $2^c = c^5$.

Note that there exists c such that $2^c = c^5$ is satisfied if and only if there exists c such that $2^c - c^5 = 0$. Thus, it suffices to show that the function $f(x) = 2^x - x^5$ has a root.

Observe that $f(x) = 2^x - x^5$ is continuous everywhere, as it is a difference of two continuous functions. Moreover, choosing inputs somewhat arbitrarily, we find that

$$f(0) = 2^0 - 0^5 = 1 \quad \text{and} \quad f(2) = 2^2 - 2^5 = -28.$$

Since $f(0) > 0$ and $f(2) < 0$, by the Intermediate Value Theorem, f has a root in $(0, 2)$. Thus, there exists $c \in (0, 2)$ such that $2^c = c^5$.

2.10.4. Prove that the equation

$$x \sin(x) = 1$$

has infinitely many solutions.

Observe that the function $f(x) = x \sin(x)$ is continuous everywhere, as it is a product of continuous functions.

Consider the mutually disjoint intervals

$$\left(0, \frac{\pi}{2}\right), \left(2\pi, 2\pi + \frac{\pi}{2}\right), \left(4\pi, 4\pi + \frac{\pi}{2}\right), \ldots,$$

which can be written as $(a_n, b_n) = \left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ for $n \geq 0$. Notice that

$$f(a_n) = 2n\pi \sin(2n\pi) = 0 < 1,$$

while

$$f(b_n) = \left(2n\pi + \frac{\pi}{2}\right) \sin\left(2n\pi + \frac{\pi}{2}\right) = 2n\pi + \frac{\pi}{2} > 1.$$

Thus by the Intermediate Value Theorem, within each interval (a_n, b_n), there exists c_n with $f(c_n) = 1$. That is, c_n is a solution to the equation $x \sin(x) = 1$. Since we have exhibited infinitely many disjoint intervals, each of which contains a solution to the equation $x \sin(x) = 1$, this equation must have infinitely many solutions.

2.10.5. Consider the function $f(x) = \dfrac{x^2 + x + 3}{x^3 + 1}$.

(a) Verify that $f(-2) < 0$ and $f(1) > 0$.

We compute $f(-2) = -\frac{5}{7}$ and $f(1) = \frac{5}{2}$. Indeed, $f(-2) < 0$ and $f(1) > 0$.

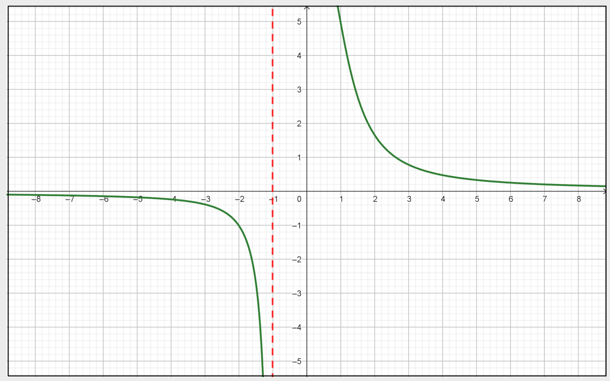
(b) Show that there are no real solutions to the equation $f(x) = 0$.

Note that $f(x) = \frac{x^2+x+3}{x^3+1} = 0$ only if $x^2 + x + 3 = 0$. However, since $x^2 + x + 3$ is a quadratic with discriminant $1^2 - 4(1)(3) = -11 < 0$, the equation $x^2 + x + 3 = 0$ has no real solutions. Thus, there are no real solutions to the equation $f(x) = 0$.

(c) Do your findings from (a) and (b) contradict the Intermediate Value Theorem? Why or why not?

Observe that f is not continuous on the interval $[-2, 1]$, as f is not defined at $x = -1$. Thus, f does not satisfy the assumptions of the Intermediate Value Theorem and hence it does not follow from (a) that f has a real root. Therefore, there is no contradiction.

(The graph of f below may shed light on how positive and negative outputs are possible without the existence of a root.)



Chapter 3

Derivatives

3.1 Average and Instantaneous Velocity

Section 3.1 Problems

3.1.1. For the displacement function, $s(t) = -9.8t^2 + 30t + 12$, determine the instantaneous velocity at $t = 1$.

$$
\begin{aligned}
v(t_0) &= \lim_{h\to 0} \frac{-9.8(1+h)^2 + 30(1+h) + 12 - (-9.8(1)^2 + 30(1) + 12)}{h} \\
&= \lim_{h\to 0} \frac{-9.8 - 19.6h - 9.8h^2 + 30 + 30h + 12 + 9.8 - 30 - 12}{h} \\
&= \lim_{h\to 0} \frac{10.4h - 9.8h^2}{h} \\
&= \lim_{h\to 0} 10.4 - 9.8h \\
&= 10.4
\end{aligned}
$$

3.1.2. The height of a tree is measured every year in the autumn. The data is shown below.

Year	0	1	2	3	4	5	6	7	8	9	10
Height (m)	9.91	11.07	12.23	13.28	15.08	16.53	18.40	20.16	22.30	24.62	27.51

(a) Estimate the rate of change over each year.

The average rate of change over each year is:

Year	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10
Change (m/yr)	1.16	1.16	1.05	1.80	1.45	1.87	1.76	2.14	2.32	2.89

(b) Tabulate the estimated rate of change divided by the average height of the tree over each year.

The average rate of change over each year divided by the average height is:

Year	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10
Rel. change (/yr)	0.11	0.10	0.08	0.13	0.09	0.11	0.09	0.10	0.10	0.11

(c) Propose an equation linking the instantaneous rate of change $\frac{dh}{dt}$ to the height of the tree $h(t)$.

The relative increases are close to one another (the average is 0.10 /year). If we assume that the average rate of change is similar to the instantaneous rate of change, then the data suggests that the derivative is related to the height as,

$$\frac{1}{h(t)}\frac{dh}{dt} = 0.1, \qquad \text{or,} \qquad \frac{dh}{dt} = 0.1h.$$

Equations involving derivatives are called *differential equations*, and these form the basic language of science and engineering.

3.1.3. Suppose the speed limit on a road is 50 km/h – when you approach a set of traffic lights, how long should the yellow light stay on to give the motorists sufficient time to stop? What about if the speed limit is 70 km/h? In this question, you will analyze a simple model for the the physics of traffic stops.

(a) When the light turns yellow, a motorist has two choices: either slam on the breaks and stop (call the time it takes to stop T_{stop}), or continue through the intersection (call the time it takes to clear the intersection T_{run}). There is a reaction time associated with the decision (call this decision time T_{reac}). The length of time for the yellow light (T_{yellow}) should be at least as long as,

$$T_{\text{yellow}} = T_{\text{reac}} + \max(T_{\text{stop}}, T_{\text{run}}), \tag{3.1}$$

to ensure that the intersection is clear when the opposing light turns green. Provide an explanation for this equation.

The driver had two choices: to stop or to run the light. The time of the yellow light should be at least as long as the maximum of these two choices:

$$T_{\text{yellow}} \geq \max\Big(T_{\text{reac}} + T_{\text{stop}}, T_{\text{reac}} + T_{\text{run}}\Big).$$

Assuming that the reaction time is the same in both cases,

$$T_{\text{yellow}} \geq T_{\text{reac}} + \max\Big(T_{\text{stop}}, T_{\text{run}}\Big).$$

(b) Assuming a typical car length of $L = 5$ m, and a typical intersection width to be $I = 10$ m, a car travelling at v_0 km/h will take $T_{\text{run}} = (I + L)/v_0$ to clear the intersection. On the other hand, assuming uniform braking, a car will take $T_{\text{stop}} = v_0/(2fg)$ to come to a stop, where $g = 9.81$ m/s^2 is the acceleration of gravity and f is the coefficient of friction between the tires and pavement. For a dry, flat stretch of road, a typical value of the friction coefficient is $f = 0.2$.

When the speed limit v_0 is very slow, $T_{\text{run}} > T_{\text{stop}}$; whereas for faster speed limits v_0 the opposite is true, $T_{\text{run}} < T_{\text{stop}}$. Using the typical values above, at what speed limit v_0^{crit} does $T_{\text{run}} = T_{\text{stop}}$?

Setting the two times equal to one another, and isolating v_0,

$$(I+L)/v_0^{\text{crit}} = v_0^{\text{crit}}/(2fg) \qquad \Longrightarrow \qquad v_0^{\text{crit}} = \sqrt{(I+L)2fg}$$

Using the typical values given, $v_0^{\text{crit}} \approx 7.7$ m/s, about 28 km/h. Typically, a stop sign would be used for speed limits below 40 km/h, so it is safe to assume that if a traffic light is being used to control an intersection, then the speed limit is faster than 28 km/h, and that $T_{\text{run}} < T_{\text{stop}}$.

(c) Most traffic lights are on roads where the speed limit exceeds the critical speed found in part b, $v_0 > v_0^{\text{crit}}$, so that $\max(T_{\text{stop}}, T_{\text{run}}) = T_{\text{stop}}$. Ontario cities can lawfully time the yellow light at $T_{\text{yellow}} = 3.7$ seconds on flat roads with a speed limit of 60 km/h. Is this consistent with Eq. 3.1 above? Assume that $T_{\text{reac}} = 1$ second.

Using the expression above,

$$T_{\text{yellow}} = 1\text{ s} + \frac{60\ \frac{\text{km}}{\text{h}} \times \frac{1\text{ h}}{3600\text{ s}} \times \frac{1000\text{ m}}{1\text{ km}}}{2 \times 0.2 \times 9.81\ \frac{\text{m}}{\text{s}^2}} = 5.25\text{ s}.$$

Even if the reaction time is omitted, it is unlikely that a driver could stop in time without skidding into the intersection.

(d) For roads on a hill, the situation is more complicated: the effective weight of the car is reduced (leading to a reduction in the friction between the tires and road), and gravity is working against braking on a down slope (or acting with braking on an up-slope). The US Institute of Transport Engineers (USITE) recommends the following estimator for T_{yellow},

$$T_{\text{yellow}} = T_{\text{reac}} + \frac{v_0}{2g(f+G)},$$

where G is the slope of the road in the direction of approach. Calculate the recommended stopping times for three different speed limits $v_0 = 50$ km/h, 70 km/h and 100 km/h under each of three different road conditions: $G = 0$ (flat road), $G = 6/100$ (recommended maximum slope for a highway, approaching uphill), and $G = -6/100$ (recommended maximum slope for a highway, approaching downhill). What happens if you try to compute the stopping time traveling 50 km/h downhill along North America's steepest road ($G = -37/100$; Canton Ave. in Pittsburgh, PA)?

For $G = -37/100$, the formula returns $T_{\text{yellow}} = -3.16$ seconds, which is not possible. The simple formula is not designed to accommodate such a steep decline. There are no street lights on Canton Ave. and traffic presumably moves very slowly down that road. For the other speeds and grades, the formula works fine, and the results are tabulated below.

Speed limit, km/h v_0	**Approach grade, m/m** G	**Illumination time, s** T_{yellow}
50	0	4.54
50	6/100	3.72
50	−6/100	6.06
70	0	5.96
70	6/100	4.81
70	−6/100	8.08
100	0	8.08
100	6/100	6.45
100	−6/100	11.11

3.2 Definition of the Derivative

Section 3.2 Problems

3.2.1. Use the definition of the derivative $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to determine the derivative of the following functions.

$$\text{i) } f(x) = x^2, \qquad \text{ii) } f(x) = \sqrt{x}, \qquad \text{iii) } f(x) = \ln x$$

Hint: For iii), recall that $\lim_{\alpha\to 0}(1+\alpha)^{1/\alpha} = e$, and try a change of variables $\alpha = \frac{h}{x}$.

i)

$$\lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0} 2x + \lim_{h\to 0} h = 2x$$

ii)

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

iii) Following the hint,

$$\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}$$
$$= \lim_{\alpha\to 0}\frac{\ln\left(\frac{x+\alpha x}{x}\right)}{\alpha x} = \frac{1}{x}\lim_{\alpha\to 0}\tfrac{1}{\alpha}\ln(1+\alpha) = \frac{1}{x}\lim_{\alpha\to 0}\ln(1+\alpha)^{1/\alpha} = \frac{1}{x}\ln e = \frac{1}{x}.$$

3.2.2. For each of the following, determine the value of $f'(a)$ using the definition of the derivative.

(a) $f(x) = x^4$, $a = 2$
(b) $f(x) = \sqrt{x-2}$, $a = 9$
(c) $f(x) = \dfrac{3}{x^2+7}$, $a = -3$
(d) (**Challenge Problem!**) $f(x) = \dfrac{1}{\sqrt{x-3}+\sqrt{x+2}}$, $a = 7$

(a)

$$\begin{aligned} f'(2) &= \lim_{h\to 0}\frac{(2+h)^4 - 2^4}{h} \\ &= \lim_{h\to 0}\frac{16+32h+24h^2+8h^3+h^4-16}{h} \\ &= \lim_{h\to 0} 32+24h+8h^2+h^3 \end{aligned}$$

$$= \quad 32$$

(b)

$$\begin{aligned}
f'(9) &= \lim_{h\to 0} \frac{\sqrt{(9+h)-2}-\sqrt{9-2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{7+h}-\sqrt{7}}{h} \cdot \frac{\sqrt{7+h}+\sqrt{7}}{\sqrt{7+h}+\sqrt{7}} \\
&= \lim_{h\to 0} \frac{7+h-7}{h\left(\sqrt{7+h}+\sqrt{7}\right)} \\
&= \lim_{h\to 0} \frac{1}{\left(\sqrt{7+h}+\sqrt{7}\right)} \\
&= \frac{1}{2\sqrt{7}}
\end{aligned}$$

(c)

$$\begin{aligned}
f'(-3) &= \lim_{h\to 0} \frac{\frac{3}{(-3+h)^2+7} - \frac{3}{(-3)^2+7}}{h} \\
&= 3\lim_{h\to 0} \left[\frac{1}{(-3+h)^2+7} - \frac{1}{16}\right] \cdot \frac{1}{h} \\
&= \frac{3}{16}\lim_{h\to 0} \left[\frac{16-\left[(-3+h)^2+7\right]}{(-3+h)^2+7}\right] \cdot \frac{1}{h} \\
&= \frac{3}{16}\lim_{h\to 0} \left[\frac{16-(9-6h+h^2+7)}{(-3+h)^2+7}\right] \cdot \frac{1}{h} \\
&= \frac{3}{16}\lim_{h\to 0} \left[\frac{6h-h^2}{(-3+h)^2+7}\right] \cdot \frac{1}{h} \\
&= \frac{3}{16}\lim_{h\to 0} \frac{6-h}{(-3+h)^2+7} \\
&= \frac{9}{128}
\end{aligned}$$

(d)

$$\begin{aligned}
f'(7) &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{(7+h)-3}+\sqrt{(7+h)+2}} - \frac{1}{\sqrt{7-3}+\sqrt{7+2}}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}+\sqrt{9+h}} - \frac{1}{5}}{h} \cdot \frac{\frac{1}{\sqrt{4+h}+\sqrt{9+h}} + \frac{1}{5}}{\frac{1}{\sqrt{4+h}+\sqrt{9+h}} + \frac{1}{5}} \\
&= \lim_{h\to 0} \frac{\frac{1}{(4+h)+2\sqrt{4+h}\sqrt{9+h}+(9+h)} - \frac{1}{25}}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}} + \frac{1}{5}\right)} \\
&= \lim_{h\to 0} \frac{\frac{1}{(13+2h)+2\sqrt{4+h}\sqrt{9+h}} \cdot \frac{(13+2h)-2\sqrt{4+h}\sqrt{9+h}}{(13+2h)-2\sqrt{4+h}\sqrt{9+h}} - \frac{1}{25}}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}} + \frac{1}{5}\right)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\frac{(13+2h)-2\sqrt{4+h}\sqrt{9+h}}{169+52h+4h^2-4(4+h)(9+h)} - \frac{1}{25}}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}}+\frac{1}{5}\right)} \\
&= \lim_{h\to 0} \frac{\frac{(13+2h)-2\sqrt{4+h}\sqrt{9+h}}{169+52h+4h^2-144-52h-4h^2} - \frac{1}{25}}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}}+\frac{1}{5}\right)} \\
&= \lim_{h\to 0} \frac{\frac{(13+2h)-2\sqrt{4+h}\sqrt{9+h}}{25} - \frac{1}{25}}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}}+\frac{1}{5}\right)} \\
&= \lim_{h\to 0} \frac{(12+2h)-2\sqrt{4+h}\sqrt{9+h}}{25h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}}+\frac{1}{5}\right)} \\
&= \frac{2}{25}\lim_{h\to 0} \frac{(6+h)-\sqrt{4+h}\sqrt{9+h}}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}}+\frac{1}{5}\right)} \cdot \frac{(6+h)+\sqrt{4+h}\sqrt{9+h}}{(6+h)+\sqrt{4+h}\sqrt{9+h}} \\
&= \frac{2}{25}\lim_{h\to 0} \frac{(6+h)^2-(4+h)(9+h)}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}}+\frac{1}{5}\right)\left(6+h+\sqrt{4+h}\sqrt{9+h}\right)} \\
&= \frac{2}{25}\lim_{h\to 0} \frac{-h}{h\left(\frac{1}{\sqrt{4+h}+\sqrt{9+h}}+\frac{1}{5}\right)\left(6+h+\sqrt{4+h}\sqrt{9+h}\right)} \\
&= \frac{2}{25}\cdot \frac{-1}{\left(\frac{1}{5}+\frac{1}{5}\right)(6+6)} \\
&= -\frac{1}{60}
\end{aligned}$$

3.2.3. For $f(x) = \dfrac{x+1}{x-1}$, find $f'(x)$ using the limit definition.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{(x+h+1)(x-1)-(x+1)(x+h-1)}{(x+h-1)(x-1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{x^2+xh+x-x-h-1-x^2-xh+x-x-h+1}{(x+h-1)(x-1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{-2h}{(x+h-1)(x-1)} \cdot \frac{1}{h} \\
&= -\frac{2}{(x-1)^2}
\end{aligned}$$

