CH 1 — Integration

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Definite Integrals

🐳 **Info — Riemann Sums**

Given $f(x)$ that is defined over $[a, b]$ with $a < b$, the area under function $f(x)$ can be found by

1. **Left-Endpoint Riemann Sum**

$$L_n = \sum_{i=0}^{n-1} f(x_i^*)\Delta x$$

- Underestimates Increasing Funcitons

2. **Right-Endpoint Riemann Sum**

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- Overestimates Increasing Functions

where

- $\Delta x = \frac{b-a}{n}$ under regular partition
- $x_i^* = a + i\Delta x = a + i\frac{b-a}{n}$

$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$ for $f(x)$ on interval $[a, b]$

Regular Parition means interval $[a, b]$ is equally partitionined into n rectangles with identical width

Example:

Estimate area under the curve for $f(x) = x^2$ on $x \in [0, 1]$

$R_n = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6n}$
$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n} = \frac{1}{3}$

🐳 **Info — Definite Integral**

$f(x)$ defined on $x \in [a, b]$ with regular partition with n subintervals

The definite integral of $f(x)$ on $[a, b]$ is defined

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

A function is integrable on $x \in [a, b]$ provided that the limit of Riemann Sum exists and has the same value regardless of the choice of x_i^*

🐳 **Info — Integrability Theorem for Continuous Functions**

Integrability: $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$

1. If f is continuous of $[a, b]$ then f is integrable on $[a, b]$
2. f is bounded on $[a, b]$ and has a **finite** number of discontinuities, then f is integrable on $[a, b]$

That is continuity implies integrability and the other way is false

Examples:

1. $f(x) = x^2$
2. $f(x) = \begin{cases} 2 \text{ if } x \leq 3 \\ 1 \text{ if } x > 3 \end{cases}$, note that $f(x)$ is discontinuous
3. $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ on $[0, 1]$
   - x_i^* is rational

     $\int_0^1 f(x) \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 1$
   - x_i^* is irrational $\int_0^1 f(x) \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 0$

   Thus not integrable

For geometric interpretation, Riemann Sums and Definite Integrals measures the "signed" area where there is no more than 1 inflection point

- A positive result of w implies the area under the curve above x-axis is w
- A negative result of w implies the area under the curve under x-axis is w

🐳 **Info — Parity of Functions and Definite Integrals**

Let $f(x)$ be bounded and integrable on $[-a, a]$

1. If $f(x)$ is odd function, then

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 0$$

2. If $f(x)$ is even function where $\int_0^a f(x) \,\mathrm{d}x = w$

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 2w$$

Examples:

1. $\int_1^3 x^2 - 3x \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n} = \mathrm{im}_{n\to\infty} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^3 - 3\left(1 + \frac{2i}{n}\right)\right] \cdot \frac{2}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^n \left(-\frac{4i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right) = -\frac{10}{3}$

2. $\int_0^5 x^3 - 2 \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(i\frac{5}{n}\right)\frac{5}{n} = \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{5}{n}\right)\left(\left(i\frac{5}{n}\right)^3 - 2\right)\right] = \frac{583}{4}$

🐳 Info — Basic Property of Definite Integral

Let $f(x), g(x)$ be integrable on $[a, b]$

1. For any $c \in \mathbb{R}$, the function $cf(x)$ is integrable and
$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$
2. The function $f + g$ is integrable and
$$\int_a^b (f+g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$
3. If $m, M \in \mathbb{R}$ and $m \leq f(x) \leq M \forall x \in [a, b]$, then
$$m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$$
4. If $f(x) \geq 0 \forall x$, then
$$\int_a^b f(x)\,\mathrm{d}x \geq 0$$
5. If $f(x) \leq g(x) \forall x \in [a, b]$, then
$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$
6. The function $|f|$ is integrable on $[a, b]$ and
$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$$
7. Bound fliping
$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$
8. $$\int_a^a f(x)\,\mathrm{d}x = 0$$

🐳 Info — Separation of Domain of Definite Integral

If $f(x)$ is also integrable on an interval containing a, b, c, then
$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

> 🐳 **Info — Average Value of Function**
>
> Let f be a function that is continuous on an interval $[a, b]$ with $a < b$. The **average calue of f on $[a, b]$** is defined as
>
> $$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x$$

Examples:

1. Determine the average value of $f(x) = 1 - x^2$ on [−1, 1]

   $f_{\text{avg}} = \frac{1}{1-(-1)} \int_{-1}^{1} f(x) \, \mathrm{d}x = \int_0^1 f(x) \, \mathrm{d}x = \lim_{x \to \infty} \sum_{i=1}^{n} \frac{1 - \left(\frac{i}{n}\right)^2}{n}$
   $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1 - \frac{i^2}{n^2} = \lim_{n \to \infty} \frac{1}{n} \left(n - \frac{1}{n^2} \left(\frac{n(n+1)(2n+1)}{6}\right)\right) = 1 - \frac{1}{3} = \frac{2}{3}$
2. Suppose that f, g are integrable on $[-1, 1]$, $\int_1^{-1} f(t) \, \mathrm{d}t = 5$, and g is an evern function with $\int_0^1 g(t) \, \mathrm{d}t = 2$.

