CH 1 — Integration

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Definite Integrals

🐳 **Info — Riemann Sums**

Given $f(x)$ that is defined over $[a, b]$ with $a < b$, the area under function $f(x)$ can be found by

1. **Left-Endpoint Riemann Sum**

$$L_n = \sum_{i=0}^{n-1} f(x_i^*)\Delta x$$

- Underestimates Increasing Funcitons

2. **Right-Endpoint Riemann Sum**

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- Overestimates Increasing Functions

where

- $\Delta x = \frac{b-a}{n}$ under regular partition
- $x_i^* = a + i\Delta x = a + i\frac{b-a}{n}$

$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$ for $f(x)$ on interval $[a, b]$

Regular Parition means interval $[a, b]$ is equally partitionined into n rectangles with identical width

Example:

Estimate area under the curve for $f(x) = x^2$ on $x \in [0, 1]$

$R_n = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6n}$
$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n} = \frac{1}{3}$

🐳 **Info — Definite Integral**

$f(x)$ defined on $x \in [a, b]$ with regular partition with n subintervals

The definite integral of $f(x)$ on $[a, b]$ is defined

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

A function is integrable on $x \in [a, b]$ provided that the limit of Riemann Sum exists and has the same value regardless of the choice of x_i^*

🐳 **Info — Integrability Theorem for Continuous Functions**

Integrability: $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$

1. If f is continuous of $[a, b]$ then f is integrable on $[a, b]$
2. f is bounded on $[a, b]$ and has a **finite** number of discontinuities, then f is integrable on $[a, b]$

That is continuity implies integrability and the other way is false

Examples:

1. $f(x) = x^2$
2. $f(x) = \begin{cases} 2 \text{ if } x \leq 3 \\ 1 \text{ if } x > 3 \end{cases}$, note that $f(x)$ is discontinuous
3. $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ on $[0, 1]$
   - x_i^* is rational

     $\int_0^1 f(x) \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 1$
   - x_i^* is irrational $\int_0^1 f(x) \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 0$

   Thus not integrable

For geometric interpretation, Riemann Sums and Definite Integrals measures the "signed" area where there is no more than 1 inflection point

- A positive result of w implies the area under the curve above x-axis is w
- A negative result of w implies the area under the curve under x-axis is w

🐳 **Info — Parity of Functions and Definite Integrals**

Let $f(x)$ be bounded and integrable on $[-a, a]$

1. If $f(x)$ is odd function, then

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 0$$

2. If $f(x)$ is even function where $\int_0^a f(x) \,\mathrm{d}x = w$

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 2w$$

Examples:

1. $\int_1^3 x^2 - 3x \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n} = \text{im}_{n\to\infty} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^3 - 3\left(1 + \frac{2i}{n}\right)\right] \cdot \frac{2}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^n \left(-\frac{4i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right) = -\frac{10}{3}$

2. $\int_0^5 x^3 - 2 \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(i\frac{5}{n}\right)\frac{5}{n} = \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{5}{n}\right)\left(\left(i\frac{5}{n}\right)^3 - 2\right)\right] = \frac{583}{4}$

🐳 Info — Basic Property of Definite Integral

Let $f(x), g(x)$ be integrable on $[a, b]$

1. For any $c \in \mathbb{R}$, the function $cf(x)$ is integrable and
$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$
2. The function $f + g$ is integrable and
$$\int_a^b (f+g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$
3. If $m, M \in \mathbb{R}$ and $m \leq f(x) \leq M \forall x \in [a, b]$, then
$$m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$$
4. If $f(x) \geq 0 \forall x$, then
$$\int_a^b f(x)\,\mathrm{d}x \geq 0$$
5. If $f(x) \leq g(x) \forall x \in [a, b]$, then
$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$
6. The function $|f|$ is integrable on $[a, b]$ and
$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$$
7. Bound fliping
$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$
8. $$\int_a^a f(x)\,\mathrm{d}x = 0$$

🐳 Info — Separation of Domain of Definite Integral

If $f(x)$ is also integrable on an interval containing a, b, c, then
$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

🐳 **Info — Average Value of Function**

Let f be a function that is continuous on an interval $[a, b]$ with $a < b$. The **average calue of f on $[a, b]$** is defined as

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x$$

Examples:

1. Determine the average value of $f(x) = 1 - x^2$ on $[-1, 1]$

   $f_{\text{avg}} = \frac{1}{1-(-1)} \int_{-1}^1 f(x) \, \mathrm{d}x = \int_0^1 f(x) \, \mathrm{d}x = \lim_{x \to \infty} \sum_{i=1}^n \frac{1-\left(\frac{i}{n}\right)^2}{n}$
   $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n 1 - \frac{i^2}{n^2} = \lim_{n \to \infty} \frac{1}{n}\left(n - \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right) = 1 - \frac{1}{3} = \frac{2}{3}$
2. Suppose that f, g are integrable on $[-1, 1]$, $\int_1^{-1} f(t) \, \mathrm{d}t = 5$, and g is an evern function with $\int_0^1 g(t) \, \mathrm{d}t = 2$.