3.2.4. If f is differentiable at x and $f(x) \neq 0$, use the definition of the derivative to prove that
$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{[f(x)]^2}.$$

From the definition,

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x+h)f(x)}$$

$$= \lim_{h\to 0} \left(\frac{-1}{f(x+h)f(x)}\right)\left(\frac{f(x+h) - f(x)}{h}\right) = \frac{-1}{[f(x)]^2}\, f'(x) \ \checkmark$$

3.2.5. If f is differentiable and $f(x) > 0$, use the definition of the derivative to prove that

$$\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}.$$

From the definition,

$$\frac{d}{dx}\left(\sqrt{f(x)}\right) = \lim_{h\to 0} \frac{\sqrt{f(x+h)} - \sqrt{f(x)}}{h}$$

To evaluate this limit, we need to rationalize the numerator by multiplying top-and-bottom by $\sqrt{f(x+h)} + \sqrt{f(x)}$,

$$\lim_{h\to 0} \frac{\sqrt{f(x+h)} - \sqrt{f(x)}}{h} \times \frac{\sqrt{f(x+h)} + \sqrt{f(x)}}{\sqrt{f(x+h)} + \sqrt{f(x)}} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h(\sqrt{f(x+h)} + \sqrt{f(x)})}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h}\right)\left(\frac{1}{\sqrt{f(x+h)} + \sqrt{f(x)}}\right) = f'(x)\, \frac{1}{2\sqrt{f(x)}} \ \checkmark$$

3.3 Derivative of Some Common Functions

Section 3.3 Problems

3.3.1. Show that $f(x) = |\sin x|$ is continuous but not differentiable for any $x \in \{k\pi, k \in \mathbb{Z}\}$.

To show continuity, we simply notice that $f(x)$ is the composition of two continuous functions, namely $|x|$ and $\sin(x)$, so it is continuous everywhere. Alternatively, we could note that $|\sin(k\pi)| = |0| = 0$ for all $k \in \mathbb{Z}$. Now let $k_0 \in \mathbb{Z}$. Then

$$\begin{aligned} \lim_{x \to k_0\pi^-} f(x) &= \lim_{x \to k_0\pi^-} |\sin(x)| \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to k_0\pi^+} f(x) &= \lim_{x \to k_0\pi^+} |\sin(x)| \\ &= 0 \end{aligned}$$

so

$$\lim_{x \to k_0\pi} f(x) = f(x) = 0$$

meaning $f(x)$ is continuous for any $k_0 \in \mathbb{Z}$.
Now the derivative, if it exists, would be

$$\begin{aligned} \lim_{h \to 0} \frac{|\sin(k_0\pi + h)| - |\sin(k_0\pi)|}{h} &= \lim_{h \to 0} \frac{|\sin(k_0\pi)\cos h + \cos(k_0\pi)\sin h| - |0|}{h} \\ &= \lim_{h \to 0} \frac{|(0) \cdot \cos h + (1) \cdot \sin h|}{h} \\ &= \lim_{h \to 0} \frac{|\sin h|}{h} \end{aligned}$$

Looking at the left-hand and right-hand separately:

$$\begin{aligned} \lim_{h \to 0^-} \frac{|\sin h|}{h} &= \lim_{h \to 0^-} \frac{-\sin h}{h} \qquad (\text{since for any } h \in (-\pi, 0)\text{, we know that } \sin h < 0). \\ &= -1 \end{aligned}$$

while

$$\begin{aligned} \lim_{h \to 0^+} \frac{|\sin h|}{h} &= \lim_{h \to 0^+} \frac{\sin h}{h} \qquad (\text{since for any } h \in (0, \pi)\text{, we know that } \sin h > 0). \\ &= 1 \end{aligned}$$

So the limit does not exist, and thus $f(x)$ is not differentiable for any $x \in \{k\pi, k \in \mathbb{Z}\}$.

3.3.2. The trigonometric functions $\sin(\omega x)$ and $\cos(\omega x)$ exhibit useful symmetries upon differentiation.

(a) Use the definition of the derivative to show that for $\omega \in \mathbb{R}$,

$$\frac{d}{dx}\sin(\omega x) = \omega\cos(\omega x), \qquad \text{and} \qquad \frac{d}{dx}\cos(\omega x) = -\omega\sin(\omega x).$$

From the definition,

$$\begin{aligned}\frac{d}{dx}\sin(\omega x) &= \lim_{h\to 0}\frac{\sin(\omega[x+h]) - \sin(\omega x)}{h}\\ &= \lim_{h\to 0}\frac{\sin(\omega x)\cos(\omega h) - \cos(\omega x)\sin(\omega h) - \sin(\omega x)}{h}.\end{aligned}$$

Multiplying top-and-bottom by ω,

$$\begin{aligned}&= \omega\lim_{h\to 0}\frac{\sin(\omega x)\Big(\cos(\omega h) - 1\Big)}{\omega h} + \frac{\cos(\omega x)\sin(\omega h)}{\omega h}\\ &= \omega\cos(\omega x).\end{aligned}$$

The derivative of $\cos(\omega x)$ is similar,

$$\begin{aligned}\frac{d}{dx}\cos(\omega x) &= \lim_{h\to 0}\frac{\cos(\omega[x+h]) - \cos(\omega x)}{h}\\ &= \lim_{h\to 0}\frac{\cos(\omega x)\cos(\omega h) - \sin(\omega x)\sin(\omega h) - \cos(\omega x)}{h}.\end{aligned}$$

Multiplying top-and-bottom by ω,

$$\begin{aligned}&= \omega\lim_{h\to 0}\frac{\cos(\omega x)\Big(\cos(\omega h) - 1\Big)}{\omega h} - \frac{\sin(\omega x)\sin(\omega h)}{\omega h}\\ &= -\omega\sin(\omega x).\end{aligned}$$

(b) Use part (a) to show that

$$\frac{d^2}{dx^2}\sin(\omega x) = -\omega^2\sin(\omega x), \qquad \text{and} \qquad \frac{d^2}{dx^2}\cos(\omega x) = -\omega^2\cos(\omega x)$$

The derivative is a linear operation, and so a constant multiple passes through the derivative operator,

$$\frac{d^2}{dx^2}\sin(\omega x) = \frac{d}{dx}\left(\frac{d}{dx}\sin(\omega x)\right) = \omega\frac{d}{dx}\cos(\omega x) = \omega\times(-\omega)\times\sin\omega x = -\omega^2\sin(\omega x)$$

Similarly,

$$\frac{d^2}{dx^2}\cos(\omega x) = \frac{d}{dx}\left(\frac{d}{dx}\cos(\omega x)\right) = -\omega\frac{d}{dx}\sin(\omega x) = -\omega\times\omega\times\cos(\omega x) = -\omega^2\cos(\omega x)$$

(c) An equation that involves a function $y(x)$ and its derivatives is called a *differential equation*. Show that $y(x) = A\cos(\omega x) + B\sin(\omega x)$ satisfies the differential equation,

$$y''(x) + \omega^2 y(x) = 0, \tag{3.2}$$

for arbitrary constants $A, B \in \mathbb{R}$.

As in part (b), the constants A and B pass through the derivative operators;

$$y''(x) = A\left(\frac{d^2}{dx^2}\cos(\omega x)\right) + B\left(\frac{d^2}{dx^2}\sin(\omega x)\right)$$

$$= A\left(-\omega^2\cos(\omega x)\right) + B\left(-\omega^2\sin(\omega x)\right) = -\omega^2\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

$$= -\omega^2 y(x) \quad \checkmark$$

(d) If, in addition to Eq. 3.2, we are told that $y(0) = 1$ and $y'(0) = 0$, determine the function $y(x)$.

The additional constraints on the function $y(x)$ (called *initial conditions*) create a system of linear equations for the unknown coefficients (A, B):

$$y(0) = A\cos(0)+B\sin(0) = A = 1, \quad \text{and} \quad y'(0) = -\omega A\sin(0)+\omega B\cos(0) = \omega B = 0,$$

so that $y(x) = \cos\omega x$ obeys the differential equation, Eq. 3.2, and the initial conditions.

We will learn more about differential equations in MATH 138!

3.3.3. The exponential function has the unique property that derivatives are transformed to scalar multiplication.

(a) Use the definition of the derivative to show that for $\lambda \in \mathbb{R}$,

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$$

From the definition,

$$\frac{d}{dx}e^{\lambda x} = \lim_{h\to 0} = \frac{e^{\lambda(x+h)} - e^{\lambda x}}{h} = e^{\lambda x}\lim_{h\to 0}\frac{e^{\lambda h} - 1}{h}$$

Multiplying top-and-bottom by λ,

$$= \lambda e^{\lambda x}\lim_{h\to 0}\frac{e^{\lambda h} - 1}{\lambda h} = \lambda e^{\lambda x}.$$

(b) Show that $y(x) = Ae^{\lambda x}$ obeys the differential equation,

$$y'(x) = \lambda y(x), \tag{3.3}$$

for arbitrary constant A.

The derivative is a linear operation, and so a constant multiple passes through the derivative operator,

$$y'(x) = \frac{d}{dx}\, Ae^{\lambda x} = A\frac{d}{dx}\, e^{\lambda x} = A\lambda e^{\lambda x}$$
$$= \lambda \times Ae^{\lambda x} = \lambda y(x) \quad \checkmark$$

(c) From Ex. 2c you should obtain an equation that looks like Eq. 3.3. Determine a value of λ and A that best fits the data.

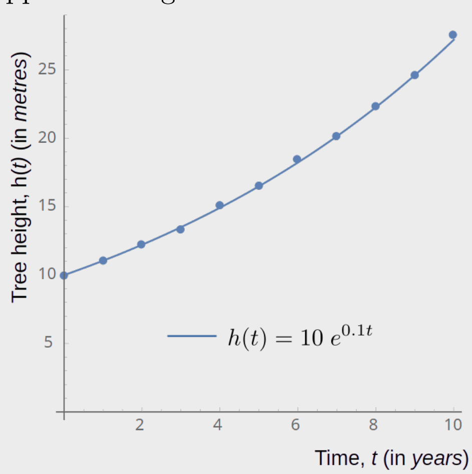
From Ex. 2c, you should find that

$$\frac{dh}{dt} \approx 0.1h(t),$$

so $\lambda = 0.1$ is a good choice. For the exponential to pass through the data, it must pass through the initial point ($h(0) = 9.91$ m). A good choice for A is something close to 9.91 (*e.g.* $A = 10$). The function

$$h(t) = 10e^{0.1t}$$

does a reasonable job approximating the data.



The dots are data from Ex. 2; the solid line is the exponential function $h(t) = 10\ e^{0.1t}$.

3.4 Derivative Rules

Section 3.4 Problems

3.4.1. Let $f(x) = \dfrac{ax+b}{ax-b}$ where $a \neq 0, b \neq 0$.

(a) Find $f'(x)$ using any method.

Using the Quotient Rule, we get $f'(x) = \dfrac{-2ab}{(ax-b)^2}$.

(b) Show that given $x \neq \frac{b}{a}$, $abf'(x) < 0$.

Since $a \neq 0$ and $b \neq 0$ we know that both $ab \neq 0$ and $f'(x) \neq 0$.
Since $a, b \in \mathbb{R}$, $a \neq 0$, $b \neq 0$ and $x \neq \frac{b}{a}$, we know that $(ax-b)^2 > 0$.
If $ab < 0$, we will get $f'(x) > 0$, but when $ab > 0$ we will get $f'(x) < 0$.
In other words, ab and $f'(x)$ always have opposite signs, therefore the product $abf'(x) < 0$.

3.4.2. In each case, find $f'(x)$ using any method.

(a) $f(x) = 5^x \sin x + (x^3 + x^2) \cos x$.

$$\begin{aligned} f'(x) &= (5^x \sin x)' + \big((x^3+x^2)\cos x\big)' \\ &= (5^x \ln 5 \sin x + 5^x \cos x) + \big((3x^2+2x)\cos x - (x^3+x^2)\sin x\big) \\ &= \sin x(5^x \ln 5 - x^3 - x^2) + \cos x(5^x + 3x^2 + 2x). \end{aligned}$$

(b) $f(x) = \dfrac{x^2+x-2}{x^3+6}$.

$$\begin{aligned} f'(x) &= \frac{(x^2+x-2)'(x^3+6) - (x^2+x-2)(x^3+6)'}{(x^3+6)^2} \\ &= \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2)}{(x^3+6)^2} \\ &= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2} \\ &= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}. \end{aligned}$$

(c) $f(x) = \sqrt{2\tan^2 x + 3}$.

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(2\tan^2 x + 3\right)^{\frac{-1}{2}} \left(2\tan^2 x + 3\right)' \\ &= \frac{1}{2}\left(2\tan^2 x + 3\right)^{\frac{-1}{2}} 4\tan x\,(\tan x)' \\ &= \frac{1}{2}\left(2\tan^2 x + 3\right)^{\frac{-1}{2}} 4\tan x\,\sec^2 x \\ &= 2\tan x\,\sec^2 x\,(2\tan^2 x + 3)^{\frac{-1}{2}}. \end{aligned}$$

3.4.3. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each case.

(a) $y = \cos(x^2)$.

The outer function is $\cos x$ and the inner function is x^2. Since $(\cos x)' = -\sin x$, by the Chain Rule,

$$\frac{dy}{dx} = -\sin(x^2)\cdot(x^2)' = -2x\sin(x^2).$$

Next, by the Product Rule and Chain Rule, we have

$$\frac{d^2y}{dx^2} = (-2x\sin(x^2))' = -2\sin(x^2) - 2x\cos(x^2)(2x) = -2\sin(x^2) - 4x^2\cos(x^2).$$

(b) $y = \cos^2 x$.

The outer function is x^2 and the inner function is $\cos x$. Since $(x^2)' = 2x$, by the Chain Rule,

$$\frac{dy}{dx} = 2\cos x(\cos x)' = -2\cos x\sin x.$$

Also, by the Product Rule, we have

$$\frac{d^2y}{dx^2} = (-2\cos x\sin x)' = 2\sin x\sin x - 2\cos x\cos x = 2\sin^2 x - 2\cos^2 x.$$

3.4.4. Using ONLY the Chain Rule and the Product Rule (and not the Reciprocal/Quotient rules), prove the following.

(a) The derivative of an even function is odd.

If f is even, then $f(x) = f(-x)$. By the Chain Rule, we have

$$f'(x) = (f(x))' = (f(-x))' = f'(-x)(-x)' = -f'(-x).$$

Thus f' is an odd function.

(b) The Quotient Rule. [Hint: $\dfrac{f(x)}{g(x)} = f(x)\,(g(x))^{-1}$].

By the Chain Rule and the Product Rule, we have

$$\begin{aligned} \left(\frac{f(x)}{g(x)}\right)' &= (f(x)g(x)^{-1})' \\ &= f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x) \\ &= \frac{f'(x)g(x)}{g(x)^2} - \frac{f(x)g'(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \end{aligned}$$

as desired.

3.4.5. If $y = f(u)$ and $u = g(x)$ where f and g are twice differentiable functions, prove that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}.$$

We have

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) \\ &= \left(\frac{d}{dx}\left(\frac{dy}{du}\right)\right)\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{(by the Product Rule)} \\ &= \left(\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right)\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} \quad \text{(by the Chain Rule)} \\ &= \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2} \end{aligned}$$

as desired.

3.5 Linear Approximation

Section 3.5 Problems

3.5.1. In each case, determine the equation of the tangent to $y = f(x)$ at the point where $x = a$.

(a) $f(x) = x^2, a = 3$

$f(x) = x^2$, so $f'(x) = 2x$ and $f'(3) = 6$. Thus the equation of the required tangent is

$$\begin{aligned} y &= f(3) + f'(3)(x-3) \\ &= 9 + 6(x-3) \\ &= 6x - 9. \end{aligned}$$

Therefore, the equation of the desired tangent line is $y = 9 + 6(x-3)$ or $y = 6x - 9$.

(b) $f(x) = \cos(x), a = -\frac{3\pi}{4}$.

$f(x) = \cos(x)$, so $f'(x) = -\sin(x)$ and $f'\left(-\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

$$\begin{aligned} y &= f\left(-\frac{3\pi}{4}\right) + f'\left(-\frac{3\pi}{4}\right)\left(x + \frac{3\pi}{4}\right) \\ &= \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x + \frac{3\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}\,(3\pi - 4)}{8}. \end{aligned}$$

Therefore, the equation of the desired tangent line is $y = -\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(x + \dfrac{3\pi}{4}\right)$ or $y = \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}\,(3\pi + 4)}{8}$.

(c) $f(x) = e^x, a = \ln \pi$.

$f(x) = e^x$, so $f'(x) = e^x$ and $f'(\ln(\pi)) = \pi$.

$$\begin{aligned} y &= f(\ln(\pi)) + f'(\ln(\pi))\,(x - \ln(\pi)) \\ &= \pi + \pi\,(x - \ln(\pi)) \\ &= \pi x + \pi(1 - \ln(\pi)). \end{aligned}$$

Therefore, the equation of the desired tangent line is $y = \pi + \pi\,(x - \ln(\pi))$ or $y = \pi x + \pi(1 - \ln(\pi))$.

(d) $f(x) = 4^x, a = -2$.

$f(x) = 4^x$, so $f'(x) = 4^x \ln(4)$ and $f'(-2) = \dfrac{\ln(4)}{16}$.

$$\begin{aligned} y &= f(-2) + f'(-2)(x+2) \\ &= \frac{1}{16} + \frac{\ln(4)}{16}(x+2) \\ &= \frac{\ln(4)}{16}x + \frac{1+2\ln(4)}{16}. \end{aligned}$$

Therefore, the equation of the desired tangent line is $y = \dfrac{1}{16} + \dfrac{\ln(4)}{16}(x+2)$ or $y = \dfrac{\ln(4)}{16}x + \dfrac{1+2\ln(4)}{16}$.

3.5.2. Let $f(x) = \sin(x)$.

(a) Determine the equation for the linear approximation to $f(x)$ at $x = a$ for any $a \in \mathbb{R}$.

