CH 4 - Derivatives

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Velocity

🐳 **Info — Average Velocity and Instantaneous Velocity**

$$v_{avg} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

$$v_{inst} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Definition of Derivatives

🐳 **Info — Average Rate of Change and Instantaneous Rate of Change (Derivative)**

$$f_{avg} = \frac{f(b) - f(a)}{b - a}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

If $f'(x)$ exists at $x = a$, then $f(x)$ is **differentiable** at $x = a$

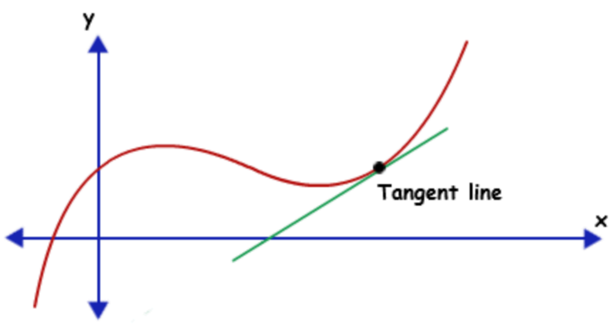
🐳 **Info — Tangent Line**

If $f(x)$ is differentiable at $x = a$, then the **tangent line** to $f(x)$ at $x = a$ is the line passing through $(a, f(a))$ with slope $f'(a)$

The equation of the tangent line

$$y = f'(a)(x - a) + f(a)$$

$(a, f(a))$ is the **point of tangency**



Examples:

Find the tangent line to $f(x) = \frac{1}{x+5}$ at $x = 3$
$f(3) = \frac{1}{8}$
$f'(3) = f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h+5} - \frac{1}{a+5}}{h} =$
$\lim_{h \to 0} \frac{1}{h} \frac{a+5-(a+h+5)}{(a+5)(a+h+5)} = \lim_{h \to 0} -\frac{1}{(a+5)(a+h+5)} = -\frac{1}{(a+5)^2} = -\frac{1}{64}$

$y = -\frac{1}{64}(x-3) + \frac{1}{8}$

🐳 **Info — Differentiability Implies Continuity**
If a function f is differentiable at $x = a$, then f is continuous at $x = a$

Proof

f is differentiable at $x = a$ then, $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists
$\lim_{h \to 0} [f(a+h) - f(a)] = 0 \Longrightarrow \lim_{h \to 0} [f(a+h) - f(a) + f(a)] = \lim_{h \to 0} f(a) \Longrightarrow$
$\lim_{h \to 0} f(a) = f(a)$

⚠️ **Warning — Continuity Not Implies Differentiability**
$f(x) = |x| = \begin{cases} -x \text{ if } x<0 \\ x \text{ if } x \geq 0 \end{cases}$

$\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \frac{h-0}{h} = 1$

$\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \frac{-h-0}{h} = -1$

Thus $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} =$ DNE but continuous.

$\therefore$ continuity does not impliy differentiability

🐳 **Info — Differentiability of Funciton**
We say that f is **differentiable** on an interval I if $f'(a)$ exists $\forall a \in I$.

We define the derivative funciton $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

We sometimes also write $f'(x)$ as $\frac{d}{dx} f(x)$, and $f'(a) = \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_a$

🐳 **Info** — Constant Function
$f(x) = c$

$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

🐳 **Info** — Linear Function
$f(x) = mx + b$

$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{f(m(a+h)+b)-f(ma+b)}{h} = \lim_{h \to 0} m\frac{h}{h} = m$

Info — Quadratic Function

$f(x) = px^2 + sx + c$

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[p(a+h)^2+s(a+h)+c]-[pa^2+sa+c]}{h} = \lim_{h\to 0} \frac{2aph+ah^2+sh}{h} = \lim_{h\to 0} 2ap + ah + s = 2ap + s$

Info — Basic Trig

$f(x) = \sin x$

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sin(a+h)-\sin(a)}{h} = \lim_{h\to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} =$

$\lim_{h\to 0} \frac{[\sin a(\cos h - 1)]}{h} + \lim_{h\to 0} \cos a \frac{\sin h}{h} = \sin a \cdot \lim_{h\to 0} \frac{\cos^2 h - 1}{h\cdot(\cos h+1)} + \cos a =$

$\sin a \cdot \lim_{h\to 0} \frac{\sin^2 h}{h\cdot(\cos h+1)} = \sin a \cdot \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{\sin h}{\cos h+1} + \cos a = \cos a$