   $\int_{-1}^1 3f(x) - g(x) \, \mathrm{d}x = 3\int_{-1}^1 f(x) \, \mathrm{d}x - \int_{-1}^1 g(x) \, \mathrm{d}x = -3\int_1^{-1} f(x) \, \mathrm{d}x - 2\int_0^1 g(x) \, \mathrm{d}x = -19$

🐳 **Info — Fundamental Theorem of Calculus (FTC - 1)**

Let $a \in \mathbb{R}$. If f is continuous on an open interval I containing a, then the funciton

$$G(x) = \int_a^x f(t) \, \mathrm{d}t$$

is differentiable $\forall x \in I$ and $G'(x) = f(x)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t) \, \mathrm{d}t = f(x)$$

General Extended FTC 1

Let f be continuous, g, h be differentiable

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{g(x)}^{h(x)} f(t) \, \mathrm{d}t = f(h(x))h'(x) - f(g(x))g'(x)$$

Proof

Given $x \in I$, from the definition of the derivative, we have
$G'(x) = \lim_{h \to \infty} \frac{G(x+h)-G(x)}{h}$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t) \, \mathrm{d}t - \int_a^x f(t) \, \mathrm{d}t}{h} = \lim_{h \to 0} \frac{\int_a^x f(t) \, \mathrm{d}t + \int_x^{x+h} f(t) \, \mathrm{d}t - \int_a^x f(t) \, \mathrm{d}t}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, \mathrm{d}t.$$

For all $h \neq 0$, sufficiently close to 0, and $h > 0$ f is continuous on $[x, x+h]$.

$\forall h, \exists c = c(h)$ in $[x, x+h]$ s.t.

$$f(c(h)) = \frac{1}{h} \int_x^{x+h} f(t) \, \mathrm{d}t$$

Since $x \leq c(h) \leq x + h$, by Squeeze Theorem, $\lim_{h \to 0} c_h = x$, thus

$$G'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, \mathrm{d}t = \lim_{h \to 0} (c_h) = f(x)$$

$\square$

Examples

1. $G(x) = \int_0^x \frac{1}{1+t^2} \, \mathrm{d}t$

   Since $f(t) = \frac{1}{1+t^2}$ is continuous on $\mathbb{R}$, by FTC 1

   $G'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{1}{1+t^2} \, \mathrm{d}t = \frac{1}{1+t^2}$

2. $H(x) = \int_2^{e^x} \cos(t^2) \, \mathrm{d}t$

   Since $f(t) = \cos(t^2)$ is continuous on $\mathbb{R}$, by FTC 1

   $H'(u) = \frac{\mathrm{d}}{\mathrm{d}u} \int_2^u \cos(t^2) \, \mathrm{d}t \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \cos(u^2) \overset{u=e^x}{=} e^x \cos(e^{2x})$

3. Assume f is continuous and g, h differentiable

   $G(x) = \int_{g(x)}^{h(x)} f(t) \, \mathrm{d}t = \int_{g(x)}^0 f(t) \, \mathrm{d}t + \int_0^{h(x)} f(t) \, \mathrm{d}t = -\int_0^{g(x)} f(t) \, \mathrm{d}t + \int_0^{h(x)} f(t) \, \mathrm{d}t$

   $G'(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \int_0^{g(x)} f(t) \, \mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{h(x)} f(t) \, \mathrm{d}t \overset{\text{by FTC 1}}{=} -f'(g(x))g'(x) + f'(h(x))h'(x)$

🐳 **Info — Fundamental Theorem of Calculus (FTC - 2)**

If f, F are continuous on $[a, b]$ and $F'(x) = f(x) \ \forall x \in (a, b)$, then