Since $f'(x) = \cos(x)$, the linear approximation at $x = a$ is $L_a(x) = \sin(a) + \cos(a)(x - a)$.

(b) Determine the equation for the linear approximation to $f(x)$ at $x = \frac{\pi}{3}$.

The linear approximation at $x = \frac{\pi}{3}$ is

$$\begin{aligned} L_{\frac{\pi}{3}}(x) &= \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) \end{aligned}$$

(c) Use your answer to (b) to approximate $\sin(1)$.

$$\begin{aligned} \sin(1) &\approx L_{\frac{\pi}{3}}(1) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(1 - \frac{\pi}{3}\right) \\ &= \frac{1+\sqrt{3}}{2} - \frac{\pi}{6} \end{aligned}$$

(d) Since $f''(x) = -\sin(x)$, we can safely say that $|f''(x)| \leq 1$. Given this fact, what is the maximum error (worst-case scenario) for your answer in (c)?

Using Theorem 6, the maximum error is $\frac{1}{2}\left(1-\frac{\pi}{3}\right)^2$ (≈ 0.001114).

3.5.3. Rich bought a yoga ball that is made of material which, when the ball is properly inflated, is a sphere with outer radius R. The manufacturer has determined that the material can tolerate a 4% "stretch" beyond specifications, meaning that if the ball is inflated in such a way that the surface area increases by more than 4% of the actual size, the material will rupture and the ball will deflate rather suddenly.
In the instructions for inflation, consumers are told to inflate the ball so that one side touches a wall and the other side touches a box placed $2R$ units away from the wall. Rich uses a ruler to measure $2R$ units from a wall, and then inflates the ball according to the instructions. If Rich's ruler and Rich's measurement skills create an error of 3% in excess of what he thinks he is measuring (that is, instead of inflating to a diameter of $2R$ it is inflated to a diameter of $1.03(2R)$), should we expect the ball to survive this initial inflation?

First, note that that the surface area of a sphere is given by the function $A(r) = 4\pi r^2$, so $A'(r) = 8\pi r$.

The recommended surface area is $A_0 = 4\pi R^2$, and the change in surface area is approximately $\Delta A = A'(R)\Delta r$, where $\Delta r = 1.03R - R$

The percent change in surface area is approximately

$$\begin{aligned}\frac{\Delta A}{A_0} &\approx \frac{A'(R)\Delta r}{4\pi R^2}\\ &= \frac{8\pi R\Delta r}{4\pi R^2}\\ &= \frac{2\cdot[1.03(R)-R]}{R}\\ &= 0.06 \text{ (or } 6\%)\end{aligned}$$

This is more than the stretch tolerance. Rich is out of luck.

3.6 Newton's Method

Section 3.6 Problems

3.6.1. Newton's method provides a powerful algorithm to extract square-roots.

(a) For $a \in \mathbb{R}$, $a > 0$, show that Newton's Method applied to the function $f(x) = x^2 - a$ generates the recursive sequence,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{3.4}$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{(x_n)^2 - a}{2x_n} \\ &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right). \end{aligned}$$

(b) Assuming that the sequence $\{x_n\}$ converges, show that the limit is $\sqrt{a}$.

Assuming that $x_n \to x^\star$ and $x_{n+1} \to x^\star$ as $n \to \infty$, the recursive sequence definition provides the constraint on the limit $x^\star$,

$$x^\star = \frac{1}{2}\left(x^\star + \frac{a}{x^\star}\right),$$

or, $(x^\star)^2 = a$ which implies $x^\star = \pm\sqrt{a}$. If the initial term x_0 is positive ($x_0 > 0$), then all subsequent terms will be positive and $\sqrt{a}$ is the only attainable limit.

(c) Make the substitution $x_n \mapsto \sqrt{a}\coth(b_n)$, where

$$\coth u = \frac{e^u + e^{-u}}{e^u - e^{-u}}$$

is the *hyperbolic cotangent*, to convert the original sequence to

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \quad \mapsto \quad b_{n+1} = 2b_n.$$

Solve this sequence for b_n (*i.e.* for some initial term b_0, find b_n as an *explicit* function of n). Use your solution to prove that the original sequence $\{x_n\}$ converges to $\sqrt{a}$, and comment on the speed of this convergence with increasing n.

Making the substitution $x_n = \sqrt{a}\coth(b_n)$, the sequence definition becomes,

$$\sqrt{a}\,\frac{e^{b_{n+1}} + e^{-b_{n+1}}}{e^{b_{n+1}} - e^{-b_{n+1}}} = \frac{1}{2}\left(\sqrt{a}\,\frac{e^{b_n} + e^{-b_n}}{e^{b_n} - e^{-b_n}} + \sqrt{a}\,\frac{e^{b_n} - e^{-b_n}}{e^{b_n} + e^{-b_n}}\right) = \sqrt{a}\,\frac{e^{2b_n} + e^{-2b_n}}{e^{2b_n} - e^{-2b_n}},$$

which is satisfied if $b_{n+1} = 2b_n$, with solution $b_n = b_0 2^n$, where b_0 is found by inverting the initial term of the original sequence $x_0 = \sqrt{a}\coth(b_0) \Rightarrow b_0 = \coth^{-1}(x_0/\sqrt{a})$. In terms of the original sequence, $x_n = \sqrt{a}\coth(b_0 2^n)$ – the substitution has turned the recursively-defined sequence into an *explicit* expression for x_n. As $n \to \infty$, $\coth(b_0 2^n) \to 1$ and $x_n \to \sqrt{a}$. That is,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{a}\coth(b_0 2^n) = \sqrt{a} \lim_{n\to\infty} \frac{e^{b_0 2^n} + e^{-b_0 2^n}}{e^{b_0 2^n} - e^{-b_0 2^n}}$$

Divide top and bottom by $e^{b_0 2^n}$,

$$\lim_{n\to\infty} x_n = \sqrt{a} \lim_{n\to\infty} \frac{e^{b_0 2^n} + e^{-b_0 2^n}}{e^{b_0 2^n} - e^{-b_0 2^n}} = \sqrt{a} \lim_{n\to\infty} \frac{1 + e^{-2b_0 2^n}}{1 - e^{-2b_0 2^n}} = \sqrt{a}.$$

Given the exponential increase in $b_0 2^n$ as $n \to \infty$, and that b_n appears in the argument of an exponential, we would expect a rapid rate of convergence.

3.6.2. Use Newton's Method (Ex. 1) to approximate each of the following, correct to 5 decimal places.

(a) $\sqrt{7}$ (use initial guess $x_1 = 3$).

Use part (a) with $a = 7$. Correct to 5 decimal places, this yields
$x_2 = 2.66667, x_3 = 2.64583, x_4 = 2.64575, x_5 = 2.64575, \ldots$, so $\sqrt{7} \approx 2.64575$.

(b) $\sqrt{\pi}$ (use initial guess $x_1 = 2$).

Use part (a) with $a = \pi$. Correct to 5 decimal places, this yields
$x_2 = 1.78540, x_3 = 1.77250, x_4 = 1.77245, x_5 = 1.77245, x_6 = 1.77245, \ldots$, so $\sqrt{\pi} \approx 1.77245$.
Note that we can use a calculator to determine that $\sqrt{\pi} \approx 1.772454$, so our estimate is extremely good.

3.6.3. Consider the function $f(x) = \dfrac{6x+1}{3x+5}$.

(a) What is the domain of f?

The domain is $x \in \mathbb{R}, x \neq -\dfrac{5}{3}$.

(b) Find $f'(x)$.

The derivative is $f'(x) = \dfrac{27}{(3x+5)^2}$.

(c) There is only one point $c \in \mathbb{R}$ where $f(c) = 0$, find it directly.

We need $0 = \dfrac{6c+1}{3c+5}$ so $c = -\dfrac{1}{6}$.

(d) Now, starting with $x_1 = 5$ (a particularly foolish choice), perform 3 iterations of Newton's Method. Use 5 decimal places.

In this case, for $n \geq 1$, we will be using

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{6x_n+1}{3x_n+5} \div \frac{27}{(3x_n+5)^2} \\
&= x_n - \frac{6x_n+1}{3x_n+5} \cdot \frac{(3x_n+5)^2}{27} \\
&= x_n - \frac{(6x_n+1)(3x_n+5)}{27} \\
&= \frac{-18x_n^2 - 6x_n - 5}{27}
\end{aligned}$$

So, we get

$$\begin{aligned}
x_1 &= 5 \\
x_2 &= \frac{-18(5)^2 - 6(5) - 5}{27} \\
&= -17.96296 \\
x_3 &= \frac{-18(-17.96296)^2 - 6(-17.96296) - 5}{27} \\
&= -211.30544 \\
x_4 &= \frac{-18(-211.30544)^2 - 6(-211.30544) - 5}{27} \\
&= -29719.88808
\end{aligned}$$

(e) It is clear that starting with $x_1 = 5$ will not lead us to the root. In fact, the sequence generated by Newton's method in this case diverges. Prove that using $x_1 = 5$ as a starting value, Newton's Method will not converge to the root of this function. (Hint: show the recursive sequence you get from Newton's Method is strictly decreasing).

Consider the recursive sequence defined by

$$x_1 = 5, x_{n+1} = \frac{-18x_n^2 - 6x_n - 5}{27}, n \geq 1$$

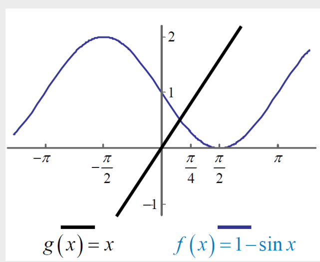
We will show that the sequence is strictly decreasing, so that it will not be able to converge to the root of the function since x_2 is already less than the root.
Let's investigate the inequality: $x_{n+1} < x_n$.

$$\begin{aligned} -18x_n^2 - 6x_n - 5 &< 27x_n \\ \Leftrightarrow\ 18x_n^2 + 33x_n + 5 &> 0 \\ \Leftrightarrow\ (3x_n + 5)(6x_n + 1) &> 0 \\ \Leftrightarrow\ x_n &\in \left(-\infty, -\frac{5}{3}\right) \cup \left(-\frac{1}{6}, \infty\right) \end{aligned}$$

Specifically, this tells us that if $x_n \in \left(-\infty, -\frac{5}{3}\right)$ we can be sure that $x_{n+1} < x_n$, so that as soon as any term in the sequence enters this interval, the next term will be less, and thus the sequence will continue to decrease, staying in that interval.
We have shown in part (d) that $x_2 \in \left(-\infty, -\frac{5}{3}\right)$, and also so from x_2 on, the sequence is decreasing, and so will never converge to the root of the function.

3.6.4. Find the root of $\sin x = 1 - x$ to 5 decimal places. Use a sketch to estimate the initial point x_0.

The roots of the equation correspond to the points of intersection between $f(x) = 1 - \sin x$ and $g(x) = x$. From the figure below, $f(x)$ and $g(x)$ intersect only once, at a point close to $x = \frac{\pi}{8}$.



The root of the equation $\sin x = 1 - x$ corresponds to the intersection point between the black and blue curves.

Newton's method generates the implicit sequence,

$$x_{n+1} = x_n + \frac{1 - \sin x_n - x_n}{1 + \cos x_n}, \quad x_0 = \tfrac{\pi}{8} = 0.392\,699\ldots$$

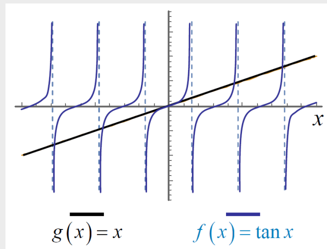
After 4 iterations,

$$\{x_n\} = \{0.392\,699\ldots, 0.509\,451\ldots, 0.510\,973\ldots, 0.510\,973\ldots\}$$

To 5 decimal places of accuracy, the root of the equation $\sin x^\star = 1 - x^\star$ is $x^\star \approx 0.51097$.

3.6.5. How many roots does the equation $\tan x = x$ have? Find the one between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ to 5 decimal places.

The roots of the equation correspond to the points of intersection between $f(x) = \tan x$ and $g(x) = x$. From the figure below, $f(x)$ and $g(x)$ intersect an infinite number of times.



The root of the equation $\tan x = x$ corresponds to the intersection points between the black and blue curves.

Aside from the trivial root at $x = 0$, the other roots lie closer and closer to the vertical asymptotes of $\tan x$ as $x \to \infty$. That is, the roots are

$$x^\star \approx \frac{\pm(2n+1)\pi}{2}, \quad (n \geq 0)$$

For the root in the interval $x^\star \in (\frac{\pi}{2}, \frac{3\pi}{2})$, a reasonable initial point is somewhere very close to $\frac{3\pi}{2}$, $x_0 = \frac{3\pi}{2} - \varepsilon$. If ε is too large, the algorithm will jump to a far-away root because the slope of the tangent line will get close to zero; whereas if ε is too small, $\tan x$ diverges and the slope of the tangent line becomes undefined. From trial-and-error, $\varepsilon = 0.1$ seems to work pretty well. In that case, Newton's method generates the implicit sequence,

$$x_{n+1} = x_n - \frac{\tan x_n - x_n}{\sec^2 x_n - 1}, \quad x_0 = \tfrac{3\pi}{2} - 0.1 = 4.612\,389\ldots$$

After 6 iterations,

$$\begin{aligned}\{x_n\} = \big\{&4.612\,389\ldots,\\ &4.558\,487\ldots,\\ &4.513\,058\ldots,\\ &4.495\,220\ldots,\\ &4.493\,425\ldots,\\ &4.493\,409\ldots,\\ &4.493\,409\ldots\big\}\end{aligned}$$

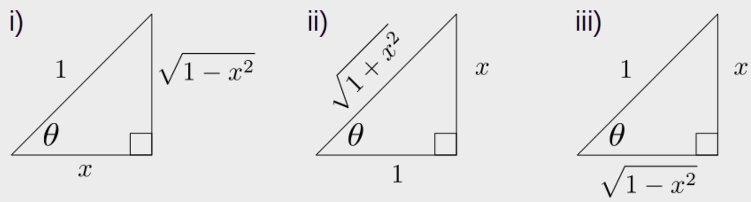
To 5 decimal places of accuracy, the root of the equation $\tan x^\star = x^\star$, $x^\star \in (\frac{\pi}{2}, \frac{3\pi}{2})$, is $x^\star \approx 4.49341$.

3.7 Derivatives of Inverse Functions

Section 3.7 Problems

3.7.1. Simplify the following.

$$\text{i) } \sin(\arccos x), \quad \text{ii) } \sin(\arctan x), \quad \text{iii) } \tan(\arcsin x).$$



i) We are looking for $\sin\theta$, where $\theta = \arccos x$. Inverting for x,

$$x = \cos\theta$$

A right-triangle corresponding to this constraint is shown in the figure above (adjacent side is x, hypotenuse is 1). Using the Pythagorean theorem to fill in the missing side, we have finally that

$$\sin(\arccos x) = \sin\theta = \sqrt{1-x^2}.$$

ii) We are looking for $\sin\theta$, where $\theta = \arctan x$. Inverting for x,

$$x = \tan\theta$$

A right-triangle corresponding to this constraint is shown in the figure above (opposite side is x, adjacent side is 1). Using the Pythagorean theorem to fill in the missing side, we have finally that

$$\sin(\arctan x) = \sin\theta = \frac{x}{\sqrt{1+x^2}}.$$

iii) We are looking for $\tan\theta$, where $\theta = \arcsin x$. Inverting for x,

$$x = \sin\theta$$

A right-triangle corresponding to this constraint is shown in the figure above (opposite side is x, hypotenuse is 1). Using the Pythagorean theorem to fill in the missing side, we have finally that

$$\tan(\arcsin x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}.$$

3.7.2. Determine $y'(x)$ for each of the following,

$$\text{i) } y(x) = x\arcsin x, \quad \text{ii) } y(x) = \arcsin\left(\tfrac{a}{x}\right) \quad (a \in \mathbb{R}), \quad \text{iii) } y(x) = \frac{\arcsin x}{\sin x}.$$

i) Using the product rule,

$$y'(x) = \arcsin x + \frac{x}{\sqrt{1-x^2}}.$$

ii) Using the chain rule,

$$y'(x) = \frac{1}{\sqrt{1-(\frac{a}{x})^2}}\left(-\frac{a}{x^2}\right).$$

iii) Using the quotient rule,

$$\begin{aligned} y'(x) &= \frac{1}{\sin x}\frac{1}{1-x^2} - \arcsin x \frac{\cos x}{(\sin x)^2} \\ &= \csc x\left(\frac{1}{\sqrt{1-x^2}} - \arcsin x \ \cot x\right) \end{aligned}$$

3.7.3. It is difficult to prove the identity

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}, \quad (x > 0) \tag{3.5}$$

directly, but we can exploit the properties of the derivative to facilitate the proof.

(a) First, show that the derivative of the inverse cotangent, arccot x, is,

$$\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}.$$

Hint: The cotangent $\cot x = \cos x / \sin x$, and obeys the identity $1+\cot^2 x = \csc^2 x$.

The inverse cotangent is defined such that $\cot(\operatorname{arccot} x) = x$. Call $g(x) = \operatorname{arccot} x$. Differentiating with respect to x and applying the chain-rule,

$$\frac{d}{dx}\cot(g(x)) = 1, \quad \text{or} \quad -\csc(g(x))^2\frac{dg}{dx} = 1.$$

Isolating for the derivative,

$$\frac{dg}{dx} = \frac{d}{dx}\operatorname{arccot} x = \frac{-1}{\csc^2(g(x))}.$$

Using the trigonometric identity provided in the hint,

$$\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{\csc^2(g(x))} = \frac{-1}{1+\cot^2(g(x))} = \frac{-1}{1+x^2}.$$

(b) Next, show that the function $f(x) = \arctan x + \operatorname{arccot} x$ has zero derivative, $f'(x) = 0$.

