# CH 2 — Differential Equations

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## Introduction to Differential Equations

🐳 **Info — Differential Equations**

A differential equation (DE), is an equation involving an unknown function and its derivatives. The term ordinary differential equation (ODE) refers to a differential equation involving single-variable functions, whereas the term partial differential equation (PDE) refers to a differential equation involving multivariable functions (i.e., functions with multiple inputs).

An ODE is expressed

$$F(x, y, y', y'', ..., y^n) = 0$$

for some $n \in \mathbb{N}$

The order of a DE is the order of the **highest derivative** that appears in the equation.

A function $y = \varphi(x)$ is a solution to the differential equation $F(x, y, y', y'', ..., y^n) = 0$ if

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), ..., \varphi^n(x)) = 0$$

The graph of a solution to a DE is called a **solution curve**

The complete collection of solutions to a DE, including any arbitrary constants, is called its general solution. A particular solution to a DE is one in which all arbitrary constants have been specified.

A differential equation together with one or more initial conditions is known as an initial value problem (IVP)

Examples:

1. First order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$
2. Second order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \ln x + y$
3. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \implies y = -\cos x + C, c \in \mathbb{R}$
4. $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x, y(0) = 2 \implies -\cos x + C \implies -\cos x + 3$
5. $y' = x + y$

   $y = -1 - x + 2e^x \implies y' = -1 + 2e^x \implies (x + y) = x + (-1 - x + 2e^x) = -1 + 2e^x = y'$
6. $y = -1 - x - 5e^x$

   $y' = -1 - 5e^x \implies x + (-1 - x - 5e^x) = -1 - 5e^x = y'$
7. $y = -5 - x + 2e^x$

   $y' = -1 + 2e^x \implies x + (-5 - x + 2e^x) = -5 + 2e^x \neq y'$

Thus $y = -1 - x + Ce^x$, for $C \in \mathbb{R}$ is always a solution

8. Determine all real numbers $k$ s.t. $x(t) = \sin(kt)$ is a solution to the second-order differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2x$

$$\begin{cases} x'(t) = k\cos(kt) \\ x''(t) = -k^2 \sin(kt) \end{cases} \Longrightarrow -k^2 \sin kt = 02 \sin kt$$

$$(k^2 - 2)\sin(kt) = 0$$

$$k = \pm\sqrt{2}, k = 0$$

## Direction Fields

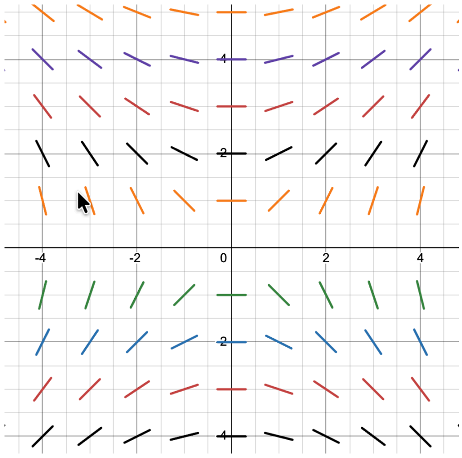
**Info — Direction Field**

A direction field for the different equation $y' = F(x)$ displays short line segments of slope $F(x, y)$ at various points in the Cartesian plane

**Tip** — Direction field plotter for DE: https://www.desmos.com/calculator/p7vd3cdmei

Examples:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$

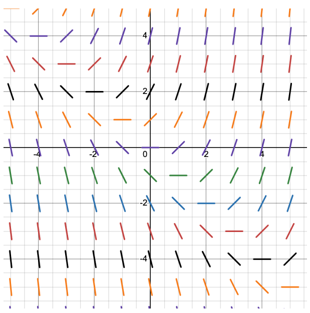



We can obtain this when plugging-in numbers.

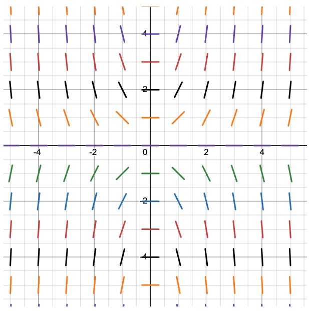
At $(-2, 1)$, $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(-2,1)} = \left.\frac{x}{y}\right|_{(x,y)=(-2,1)} \Longrightarrow -\frac{2}{1} = -2$

Hence a slope of $-2$ at $(-2, 1)$ and other points have the same method

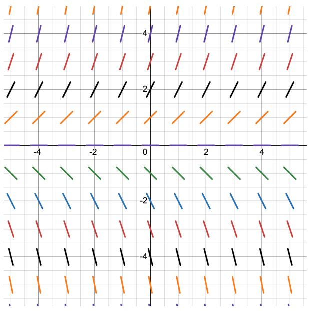
2. $y'x = x + y$



3. $y' = xy$



4. $y' = y$



## Separable Differential Equations

> 🐳 **Info — Separable Differential Equation**
>
> A first order differential equation is said to be separable if it is written in format
>
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(x)$$

Examples:

1. $y' = \sin x$
2. $y' = \frac{x}{y}$
3. $y' = 5y$
4. $y^2 y' = 2y + xy$
5. $y' = x + y$ is not separable but linear, see in Linear First Order DEs

💡 **Tip — Solving Separable DE** We have two cases

1. Determine any solution $y$ with $h(y) = 0$
2. Find the solutions $y$ where $h(y) \not\equiv 0$ by evaluating

   $$\int \frac{1}{h(y)} \, \mathrm{d}y = \int g(x) \, \mathrm{d}x$$

   If possible, isolate $y$ as a function of $x$ in the resulting equation.

The general solution is the collection of all solutions obtained in Case 1 together with all solutions obtained in Case 2.

Examples:

1. $y' = y$

   Case 1: $y = 0$

   Case 2:

   $$\int \frac{1}{y} \, \mathrm{d}y = \int 1 \, \mathrm{d}x$$
   $$\ln|y| = x + C_1$$
   $$|y| = e^{x+C_1} y = \pm e^{C_1} e^x = \pm C e^x, \ C \in \mathbb{R}$$

   The general solution is $y = A e^x, \ A \in \mathbb{R}$

2. $y' = \frac{x}{y}$

   Case 1: $h(y) = \frac{1}{y} \neq 0$

   Case 2:

   $$\int y \, \mathrm{d}y = \int x \, \mathrm{d}x$$
   $$\frac{y^2}{2} = \frac{x^2}{2} + C$$
   $$y^2 = x^2 + 2C$$
   $$y = \pm\sqrt{x^2 + 2C}$$

   The general solution $y = \pm\sqrt{x^2 + D}, \ D \in \mathbb{R}$

## Linear First-Order Differential Equations