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a)$$

Proof

Let F be any antiderivative of f. Then $F(x)$ and the antiderivative $G(x) = \int_a^x f(t) \, \mathrm{d}t$ have the relation that $G(x) = F(x) + C$

$$\begin{aligned} \int_a^b f(x) \, \mathrm{d}x &= \int_a^b f(t) \, \mathrm{d}t - \int_a^a f(t) \, \mathrm{d}t \\ &= G(b) - G(a) \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) \end{aligned}$$

$\square$

Example

If $H(x) = \int_5^x x^2 \, \mathrm{d}x$, $\int_1^2 = H(2) - H(1) = \int_5^2 x^2 \, \mathrm{d}x - \int_5^1 x^2 \, \mathrm{d}x = \int_1^2 x^2 \, \mathrm{d}x$

🐳 **Info — Basic Integraion Rules**

$$\int x^r \, \mathrm{d}x = \frac{x^{r+1}}{r+1} + C \ \forall r \neq -1$$

$$\int x^{-1} \, \mathrm{d}x = \ln|x| + C$$

$$\int \sin x \, \mathrm{d}x = -\cos x + C$$

$$\int \cos x \, \mathrm{d}x = \sin x + C$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + C$$

$$\int -\csc^2 x \, \mathrm{d}x = \cot x + C$$

$$\int \sec x \tan x \, \mathrm{d}x = \sec x + C$$

$$\int \csc x \cot x \, \mathrm{d}x = -\csc x + C$$

$$\int e^x \, \mathrm{d}x = e^x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arccos x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arcsin x + C$$

$$\int \frac{1}{1+x^2} \, \mathrm{d}x = \arctan x + C$$

$$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + C$$

$$\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, \mathrm{d}x = -\ln|\csc x + \cot x| + C$$

Examples:

1. $\int e^{5x} \, \mathrm{d}x = \frac{e^{5x}}{5} + C$
2. $\int \frac{t}{t+1} \, \mathrm{d}t = \int 1 - \frac{1}{t+1} \, \mathrm{d}t = t - \ln|t+1| + C$

Examples:

1. $\displaystyle\int_0^4 2x^2 - x\,\mathrm{d}x = \frac{2}{3}x^3 - \frac{x^2}{2}\bigg|_0^4 = \frac{2}{3}(4)^2 - \frac{4^3}{2} - 0 = \frac{128}{3} - 8 = \frac{104}{3}$

2. $$\begin{aligned}\int_1^3 \frac{x+|x-2|}{x}\,\mathrm{d}x &= \int_1^2 \frac{2}{x}\,\mathrm{d}x + \int_2^3 \frac{2x-2}{x}\,\mathrm{d}x \\ &= 2\ln|x|\Big|_1^2 + \int_2^3 2 - \frac{2}{x}\,\mathrm{d}x \\ &= 2\ln 2 + 2x - 2\ln|x|\Big|_2^3 \\ &= 2\ln 2 + 6 - 2\ln 3 - 4 + 2\ln 2 \\ &= 4\ln 2 + 2 - 2\ln 3\end{aligned}$$

Substitution Rule / U-Substitution

🐳 **Info — U-Substitution**

Let f, g be functions s.t. $g'(x)$ is continuous on $[a, b]$ and f is continuous on range of g

$$\int f(g(x))g(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\Big|_{u=g(x)}$$

Example:

1. $\int 2x\sqrt{1+x^2}\,\mathrm{d}x$

   Let $u = 1 + x^2 \Longrightarrow \mathrm{d}u = 2x\,\mathrm{d}x \Longrightarrow \mathrm{d}x = \frac{\mathrm{d}u}{2x} \overset{u=1+x^2}{\Longrightarrow} \int u^{\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} \overset{u=1+x^2}{\Longrightarrow} +C$

   $= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$

2. $\int x^2 e^{x^3}\,\mathrm{d}x$

   Let $u = x^3 \Longrightarrow \mathrm{d}u = 3x^2\,\mathrm{d}x \Longrightarrow \mathrm{d}x = \frac{\mathrm{d}u}{3x^2} \overset{u=x^3}{\Longrightarrow} \int \frac{1}{3}e^u\,\mathrm{d}u = \frac{1}{3}e^u \overset{u=x^3}{\Longrightarrow} +C = \frac{1}{3}e^{x^3} + C$

3. $\int \frac{\cos(\ln x)}{x}\,\mathrm{d}x$

   Let $u = \ln x \Longrightarrow \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \Longrightarrow \mathrm{d}x = x\,\mathrm{d}u \overset{u=\ln x}{\Longrightarrow} \int \cos u\,\mathrm{d}u = \sin u \overset{u=\ln x}{\Longrightarrow} +C = \sin(\ln x) + C$