From part (a), and from the examples we did in lectures,

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \quad \text{and} \quad \frac{d}{dx}\text{arccot } x = \frac{-1}{1+\cot^2(g(x))} = \frac{-1}{1+x^2},$$

so that $f'(x) = 0$.

(c) Use part (b) to find a suitable x that proves the identity (3.5).

Because the derivative $f'(x) = 0$, we know that $f(x)$ must be a *constant* on its domain. There are two straightforward choices for x that allow that constant to be determined,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to\infty} f(x) = \frac{\pi}{2}.$$

Other choices are possible, but any choice will result in $f(x) = \frac{\pi}{2}$.

3.7.4. Prove the identity

$$2\arcsin x = \arccos\left(1-2x^2\right) \qquad (x \geq 0)$$

Hint: Do Ex. 3 first.

Following the hint, we want to first show that

$$f(x) = 2\arcsin x - \arccos\left(1-2x^2\right),$$

has a vanishing derivative $f'(x) = 0$, and then find a convenient point $x_0 \geq 0$ with $f(x_0) = 0$ to prove that $f(x) = 0$ for $x \geq 0$. The $\arcsin x$ and $\arccos x$ derivatives were done in lectures; for the derivative of $\arccos\left(1-2x^2\right)$, we will apply the chain rule to $\arccos y$, where $y(x) = 1-2x^2$.

$$\frac{d}{dx}\left(2\arcsin x\right) = \frac{2}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}\arccos\left(1-2x^2\right) = \frac{d\arccos y}{dy}\frac{d(1-2x^2)}{dx} = \frac{-1}{\sqrt{1-(1-2x^2)^2}}(-4x) = \frac{2x}{\sqrt{x^2-x^4}}.$$

Altogether,

$$\frac{df}{dx} = \frac{2}{\sqrt{1-x^2}} - \frac{2}{\sqrt{1-x^2}} = 0.$$

Therefore, $f(x)$ is a *constant* for $x \geq 0$. Substituting $x = 0$, $f(0) = 2\arcsin(0) - \arccos(1) = 0 - 0 = 0$, and we conclude that $f(x) = 0$ for all $x \geq 0$, which proves the identity ✓.

3.8 Implicit and Logarithmic Differentiation

Section 3.8 Problems

3.8.1. For each of the following, determine $\frac{dy}{dx}$, assuming that $f(x)$ and $g(x)$ are positive, differentiable functions.

$$\text{a) } y = \ln\left(f\left(x^2\right)\right) \qquad \text{b) } y = [f(x)]^{g(x)} \qquad \text{c) } y = f\left(\sqrt{x}\ \ln\left(g(x)\right)\right)$$

a) The function y can be written as a composition $y = \ln f(h(x))$, with $h(x) = x^2$. Applying the chain rule,

$$\frac{dy}{dx} = \frac{d\ln f}{df}\frac{df}{dh}\frac{dh}{dx} = \frac{1}{f\left(x^2\right)}\, f'\left(x^2\right)\, 2x = 2x\,\frac{f'\left(x^2\right)}{f\left(x^2\right)}$$

b) Taking the logarithm, and differentiating using the product rule,

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(g(x)\ \ln f(x)\right) = \ln f\,\frac{dg}{dx} + g\,\frac{1}{f}\,\frac{df}{dx}$$

Isolating for the derivative $\frac{dy}{dx}$,

$$\frac{dy}{dx} = [f(x)]^{g(x)}\left[\ln f\,\frac{dg}{dx} + \frac{g}{f}\,\frac{df}{dx}\right]$$

c) Applying the chain rule,

$$\frac{dy}{dx} = f'(\sqrt{x}\ln(g(x)))\,\left[\frac{1}{2\sqrt{x}}\ \ln g + \sqrt{x}\,\frac{1}{g}\,\frac{dg}{dx}\right]$$

3.8.2. Find $\frac{dy}{dx}$ for $\arcsin(x^2y) + xy = 1$.

Differentiating both sides (implicitly) with respect to x, we get

$$\frac{1}{\sqrt{1-(x^2y)^2}}\,(x^2y)' + (xy)' = 0,$$

$$\Rightarrow \frac{1}{\sqrt{1-x^4y^2}}\,(2xy + x^2y') + (y + xy') = 0.$$

It follows that

$$\left(\frac{x^2}{\sqrt{1-x^4y^2}} + x\right)y' = \frac{-2xy}{\sqrt{1-x^4y^2}} - y.$$

Thus

$$\frac{dy}{dx} = y' = \frac{\frac{-2xy}{\sqrt{1-x^4y^2}} - y}{\frac{x^2}{\sqrt{1-x^4y^2}} + x}.$$

3.8.3. If $y = (\arcsin(x))^2$ and $0 < x < 1$, then prove that

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y,$$

and thereby deduce that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$

Hint: To deduce the second expression, try differentiating the first expression with respect to x.

$y = (\arcsin x)^2 = \arcsin^2 x$. Applying the chain rule,

$$\frac{dy}{dx} = 2(\arcsin x)\frac{1}{\sqrt{1-x^2}}.$$

Squaring,

$$\left(\frac{dy}{dx}\right)^2 = 4(\arcsin^2 x)\frac{1}{1-x^2} \quad \text{or } (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y.$$

Differentiating this expression implicitly,

$$\frac{d}{dy}\left[(1-x^2)\left(\frac{dy}{dx}\right)^2\right] = -2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4\frac{dy}{dx}.$$

Over the domain $0 < x < 1$, $\frac{dy}{dx} \neq 0$, so dividing through by $2\frac{dy}{dx}$,

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 2 \quad \text{or } (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$

3.8.4. Find the equation of the tangent line at the point $(0, -1)$ to the curve defined by $xy + y^3 = \arctan(x) - 1$.

Using implicit differentiation, we have

$$y + xy' + 3y^2y' = \frac{1}{1+x^2}.$$

Hence,

$$(x + 3y^2)y' = \frac{1}{1+x^2} - y.$$

Put $x = 0$ and $y = -1$, we get $3y' = 1 - (-1)$, i.e., $y' = \frac{2}{3}$. Thus the tangent line is

$$y = -1 + \frac{2}{3}x.$$

3.8.5. Use the logarithmic differentiation to find $\frac{dy}{dx}$

(a) $y = x^{\sin x}$ with $x > 0$.

We have

$$\ln y = \sin x \ln x.$$

Differentiate both sides, we get

$$\frac{y'}{y} = \cos x \ln x + \sin x \cdot \frac{1}{x}.$$

Hence,

$$y' = x^{\sin x}\Big(\cos x \ln x + \sin x \cdot \frac{1}{x}\Big).$$

(b) $y = (2x)^{x^{\frac{1}{3}}}$.

Write $|y| = |(2x)^{x^{\frac{1}{3}}}| = |2x|^{x^{\frac{1}{3}}}$.
By taking logarithms on both sides, we get

$$\ln|y| = \ln\left(|2x|^{x^{\frac{1}{3}}}\right) = x^{\frac{1}{3}}\ \ln|2x|.$$

By differentiating,

$$\frac{y'}{y} = \frac{1}{3}x^{\frac{-2}{3}}\ \ln|2x| + x^{\frac{1}{3}}\ \frac{1}{2x}\ 2 = \frac{1}{3}x^{\frac{-2}{3}}\ \ln|2x| + x^{\frac{1}{3}-1} = x^{-\frac{2}{3}}\Big(\frac{1}{3}\ \ln|2x| + 1\Big).$$

Thus

$$y' = y\,x^{-\frac{2}{3}}\Big(\frac{1}{3}\ \ln|2x| + 1\Big) = (2x)^{x^{\frac{1}{3}}}\ x^{-\frac{2}{3}}\Big(\frac{1}{3}\ \ln|2x| + 1\Big).$$

(c) $y = 2^{\sin(\sec x)}$.

$$\begin{aligned} y'(x) &= 2^{\sin(\sec x)}\ \ln 2\ \big(\sin(\sec x)\big)' \\ &= 2^{\sin(\sec x)}\ \ln 2\ \big(\cos(\sec x)\big)\ (\sec x)' \\ &= 2^{\sin(\sec x)}\ \ln 2\ \big(\cos(\sec x)\big)\ (\sec x\ \tan x). \end{aligned}$$

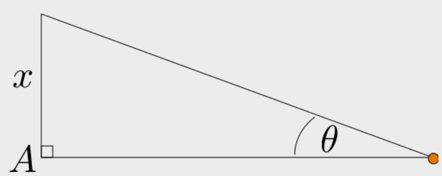
Chapter 4

Applications of the Derivative

4.1 Related Rates

Section 4.1 Problems

4.1.1. A rotating beacon is located 500 metres out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. At a point 10 metres along the shore from point A, how fast is beam of light moving? (Assume that the shore line is straight and runs perpendicular to the line connecting A and the beacon.)



Denote the distance along the shore line from A by x. The angle between the line connecting the beacon to A and the light beam through x is denoted by θ. From the geometry of the problem.

$$\frac{x}{2} = \tan\theta, \qquad \Longrightarrow \qquad x = 2\tan\theta.$$

taking the derivative of both sides with respect to time t,

$$\frac{dx}{dt} = 2\sec^2\theta\,\frac{d\theta}{dt}.$$

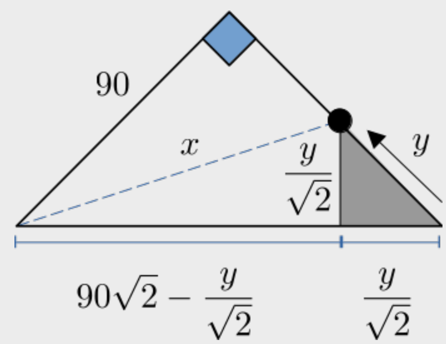
We are given that the point x is 10 metres from the point A so that $\theta = \arctan(1/50)$. We are further given that the beacon rotates at 10 rev/min, or

$$\frac{d\theta}{dt} = \frac{10\ \text{rev}}{\text{min}}\frac{2\pi\ \text{rad}}{1\ \text{rev}} = 20\pi\,\frac{\text{rad}}{\text{min}}.$$

Altogether, at a point 10 metres from A, the beam is moving at

$$\frac{dx}{dt} = 2\sec^2\left(\arctan(\tfrac{1}{50})\right) \times 20\pi \approx 40.02\pi\,\frac{\text{m}}{\text{min}} \approx 7.54\,\frac{\text{km}}{\text{h}}$$

4.1.2. A baseball diamond is a square 90 feet on a side. A player runs from first base to second base at 15 feet/sec. At what rate is the player's distance from third base decreasing when they are half-way from first to second base?



The line connecting third and first base creates a right-isosceles triangle (with hypotenuse $90\sqrt{2}$). The runners distance along the baseline is denoted by y and the distance to third base is denoted by x. From the figure, the grey triangle is again a right-isosceles triangle, and thereby provides the base and height of a right triangle with the desired distance x as a hypotenuse. The Pythagorean theorem connects x and y,

$$x^2 = (90\sqrt{2} - y/\sqrt{2})^2 + (y/\sqrt{2})^2 \quad \Longrightarrow \quad x = \sqrt{16200 - 180y + y^2}$$

Differentiating each side with respect to time,

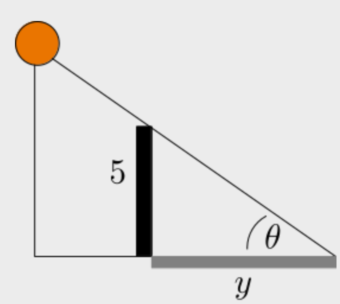
$$\frac{dx}{dt} = \frac{(90 - y)}{\sqrt{16200 - 180y + y^2}} \frac{dy}{dt}$$

When the runner is half-way between first and second base, $y = 45$ feet and we are given that their speed is $y'(t) = 15$ feet/sec, and so

$$\frac{dx}{dt} = -3\sqrt{5} \text{ feet/sec.}$$

The distance between the runner and third base is decreasing at a rate of $3\sqrt{5} \approx 6.71$ feet/sec.

4.1.3. The sun is setting at the rate of $\frac{1}{250}$ rad/min, and appears to be dropping perpendicular to the horizon. How fast is the shadow of a 5 meter wall lengthening at the moment when the shadow is 10 meters long?



From the figure, the length of the shadow, y, is related to the angle θ via the tangent,

$$\frac{5}{y} = \tan\theta \qquad \Longrightarrow \qquad y = 5\cot\theta.$$

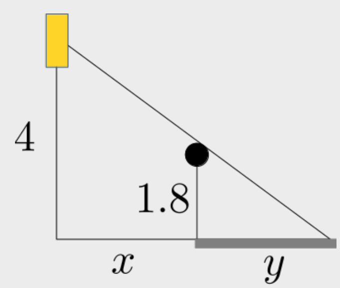
Differentiating both with respect to time,

$$\frac{dy}{dt} = -5\csc^2\theta\,\frac{d\theta}{dt}.$$

When the shadow is 10 m long, $\theta = \arctan(5/10) = \arctan(1/2)$, and $\csc\left(\arctan(1/2)\right)^2 = 5$, so that

$$\frac{dy}{dt} = -5 \times 5 \times \frac{1}{250} = 0.1 \text{ m/min}.$$

4.1.4. A person 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is their shadow shortening? At what rate is the tip of their shadow moving?



From the figure, the tip of the shadow and the street light forms a triangle that is similar to the shadow and the person walking; consequently, we have

$$\frac{4.8}{x+y} = \frac{1.8}{y} \qquad \Longrightarrow \qquad y = \frac{9}{11}\,x.$$

Differentiating both sides with respect to time,

$$\frac{dy}{dt} = \frac{9}{11}\frac{dx}{dt} = \frac{9}{11} \times 1 = \frac{9}{11} \text{ m/s}.$$

The shadow is shortening at a rate of 9/11 m/s. The tip of the shadow is moving because of changes in x and y; the tip of the shadow moves at a rate

$$\frac{d(x+y)}{dt} = 1 + \frac{9}{11} = \frac{20}{11} \text{ m/s}.$$

4.1.5. A picture of a sea horse is stamped on the surface of a latex balloon, and occupies 1/20 of the surface area. The balloon is inflated at a rate of 10 cm^3/sec. When the balloon has a radius of 5 cm, how quickly is the area of the sea horse increasing? (Assume that the balloon is a sphere.)

The surface of the balloon stretches, and so the sea horse will always occupy 1/20 of the surface area of the balloon. Denote the area of the sea horse by A_{SH} and the surface area of the balloon by A_B, then

$$\frac{dA_{SH}}{dt} = \frac{1}{20}\frac{dA_B}{dt}.$$

For a spherical balloon, the surface area A_B and volume V_B are given by,

$$A_B = 4\pi r^2, \qquad V_B = \frac{4}{3}\pi r^3,$$

where r is the radius of the balloon. Differentiating with respect to time t, and isolating $r'(t)$ from $V_B'(t)$,

$$\frac{dV_B}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \Longrightarrow \qquad \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV_B}{dt},$$

$$\frac{dA_B}{dt} = 8\pi r \frac{dr}{dt} = \frac{8\pi r}{4\pi r^2}\frac{dV_B}{dt} = \frac{2}{r}\frac{dV_B}{dt}.$$

Given that $V_B'(t) = 10$ cm^3/sec and $r = 5$ cm, the area of the sea horse is increasing at a rate,

$$\frac{dA_{SH}}{dt} = \frac{1}{20}\frac{dA_B}{dt} = \frac{1}{20} \times \frac{2}{5} \times 10 = \frac{1}{5}\frac{\text{cm}^2}{\text{sec}}.$$

4.2 Extrema

Section 4.2 Problems

4.2.1. Find all critical points for the following functions.

(a) $f(x) = x + \dfrac{1}{x}$.

$f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2} = \dfrac{(x-1)(x+1)}{x^2}$. Clearly $f'(x) = 0$ when $x = \pm 1$, and $f'(x)$ does not exist when $x = 0$. Note that $x = 0$ is not in the domain of f, so it is not a critical point. Both $x = \pm 1$ are in the domain of f and therefore the critical points of f are $x = \pm 1$.

(b) $f(x) = \dfrac{(x-1)^3}{(x+1)^4}$.

$f'(x) = \dfrac{(x-1)^2(7-x)}{(x+1)^5}$. Clearly $f'(x) = 0$ when $x = 1, 7$, and $f'(x)$ does not exist when $x = -1$. Note that $x = -1$ is not in the domain of f, so it is not a critical point. Both $x = 1$ and $x = 7$ are in the domain of f and therefore the critical points of f are $x = 1, 7$.

4.2.2. Find the global maximum and minimum of $f(x) = 2\cos x + \sin 2x$ for $x \in \left[0, \frac{\pi}{2}\right]$.

Note that f is continuous on $[0, \pi/2]$, so by the Extreme Value Theorem, f attains its global maximum and minimum. We have

$$\begin{aligned} f'(x) &= -2\sin x + 2\cos 2x \\ &= -2\sin x + 2(1 - 2\sin^2 x) \\ &= -2(2\sin^2 x + \sin x - 1) \\ &= -2(2\sin x - 1)(\sin x + 1). \end{aligned}$$

Thus $f'(x)$ exists for all $x \in \mathbb{R}$, and so the only critical points of $f(x)$ is when $f'(x) = 0$, i.e., $\sin x = 1/2$ or $\sin x = -1$. Note that since $\sin x \geq 0$ for $x \in [0, \pi/2]$, we only have $\sin x = 1/2$. Thus $x = \pi/6$ is the only critical point. Note that

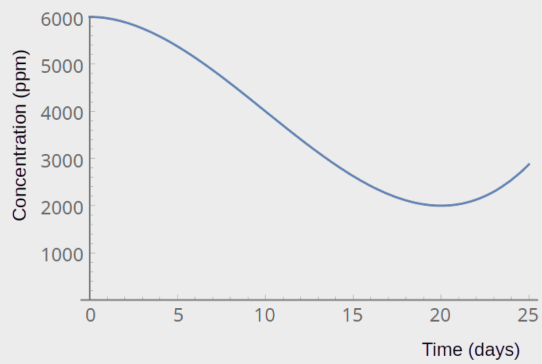
$$f(\pi/6) = \sqrt{3} + \sqrt{3}/2 = 3\sqrt{3}/2 \approx 2.60.$$

Also, the values at the end points are

$$f(0) = 2 \quad \text{and} \quad f(\pi/2) = 0.$$

Comparing the above values, we see that the global maximum value is $f(\pi/6) = 3\sqrt{3}/2$ and the global minimum value is $f(\pi/2) = 0$.

