

CH 4 - Mathematical Induction

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Notations

$$\sum_{i=m}^n x_i = x_m + x_{m+1} + x_{m+2} + \dots + x_{n-1} + x_n$$

$$\prod_{i=m}^n x_i = x_m \cdot x_{m+1} \cdot x_{m+2} \cdot \dots \cdot x_{n-1} \cdot x_n$$

Properties

Constant multiplication

$$\sum_{i=m}^k cx_i = c \cdot \sum_{i=m}^k x_i$$

Addition/Subtraction

$$\sum_{i=m}^k x_i \pm \sum_{i=m}^k y_i = \sum_{i=m}^k x_i \pm \sum_{i=m}^k y_i$$

Index Shift

$$\sum_{i=m}^k x_i = \sum_{m \pm n}^{k \pm n} x_{i \mp n}$$

Breaking Sum

$$\sum_{i=m}^k x_i = \sum_{i=m}^r x_i + \sum_{i=r+1}^k x_i$$

Recurrence Relation

A sequence of values by giving one or more initial terms, together with an equation expressing each subsequent term in terms of earlier ones. (i.e. $s_1 = 1, s_n = s_{n-1} + n$ is the same as $\sum_{i=1}^n i$)

Proof by Induction

An **axiom** of a mathematical system is a statement that is assumed to be true. No proof is given. From axioms we derive proposition and theorems.



Info – Axiom

Principle of Mathematical Induction

Let $P(n)$ be an open sentence that depends on $n \in \mathbb{N}$

If statements 1 and 2 are both true:

1. $P(1)$
2. $\forall k \in \mathbb{N}$, if $P(k)$, then $P(k + 1)$

Then statement 3 is true:

3. $\forall n \in \mathbb{N}, P(n)$

Examples

1. Let $P(n)$ be the open sentence $\sum_{i=1}^n i(i + 1) = \frac{1}{3}n(n + 1)(n + 2)$

Prove that $P(n)$ is true $\forall n \in \mathbb{N}$

Proof

We will use the Principle of Mathematical Induction

Base case:

When $n = 1$

$$\sum_{i=1}^n i(i + 1) = \sum_{i=1}^1 i(i + 1) = 1(2) = 2$$

$$\frac{1}{3}n(n + 1)(n + 2) = \frac{1}{3}(1)(2)(3) = 2$$

$\therefore P(1)$ is true.

Induction:

Inductive hypothesis:

Let $k \in \mathbb{N}$.

Assume that $P(k)$ is true.

Then we have $\sum_{i=1}^k i(i + 1) = \frac{1}{3}k(k + 1)(k + 2)$

Now,

$$\sum_{i=1}^{k+1} i(i + 1) = \sum_{i=1}^k i(i + 1) + (k + 1)((k + 1) + 1) \quad (\text{by Breaking Sum Property})$$

$$= \frac{1}{3}(k)(k + 1)(k + 1 + 1) + (k + 1)(k + 2) = \frac{1}{3}(k + 1)(k + 1 + 1)(k + 1 + 2)$$

By induction we have shown that $P(n)$ is true $\forall n \in \mathbb{N}$

□

2. Prove that $n! > 2^n, \forall$ positive integers $n \geq 4$.

Proof

We will use the Principle of Mathematical Induction

We are trying to prove the open sentence $P(n) : n! \geq 2^n$ is true $\forall n \in \mathbb{Z}$ s.t. $n \geq 4$

Base case:

When $n = 4$

$$n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$2^n = 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

We have that $24 > 16$

$\therefore P(4)$ is true.

Induction:

Inductive hypothesis:

Let $k \in \mathbb{N}$ and that $k > 4$ is true

Assume that $P(k)$ is true

Then

$$(k+1)! = (k+1)k! > (k+1)2^k \quad (\text{By Inductive Hypothesis})$$

$$> (1+1)2^k = 2 \cdot 2^k = 2^{k+1}$$

Thus, by induction, we have shown that $n! \geq 2^n$ is true $\forall n \in \mathbb{Z}$ s.t. $n \geq 4$

□

3. Use induction to prove that $6 \mid (2n^3 + 3n^2 + n) \forall n \in \mathbb{N}$

Quiz 4 Cutoff

Proof by Strong Induction

Info —

Principle of Strong Induction

Let $P(n)$ be an open sentence that depends on $n \in \mathbb{N}$

If statements 1 and 2 are both true:

1. $P(1)$
2. $\forall k \in \mathbb{N}$, if $P(1) \wedge P(2) \wedge \dots \wedge P(k)$, then $P(k+1)$

Then statement 3 is true:

3. $\forall n \in \mathbb{N}, P(n)$

Example:

1. Consider the sequence defined recursively by

$$x_1 = 4, x_2 = 68 \text{ and } x_m = 2x_{m-1} + 15x_{m-2} \forall m \geq 3$$

Prove that $x_n = 2(-3)^n + 10(5)^{n-1} \forall n \in \mathbb{N}$

Proof

We proceed by strong induction.

Base cases:

$$\text{When } n = 1, x_1 = 4 \text{ and } 2(-3)^1 + 10(5)^1 = -6 + 10 = 4$$

So the claim holds for $n = 1$

$$\text{When } n = 2, x_2 = 68 \text{ and } 2(-3)^2 + 10(5)^2 = 18 + 50 = 68$$

So the claim holds for $n = 2$

Induction :

Assume the claim holds for $n = 1, n = 2, \dots, n = k$ for some $k \in \mathbb{N}$ with $k \geq 2$

Assume

$$x_k = 2(-3)^k + 10(5)^{k-2}$$

$$x_{k-1} = 2(-3)^{k-1} + 10(5)^{k-2}$$

$$x_{k-2} = 2(-3)^{k-2} + 10(5)^{k-2}$$

Then

$$\begin{aligned}
 x_{k+1} &= 2x_k + 15x_{k-1} \\
 &= 2(2(-3)^{k-1} + 10(5)^{k-2}) + 15(2(-3)^{k-1} + 10(5)^{k-2}) \quad (\text{by I.H}) \\
 &= 4(-3)^k + 20(5)^{k-1} + 5 \cdot 3 \cdot 2(-3)^{k-1} + 10 \cdot 5 \cdot 3 \cdot (5)^{k-2} \\
 &= 4(-3)^k + 5 \cdot 4 \cdot 5^{k-1} - 5 \cdot 2(-3)^k + 2 \cdot 3 \cdot 5^k \\
 &= (-3)^{k(4-10)} + 5^{k(4+6)} \\
 &= 2(-3)^{k+1} + 105^k
 \end{aligned}$$

$\therefore x_{k+1} = 2(-3)^{k+1} + 105^k$ so the claim holds for $n = k + 1$ \therefore

by strong induction the claim holds $\forall n \in \mathbb{N}$

□