4. $\int \frac{x}{3\sqrt[3]{x+2}}\,\mathrm{d}x$

   Let $u = x + 2 \Longrightarrow \mathrm{d}u = \mathrm{d}x$

   $\overset{u=x+2}{\Longrightarrow} \int \frac{u-2}{\sqrt[3]{u}}\,\mathrm{d}u = \int u^{\frac{2}{3}}\,\mathrm{d}u - 2\int u^{-\frac{1}{3}}\,\mathrm{d}u = \frac{3}{5}u^{\frac{5}{3}} + 3u^{\frac{2}{3}} \overset{u=x+2}{\Longrightarrow} +C = (x+2)^{\frac{5}{3}} + 3(x+2)^{\frac{2}{3}} + C$

5. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,\mathrm{d}x$

   Let $u = 1 + \sin x \Longrightarrow \mathrm{d}u = \cos x\,\mathrm{d}x \Longrightarrow \mathrm{d}x = \frac{\mathrm{d}u}{\cos x} \Longrightarrow \int_1^2 \frac{1}{u}\,\mathrm{d}u = \ln u\Big|_1^2 = \ln 2$

6. $\int_0^{\frac{\pi}{3}} \tan x\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,\mathrm{d}x$

   Let $u = \cos x \Longrightarrow \mathrm{d}u = -\sin x\,\mathrm{d}x \Longrightarrow \int_1^{\frac{1}{3}} -\frac{1}{u}\,\mathrm{d}u = \ln u\Big|_{\frac{1}{3}}^{1} = -\ln\frac{1}{3}$

🐳 **Info — $f(ax)$**

Let $a \in \mathbb{R}, a \neq 0$ If $\int f(x)\,\mathrm{d}x = F(x) + C$, then

$$\int f(ax)\,\mathrm{d}x = \frac{1}{a}F(ax) + C$$

Trigonometry Substitution

🐳 **Info — Trig-Sub**

Integral contains	Substitution	Domain for θ	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{x^2 + a^2}$	$x = a\tan\theta$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\sec^2\theta - 1 = \tan^2\theta$

💡 **Tip — Half-Angle**

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

Example:

1. $\int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$

   Let $u = \arcsin x \Longrightarrow x = \sin u, \forall u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Longrightarrow \mathrm{d}x = \cos u\,\mathrm{d}u \Longrightarrow$

   $\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u}\cos u\,\mathrm{d}u = \int_0^{\frac{\pi}{2}} \cos^2 u\,\mathrm{d}u = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos(2u))\,\mathrm{d}u \frac{1}{2}(u + \sin(2u))\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

2. $\int \sqrt{1 - x^2}\,\mathrm{d}x$

   Let $u = \arcsin x \Longrightarrow x = \sin u\ \forall u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Longrightarrow \mathrm{d}x = \cos u\,\mathrm{d}u \Longrightarrow$

   $\int \sqrt{1 - \sin^2 u}\cos u\,\mathrm{d}u = \int \cos^2 u\,\mathrm{d}u = \int \frac{1+\cos 2x}{2}\,\mathrm{d}u \overset{u=\arcsin x}{\Longrightarrow} \frac{1}{2}\left(\arcsin x + \sqrt{1 - x^2}\right)) + C$

3. $\int \frac{1}{\sqrt{x^2+9}}$

   Let $x = 3\tan\theta\ \forall\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Longrightarrow \mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta \Longrightarrow \int \frac{1}{\sqrt{9\tan^\theta + 9}} 3\sec^2\theta\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta$

   $\int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\,\mathrm{d}\theta = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,\mathrm{d}\theta \overset{u=\sec\theta+\tan\theta}{\Longrightarrow} \int \frac{1}{u}\,\mathrm{d}u = \ln|\sec\theta + \tan\theta| + C$

   $\overset{\theta=\arctan\frac{x}{3}}{\Longrightarrow} \ln|\sec\arctan\frac{x}{3} + \tan\arctan\frac{x}{3}| + C = \ln\left(\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right) + C$

4. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x$

Let $x = \sec\theta\ \forall\theta \in [0, \frac{\pi}{2}) \cup [\pi, 3\frac{\pi}{2})$

$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{3}} \tan^2\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{3}} \sec^2\theta - 1\,\mathrm{d}\theta = \tan\theta - \theta\Big|_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}$

