

CH 3 - Matrices and Linear Mapping

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Info – Matrix Definition

A $m \times n$ **matrix A** is a rectangular array with m rows and n columns. We denote the entry in the i -th row and j -th column by a_{ij} or $(A)_{ij}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{jn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Two $m \times n$ matrices A, B are equal if $a_{ij} = b_{ij}, \forall 1 \leq i \leq m, 1 \leq j \leq n$.

The set of all $m \times n$ matrices with real entries is denoted by $M_{m \times n}(\mathbb{R})$

Let $A, B \in M_{m \times n}(\mathbb{R}), c \in \mathbb{R}$

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$$

$$(cA)_{ij} = c(A)_{ij}$$

Example:

$$\begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 6 & -3 \end{bmatrix} + 7 \begin{bmatrix} \pi & -1 \\ 6 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2+7\pi & -6 \\ 42 & 26 \\ 6 & -3 \end{bmatrix}$$

Info – Matrix Properties

If $A, B, C \in M_{m \times n}(\mathbb{R}), c, d \in \mathbb{R}$ then

1. $A + B \in M_{m \times n}(\mathbb{R})$
2. $(A + B) + C = A + (B + C)$
3. $A + B = B + A$
4. There exists a matrix $O_{m,n} \in M_{m \times n}(\mathbb{R})$, s.t. $A + O_{m,n} = A \quad \forall A$
5. For every $A \in M_{m \times n}(\mathbb{R})$, $\exists (-A) \in M_{m \times n}(\mathbb{R})$ s.t. $A + (-A) = O_{m,n}$
6. $cA \in M_{m \times n}(\mathbb{R})$
7. $c(dA) = (cd)A$
8. $(c + d)A = cA + dA$
9. $c(A + B) = cA + cB$
10. $1A = A$

Info – Transpose

The **transpose** of a $m \times n$ matrix A is the $n \times m$ matrix A^T whose ij -th entry is the ji -th entry of A . $(A^T)_{ij} = (A)_{ji}$

If $A, B \in M_{m \times n}(\mathbb{R}), c \in \mathbb{R}$

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = cA^T$

Proof

$$((cA)^T)_{ij} = (cA)_{ji} = c(A)_{ji} = c(A^T)_{ij} = (cA^T)_{ij}$$

□

Note: we can regard vectors in \mathbb{R}^n as matrices in $M_{n \times 1}(\mathbb{R})$

Examples:

$$\begin{aligned} 1. \quad A &= \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 6 & -3 \end{bmatrix} \\ A^T &= \begin{bmatrix} 2 & 0 & 6 \\ 1 & 5 & -3 \end{bmatrix} \\ 2. \quad \vec{x} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ \vec{x}^T &= [2 \ 1 \ 3] \end{aligned}$$

Matrix Multiplications

Note that

- $A = \begin{bmatrix} \overrightarrow{a_1}^T \\ \overrightarrow{a_2}^T \\ \dots \\ \overrightarrow{a_m}^T \end{bmatrix}$
- $\vec{a}_i = \vec{x} = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \dots \\ 1_{in} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$

Thus we can solve the linear system with augmented matrix $[A \mid \vec{b}]$ as $\begin{bmatrix} \overrightarrow{a_1} \cdot \vec{x} \\ \overrightarrow{a_2} \cdot \vec{x} \\ \dots \\ \overrightarrow{a_m} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$

Info – Matrix-Vector Multiplication

1. Let A be an $m \times n$ matrix whose rows are denoted \vec{a}_i^T for $1 \leq i \leq m$. For any $\vec{x} \in \mathbb{R}^n$

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \dots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$$

2. Let $A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$ be $m \times n$ matrix, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$A\vec{x} = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \dots + \vec{a}_n x_n$$

Note that \vec{a}_i refers to the i -th column of the matrix A

Properties

If $A \in M_{m \times n}(\mathbb{R})$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, $c \in \mathbb{R}$

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
2. $A(c\vec{x}) = c(A\vec{x})$

Examples:

1. Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \\ 0 & -3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} [2] \cdot [3] \\ [1] \cdot [-1] \\ [5] \cdot [3] \\ [7] \cdot [-1] \\ [0] \cdot [3] \\ [-3] \cdot [-1] \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

2. Compute E1 in another way:

$$3 \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 6 & 3 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} -9 \\ -1 \\ 7 \\ -3 \end{bmatrix}$$

Info – Column Extraction Theorem

If \vec{e}_i is the i -th standard basis and $A = [\vec{a}_1 \dots \vec{a}_n]$ then

$$A\vec{e}_i = \vec{a}_i$$

Example:

Let A be the coefficient matrix of the homogeneous linear system $[A \mid \vec{0}]$ or $A\vec{x} = \vec{0}$

Let S be the solution set. Prove that S is a subspace of \mathbb{R}^n

Proof

1. $\vec{x} = \vec{0}$ is a solution to $[A \mid \vec{0}]$, $\vec{0} \in S$
2. For $\vec{x}, \vec{y} \in S$, we have $A\vec{x} = \vec{0}, A\vec{y} = \vec{0}$. Then $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0} \in S$
3. For $\vec{x} \in S, c \in \mathbb{R}, A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0} \in S$

By subspace test, S is a subspace of \mathbb{R}^n

□

Info – Transpose - Dot Product Theorem

If $\vec{x}, \vec{y} \in \mathbb{R}^n$

$$\vec{x}^T \vec{y} = \vec{x} \cdot \vec{y}$$

Info – Matrix-Matrix Multiplication

Let A be a $m \times n$ matrix, B be $n \times p$ matrix. Then AB is the $m \times p$ matrix defined by

$$AB = [A\vec{b}_1 \ A\vec{b}_2 \ \dots \ A\vec{b}_p]$$

where $B = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_p], A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_p]$

Consider the ij th entry of AB , that is the entry of AB in the i -th row and j -th column, that is, $(AB)_{ij}$

1. The j -th column of AB is $A\vec{b}_j$, and thus its i -th entry will be the ij -th entry of AB

2. If we write $A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \dots \\ \vec{a}_m^T \end{bmatrix}$, then $A\vec{b}_j = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \dots \\ \vec{a}_m^T \end{bmatrix} \vec{b}_j = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 \\ \vec{a}_2 \cdot \vec{b}_2 \\ \dots \\ \vec{a}_m \cdot \vec{b}_j \end{bmatrix}$

Then the i -th entry of this vector is $(AB)_{ij} = \vec{a}_i \cdot \vec{b}_j$

Properties

If A, B, C be matrices of the correct size so that the required products are defined, $t \in \mathbb{R}$,

1. $A(B + C) = AB + AC$
2. $(A + B)C = AC + BC$
3. $t(AB) = (tA)(B) = A(tB)$
4. $A(BC) = (ABC)$
5. $(AB)^T = B^T A^T$

Note that AB is not BA generally speaking. There might be special cases.

Examples:

$$1. \text{ Let } A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 5 & 6 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 & 6 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} (2)(3)+(1)(0) & (2)(5)+(1)(1) & (2)(6)+(1)(-1) \\ (1)(3)+(-2)(0) & (1)(5)+(-2)(1) & (1)(6)+(-2)(-1) \\ (3)(3)+(4)(0) & (3)(5)+(4)(1) & (3)(5)+(4)(-1) \end{bmatrix} = \begin{bmatrix} 6 & 11 & 11 \\ 3 & 4 & 8 \\ 9 & 19 & 14 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 5 & 6 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (3)(2)+(5)(1)+(6)(3) & (3)(1)+(5)(-2)+(6)(4) \\ (0)(2)+(1)(1)+(-1)(3) & (0)(1)+(-1)(-2)+(1)(4) \end{bmatrix} = \begin{bmatrix} 29 & 17 \\ -2 & -6 \end{bmatrix}$$

$$2. \text{ Let } A = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ -6 & 20 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 17 \end{bmatrix}$$