4.2.3. Organic waste deposited in a lake at $t = 0$ decreases the oxygen content of the water. Suppose the oxygen content is $C(t) = t^3 - 30t^2 + 6000$ ppm for $0 \leq t \leq 25$ days. Find the maximum and minimum oxygen content during that time.



The critical points $t^\star$ correspond to those points where the first derivative vanishes (or the function is undefined – but this is not relevant for our polynomial function). Evaluating the first derivative,

$$C'(t) = 3t^2 - 60t = 3t(t - 20),$$

which vanishes, $C'(t^\star) = 0$, when $t^\star = 0, 20$. Evaluating the function at the critical points, as well as at the end-points of the domain,

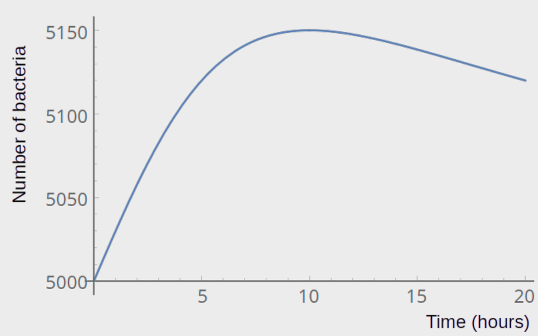
$$C(0) = 6000, \qquad C(20) = 2000, \qquad C(25) = 2875.$$

Over the given domain $0 \leq t \leq 25$ days, the global maximum is $C(0) = 6000$ ppm, and the global minimum is $C(20) = 2000$ ppm (as can be seen in the plot above).

4.2.4. A bacterial population grows according to

$$N(t) = 5000 + \frac{3000t}{100 + t^2},$$

with time t measured in hours. Determine the maximum population size for $t \geq 0$.



The critical points $t^\star$ correspond to those points where the first derivative vanishes (or the function is undefined – but this is not relevant for our rational function).

Evaluating the first derivative,

$$N'(t) = -\frac{6000t^2}{(100+t^2)^2} + \frac{3000(100+t^2)}{(100+t^2)^2} = \frac{3000(100-t^2)}{(100+t^2)^2},$$

which vanishes, $N'(t^\star) = 0$, when the numerator vanishes: $t^\star = \pm 10$. The only critical point within the domain is $t^\star = 10$. The domain is semi-infinite, so the right-most end-point must be evaluated via a limit; evaluating the function at the critical points, as well as at the end-points of the domain,

$$N(0) = 5000, \qquad N(10) = 5150, \qquad \lim_{t\to\infty} N(t) = 5000.$$

Over the given domain $t \geq 0$ hours, the global maximum is $N(10) = 5150$, and the global minimum is attained at $N(0) = 5000$ and as a limiting asymptote when $t \to \infty$.

4.3 Mean Value Theorem

Section 4.3 Problems

4.3.1. The Netherlands was the first country to introduce average-speed cameras for speed-limit enforcement. The approach has subsequently spread to several European countries and through the Middle East. In the simplest implementation, two cameras are deployed a large distance apart along a highway. As cars pass by the cameras, their licence plates are recorded. By comparing the time-stamps of identifications made by each camera, the average velocity of the vehicle is determined.

(a) Prove that if the average velocity of the vehicle exceeds the posted speed limit, then the vehicle has exceeded the speed limit somewhere along the highway between the cameras.

The result follows from the Mean Value Theorem. Denote by $f(t)$ the vehicles position along the highway. The average-speed cameras are passed at times $t = a$ and $t = b$. At some time in between, $a < c < b$,

$$\underbrace{f'(c)}_{\text{speed}} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{average speed}} > \underbrace{S}_{\substack{\text{speed}\\ \text{limit}}}$$

If the average speed exceeds the speed limit S, then, for at least a moment $t = c$, the instantaneous speed of the vehicle exceeded the speed limit.

(b) Prove that the converse is not true: although the average speed is well-below the posted speed limit, a vehicle may have exceeded the speed limit somewhere along the highway.

A vehicle could greatly-exceed the speed limit as it passes the first camera, then pull over to the side of the highway and stop for long enough to bring the average speed back below the posted limit.

4.3.2. Prove that

$$|\sin b - \sin a| \leq |b - a|.$$

Using the Mean Value Theorem applied to $f(x) = \sin x$, we have for some point $a < c < b$,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \Longrightarrow \qquad \cos c = \frac{\sin b - \sin a}{b - a}.$$

Taking the absolute value of both sides, and using the fact that cosine is bounded $|\cos x| \leq 1$,

$$1 \geq |\cos c| = \left| \frac{\sin b - \sin a}{b - a} \right| = \frac{|\sin b - \sin a|}{|b - a|}.$$

Multiplying through by the positive quantity $|b - a|$,

$$|b - a| \geq |\sin b - \sin a|.$$

4.3.3. The Mean-Value Theorem can be used to compare the geometric and arithmetic means.

(a) Use the MVT to show that

$$\sqrt{b}-\sqrt{a}<\frac{b-a}{2\sqrt{a}}$$

for $0<a<b$.

Since $f(x)=\sqrt{x}$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, then we can apply the MVT to $f(x)=\sqrt{x}$ on the interval $[a,b]$. We get

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

for some $c\in(a,b)$, that is

$$\frac{1}{2\sqrt{c}}=\frac{\sqrt{b}-\sqrt{a}}{b-a}.$$

But since $c>a$ we get $\dfrac{1}{2\sqrt{c}}<\dfrac{1}{2\sqrt{a}}$ and therefore

$$\frac{\sqrt{b}-\sqrt{a}}{b-a}<\frac{1}{2\sqrt{a}}\Rightarrow\sqrt{b}-\sqrt{a}<\frac{b-a}{2\sqrt{a}},$$

as desired.

(b) Use part (a) to show that, for $0<a<b$, the geometric mean $\sqrt{ab}$ is always smaller than the arthicmetic mean $\frac{1}{2}(a+b)$, that is, show that

$$\sqrt{ab}<\frac{a+b}{2}.$$

Let's multiply the inequality in part (a) by $2\sqrt{a}$ to get

$$2\sqrt{a}\sqrt{b}-2a<b-a\Rightarrow 2\sqrt{a}\sqrt{b}<b+a\Rightarrow\sqrt{ab}<\frac{b+a}{2},$$

as desired.

4.3.4. Show that the function $f(x)=2x^5+2x+1$ has exactly one root without sketching the graph of the function.

Hint: assume there is more than 1 root and use the MVT to build a contradiction.

We first show that there can be at most one root. The argument is simple. Assume there is more than one root. Then this means that there are two distinct values x_1 and x_2 such that $f(x_1)=f(x_2)=0$. Since the function f is differentiable then we get by the MVT (or by Rolle's theorem) that there must be a number c between x_1 and x_2 such that

$$f'(c)=\frac{f(x_1)-f(x_2)}{x_1-x_2}=0$$

However, we have that
$$f'(x) = 10x^4 + 2$$
which is always positive and so $f'(x) \neq 0$ for any x.
And so it is not possible to have 2 (or more) values such that $f(x) = 0$.
This means we may have either 0 or 1 root.
To show we have exactly 1 root we use the IVT. Note that for positive x the function $f(x)$ is positive. For example at $x = 1$ we get
$$f(1) = 5$$
For negative x the function quickly becomes dominated by x^5. For example at $x = -2$ we get
$$f(-2) = -67$$
Finally, since f is continuous (since it's a polynomial) by the IVT there must be a number b on the interval $(-2, 1)$ such that $f(b) = 0$.
Since we established there can't be more than 1 root then there must be exactly 1 root.

4.3.5. Let $f(x)$ be differentiable on (a, b) and $f'(x)$ be continuous on (a, b). Assume there are three points x_1, x_2 and x_3 with each $x_i \in (a, b)$ and with $x_1 < x_2 < x_3$ such that
$$f(x_1) < f(x_2) \quad \text{and} \quad f(x_2) > f(x_3)$$
Use the MVT and the IVT to show that there must be a point $c \in (x_1, x_3)$ such that $f'(c) = 0$.

From the MVT we have that there is a point $s \in (x_1, x_2)$ such that
$$f'(s) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
Since we are told that $f(x_1) < f(x_2)$ this means that $f(x_2) - f(x_1) > 0$ and since $x_2 > x_1$ we have that
$$f'(s) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$
Similarly, the MVT says that there is a point $t \in (x_2, x_3)$ such that
$$f'(t) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$
Since we are told that $f(x_3) < f(x_2)$ this means that $f(x_3) - f(x_2) < 0$ and since $x_3 > x_2$ we have that
$$f'(t) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} < 0$$
Thus we have established that $f'(s) > 0$ and $f'(t) < 0$. Since we know that $f'(x)$ is continuous then the IVT states that there must be a value $c \in (s, t)$ such that $f'(c) = 0$.
Since $x_1 < s < x_2 < t < x_3$ and $c \in (s, t)$ we get that $c \in (x_1, x_3)$.

4.3.6. If $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$, then we say the function f is a *Lipschitz* function.

(a) Use the Mean Value Theorem to prove that if f is a differentiable function and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$ then f is *Lipschitz*.

Suppose $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, and let $x, y \in \mathbb{R}$ be given. If $x = y$ then there is nothing to prove, so let $x \neq y$. Since $f(x)$ is differentiable on $\mathbb{R}$ then it is differentiable on any open interval with endpoints x and y (and hence continuous on the closed interval between x and y). Then, by the Mean Value Theorem, there exists $c \in \mathbb{R}$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(c),$$

which means

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)|$$

and thus

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|$$

as desired.

(b) Is the converse of part (a) true? Prove it or give a counterexample.

Yes, it is true. If we know that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$, then, since the absolute value function is continuous,

$$|f'(x)| = \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} \leq \lim_{h \to 0} \frac{M|x + h - x|}{|h|} = M$$

as desired.

4.4 Antiderivatives

Section 4.4 Problems

4.4.1. Galileo studied the dynamics of falling objects by rolling a brass ball down an inclined plane. He adjusted the spacing of bumps along the plane until the rolling ball clicked at a steady rate. He found that the spacing of the bumps were roughly a sequence of squares so that the total elapsed distance $(x - x_0)$ was related to the time between clicks t as,

$$(x - x_0) \propto t^2, \qquad \text{(Galileo's Law)}.$$

(a) Show that Galileo's Law implies that the ball experiences a constant rate of acceleration. Note that acceleration is the second derivative of position, i.e., $a(t) = x''(t)$.

Using a proportionality constant k,

$$x(t) = x_0 + kt^2.$$

Differentiating twice with respect to time,

$$x'(t) = 2kt, \ x''(t) = 2k,$$

and so the rate fo acceleration is constant $(2k)$.

(b) Conversely, suppose the ball experiences a constant rate of acceleration a,

$$\frac{d^2x}{dt^2} = a,$$

where x is the position of the ball (initially at $x(0) = x_0$ and starting with zero velocity $x'(0) = 0$). Derive Galileo's Law.

The anti derivative of a constant is the linear function,

$$x''(t) = a \qquad \Longrightarrow \qquad x'(t) = C + at.$$

We are given that the ball is initially at rest, $x'(0) = 0$, and so the constant of integration is zero, $C = 0$. The anti derivative of a linear function is the quadratic,

$$x'(t) = at \qquad \Longrightarrow \qquad x(t) = D + \frac{a}{2}\, t^2.$$

We are given that the ball is initially at x_0, $x(0) = x_0$, so the integration constant is $D = x_0$. Altogether,

$$x(t) = x_0 + \frac{a}{2}\, t^2,$$

which is Galileo's Law.

4.4.2. The anti-derivatives of the basic trigonometric functions are straightforward,

$$\frac{d}{dx}\Big(-\cos x + C\Big) = \sin x, \quad \frac{d}{dx}\Big(\sin x + C\Big) = \cos x,$$

but the anti-derivative of their quotient $\tan x$ is more complicated. Given that

$$\frac{d}{dx}\Big(-\ln\left[f(x)\right]+C\Big)=\tan x=\frac{\sin x}{\cos x},$$

determine the function $f(x)$, and thereby determine the anti-derivative of $\tan x$.

Differentiating the given expression,

$$\frac{d}{dx}\Big(-\ln\left[f(x)\right]+C\Big)=-\frac{f'(x)}{f(x)}.$$

Comparing with $\tan x$,

$$\frac{-f'(x)}{f(x)}=\frac{\sin x}{\cos x} \qquad \Longrightarrow \qquad f(x)=\cos x,$$

and so the anti-derivative of $\tan x$ is $-\ln[\cos x]+C$.

4.5 The First Derivative and the Shape of a Function

Section 4.5 Problems

4.5.1. Find the intervals over which the following functions are increasing/decreasing.

(a) $f(x) = x^4 - 8x^2$

Here we have $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$ which has $f'(x) = 0$ when $x = \pm 2$ or $x = 0$. Note that $x^2 - 4 > 0$ only for $x < -2$ or $x > 2$, and $4x > 0$ for $x > 0$. So we get the following table:

$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
x		-2		0		2	

Which means that $f(x)$ is decreasing on $(-\infty, -2]$ and $[0, 2]$ and increasing on $[-2, 0]$ and $[2, \infty)$.

(b) $f(x) = \dfrac{1}{x^2 - 1}$

Here we have $f'(x) = \dfrac{-2x}{(x^2 - 1)^2}$, so $f'(x) = 0$ at $x = 0$ and does not exist at $x = \pm 1$. However, only $x = 0$ is in the domain of f, so $f(x)$ has a single critical point at $x = 0$. However, we should check around ± 1 as well. In this case $(x^2 - 1)^2 \geq 0$ always, and $-2x > 0$ if $x < 0$. So we get:

$f'(x)$	$+$	DNE	$+$	0	$-$	DNE	$-$
x		-1		0		1	

Which means that $f(x)$ is increasing on $(-\infty, -1)$ and $(-1, 0]$ and decreasing on $[0, 1)$ and $(1, \infty)$.
Note that it is incorrect to say that f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ as the function is not defined at $x = \pm 1$.

(c) $f(x) = e^x + e^{-x+1}$

Here we have $f'(x) = e^x - e^{-x+1}$. This is defined everywhere so we just need to look for when $f'(x) = 0$

$$\begin{aligned} f'(x) = 0 \quad &\Rightarrow \quad e^x - e^{-x+1} = 0 \\ &\Rightarrow \quad e^{-x+1} = e^x \\ &\Rightarrow \quad e = e^{2x} \\ &\Rightarrow \quad 2x = 1 \\ &\Rightarrow \quad x = \frac{1}{2}. \end{aligned}$$

Calculating f' everywhere else yields (using $x = 0$ and $x = 1$ as test points)

$f'(x)$	$-$	0	$+$
x		$\frac{1}{2}$	

Which means that $f(x)$ is decreasing on $(-\infty, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, \infty)$.

(d) $f(x) = x^4 - 4x^3 + 16x - 7$

Here we have $f'(x) = 4x^3 - 12x^2 + 16 = 4(x^3 - 3x^2 + 4) = 4(x-2)^2(x+1)$ which has $f'(x) = 0$ at $x = 2$ and $x = -1$. It is clear that the sign of $x + 1$ determines the sign of f', so we get:

$f'(x)$	$-$	0	$+$	0	$+$
x		-1		2	

Which means that $f(x)$ is decreasing on $(-\infty, -1]$ and increasing on $[-1, \infty)$. Note that saying $f(x)$ is increasing on $[-1, 2]$ and $[2, \infty)$ is not incorrect, it is just not including all of the information.

4.5.2. Show that if f is increasing and differentiable on (a, b) then $f'(x) \geq 0$ for all $x \in (a, b)$. *Hint*: You may wish to use the result:

$$\text{If } g(x) > 0 \text{ for all } x \neq a \text{ and } \lim_{x \to a} g(x) = L \text{ then } L \geq 0.$$

For any $t \in (a, b)$ we have, by definition

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Since f is increasing this means that, for $h > 0$, we have $f(t+h) - f(t) > 0$ and so

$$\frac{f(t+h) - f(t)}{h} > 0$$

and for $h < 0$ we have $f(t+h) - f(t) < 0$ and so

$$\frac{f(t+h) - f(t)}{h} > 0.$$

Thus, for $h > 0$ or $h < 0$, we have that

$$\frac{f(t+h) - f(t)}{h} > 0.$$

If we let $g(h) = \dfrac{f(t+h) - f(t)}{h}$, then computing $f'(t)$ is equivalent to finding

$$\lim_{h \to 0} g(h) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

We know that f is differentiable so this limit exists. Furthermore, we know that $g(h) > 0$ (except at $h = 0$) and so by the result given it must be that

$$\lim_{h \to 0} g(h) \geq 0$$

i.e.

$$f'(t) \geq 0$$

and this holds for any $t \in (a, b)$.

4.6 The Second Derivative and the Shape of a Function

Section 4.6 Problems

4.6.1. For each of the following functions find the intervals where the function is concave up or concave down and state any points of inflection.

(a) $f(x) = x^4 + 6x^3 - 60x^2 + 6x - 60$

This is a polynomial that is defined everywhere. We have that

$$f'(x) = 4x^3 + 18x^2 - 120x + 6$$

$$f''(x) = 12x^2 + 36x - 120$$

and so $f''(x)$ is defined everywhere also. Setting $f''(x) = 0$ we get

$$\begin{aligned} f''(x) = 0 \quad &\Leftrightarrow \quad 0 = 12x^2 + 36x - 120 \\ &\Leftrightarrow \quad 0 = x^2 + 3x - 10 \\ &\Leftrightarrow \quad 0 = (x+5)(x-2) \end{aligned}$$

and so we have 2 possible inflection points at $x = -5$ and $x = 2$.
Note that we can rewrite $f''(x)$ as

$$f''(x) = 12(x+5)(x-2)$$

Using this:
For $x < -5$ we get that $f''(x) > 0$.
For $-5 < x < 2$ we get that $f''(x) < 0$.
For $x > 2$ we get $f''(x) > 0$.
And so f is concave up on $(-\infty, -5]$ and $[2, \infty)$
Whereas f is concave down on $[-5, 2]$.
The points of inflection are $(-5, f(-5))$ and $(2, f(2))$, that is, $(-5, -1715)$ and $(2, -224)$.