5. $\int \frac{1}{(5-4x-x^2)^{\frac{5}{2}}}\,\mathrm{d}x = \int \frac{1}{(1-(2-x)^2)^{\frac{5}{2}}}\,\mathrm{d}x$

Let $2 - x = \sin\theta,\ \ \forall\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Longrightarrow \mathrm{d}x = -\cos\theta\,\mathrm{d}\theta$

$= \int \frac{-\cos\theta}{(1-(\sin\theta)^2)^{\frac{5}{2}}}\,\mathrm{d}\theta = \int \frac{-\cos\theta}{\cos^5\theta}\,\mathrm{d}\theta = \int -\sec^4\theta\,\mathrm{d}\theta = -\int (\sec^2\theta)^2\,\mathrm{d}\theta = -\frac{2}{7}\sec^{\frac{7}{2}}\theta + C =$
$-\frac{2}{7}\left(\frac{1}{\sqrt{5-4x-x^2}}\right)^{\frac{7}{2}} + C$

Integration By Parts

🐋 Info — Integration by Parts

If f and g are differentiable functions of x, then

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

The rule of choosing parts being $f(x)$ is

1. Logarithmic
2. Inverse trigonometric
3. Algebraic
4. Trigonometric
5. Exponential

Examples

1. $\int xe^x\,\mathrm{d}x$

Let $\begin{cases} u=x \\ \mathrm{d}v=e^x\,\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}u=\mathrm{d}x \\ v=e^x \end{cases} \Longrightarrow xe^x - \int e^x = xe^x - e^x + C$

2. $\int x^2\sin x\,\mathrm{d}x$

Let $\begin{cases} u=x^2 \\ \mathrm{d}v=\sin x\,\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}u=2x\,\mathrm{d}x \\ v=-\cos x \end{cases} \Longrightarrow -x^2\cos x + 2\int x\cos x\,\mathrm{d}x =$
$-x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C$

3. $\int x\ln x\,\mathrm{d}x$

Let $\begin{cases} u=\ln x \\ \mathrm{d}v=x\,\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}u=\frac{1}{x}\,\mathrm{d}x \\ v=\frac{x^2}{2} \end{cases} \Longrightarrow \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

4. $\int_1^e \ln x\,\mathrm{d}$

Let $\begin{cases} u=\ln x \\ \mathrm{d}v=\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}u=\frac{1}{x}\,\mathrm{d}x \\ x \end{cases} \Longrightarrow x\ln x\Big|_1^e - \int_1^e \mathrm{d}x = e - (e-1) = 1$

5. $\int_0^1 e^x\cos x\,\mathrm{d}x$

Let $\begin{cases} u=\cos x \\ \mathrm{d}v=e^x\,\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}u=-\sin x\,\mathrm{d}x \\ e^x \end{cases} = e^x\cos x\Big|_0^1 + \int_0^1 e^x\sin x$

Let $\begin{cases} a = \sin x \\ \mathrm{d}b = e^x \,\mathrm{d}x \end{cases} \Longrightarrow \begin{cases} \mathrm{d}a = \cos x \,\mathrm{d}x \\ b = e^x \end{cases} = e^x \sin x \Big|_0^1 - \int_0^1 e^x \cos x \,\mathrm{d}x$

$\int_0^1 e^x \cos x \,\mathrm{d}x = e^x \cos x \Big|_0^1 + e^x \sin x \Big|_0^1 - \int_0^1 e^x \cos x$

$\Longrightarrow \int_0^1 e^x \cos x \,\mathrm{d}x = \frac{1}{2}\left(e^x \cos x \Big|_0^1 + e^x \sin x \Big|_0^1 \right) = \frac{1}{2} e \sin(1) + \frac{1}{2} e \cos(1) - \frac{1}{2}$

💡 **Tip — Reduction Formula for $\int \cos^n x \,\mathrm{d}x$**

For any $n \geq 2$

$$\int \cos^n x \,\mathrm{d}x = \frac{1}{n} \cos^{n-1} \sin x + \frac{n-1}{n} \int \cos^{n-2} x \,\mathrm{d}x$$

Partial Fractions

🐳 **Info — Partial Fractions**

For rational functions which is $\frac{P(x)}{Q(x)}$ $\forall P(x), Q(x)$ be polynomials and $P(x) \neq 0$

