

# CH 1 — Vectors in Euclidean Space

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## Vector Addition and Scalar Multiplication

### Info — Vector

The set  $\mathbb{R}^n$  is defined as  $\left\{ \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$

A **vector** is an element  $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  of  $\mathbb{R}^n$

The row notation of  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  is  $\vec{v} = [v_1 \ v_2 \ v_3]^T$

### Info — Equality

We say that vectors  $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^m$  and  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  in  $\mathbb{R}^n$  are **equal**

if  $n = m$  and  $u_i = v_i \forall i = 1, 2, \dots, n$ .

We denote it:  $\vec{w} = \vec{v}$

### Info — Addition and Properties

Let  $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ .

Then  $\vec{w} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$

1.  $\vec{w} + \vec{v} = \vec{v} + \vec{w}$
2.  $\vec{w} + \vec{v} + \vec{w} = \vec{w} + (\vec{v} + \vec{w})$
3. There is a zero **vector**,  $\vec{0} = [0 \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{R}^n$
4.  $\vec{v} + \vec{0} = \vec{v}$

### Info – Additive Inverse

Let  $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ . The additive inverse of  $\vec{w}$  denoted  $-\vec{w}$  is defined as

$$-\vec{w} = \begin{bmatrix} -u_1 \\ -u_2 \\ \dots \\ -u_n \end{bmatrix}$$

$$\vec{w} - \vec{w} = \vec{w} + (-\vec{w}) = \vec{0}$$

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \\ \dots \\ v_n - u_n \end{bmatrix}$$

### Info – Scalar Multiplication

Let  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ ,  $\vec{w} \in \mathbb{R}^n$ ,  $c, d \in \mathbb{R}$ . Then the scalar product  $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \dots \\ cv_n \end{bmatrix}$

1.  $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
2.  $c(\vec{w} + \vec{v}) = c\vec{w} + c\vec{v}$
3.  $0\vec{w} = \vec{0}$
4. If  $c\vec{v} = \vec{0}$  then  $c = 0 \vee \vec{v} = \vec{0}$

### Info – Linear Combination

For  $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ , and  $c_1, \dots, c_k \in \mathbb{R}$  we call the expression

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$$

a **linear combination** of  $\vec{v}_1, \dots, \vec{v}_k$ .

Examples:

1. Let  $\vec{u} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$  then  $2\vec{u} - 3\vec{v} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$
2. Let  $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ . Is  $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$  a linear combination of  $\vec{u}$  and  $\vec{v}$ ?

We set  $\vec{x} = c_1\vec{u} + c_2\vec{v}$  and try to solve for  $c_1, c_2$

That is  $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 \end{bmatrix}$ , we obtain  $c_1 = 2$ ,  $c_2 = \frac{1}{2}$ . So  $\vec{x}$  is a linear combination of  $\vec{u}, \vec{v}$

## Bases



### Info – Standard Basis

In  $\mathbb{R}^n$ , let  $\vec{e}_i$  be the vector whose  $i^{\text{th}}$  component is 1 with all other components 0. The set  $E = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is called the **standard basis for  $\mathbb{R}^n$**

(i.e.  $\mathbb{R}^3$  is  $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ )

If  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_n \vec{e}_n$  then we call  $v_1, v_2, \dots, v_n$  the **components of  $\vec{v}$**

## Vectors in $\mathbb{C}^n$



### Info – Vectors in $\mathbb{C}^n$

The set  $\mathbb{C}^n$  is defined as  $\left\{ \vec{z} = \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix} : z_1, \dots, z_n \in \mathbb{C} \right\}$

The **vector** is an element  $\vec{z} = \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix}$  of  $\mathbb{C}^n$

In  $\mathbb{C}^n$ , let  $\vec{e}_i$  be the vector whose  $i^{\text{th}}$  component is 1 with all other components 0. The set  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is called the **standard basis for  $\mathbb{C}^n$**