(b) $f(x) = 1 + \dfrac{1}{x} - \dfrac{1}{x^3}$

This function is not defined at $x = 0$ so we need to consider this when determining concavity.
We have that

$$f'(x) = -\frac{1}{x^2} + \frac{3}{x^4}$$

$$f''(x) = \frac{2}{x^3} - \frac{12}{x^5}$$

As mentioned above, this is not defined at $x = 0$. Setting $f''(x)$ equal to 0 however we get that

$$f''(x) = 0 \quad \Leftrightarrow \quad 0 = \frac{2}{x^3} - \frac{12}{x^5}$$

$$\begin{aligned} &\Leftrightarrow \quad 0 = 2x^2 - 12 \\ &\Leftrightarrow \quad 0 = x^2 - 6 \end{aligned}$$

and so the possible inflection points are at $x = \pm\sqrt{6}$.
Rewriting $f''(x)$ we have

$$f''(x) = \frac{2}{x^5}(x^2 - 6)$$

Using this (and recalling that $x = 0$ is special) we have:
For $x < -\sqrt{6}$ we get $f''(x) < 0$.
For $-\sqrt{6} < x < 0$ we get that $f''(x) > 0$.
For $0 < x < \sqrt{6}$ we get that $f''(x) < 0$.
For $x > \sqrt{6}$ we get that $f''(x) > 0$.
Therefore,
f is concave up on $[-\sqrt{6}, 0)$ and $[\sqrt{6}, \infty)$.
f is concave down on $(-\infty, -\sqrt{6}]$ and $(0, \sqrt{6}]$.
With two points of inflection $(-\sqrt{6}, f(-\sqrt{6}))$ and $(\sqrt{6}, f(\sqrt{6}))$, that is, $(-\sqrt{6}, 1 - \frac{5\sqrt{6}}{36})$ and $(\sqrt{6}, 1 + \frac{5\sqrt{6}}{36}))$. Note that $x = 0$ is not a point of inflection because it is not in the domain of f.

(c) $f(x) = \dfrac{x^2}{x + x^3}$

This function is undefined for $x = 0$. However for all $x \neq 0$ the function is equivalent to $g(x) = \frac{x}{1+x^2}$ so we will examine this function and consider $x = 0$ separately.
We get that

$$g'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$g''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$$

We see that $g''(x) = 0$ when $x = 0, -\sqrt{3}, \sqrt{3}$. We already know that $f(0)$ is undefined so $x = 0$ cannot be a point of inflection.
Nonetheless we consider the various intervals below:
For $x < -\sqrt{3}, f''(x) = g''(x) < 0$
For $-\sqrt{3} < x < 0, f''(x) = g''(x) > 0$
For $0 < x < \sqrt{3}, f''(x) = g''(x) < 0$
For $x > \sqrt{3}, f''(x) = g''(x) > 0$
Therefore,
f is concave down on $(-\infty, -\sqrt{3}]$ and$(0, \sqrt{3}]$.
f is concave up on $[-\sqrt{3}, 0)$ and$[\sqrt{3}, \infty)$.
It has two points of inflection $(-\sqrt{3}, \frac{-3}{\sqrt{3}+3\sqrt{3}})$ and $(\sqrt{3}, \frac{3}{\sqrt{3}+3\sqrt{3}})$.

(d) $f(x) = \dfrac{\sin(x)}{1 + \cos(x)}$

This function is undefined for all $x = (2k+1)\pi$ for $k \in \mathbb{Z}$.
Otherwise,

$$f'(x) = \frac{\cos(x)(1+\cos(x)) + \sin^2(x)}{(1+\cos(x))^2} = \frac{1}{1+\cos(x)}$$

$$f''(x) = \frac{\sin(x)}{(1+\cos(x))^2}$$

Here $f''(x)$ is not defined precisely at the same points where $f(x)$ is not defined, for $x = (2k+1)\pi$ for $k \in \mathbb{Z}$.
We also have that $f''(x) = 0$ when $x = 2k\pi$ for $k \in \mathbb{Z}$.
We thus consider intervals of the form ...$(-3\pi, -2\pi), (-2\pi, -\pi), (-\pi, 0), (0, \pi), (\pi, 2\pi)$.....
We write these as $I_n = (n\pi, (n+1)\pi)$ for $n \in \mathbb{Z}$.
The function $f''(x)$ has a denominator that is always positive so the function f is concave up whenever $\sin(x)$ is positive and concave down when $\sin(x)$ is negative.
Specifically:
$f''(x) < 0$ when $x \in I_{2n+1}$ for $n \in \mathbb{Z}$ and thus f is concave down on

$$\ldots(-3\pi, -2\pi], (-\pi, 0], (\pi, 2\pi]\ldots$$

$f''(x) > 0$ when $x \in I_{2n}$ for $n \in \mathbb{Z}$ and thus f is concave up on

$$\ldots[-2\pi, -\pi), [0, \pi), [2\pi, 3\pi)\ldots$$

The points of inflection occur at all $x = 2k\pi$ for $k \in \mathbb{Z}$ (with corresponding y value of 0).

(e) $f(x) = x^2 - 9x^{\frac{1}{3}}$

Note that this function is defined for all x.
We have

$$f'(x) = 2x - \frac{3}{x^{\frac{2}{3}}}$$

$$f''(x) = 2 + \frac{2}{x^{\frac{5}{3}}}$$

From this we get that $f''(x)$ is undefined at $x = 0$. Furthermore, setting $f''(x) = 0$ we get

$$0 = 2 + \frac{2}{x^{\frac{5}{3}}}$$

$$0 = x^{\frac{5}{3}} + 1$$

which has a single (real) solution of $x = -1$.
Using $f''(x) = 2\left(1 + \frac{1}{x^{\frac{5}{3}}}\right)$ we see that:
for $x < -1$, we get $f''(x) > 0$
for $-1 < x < 0$, we get $f''(x) < 0$
for $x > 0$, we get $f''(x) > 0$

Therefore,
f is concave up on $(-\infty, -1]$ and $[0, \infty)$
f is concave down on $[-1, 0]$
There are two points of inflection at $(-1, 10)$ and at $(0, 0)$.

(f) $f(x) = x\ln(x^2 - 2x)$

Note this function is undefined when $x^2 - 2x \le 0$ which is equivalent to $x(x-2) \le 0$ which occurs when $0 \le x \le 2$.
We have

$$f'(x) = \ln(x^2 - 2x) + \frac{2(x-1)}{x-2}$$

$$f''(x) = \frac{2(x^2 - 4x + 2)}{x(x-2)^2}$$

We see that $f''(x)$ is undefined at $x = 0$ and $x = 2$ but so is the original function so no new information is gained.
Setting $f''(x) = 0$ requires that $x^2 - 4x + 2 = 0$ which happens at $x = 2 \pm \sqrt{2}$. However the function is undefined at $x = 2 - \sqrt{2}$ so we only consider $x = 2 + \sqrt{2}$.
We thus have:
for $x < 0$, we see that $f''(x) < 0$
for $2 < x < 2 + \sqrt{2}$, we see that $f''(x) < 0$
for $x > 2 + \sqrt{2}$ we see that $f''(x) > 0$.
Thus:
f is concave up $[2 + \sqrt{2}, \infty)$
f is concave down on $(-\infty, 0)$ and $(2, 2 + \sqrt{2}]$
There is one point of inflection at $(2 + \sqrt{2}, f(2 + \sqrt{2}))$, that is
$(2 + \sqrt{2}, (2 + \sqrt{2})\ln((2 + \sqrt{2})^2 - 2(2 + \sqrt{2})))$.

4.6.2. Consider the function

$$f(x) = \begin{cases} \dfrac{x^3}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Compute $f'(x)$ for all possible x. Your answer can be piecewise if necessary.

For $x > 0$ we have that $f(x) = x^2$ and so $f'(x) = 2x$.
For $x < 0$ we have that $f(x) = -x^2$ and so $f'(x) = -2x$
At $x = 0$, by definition we have that

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

This requires 2 cases given the definition of f. We have

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{h^2 - 0}{h} \\ &= \lim_{h\to 0^+} h \end{aligned}$$

$$= 0$$

Similarly

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{-h^2-0}{h} \\
&= \lim_{h\to 0^-} -h \\
&= 0
\end{aligned}$$

And so, by definition, $f'(0) = 0$. (Note that it is also ok to take $\lim_{x\to 0} f'(x)$ though it may not be obvious when this is allowed).
Thus we have that

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

or

$$f'(x) = 2|x|$$

(Note that we could swap the strict inequalities. That is, replace $x \geq 0$ with $x > 0$ and $x < 0$ with $x \leq 0$).

(b) Compute $f''(x)$ for all possible x. Your answer can be piecewise if necessary.

For $x \geq 0$ we have that $f'(x) = 2x$ so that, for $x > 0$, $f''(x) = 2$.
For $x < 0$ we have that $f'(x) = -2x$ so that, for $x < 0$, $f''(x) = -2$.
At this point it seems like we won't be able to get a value for $f''(0)$, however the definition is the safest way to find out.

$$\begin{aligned}
f''(0) &= \lim_{h\to 0} \frac{f'(0+h)-f'(0)}{h} \\
&= \lim_{h\to 0} \frac{2|h|}{h} \\
\Rightarrow &\quad f''(0) \quad \text{does not exist}
\end{aligned}$$

Thus we have

$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}$$

(c) Does f have an inflection point at $x = 0$?

Note first that f is continuous at $x = 0$ since $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0 = f(0)$.
Also, since $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$ then the function $f(x)$ is concave down for $x < 0$ and concave up for $x > 0$.
Therefore, since f changes concavity through $x = 0$ and f is continuous at $x = 0$, we have by definition that $x = 0$ is an inflection point for f.

4.6.3. Let f and g be twice differentiable functions (that is, both $f''(x)$ and $g''(x)$ exist for all $x \in \mathbb{R}$). Prove or disprove (that is, find a counterexample to) the following statements:

(a) If the graph of f is concave up on $\mathbb{R}$, the graph of g is concave down on $\mathbb{R}$, and $f(x) > g(x)$ for all $x \in \mathbb{R}$, then the graph of $f + g$ is concave up on $\mathbb{R}$.

This is false. Consider $f(x) = x^2 + 1$ and $g(x) = -2x^2$. Then the graph of f is concave up on $\mathbb{R}$, the graph of g is concave down on $\mathbb{R}$, and $f(x) > g(x)$ for all $x \in \mathbb{R}$, but $f(x) + g(x) = 1 - x^2$, which is concave down.

(b) If the graph of f is concave up on $\mathbb{R}$, and the graph of g is concave down on $\mathbb{R}$, then the graph of $f + g$ is neither concave up nor concave down on all of $\mathbb{R}$.

This is false. Consider $f(x) = 2x^2$ and $g(x) = -x^2$. Then the graph of f is concave up on $\mathbb{R}$, and the graph of g is concave down on $\mathbb{R}$, but $f(x) + g(x) = x^2$ is concave up on $\mathbb{R}$.

4.6.4. Prove that if f is a twice differentiable function such that $f''(x) \neq 0$ for all $x \in \mathbb{R}$, f is positive on $\mathbb{R}$, and the graph of f is concave up on $\mathbb{R}$, then the graph of $g = (f)^2$ is concave up on $\mathbb{R}$.

Suppose that f is a twice differentiable function such that $f''(x) \neq 0$ for all $x \in \mathbb{R}$, f is positive on $\mathbb{R}$, and the graph of f is concave up on $\mathbb{R}$. Since the graph of f is concave up on $\mathbb{R}$ and $f''(x) \neq 0$ we get that $f''(x) > 0$ for all $x \in \mathbb{R}$.
Next, consider $g(x) = (f(x))^2$. Then $g'(x) = 2f(x)f'(x)$, and $g''(x) = 2f''(x)f(x) + 2(f'(x))^2 > 0$ since $f(x) > 0$, $f''(x) > 0$, and $(f'(x))^2 \geq 0$. Therefore, the graph of g is concave up on $\mathbb{R}$.

4.7 Classifying Critical Points

Section 4.7 Problems

4.7.1. Locate and classify the critical points of each of the following functions:
(a) $f(x) = 3x^5 - 20x^3 + 15$

Here we have

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x-2)(x+2)$$

which has critical points at $x = 0, -2, 2$.
Using the first derivative test:
For $x < -2$ we have $f'(x) > 0$
For $-2 < x < 0$ we have $f'(x) < 0$
For $0 < x < 2$ we have $f'(x) < 0$
For $x > 2$ we have $f'(x) > 0$.

Thus the critical point:
$(2, f(2))$ is a local min
$(-2, f(-2))$ is a local max
$(0, f(0))$ is neither a max nor a min.

$\boxed{\text{Alternate solution}}$
Computing $f''(x)$ gives

$$f''(x) = 60x^3 - 120x = 60x(x^2 - 1) = 60x(x-1)(x+1)$$

So that:

$f''(-2) < 0$ meaning the graph of f is concave down and so $(-2, f(-2))$ is a local max
$f''(0) = 0$ so we can't conclude anything (need first derivative test here)
$f''(2) > 0$ meaning the graph of f is concave up and so $(2, f(2))$ is a local min.

(b) $f(x) = \dfrac{x+1}{x-1} - x$

Note that this function is not defined at $x = 1$ and so it cannot contain a critical point there.
We have

$$f'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} - 1 = -\frac{x^2 - 2x + 3}{(x-1)^2}$$

This means that $f'(x)$ does not exist at $x = 1$ (though neither does $f(x)$ so nothing

gained). It also means that $f'(x) = 0$ when $x^2 - 2x + 3 = 0$ which is impossible (try the quadratic formula).
Therefore, there are no critical points of this function.

(c) $f(x) = \dfrac{x+1}{x-1} + x$

Note that this function is not defined at $x = 1$ and so it cannot contain a critical point there.
We have

$$f'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} + 1 = \frac{x^2-2x-1}{(x-1)^2}$$

This means that $f'(x)$ does not exist at $x = 1$ (though neither does $f(x)$ so nothing gained). It also means that $f'(x) = 0$ when $x^2 - 2x - 1 = 0$ which happens at $x = 1 \pm \sqrt{2}$.
Using the first derivative test we note that the denominator of $f'(x)$ is always positive so that we only need to know the sign of $x^2 - 2x - 1$ which is a parabola opening upward. That is:
For $x < 1 - \sqrt{2}$, we get $f'(x) > 0$
For $1 - \sqrt{2} < x < 1 + \sqrt{2}$, we get $f'(x) < 0$
For $x > 1 + \sqrt{2}$, we get $f'(x) > 0$.

Thus the critical point:
$(1 - \sqrt{2}, f(1 - \sqrt{2}))$ is local max
$(1 + \sqrt{2}, f(1 + \sqrt{2}))$ is a local min

$\boxed{\text{Alternate solution}}$
We have

$$f''(x) = \frac{4}{(x-1)^3}$$

so that for any $x < 1$ the function is concave down and for any $x > 1$ the function is concave up.
Thus at $x = 1 + \sqrt{2}$ the function is concave up and we must have a local min and at $x = 1 - \sqrt{2}$ the function is concave down we must have a local max.

(d) $f(x) = 12x^{\frac{1}{3}} + |x|$

For $x > 0$ we have $f(x) = 12x^{\frac{1}{3}} + x$ so that

$$f'(x) = \frac{4}{x^{\frac{2}{3}}} + 1$$

which is undefined at $x = 0$ (though we are assuming $x > 0$) and, when setting

$f'(x) = 0$ gives

$$\begin{aligned} 0 &= \frac{4}{x^{\frac{2}{3}}} + 1 \\ x^{\frac{2}{3}} &= -4 \\ x^2 &= -64 \end{aligned}$$

which does not have any real solutions.
For $x < 0$ we have $f(x) = 12x^{\frac{1}{3}} - x$ so that

$$f'(x) = \frac{4}{x^{\frac{2}{3}}} - 1$$

which is also undefined at $x = 0$ (though once again we are assuming $x < 0$). Setting $f'(x) = 0$ in this case gives

$$\begin{aligned} 0 &= \frac{4}{x^{\frac{2}{3}}} - 1 \\ x^{\frac{2}{3}} &= 4 \\ x^2 &= 64 \\ x &= \pm 8 \end{aligned}$$

However since we assumed $x < 0$ then only $x = -8$ is valid.
So our only critical point is $x = -8$ though based on the derivatives at each side of $x = 0$ it might be worth investigating $f'(0)$ as well.
By definition

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{12h^{\frac{1}{3}} + |h|}{h} \\ &= \lim_{h \to 0} \frac{12}{h^{\frac{2}{3}}} + \frac{|h|}{h} \end{aligned}$$

From here we see that as $h \to 0^+$ we get that

$$\lim_{h \to 0^+} \frac{12}{h^{\frac{2}{3}}} + \frac{|h|}{h} = \lim_{h \to 0^+} \frac{12}{h^{\frac{2}{3}}} + 1 = \infty$$

Similarly, because of the h^2 in the denominator

$$\lim_{h \to 0^-} \frac{12}{h^{\frac{2}{3}}} + \frac{|h|}{h} = \lim_{h \to 0^-} \frac{12}{h^{\frac{2}{3}}} + 1 = \infty$$

So the derivative doesn't exist (it's infinity!) but since $f(0)$ exists we should consider $x = 0$ as a critical point as well.