1. Perform long division to ensure the topside is in irreducibe linear/quadratic term i.e.

$$\frac{x^2 + 1}{x^2 + 3x + 4}$$

2. Rewrite the rational function into individual irreducible linear/quadratic terms (there are exceptions for duplicate terms)

$$\frac{Ux^2 + Vx + W}{Ex^2 + Fx + G} = \frac{Ax}{(Kx + L)^k} + \frac{Ax}{(Kx + L)^{k-1}} + \ldots + \frac{B}{Hx + I} + \frac{C}{Jx + M}$$

3. Solve for system of equations to determine A, B, C

4. Integrate individually

💡 **Tip — Breakdown of Irreducible Linear/Quadratic terms**

Factor of $Q(x)$	Term in the partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^n$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_n}{(ax+b)^n}$
$\underbrace{ax^2+bx+c}_{\text{irreducible}}$	$\dfrac{Ax+B}{ax^2+bx+c}$
$(\underbrace{ax^2+bx+c}_{\text{irreducible}})^n$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

Note that n is the time the term that has appeared in the fully factorized numerator and denominator.

Examples

1. $\int \frac{x^4+x^3+2x^2+4x+2}{x^3+x}\,\mathrm{d}x$
2. $\int \frac{1}{x(x-1)^2}\,\mathrm{d}x$
3. $\int \frac{x^2+2}{4x^5+4x^3+x}\,\mathrm{d}x$

Improper Integrals

🐳 **Info — Improper Integral (Type I)**

1. Let f be integrable on $[a, t]$ $\forall t \geq a$.

   If $\lim_{t \to \infty} \int_a^t f(x)\,\mathrm{d}x$ exists, we say that Type I improper integral $\int_a^\infty f(x)\,\mathrm{d}x$ converges and define

   $$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t \to \infty} \int_a^t f(x)\,\mathrm{d}x$$

   If $\lim_{t \to \infty} \int_a^t f(x)\,\mathrm{d}x$ does not exist, we say $\int_a^\infty f(x)\,\mathrm{d}x$ diverges

2. Let f be integrable on $[t, b]$ $\forall t \leq b$.

   If $\lim_{t \to -\infty} \int_a^t f(x)\,\mathrm{d}x$ exists, we say that Type I improper integral $\int_a^{-\infty} f(x)\,\mathrm{d}x$ converges and define

   $$\int_a^{-\infty} f(x)\,\mathrm{d}x = \lim_{t \to -\infty} \int_a^t f(x)\,\mathrm{d}x$$

   If $\lim_{t \to -\infty} \int_a^t f(x)\,\mathrm{d}x$ does not exist, we say $\int_a^{-\infty} f(x)\,\mathrm{d}x$ diverges

3. Let f be integrable on $[a, b]$ $\forall a, b \in \mathbb{R}$ with $a \leq b$.

   $\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^0 f(x)\,\mathrm{d}x + \int_0^\infty f(x)\,\mathrm{d}x$

   If $\int_{-\infty}^0 f(x)\,\mathrm{d}x$ and $\int_0^\infty f(x)\,\mathrm{d}x$ both converges, then $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ converges.

   If one of the improper integrals diverges, $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ diverges

💡 **Tip — p-Integrals**

1. If $p > 1$, the improper integral $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ converges and

   $$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \frac{1}{p-1}$$

   If $p \leq 1$, the improper integral $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ diverges.

2. If $p < 1$, the improper integral $\int_0^1 \frac{1}{x^p}\,\mathrm{d}x$ converges and

   $$\int_0^1 \frac{1}{x^p}\,\mathrm{d}x = \frac{1}{p-1}$$

   If $p \geq 1$, the improper integral $\int_0^1 \frac{1}{x^p}\,\mathrm{d}x$ diverges.

Examples:

1. $\int_{-\infty}^\infty \sin x\,\mathrm{d}x = \int_{-\infty}^0 \sin x\,\mathrm{d}x + \int_0^\infty \sin x\,\mathrm{d}x$

   $$\forall t \in \mathbb{R}, \int_0^t \sin x\,\mathrm{d}x = -\cos x\Big|_0^t = 1 - \cos t \implies \lim_{t \to \infty} 1 - \cos t = \text{DNE}$$

   Therefore $\int_{-\infty}^\infty \sin x\,\mathrm{d}x$ diverges

Note that $\lim_{t \to \infty} \int_t^t \sin x \, \mathrm{d}x = 0 \neq \int_{-\infty}^{\infty} \sin x \, \mathrm{d}x$

The explanation is that there ∞ is not a number, there are always ∞ that is bigger. However, if we have $t = \infty \implies -\infty = t$ enforces the same ∞, resulting into a symmetry