   $\int_{-1}^{1} 3f(x) - g(x) \, \mathrm{d}x = 3 \int_{-1}^{1} f(x) \, \mathrm{d}x - \int_{-1}^{1} g(x) \, \mathrm{d}x = -3 \int_1^{-1} f(x) \, \mathrm{d}x - 2 \int_0^1 g(x) \, \mathrm{d}x = -19$

> 🐳 **Info — Fundamental Theorem of Calculus (FTC - 1)**
>
> Let $a \in \mathbb{R}$. If f is continuous on an open interval I containing a, then the funciton
>
> $$G(x) = \int_a^x f(t) \, \mathrm{d}t$$
>
> is differentiable $\forall x \in I$ and $G'(x) = f(x)$. That is,
>
> $$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t) \, \mathrm{d}t = f(x)$$
>
> **General Extended FTC 1**
>
> Let f be continuous, g, h be differentiable
>
> $$\frac{\mathrm{d}}{\mathrm{d}x} \int_{g(x)}^{h(x)} f(t) \, \mathrm{d}t = f'(h(x))h'(x) - f'(g(x))g'(x)$$

Proof

Given $x \in I$, from the definition of the derivative, we have
$G'(x) = \lim_{h \to \infty} \frac{G(x+h) - G(x)}{h}$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t) \, \mathrm{d}t - \int_a^x f(t) \, \mathrm{d}t}{h} = \lim_{h \to 0} \frac{\int_a^x f(t) \, \mathrm{d}t + \int_x^{x+h} f(t) \, \mathrm{d}t - \int_a^x f(t) \, \mathrm{d}t}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, \mathrm{d}t.$$

For all $h \neq 0$, sufficiently close to 0, and $h > 0$ f is continuous on $[x, x + h]$.

$\forall h, \exists c = c(h)$ in $[x, x + h]$ s.t.

$$f(c(h)) = \frac{1}{h} \int_x^{x+h} f(t) \, \mathrm{d}t$$

Since $x \leq c(h) \leq x + h$, by Squeeze Theorem, $\lim_{h \to 0} c_h = x$, thus

$$G'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, \mathrm{d}t = \lim_{h \to 0} (c_h) = f(x)$$

□

Examples

1. $G(x) = \int_0^x \frac{1}{1+t^2} \, \mathrm{d}t$

   Since $f(t) = \frac{1}{1+t^2}$ is continuous on $\mathbb{R}$, by FTC 1

   $G'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{1}{1+t^2} \, \mathrm{d}t = \frac{1}{1+t^2}$
2. $H(x) = \int_2^{e^x} \cos(t^2) \, \mathrm{d}t$

   Since $f(t) = \cos(t^2)$ is continuous on $\mathbb{R}$, by FTC 1

   $H'(u) = \frac{\mathrm{d}}{\mathrm{d}u} \int_2^u \cos(t^2) \, \mathrm{d}t \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \cos(u^2) \overset{u = e^x}{=} e^x \cos(e^{2x})$
3. Assume f is continuous and g, h differentiable

   $G(x) = \int_{g(x)}^{h(x)} f(t) \, \mathrm{d}t = \int_{g(x)}^0 f(t) \, \mathrm{d}t + \int_0^{h(x)} f(t) \, \mathrm{d}t = -\int_0^{g(x)} f(t) \, \mathrm{d}t + \int_0^{h(x)} f(t) \, \mathrm{d}t$

   $G'(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \int_0^{g(x)} f(t) \, \mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{h(x)} f(t) \, \mathrm{d}t \overset{\text{by FTC 1}}{=} -f'(g(x))g'(x) + f'(h(x))h'(x)$

🐳 **Info — Fundamental Theorem of Calculus (FTC - 2)**

If f, F are continuous on $[a, b]$ and $F'(x) = f(x) \ \forall x \in (a, b)$, then