This means, using the first derivative test:

For $x < -8$ we get that $f'(x) = \frac{4}{x^{\frac{2}{3}}} - 1 < 0$
For $-8 < x < 0$ we get that $f'(x) = \frac{4}{x^{\frac{2}{3}}} - 1 > 0$
For $x > 0$ we get that $f'(x) = \frac{4}{x^{\frac{2}{3}}} + 1 > 0$

And thus the critical point:
$(-8, f(-8))$ is a local min
$(0, f(0))$ is neither a max or a min.

(e) $f(x) = 4e^x - e^{2x} - x$

This function is differentiable everywhere. We have

$$f'(x) = 4e^x - 2e^{2x} - 1$$

The only possible critical points occur when $f'(x) = 0$. Doing so yields

$$\begin{aligned} 0 &= 4e^x - 2e^{2x} - 1 \\ &= 2e^{2x} - 4e^x + 1 \quad \text{(note this is a quadratic in } e^x\text{)} \\ \Leftrightarrow \quad e^x &= \frac{4 \pm \sqrt{16-8}}{4} \\ &= 1 \pm \frac{1}{\sqrt{2}} \end{aligned}$$

And so we have $e^x = 1 \pm \frac{1}{\sqrt{2}}$ meaning

$$x = \ln\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

To classify these we use the second derivative test (though the first derivative test would also work).
We have

$$f''(x) = 4e^x - 4e^{2x} = 4e^x(1 - e^x)$$

From this we see that $f''(x) = 0$ only when $1 - e^x = 0$ which happens at $x = 0$. Otherwise, for $x < 0$ we have $f''(x) > 0$ and for $x > 0$ we have $f''(x) < 0$. That is:

f is concave up for $x < 0$
f is concave down for $x > 0$
Now, since $x = \ln\left(1 + \frac{1}{\sqrt{2}}\right) > 0$ and $x = \ln\left(1 - \frac{1}{\sqrt{2}}\right) < 0$ we have that:

$x = \ln\left(1 + \frac{1}{\sqrt{2}}\right)$ is a local max
$x = \ln\left(1 - \frac{1}{\sqrt{2}}\right)$ is a local min.

(f) $f(x) = \ln(x+1) + x^2 - 6x$

This function is defined for all $x > -1$. The first derivative gives

$$f'(x) = \frac{1}{x+1} + 2x - 6$$

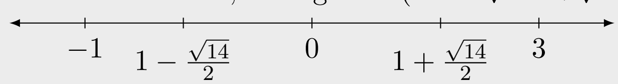
which is defined for all $x \neq -1$ (though the function is not defined there so it can't be a critical point).
Setting $f'(x) = 0$ yields

$$\begin{aligned} 0 &= \frac{1}{x+1} + 2x - 6 \\ \Leftrightarrow \quad 6 - 2x &= \frac{1}{x+1} \\ \Leftrightarrow \quad (6-2x)(x+1) &= 1 \\ \Leftrightarrow \quad -2x^2 + 4x + 5 &= 0 \end{aligned}$$

From this we get that

$$x = \frac{-4 \pm \sqrt{56}}{-4} = 1 \pm \frac{\sqrt{14}}{2}$$

Here we use the first derivative test, noting that (since $\sqrt{14} < \sqrt{16}$)



That is, using $f'(x) = \frac{1}{x+1} + 2x - 6$ we have:
for $x < 1 - \frac{\sqrt{14}}{2}$ we get $f'(x) > 0$ (as $x \to -1^+$ the expression $\frac{1}{x+1}$ can be made arbitrarily large)
for $1 - \frac{\sqrt{14}}{2} < x < 1 + \frac{\sqrt{14}}{2}$ we have $f'(x) < 0$
for $x > 1 + \frac{\sqrt{14}}{2}$ we have $f'(x) > 0$
This means that
$(1 - \frac{\sqrt{14}}{2}, f(1 - \frac{\sqrt{14}}{2}))$ is a local max
$(1 + \frac{\sqrt{14}}{2}, f(1 + \frac{\sqrt{14}}{2}))$ is a local min.

(g) $f(x) = |x| + \sin^2(x)$

$\boxed{x > 0}$
For $x > 0$ we have $f(x) = x + \sin^2(x)$ which gives

$$f'(x) = 1 + 2\sin(x)\cos(x) = 1 + \sin(2x).$$

Setting this to zero gives

$$\begin{aligned} 0 &= 1 + \sin(2x) \\ \Rightarrow \quad -1 &= \sin(2x) \end{aligned}$$

This implies that $2x = \frac{(4k+3)\pi}{2}$ for $k \in \mathbb{Z}$. However we are assuming $x > 0$ so we get the critical points

$$x = \frac{(4k+3)\pi}{4} \qquad k \geq 0 \in \mathbb{Z}$$

Note that $f''(x) = 2\cos(2x)$ and so the second derivative test tells us nothing here as $f''(x) = 0$ for the points above.
Using the first derivative test we see that since $f'(x) = 1 + \sin(2x)$ and since $1 + \sin(2x) \geq 0$ it is impossible for $f'(x) < 0$ and thus $f'(x) \geq 0$ for all $x > 0$.
Thus none of the critical points are local maxes or mins for $x > 0$.
$\boxed{x < 0}$
Since the function $f(x)$ is even, we conclude that for $x < 0$ we get a similar situation (except $f'(x) = -1 + \sin(2x) \leq 0$ for all $x < 0$) and thus the critical points (that is, where $\sin(2x) = 1$) would be

$$x = \frac{(4k+1)\pi}{4} \qquad k < 0 \in \mathbb{Z}$$

and none would be local maxes and mins.
$\boxed{x = 0}$
At $x = 0$ we have, by definition

$$\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{|h| + \sin^2(h)}{h} \\
&= \lim_{h \to 0} \frac{|h|}{h} + \lim_{h \to 0} \sin(h) \cdot \lim_{h \to 0} \frac{\sin(h)}{h} \\
&= \lim_{h \to 0} \frac{|h|}{h} + 0 \cdot 1 \\
&= \lim_{h \to 0} \frac{|h|}{h}
\end{aligned}$$

which we know doesn't exist. And so $f'(0)$ does not exist, thus $x = 0$ is a critical point of $f(x)$.
Furthermore, we already established that $f'(x) \leq 0$ for $x < 0$ and $f'(x) \geq 0$ for $x > 0$ (with equality only happening at the critical points) and so we have established that $x = 0$ is local min.

4.7.2. Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic equation.

(a) Find $a, b, c, d \in \mathbb{R}$ such that $f(x)$ has a local maximum at $(-2, 3)$ and local minimum at $(1, 0)$.

Note that
$$f'(x) = 3ax^2 + 2bx + c$$
is a polynomial which is defined everywhere in $\mathbb{R}$. Hence, a local max or minimum only occurs when $f'(x) = 0$.
We have $f'(-2) = 0 = f'(1)$. So f' must be a multiple of $(x+2)(x-1)$. Also, we know the leading coefficient is $3a$, so we get

$$f'(x) = 3a(x+2)(x-1) = 3ax^2 + 3ax - 6a.$$

Thus $2b = 3a$, i.e., $b = \frac{3}{2}a$, and $c = -6a$.

So
$$f(x) = ax^3 + \frac{3}{2}ax^2 - 6ax + d.$$
Since $f(-2) = 3$ and $f(1) = 0$, we get
$$-8a + \frac{3}{2}4a - 6a(-2) + d = 0 \quad \text{and} \quad a + \frac{3}{2}a - 6a + d = 0,$$
i.e.,
$$10a + d = 3 \quad (1) \quad \text{and} \quad \frac{-7}{2}a + d = 0 \quad (2).$$
By considering (1) - (2), we get $\frac{27}{2}a = 3$, i.e., $a = \frac{2}{9}$.
Then it follows that $b = \frac{1}{3}, c = \frac{-4}{3}, d = \frac{7}{9}$.
Thus
$$f(x) = \frac{2}{9}x^3 + \frac{1}{3}x^2 - \frac{4}{3}x + \frac{7}{9}.$$

(b) Use (a) to find the intervals of concavity and the inflection points of $f(x)$.

$$f'(x) = \frac{2}{3}x^2 + \frac{2}{3}x - \frac{4}{3} \qquad \text{and} \qquad f''(x) = \frac{4}{3}x + \frac{2}{3}.$$
Hence, $f''(x) > 0$ if $\frac{4}{3}x + \frac{2}{3} > 0$, i.e., f is concave up if $x \geq -\frac{1}{2}$.
$f''(x) < 0$ if $\frac{4}{3}x + \frac{2}{3} < 0$, i.e., f is concave down if $x \leq -\frac{1}{2}$.
Thus $f(x)$ has an inflection point at $x = -\frac{1}{2}$.

4.8 L'Hôpital's Rule

Section 4.8 Problems

4.8.1. Evaluate the following indeterminate forms using l'Hôpital's rule.

$$\text{a)}\ \lim_{x\to\pi}\frac{x\sin x}{x-\pi},\qquad \text{b)}\ \lim_{x\to\frac{\pi}{2}}\frac{\ln(2-\sin x)}{\ln(1+\cos x)},\qquad \text{c)}\ \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}},$$

$$\text{d)}\ \lim_{x\to 0}x^{\varepsilon}\ln x\ \ (\varepsilon>0),\qquad \text{e)}\ \lim_{x\to\infty}x^{r}e^{-x}\ \ (r\in\mathbb{R}, r>0).$$

a) By l'Hôpital's rule,

$$\lim_{x\to\pi}\frac{x\sin x}{x-\pi}=\lim_{x\to\pi}\frac{\sin x+x\cos x}{1}$$

which is continuous at $x=\pi$. Consequently,

$$\lim_{x\to\pi}\frac{x\sin x}{x-\pi}=\lim_{x\to\pi}\frac{\sin x+x\cos x}{1}=\sin\pi+\pi\cos\pi=-\pi$$

b) By l'Hôpital's rule,

$$\lim_{x\to\frac{\pi}{2}}\frac{\ln(2-\sin x)}{\ln(1+\cos x)}=\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos x}{2-\sin x}\right)\times\left(-\frac{1+\cos x}{\sin x}\right)$$

Both terms have a well-defined limit, and so the limit of the product is a product of the limits,

$$\lim_{x\to\frac{\pi}{2}}\frac{\ln(2-\sin x)}{\ln(1+\cos x)}$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos x}{2-\sin x}\right)\times\left(-\frac{1+\cos x}{\sin x}\right)=(0)\times(-1)=0$$

c) By l'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{\frac{1}{2}x^{-1/2}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

d) Re-writing the limit,

$$\lim_{x\to 0}x^{\varepsilon}\ln x=\lim_{x\to 0}\frac{\ln x}{x^{-\varepsilon}}$$

By l'Hôpital's rule,

$$\lim_{x\to 0}x^{\varepsilon}\ln x=\lim_{x\to 0}\frac{\ln x}{x^{-\varepsilon}}=\lim_{x\to 0}\frac{1/x}{-\varepsilon x^{-\varepsilon-1}}=\lim_{x\to 0}\frac{-x^{\varepsilon}}{\varepsilon}=0$$

for $\varepsilon>0$.

e) It is helpful to re-write the limit,

$$\lim_{x\to\infty}x^{r}e^{-x}=\lim_{x\to\infty}\frac{x^{r}}{e^{x}}$$

For the real number $r > 0$, there is an integer N so that $r \le N$ (or, $r - N \le 0$). Applying l'Hôpital's rule N times,

$$\lim_{x\to\infty} x^r e^{-x} = \lim_{x\to\infty} \frac{x^r}{e^x}$$

$$= \lim_{x\to\infty} \frac{r(r-1)(r-2)\cdots(r-N+1)x^{r-N}}{e^x}$$

$$= \lim_{x\to\infty} \frac{r(r-1)(r-2)\cdots(r-N+1)}{e^x x^{N-r}} = 0$$

4.8.2. Evaluate

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2},$$

where f is a twice-differentiable function.

The limit is an indeterminate form (0/0); applying l'Hôpital's rule (with respect to h),

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

$$= \lim_{h\to 0} \frac{f'(x+h) - f'(x) + f'(x) - f'(x-h)}{2h}$$

$$= \frac{1}{2} \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} + \frac{f'(x) - f'(x-h)}{h}$$

$$= \frac{1}{2}\Big(f''(x) + f''(x)\Big) = f''(x).$$

This limit is called the *Newton quotient for the second derivative*, and forms the basis for a centred-difference scheme approximating the second derivative.

4.8.3. What is wrong with the following use of l'Hôpital's rule:

$$\lim_{x\to 1} = \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x\to 1} \frac{6x}{2} = 3$$

(The limit is actually -4).

The second limit is not an indeterminate form. Recall that the derivation of l'Hôpital's rule is predicated on the numerator and the denominator vanishing at the limit point so that the quotient can be replaced by the quotient of derivatives, *i.e.*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f(x_0) + f'(x_0)(x - x_0)}{g(x_0) + g'(x_0)(x - x_0)} = \frac{f'(x_0)}{g'(x_0)}$$

if (and only if) $f(x_0) = g(x_0) = 0$. It is true that,

$$\lim_{x\to 1} = \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3}$$

but this last quotient is continuous at the limit point $x = 1$, and so

$$\lim_{x\to 1} = \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3} = \frac{3 \times (1)^2 + 1}{2 \times (1) - 3} = -4$$

4.9 Curve Sketching

Section 4.9 Problems

4.9.1. For each of the following functions perform all the steps we outlined to sketch the curve with calculus, and then sketch the curve. This includes listing the domain, any asymptotes, intervals of increase/decrease, intervals of concavity, and along with the sketch plotting any critical points, points of inflection, and intercepts.

(a) $f(x) = x^3 - 6x^2 + 9x$.

1. Domain is $\mathbb{R}$

2. The $y-$intercept $(x = 0)$ is $y = 0$, that is, the point $(0, 0)$. The $x-$intercepts $(y = 0)$ are: $0 = x^3 - 6x^2 + 9x = x(x - 3)^2$ so $x = 0, 3$.

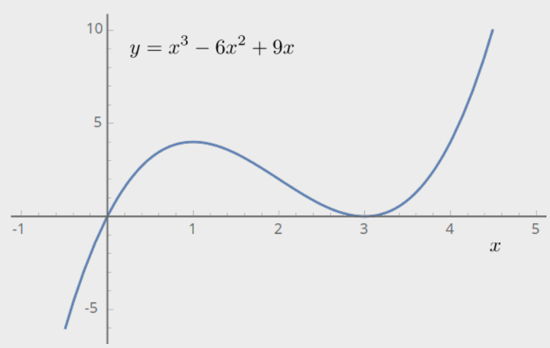
3. This function has no asymptotes.

4. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$. So $f'(x) = 0$ when $x = 1, 3$. These are both critical points. For later, the points are $(1, 4)$ and $(3, 0)$.

5. $f''(x) = 6x - 12 = 6(x - 2)$ so $f''(x)$=0 when $x = 2$.

6. Now, we test all of the intervals in a big table:

So we can see that f is increasing on the intervals $(-\infty, 1]$ and $[3, \infty)$, decreasing on $[1, 3]$, concave up on $[2, \infty)$, and concave down on $(-\infty, 2]$. The function has a point of inflection at $x = 2$, a local maximum at $x = 1$, and a local minimum at $x = 3$. The Sketch:



(b) $f(x) = \dfrac{x^2 + 1}{x^2 - 9}$.

1. Domain is $x \neq \pm 3$.

2. The $y-$intercept $(x = 0)$ is $y = \dfrac{-1}{9}$ (so the point $\left(0, \dfrac{-1}{9}\right)$. The function

has no x-intercepts.

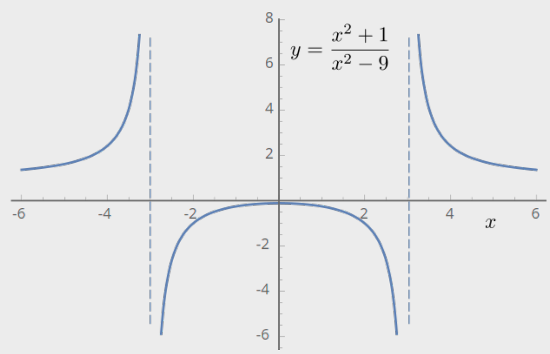
3. This function has vertical asymptotes at $x = \pm 3$ and a horizontal asymptote at $x = 1$ since $\lim_{x\to\pm\infty} \frac{x^2+1}{x^2-9} = 1$.

4. $f'(x) = \frac{-20x}{(x^2-9)^2}$. So $f'(x) = 0$ when $x = 0$ and is undefined at $x = \pm 3$. Only $x = 0$ is a critical point. For later, the point is $\left(0, \frac{-1}{9}\right)$.

5. $f''(x) = \frac{60x^2+180}{(x^2-9)^3}$ so $f''(x)$ is never zero, and is undefined when $x = \pm 3$.

6. Now, we test all of the intervals in a big table:

So we can see that f is increasing on the intervals $(-\infty, -3)$ and $(-3, 0]$, decreasing on $[0, 3)$ and $(3, \infty)$, concave up on $(-\infty, -3)$ and $(3, \infty)$, and concave down on $(3, 3)$. The function has no points of inflection, and a local maximum at $x = 0$.
The Sketch:



(c) $f(x) = \ln(18 - 2x^2)$.

We present an alternate (but totally acceptable) solution where we test the intervals of increase/decrease and concavity right away (instead of making a big table).
1. $f(x)$ is defined when $18 - 2x^2 > 0$, i.e., $2x^2 < 18$, i.e., $x^2 < 9$. So the domain of f is $(-3, 3)$.

2. if $x = 0$, then $f(0) = \ln 18$. Also, $f(x) = 0$ if $18 - 2x^2 = 1$, i.e., $2x^2 = 17$, i.e., $x = \pm\sqrt{\frac{17}{2}}$.