2. $\int_{-\infty}^{0} \frac{1}{1+4x^2} \, \mathrm{d}x$

$$\forall t \in \mathbb{R}, \int_t^0 \frac{1}{1+4x^2} = \frac{1}{2} \arctan(2x) \Big|_t^0 = -\frac{1}{2}(\arctan 2t)$$

$\lim_{t \to -\infty} -\frac{1}{2}(\arctan 2t) = \frac{\pi}{4}$

Therefore, the improper integral converges

🐳 **Info — Improper Integrals (Type II)**

Let $a, b \in \mathbb{R}, a < b$

1. Let f b integrable on $[t, b]$ $\forall t \in (a, b]$ and suppose that f has an infinite discontinuity at $x = a$. If $\lim_{x \to a^+} \int_t^b f(x) \, \mathrm{d}x$ exists, we say that the Type II improper integral $\int_a^b f(x) \, \mathrm{d}x$ converges and define

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{t \to a^+} \int_t^b f(x) \, \mathrm{d}x$$

   If $\lim_{t \to a^+} \int_t^b f(x) \, \mathrm{d}x$ does not exist, we say $\int_a^b f(x) \, \mathrm{d}x$ diverges

2. Let f b integrable on $[a, t]$ $\forall t \in [a, b)$ and suppose that f has an infinite discontinuity at $x = b$. If $\lim_{x \to b^-} \int_a^t f(x) \, \mathrm{d}x$ exists, we say that the Type II improper integral $\int_a^b f(x) \, \mathrm{d}x$ converges and define

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x) \, \mathrm{d}x$$

   If $\lim_{t \to b^-} \int_a^t f(x) \, \mathrm{d}x$ does not exist, we say $\int_a^b f(x) \, \mathrm{d}x$ diverges

3. Suppose that f has an infinite discontinuity at $x = c, a < c < b$. We say that the Type II improper integral $\int_a^b f(x) \, \mathrm{d}x$ converges if both $\int_a^c f(x) \, \mathrm{d}x$ and $\int_c^b f(x) \, \mathrm{d}x$ converges. We can have that

$$\int_a^b f(x) \, \mathrm{d}x = \int_a^c f(x) \, \mathrm{d}x + \int_c^b f(x) \, \mathrm{d}x$$

   If one of the improper integrals diverges, $\int_a^b f(x) \, \mathrm{d}x$ diverges

Example:

$$
\begin{aligned}
\int_0^3 \frac{1}{\sqrt{|x-2|}}\,\mathrm{d}x &= \int_0^2 \frac{1}{\sqrt{-(x-2)}}\,\mathrm{d}x + \int_2^3 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x \\
&= \lim_{t\to 2^-} \int_0^t \frac{1}{\sqrt{2-x}}\,\mathrm{d}x + \lim_{s\to 2^+} \int_t^3 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x \\
&= \lim_{t\to 2^-} -2\sqrt{2-x}\Big|_0^t + \lim_{s\to 2^+} 2\sqrt{x-2}\Big|_s^3 \\
&= \lim_{t\to 2^-} \left(-2\sqrt{2-t} + 2\sqrt{2}\right) + \lim_{s\to 2^+} \left(2 - 2\sqrt{s-2}\right) \\
&= 2\sqrt{2} + 2
\end{aligned}
$$

The integral converges

Area

Info — Area Between Curves

Let f, g be integrable functions defined on $[a, b]$. If $f(x) \geq g(x)\ \forall x \in [a, b]$, then the area between the graphs of f and g for $x \in [a, b]$ is given by

$$\int_a^b (f(x) - g(x))\,\mathrm{d}x$$

An alternate way of expressing is let $y = y_{\text{upper}(x)}$ be the upper curve and $y = y_{\text{lower}(x)}$ for $x \in [a, b]$.

$$\text{Area} = \int_{x=a}^{x=b} \left(y_{\text{upper}(x)} - y_{\text{lower}(x)}\right)\mathrm{d}x$$

Info — Area Between Curves interms of Y

If a region is bounded between a rightmost curve $x = x_{r(y)}$ and a leftmost curve $x = x_{l(x)}$ for $y \in [c, d]$. The area of the region can be expressed as

