CH 10 - Complex Numbers

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Standard Form

🐳 **Info — Definition of Complex Numbers**

A **complex number** z in **standard form** is an expression of the form $z = x + yi$ where $x, y \in \mathbb{R}$.

The real number x is called the **real part** of z, and is written $\mathfrak{R}(x)$.

The real number y is called the **imaginary part** of z, and is written $\mathfrak{I}(z)$.

The set of complex numbers is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$$

The complex number $z = x + yi$ and $w = u + vi$ are equal $(z = w)$ if and only if $x = u, y = v$

A complex number z is said to be purely real if $y = 0$ (i.e. $1 = 1 + 0i$)

A complex number z is said to be purely imaginary if $x = 0$ (i.e. $i = 0 + 1i$)

0 is purely real and purely imaginary (i.e. $0 = 0 + 0i$)

🐳 **Info — Complex Arithmetics**

Let $z = a + bi$ and $w = c + di$ be complex numbers. Then the

Addition is defined as

$$z + w = (a + c) + (b + d)i$$

Multiplication is defined as

$$zw = (ac - bd) + (ad + bc)i$$

Examples:

Let $z = 2 + 3i, w = -1 + 7i$

1. $z + w = (2 - 1) + (3 + 7)i = 1 + 10i$
2. $zw = (-2 - 21) + (14 - 3)i = -23 + 11i$
3. $i^2 = ii = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + (0 + 0)i = -1$

$\therefore i^2 = -1$

From (3), we can derive a easier way of multiplication

💡 **Tip** — Multiplication trick

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac-bd) + (ad+bc)i$$

Properties of Complex Arithmetics

1. $z + 0 = 0 + z = z$
2. $z0 = 0z = 0$
3. $z + (-1)z = (-1)z + z = 0$
4. $z1 = 1z = z$

🐳 **Info — Multiplicative Inverse**

For all complex numbers z, the multiplicative inverse of z exists if and only if $z \neq 0$. Moreover, for $a + bi \neq 0$, the multiplicative inverse is unique given by

$$z^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = \frac{a-bi}{a^2+b^2}$$

Examples:

1. $\frac{(1-2i)-(3+4i)}{(5-6i)} = [(1-2i)-(3+4i)] \cdot (5-6i)^{-1} = (-2-6i) \cdot \frac{1}{25+36} \cdot (5+6i)$

$= \frac{1}{61} \cdot (-10 - 30i - 12i + 36) = \frac{26-42i}{61} = \frac{26}{61} - \frac{42}{61}i$

2. $i^{2025} = \left(i^4\right)^{506} \cdot i = (-1)^{506} \cdot i = i$

💡 **Tip — i^n Solutions**

$$i^n = \begin{cases} 1, n \equiv 0 \pmod 4 \\ i, n \equiv 1 \pmod 4 \\ -1, n \equiv 2 \pmod 4 \\ -i, n \equiv 3 \pmod 4 \end{cases}$$

🐳 **Info — Properties of Complex Arithmetics**

In complex arithmetic, the following properties are vlaid for $u, v, z \in \mathbb{C}$

1. Associativity of addition: $(u+v)+z = u+(v+z)$
2. Communtativity of addition: $u+v = v+u$
3. Additive identity: $0 = 0 + 0i$ has the property that $z + 0 = z$
4. Additive inverses: If $z = a + bi$, then there exists an additive inverse of z, written $-z$, with the property that $z + (-z) = 0$. The additive inverse of $z = a + bi$ is $-z = -a - bi$
5. Associativity of multiplication: $(uv)z = u(vz)$
6. Communtativity of multiplication: $uv = vu$
7. Multiplicative identity: $1 = 1 + 0i$ has property that $z1 = z$
8. Multiplicative inverses: If $z = a + bi \neq 0$, then there exists a multiplicative inverse of z, written z^{-1}, with the property that $zz^{-1} = 1$. The multiplicative inverse of $z = a + bi \neq 0$ is $z^{-1} = \frac{a-bi}{a^2+b^2}$
9. Distributivity: $z(u+v) = zu + zv$

Example:

Proof of PCA Part 5: $(uv)z = u(vz)\ \forall u, v, z \in \mathbb{C}$

Let $u = a + bi, v = c + di, z = x + yi$ where $a, b, c, d, x, y \in \mathbb{R}$ so that $u, v, z \in \mathbb{C}$

$(uv)z = [(a + bi)(c + di)](x + yi) = (ac - bd + (ad + bc)i)(x + yi) =$

$= ((ac - bd)x - (ad + bc)y) + ((ac - bd)y + (ad + bc)x)i$

$= (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$u(vz) = (a + bi)[(c + di)(x + yi)] = (a + bi)(cx - dy) + (cy + dx)i$

$= (a(cx - dy) - b(cy + dx)) + (a(cy + dx) + b(cx - dy))i$

$= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i$

Thus $(uv)z = u(vz) = (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$

$\square$

🐳 **Info** — Other Arithemtic of Complex Numbers

For $z \in \mathbb{C}$

1. $z^0 = 1$
2. $z^1 = z$
3. $z^{k+1} = z^k z\ \forall k \in \mathbb{N}$
4. $(z^n)^m = z^{nm}$ and $z^n z^m = z^{n+m}\ \forall n, m \in \mathbb{N} \cup \{0\}$

For $n \notin \mathbb{N} \cup \{0\}$ will be discussed in later lectures

Examples:

Find a real solution to $6z^3 + \left(1 + 3\sqrt{2}i\right)z^2 - \left(11 - 2\sqrt{2}\right)i - 6 = 0$

Let $z = x \in \mathbb{R}$, that is $z = x + 0i$

$\implies 6x^3 + \left(1 + 3\sqrt{2}i\right)x^2 - \left(11 - \sqrt{2}i\right)x - 6 = 0$

$\implies (6x^3 + x^2 - 11x - 6) + \left(3\sqrt{2}x^2 + 2\sqrt{2}x\right)i = 0 + 0i$

$\implies (6x^3 + x^2 - 11x - 6) = 0$ and $3\sqrt{2}x^2 + 2\sqrt{2}x = 0$

$x = 0$ or $x = -\frac{2}{3}$ for the imaginary part.

However, $x = 0$ does not satisfy the real part.

$\therefore x = -\frac{2}{3} \implies z = -\frac{2}{3} + 0i$

Conjugate and Modulus

🐳 **Info** — **Complex Conjugate**

The complex **conjugate** of a complext number $z = x + yi$ written $\overline{z}$ is the complex number

$$\overline{z} = x - yi$$