3. Note that as $x \to 3^-$ or $x \to (-3)^+$, $18 - 2x^2 \to 0^+$. Hence

$$\lim_{x\to 3^-} \ln(18 - 2x^2) = -\infty \qquad \text{and} \qquad \lim_{x\to -3^+} \ln(18 - 2x^2) = -\infty.$$

Therefore, f has vertical asymptotes at $x = \pm 3$ and no horizontal asymptotes.
4.
$$f'(x) = \frac{1}{18 - 2x^2}(18 - 2x^2)' = \frac{-4x}{18 - 2x^2} = \frac{-2x}{9 - x^2}.$$

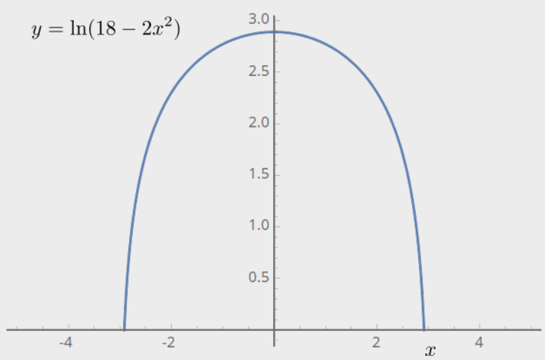
Since $9 - x^2 > 0$ for all $|x| < 3$, $f'(x) > 0$ when $x < 0$ and $f'(x) < 0$ when $x > 0$. Thus $f(x)$ is increasing on $(-3, 0)$ and $f(x)$ is decreasing on $(0, 3)$.
Critical numbers: $f'(x) = 0$ when $x = 0$ and $f'(x)$ is well-defined for all $x \in (-3, 3)$. So $x = 0$ is the only critical point. Also, since $f'(x)$ changes signs from positive to negative at $x = 0$, we see that 0 is a local maximum.

5.

$$f''(x) = \frac{-2(9 - x^2) - (-2x)(-2x)}{(9 - x^2)^2} = \frac{-18 - 2x^2}{(9 - x^2)^2}.$$

Since $(9 - x^2)^2 > 0$ and $-18 - 2x^2 < 0$, $f''(x) < 0$ for all $x \in (-3, 3)$.
Thus $f(x)$ is concave down on $(-3, 3)$ and there are no inflection points.

The Sketch:



4.9.2. Given the function $f(x) = \frac{1}{x^3 - x}$ which of the following properties are true:

(a) There is a point of inflection at $x = -1$.

False

(b) f is concave up from $[2, 5]$.

True

(c) $\displaystyle\sum_{c \in D} f(c) = 0$ where D is the set of local extrema.

True

(d) f is decreasing on $[\frac{1}{\sqrt{3}}, \infty)$.

False

(e) f is concave down from $[0, 1]$.

False

(f) There are more vertical asymptotes than horizontal asymptotes.

True

(g) f has no x or y intercepts.

True

(h) $f(-x) = f(x)$ for all x in the domain of f.

False

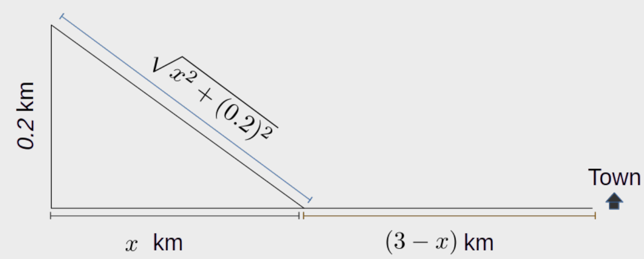
(i) Let $b > 1$ and $a < -1$, there is a $c \in (-1, 1)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

True

4.10 Optimization

Section 4.10 Problems

4.10.1. A swimmer is 200 m out in the ocean and would like to arrive at a town 3 km down the coast. The swimmer will swim to shore, then jog along the beach to the town. She can swim at 2.5 km/h and jog at 10 km/h. To what point along the shore should she swim so that the total time it takes to get to town is a minimum? What about if the time it takes is a maximum?



If x is the distance along the shore that the swimmer alights, then from the geometry of the problem, the total swim distance is $\sqrt{x^2+(0.2)^2}$ km and the total jog distance is $3-x$ km. Given the difference in swim and jog speeds, the total time $T(x)$ (in hours) it takes to arrive at the town is,

$$T(x) = \frac{\sqrt{x^2+(0.2)^2}}{2.5} + \frac{(3-x)}{10}.$$

The extrema will occur at critical points or at the boundaries. Here, $x \in [0,3]$. The critical points are determined by the first-derivative,

$$T'(x) = \frac{0.4x}{\sqrt{x^2+0.04}} - \frac{1}{10}.$$

The derivative is nowhere undefined, but becomes zero at the critical points $x^\star$

$$T'(x^\star) = \frac{0.4x^\star}{\sqrt{(x^\star)^2+0.04}} - \frac{1}{10} = 0.$$

Simplifying,

$$\frac{0.4x^\star}{\sqrt{(x^\star)^2+0.04}} = \frac{1}{10},$$

which can only be satisfied by $x^\star > 0$. To determine this point, square both sides to find

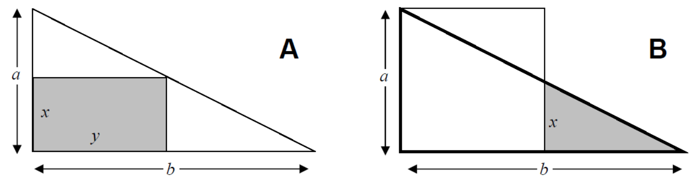
$$15(x^\star)^2 = 0.04 \implies x^\star = \sqrt{\frac{0.04}{15}} \approx 0.0516 \text{ km}$$

because the critical point is positive. Evaluating the total time at the boundaries and the critical point,

$$T(0) = 0.380\text{h}, \quad T(x^\star) = 0.377\text{h}, \quad T(3) = 1.20\text{h}.$$

The minimum time is achieved if the swimmer alights a distance 52 m along the shore (resulting in a savings of 0.2 s); the maximum is achieved if the swimmer swims the whole way to the town ($x = 3$ km).

4.10.2. The following geometry problems appear on small wooden tablets hung in Japanese temples (Sangaku).



A. (For Ex. 2a) Given side lengths a and b, find x and y to maximize the area of the inscribed rectangle. **B.** (For Ex. 2b) Maximize the area of the shaded triangle as a function of a, assuming the base b is fixed.

(a) A rectangle shares two edges with the perpendicular sides of a right angled triangle; the fourth vertex lies on the hypotenuse. Find the side lengths of the inscribed rectangle that result in the maximum area

The two white triangles are similar to one-another, and similar to the larger triangle. Consequently, the ratio of their perpendicular sides are equal,

$$\frac{a}{b} = \frac{x}{(b-y)} = \frac{(a-x)}{y}$$

Isolating x from the first two terms above,

$$x = \frac{a(b-y)}{b} \tag{4.1}$$

The area $A(y)$ of the inscribed rectangle is then,

$$A(y) = xy = \frac{a(b-y)}{b}\, y = \frac{aby - ay^2}{b}$$

The extremal points are determined from the zeros of the derivative of $A(y)$,

$$\frac{dA}{dy} = ab - 2ay$$

which vanishes when $y^\star = \frac{ab}{2a} = \frac{b}{2}$. From Eq. 4.1, the height of the rectangle with maximum area is $x^\star = \frac{a}{2}$.

(b) A right-angled triangle, with perpendicular side-lengths a and b, shares an edge with a square of side-length a. Assume that the edge-length a is variable, and find the value of a in terms of b that maximizes the shaded portion of the triangle.

The gray triangle is similar to the large triangle, so that,

$$\frac{a}{b} = \frac{x}{b-a}, \qquad \text{or} \qquad x = \frac{(b-a)a}{b}$$

The area of the gray triangle is

$$A(a) = \frac{1}{2}x(b-a) = \frac{1}{2}\frac{(b-a)a}{b}(b-a) = \frac{1}{2}(b-a)^2\frac{a}{b}$$

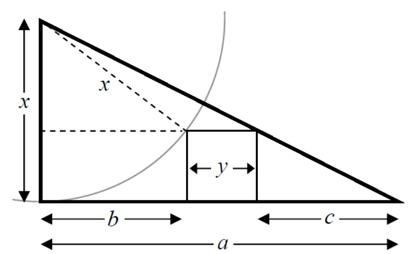
The extremal points are determined from the zeros of the derivative of $A(a)$,

$$\frac{dA}{da} = \frac{(b-a)^2}{2b} + \frac{a}{b}(-1)(b-a)$$

The derivative vanishes for $a = a^\star$,

$$\frac{(b-a^\star)}{2b} - \frac{a^\star}{b} = 0, \qquad \text{or} \qquad a^\star = \frac{b}{3}.$$

The maximal area is then $A(a^\star) = \frac{2}{27}b^2$.



(For Ex. 2c) Maximize y as a function of x assuming that the base a is fixed.

(c) In a given right triangle, draw a circle of radius x corresponding to one of the perpendicular side-lengths, and centered at the vertex. Consider a square, one of whose sides lies on the other perpendicular side and touches both the circle and the hypotenuse. If y is the length of the side of the square, and the common side-length a is held constant, find the maximum value of y as a function of x.

Ultimately, we want an expression for $y(x, a)$ in terms of x and a alone. That will require three constraints – two to eliminate the side lengths b and c, and a third to connect y with x and a. The right-most triangle is similar to the larger triangle, so

$$\frac{x}{a} = \frac{y}{c} \tag{4.2}$$

and the total base-length is,

$$a = y + b + c \tag{4.3}$$

Combining Eqs. 4.2 and 4.3 to eliminate c,

$$y\left(\frac{a}{x}\right) = a - y - b \tag{4.4}$$

To eliminate b, notice that the triangle formed by the dotted lines is right-angled, and so obeys the Pythagorean theorem,

$$b^2 = x^2 - (x-y)^2, \qquad \text{or} \qquad b = \sqrt{2xy - y^2}. \tag{4.5}$$

With substitution of Eq. 4.5 into Eq. 4.4 to eliminate b, we are left with an equation relating y to x and a,

$$a - y - y\left(\frac{a}{x}\right) = \sqrt{2xy - y^2}$$

Squaring both sides results in a quadratic equation for y,

$$\left[1 + \left(1 + \frac{a}{x}\right)^2\right] y^2 - 2\left[a\left(1 + \frac{a}{x}\right) + x\right] y + a^2 = 0$$

This equation has two roots: $y = x$, which doesn't fit in the given triangle, and

$$y = \frac{a^2 x}{a^2 + 2ax + 2x^2}$$

which (finally) is the function relating the side-length y to x and a alone. The extremal points are determined from the zeros of the derivative of $y(x)$ (with a held fixed),

$$\frac{dy}{dx} = \frac{a^2}{a^2 + 2ax + 2x^2} + \frac{a^2 x(2a + 4x)}{(a^2 + 2ax + 2x^2)^2} = \frac{a^2(a^2 - 2x^2)}{(a^2 + 2ax + 2x^2)^2}$$

The extremal point is $x = x^\star$ causes the numerator to vanish,

$$(a^2 - 2(x^\star)^2) = 0 \quad \text{or} \quad x^\star = \frac{a}{\sqrt{2}}$$

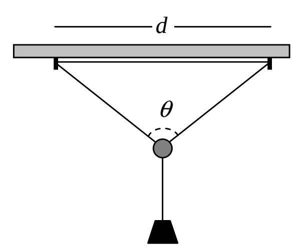
Japanese temple problems (Sangaku) provide a huge variety of interesting mathematical problems. More can be found in:

- Hidetoshi Fukagawa and Dan Pedoe. *Japanese temple geometry problems = Sangaku.* Winnipeg: Charles Babbage, 1989.
- Hidetoshi Fukagawa and Tony Rothman. *Sacred Mathematics: Japanese Temple Geometry.* Princeton: Princeton University Press, 2008.

4.10.3. Consider a rope of length ℓ with a mass W attached at one end, and a pulley attached at the other. The rope is suspended from two hooks (a distance d apart), and the weight is passed through the pulley (see figure below). After the weight is released and comes to equilibrium, determine the angle between ropes at the pulley, θ, *assuming that the distance between the weight and the ceiling is a maximum.* From the symmetry of the system, you can assume that the weight comes to rest half-way between the hooks.

Hint: First show that the distance of the weight from the ceiling is given by $L(\theta) = \ell - d - \dfrac{d}{\sin(\theta/2)} + \dfrac{d}{2\tan(\theta/2)}$.

(For Ex. 3) Assume the distance between the ceiling and the weight is a maximum at equilibrium.



There is some preliminary geometry that must be done before to find an expression for the total distance of the weight from the ceiling.
First, the total length of the rope can be divided into several segments: a perpendic-

ular length ℓ_1 connecting the weight to the pulley, two angular lengths ℓ_2 connecting the pulley to the hooks, and finally a length d between the two hooks. Altogether, the total rope length is written,

$$\ell = \ell_1 + 2\ell_2 + d. \tag{4.6}$$

The angular length ℓ_2 is related to the angle θ at the pulley by simple trigonometry,

$$\sin\left[\frac{\theta}{2}\right] = \frac{d/2}{\ell_2} \qquad \text{or} \qquad \ell_2 = \frac{d}{2\sin(\theta/2)} \tag{4.7}$$

The distance of the the pulley from the ceiling, h, can likewise be written in terms of d and θ,

$$\tan\left[\frac{\theta}{2}\right] = \frac{d/2}{h} \qquad \text{or} \qquad h = \frac{d}{2\tan(\theta/2)} \tag{4.8}$$

Finally, the total distance of the weight from the ceiling L is $L = h + \ell_1$. Using Eq. (3) to re-write h in terms of the angle θ,

$$L = \ell_1 + \frac{d}{2\tan(\theta/2)}.$$

Eq. (1) allows us to re-write the length ℓ_1 in terms of the total rope length ℓ and the angular length ℓ_2, $\ell_1 = \ell - 2\ell_2 - d$. And Eq. (2) allows us to re-write ℓ_2 in terms of the angle θ so that the total length is written,

$$L(\theta) = \ell - d - \frac{d}{\sin(\theta/2)} + \frac{d}{2\tan(\theta/2)}.$$

To find the maximum of $L(\theta)$, we take the derivative,

$$\frac{dL}{d\theta} = -d\frac{\cos(\theta/2)}{2\sin^2(\theta/2)} + \frac{d\sec^2(\theta/2)}{4\tan^2(\theta/2)} = \frac{-d}{2\sin^2(\theta/2)}\left[\cos(\theta/2) - \frac{1}{2}\right].$$

Setting the derivative equal to zero at the critical point, we have $\cos(\theta/2) = 1/2$, or $\theta = 2\pi/3$ or $120°$. What is remarkable is that this angle is independent of the length of the rope ℓ and the distance between the hooks d. This delightful problem appears in the first calculus textbook *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* ostensibly by Guillaume de l'Hôpital, though thought to have been largely written by Johann Bernoulli.

4.10.4. Consider the function $f(x) = x^x$, $x \geq 0$.

(a) The point $x = 0$ is undefined in the original function definition; determine the value of $f(0)$ that ensures $f(x) = x^x$ is continuous at $x = 0$.

To evaluate the limit, replace $x = 1/n$ and examine the limit as $n \to \infty$,

$$\lim_{x\to 0} x^x = \lim_{n\to 0} \frac{1}{n^{1/n}} = 1.$$

To make the function continuous, define $f(x)$ as,

$$f(x) = \begin{cases} x^x, & x > 0 \\ 1, & x = 0 \end{cases}$$

(b) Find the point $x^\star$ where the function achieves its minimum. *Hint:* $a = e^{\ln a}$ when $a > 0$.

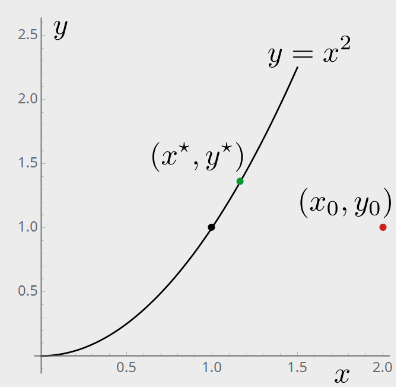
Following the hint, $f(x) = x^x = e^{\ln x^x} = e^{x \ln x}$. Then,

$$\frac{df}{dx} = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1).$$

The minimum of x^x corresponds with the point where the derivative vanishes. The function x^x is always positive for $x > 0$, and so we must find the point $x^\star$ where

$$\ln x^\star + 1 = 0 \quad \text{or } \ln x^\star = -1 \quad \text{or } x^\star = e^{-1} \approx 0.3679.$$

4.10.5. What is the point on the parabola $y = x^2$ that is closest to point $(x_0, y_0) = (2, 1)$? Express your answer correct to two decimals. *Hint:* Minimizing the distance is equivalent to minimizing the distance-squared, simplifying the algebra.



Following the hint, we want to minimize

$$d^2(x, y) = (x - x_0)^2 + (y - y_0)^2$$

where the point (x, y) lies on the parabola. That is, with $y = x^2$, we want to find the minimum of

$$f(x) = (x - 2)^2 + (x^2 - 1)^2$$

Taking the first derivative,

$$f'(x) = 4x^3 - 2x - 4.$$

The first derivative is a cubic, so it is not straightforward to determine the critical points. We can use Newton's method; denote the critical point by $x^\star$, Newton's method provides the recursive sequence,

$$x^\star_{n+1} = x^\star_n - \frac{2 + x^\star_n - 2(x^\star_n)^3}{1 - 6(x^\star_n)^2} = x^\star_n - \frac{-4 - 2x^\star_n + 4(x^\star_n)^3}{-2 + 12(x^\star_n)^2}$$

Initializing at a point close to $(x_0, y_0) = (2, 1)$, for example $x_0^\star = 1$, Newton's method returns,

$$\{x_n^\star\} = \{1., 1.2, 1.16649, 1.16537, 1.16537, \ldots\}$$

so, correct to two decimals, the critical point is $x^\star \approx 1.17$. You can verify that this is indeed a minimum: taking the second derivative, $f''(x) = -2+12x^2$, which is positive for all $x > \sqrt{2/12} \approx 0.41$. The point on the parabola closest to $(x_0, y_0) = (2, 1)$ is then $(x^\star, y^\star) = (1.17, 1.36)$.