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a)$$

Proof

Let F be any antiderivative of f. Then $F(x)$ and the antiderivative $G(x) = \int_a^x f(t) \, \mathrm{d}t$ have the relation that $G(x) = F(x) + C$

$$\begin{aligned} \int_a^b f(x) \, \mathrm{d}x &= \int_a^b f(t) \, \mathrm{d}t - \int_a^a f(t) \, \mathrm{d}t \\ &= G(b) - G(a) \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) \end{aligned}$$

□

Example

If $H(x) = \int_5^x x^2 \, \mathrm{d}x$, $\int_1^2 = H(2) - H(1) = \int_5^2 x^2 \, \mathrm{d}x - \int_5^1 x^2 \, \mathrm{d}x = \int_1^2 x^2 \, \mathrm{d}x$

🐳 **Info — Basic Integraion Rules**

$$\int x^r \, \mathrm{d}x = \frac{x^{r+1}}{r+1} + C \;\forall r \neq -1$$

$$\int x^-1 \, \mathrm{d}x = \ln|x| + C$$

$$\int \sin x \, \mathrm{d}x = -\cos x + C$$

$$\int \cos x \, \mathrm{d}x = \sin x + C$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + C$$

$$\int -\csc^2 x \, \mathrm{d}x = \cot x + C$$

$$\int \sec x \tan x \, \mathrm{d}x = \sec x + C$$

$$\int \csc x \cot x \, \mathrm{d}x = -\csc x + C$$

$$\int e^x \, \mathrm{d}x = e^x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arccos x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arcsin x + C$$

$$\int \frac{1}{1+x^2} \, \mathrm{d}x = \arctan x + C$$

$$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + C$$

Examples:

1. $\int e^{5x} \, \mathrm{d}x = \frac{e^{5x}}{5} + C$
2. $\int \frac{t}{t+1} \, \mathrm{d}t = \int 1 - \frac{1}{t+1} \, \mathrm{d}t = t - \ln|t+1| + C$

Examples:

1. $$\int_0^4 2x^2 - x \, \mathrm{d}x = \frac{2}{3}x^3 - \frac{x^2}{2}\Big|_0^4 = \frac{2}{3}(4)^2 - \frac{4^3}{2} - 0 = \frac{128}{3} - 8 = \frac{104}{3}$$

2. $$\int_1^3 \frac{x+|x-2|}{x}\,\mathrm{d}x = \int_1^2 \frac{2}{x}\,\mathrm{d}x + \int_2^3 \frac{2x-2}{x}\,\mathrm{d}x$$
$$= 2\ln|x|\Big|_1^2 + \int_2^3 2 - \frac{2}{x}\,\mathrm{d}x$$
$$= 2\ln 2 + 2x - 2\ln|x|\Big|_2^3$$
$$= 2\ln 2 + 6 - 2\ln 3 - 4 + 2\ln 2$$
$$= 4\ln 2 + 2 - 2\ln 3$$

Substitution Rule / U-Substitution

🐳 **Info** — **U-Substitution**

Let f, g be functions s.t. $g'(x)$ is continuous on $a, b]$ and f is continuous on range of g

$$\int f(g(x))g(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Example:

1. $\int 2x\sqrt{1+x^2}\,\mathrm{d}x$

   Let $u = 1 + x^2 \implies \mathrm{d}u = 2x\,\mathrm{d}x \implies \mathrm{d}x = \frac{\mathrm{d}u}{2x}$

   $\overset{u=1+x^2}{\implies} \int u^{\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1+x^2} + C = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$

2. $\int x^2 e^{x^3}\,\mathrm{d}x$

   Let $u = x^3 \implies \mathrm{d}u = 3x^2\,\mathrm{d}x \implies \mathrm{d}x = \frac{\mathrm{d}u}{3x^2}$

   $\overset{u=x^3}{\implies} \int \frac{1}{3}e^u\,\mathrm{d}u = \frac{1}{3}e^u\Big|_{u=x^3} + C = \frac{1}{3}e^{x^3} + C$

3. $\int \frac{\cos(\ln x)}{x}\,\mathrm{d}x$

   Let $u = \ln x \implies \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \implies \mathrm{d}x = x\,\mathrm{d}u$

   $\overset{u=\ln x}{\implies} \int \cos u\,\mathrm{d}u = \sin u\Big|_{u=\ln x} + C = \sin(\ln x) + C$

4. $\int \frac{x}{3\sqrt[3]{x+2}}\,\mathrm{d}x$

   Let $u = x + 2 \implies \mathrm{d}u = \mathrm{d}x$

   $\overset{u=x+2}{\implies} \int \frac{u-2}{\sqrt[3]{u}}\,\mathrm{d}$

Integration by Parts