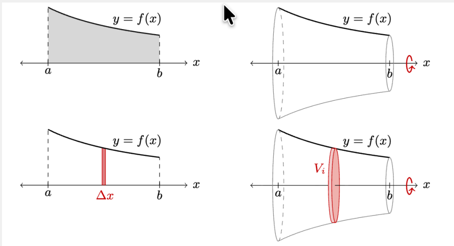
$$A = \int_{y=c}^{y=d} \left(x_{r(y)} - x_{l(y)}\right)\mathrm{d}y$$

Volume

Washer Method

🐳 **Info — Disk Method**

Let $f(x)$ be an integrable function on $[a, b]$



We have the function $f(x)$ rotate around the x-axis

Notice Δx is the thickness of the disk expressed as $\boldsymbol{A(x) = \pi f(x)^2}$

Hence the volume using the disk method is

$$V = \int_a^b A(x)\,\mathrm{d}x = \int_a^b \pi f(x)^2$$

The Washer Method is when there is a vaccum area inside the disk. A trick we can do is have to compute the outer disk volume subtract the inner disk volume

🐳 **Info — Washer Method**

Let $R(x), r(x)$ be integrable functions on $[a, b]$ where $R(x) \geq r(x)\ \forall x \in [a, b]$

The volume obtained by the area bounded by $R(x), r(x)$, around the x-axis is

$$\int_a^b \pi(R(x)^2 - r(x)^2)\,\mathrm{d}x$$

The idea is to have the **outer function squared subtract the inner function squared** to be the Area function

Example:

Consider the region bounded by the curve $y = 2x - x^2$ and the lines $y = 0, x = 0, x = 1$

1. The volume of the solid obtained by revolving about the x-axis

   $\int_0^1 \pi(2x - x^2)^2\,\mathrm{d}x = \pi \int_0^1 (4x^2 - 4x^3 + x^4)\,\mathrm{d}x = \pi\left(\frac{4}{3}x^3 - x^4 + \frac{x^5}{5}\Big|_0^1\right) = \frac{8\pi}{15}$

2. The volume of the solid obtained by revolving about the y-axis

   $\int_0^1 \pi\left(1 - (1 - \sqrt{1-y})^2\right)\mathrm{d}y = \frac{5\pi}{6}$

3. The volume of the solid obtained by the region enclosed by $y = x, y = x^2$ about $y = 1$

   $\int_0^1 \pi\left[(1 - x^2)^2 - (1 - x)^2\right]\mathrm{d}x$

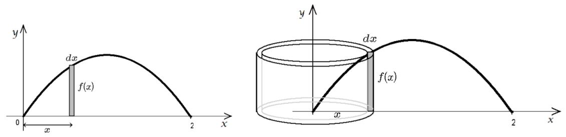
4. The volume of the solid obtained by the region enclosed by $y = x, y = x^2$ about $y = -1$

$\int_0^1 \pi \left[(x+1)^2 - \left(x^2+1\right)^2 \right] \mathrm{d}x$

Cylindrical Shell Method

🐳 Info — Cylindrical Shell

Let $f(x)$ be an integrable function on $[a, b]$



The volume of such a cylindrical shell is approximated by

$$V_i = \text{circumference} \cdot \text{height} \cdot \text{thickness}$$

We have the function $f(x)$ rotate around the $x = p$.

The formula

$$V = \int_a^b 2\pi r(x) f(x) \,\mathrm{d}x, \quad r(x) = (x - p)$$

💡 Tip — Summary

Below we discuss how to choose between the method of disks/washers and the method of cylindrical shells for calculating a volume of a solid of revolution.

1. When revolving a region R about a horizontal axis...
   (i) if R can be described in terms of functions of x, the volume of the resulting solid can be calculated using the disks/washers:
   $$V = \int_{x=a}^{x=b} \left(\pi \left(r_{\text{outer}}\right)^2 - \pi \left(r_{\text{inner}}\right)^2 \right) dx.$$
   (ii) if R can be described in terms of functions of y, the volume of the resulting solid can be calculated using cylindrical shells:
   $$V = \int_{y=c}^{y=d} 2\pi r h \, dy.$$
2. When revolving a region R about a vertical axis...
   (i) if R can be described in terms of functions of x, the volume of the resulting solid can be calculated using the cylindrical shells:
   $$V = \int_{x=a}^{x=b} 2\pi r h \, dx.$$
   (ii) if R can be described in terms of functions of y, the volume of the resulting solid can be calculated using disks/washers:
   $$V = \int_{y=c}^{y=d} \left(\pi \left(r_{\text{outer}}\right)^2 - \pi \left(r_{\text{inner}}\right)^2 \right) dy.$$