## Dot Product



### Info – Dot Product

Let  $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ ,  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$  be vectors in  $\mathbb{R}^n$ . We defined their **dot product** by

$$\vec{w} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

1.  $\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$
2.  $(\vec{w} + \vec{v}) \cdot \vec{u} = \vec{w} \cdot \vec{u} + \vec{v} \cdot \vec{u}$
3.  $(c\vec{w}) \cdot \vec{v} = c(\vec{w} \cdot \vec{v})$
4.  $\vec{w} \cdot \vec{w} \geq 0$ , with  $\vec{w} \cdot \vec{w} = 0 \iff \vec{w} = 0$

### Info – Vector Unit Basics

Let  $\vec{v}, \vec{w} \in \mathbb{R}^n$

1. The **length** of vector  $\vec{w}$  is  $\|\vec{w}\| = \sqrt{\vec{w} \cdot \vec{w}}$
2. If  $c \in \mathbb{R}, \vec{w} \in \mathbb{R}^n$ , then  $\|c\vec{w}\| = |c| \|\vec{w}\|$
3.  $\vec{v}$  is a **unit vector** if  $\|\vec{v}\| = 1$
4. **Normalization** is when some  $\vec{v}$  is a non-zero vector,

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

in the direction of  $\vec{v}$  by scaling  $\vec{v}$

5. With  $\vec{w}, \vec{v}$  non-zero vectors. The angle  $\theta, 0 \leq \theta \leq \pi$  between  $\vec{v}$  is such that

$$\vec{w} \cdot \vec{v} = \|\vec{w}\| \|\vec{v}\| \cos \theta \text{ that is } \theta = \arccos\left(\frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\| \|\vec{v}\|}\right)$$

6.  $\vec{w}, \vec{v}$  are **orthogonal/perpendicular** if  $\vec{w} \cdot \vec{v} = 0$

## Projection

### Info – Projection

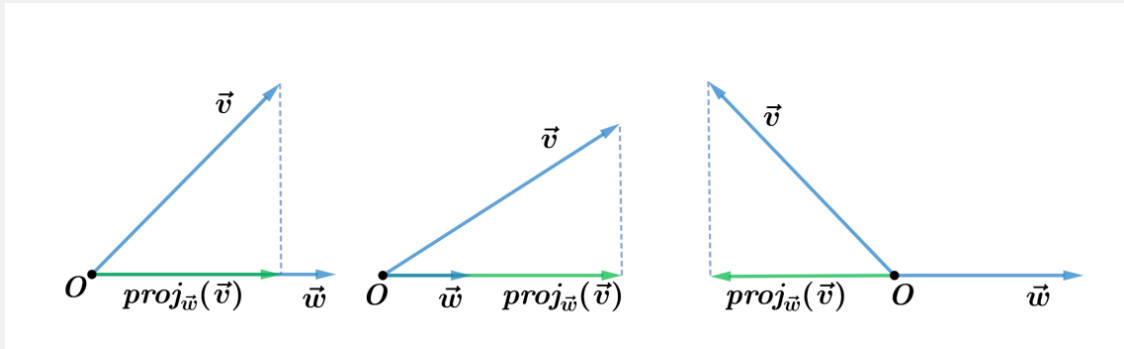
Let  $\vec{v}, \vec{w} \in \mathbb{R}^n$  with  $\vec{w} \neq 0$ .

1. The **projection** of  $\vec{v}$  onto  $\vec{w}$  is defined by

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

We also refer to this as the **projection of  $\vec{v}$  in the  $\vec{w}$  direction**

Illustration of  $\text{proj}_{\vec{w}}(\vec{v})$ :



2. We refer to the quantity

$$\|\vec{v}\| \cos \theta = \vec{v} \cdot \hat{w}$$

as the **component** (or scalar component) **of  $\vec{v}$  along  $\vec{w}$**

3. The **perpendicular** of  $\vec{v}$  onto  $\vec{w}$  is defined by  $\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$
4. The projection and the perpendicular of a vector  $\vec{v}$  onto  $\vec{w}$  are orthogonal; that is

$$\text{perp}_{\vec{w}}(\vec{v}) \cdot \text{proj}_{\vec{w}}(\vec{v}) = 0$$

## Standard Inner Product in $\mathbb{C}^n$

### Info — Standard inner product

Let  $c \in \mathbb{C}$  and  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$

The **standard inner product** of  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$  is

$$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w_1} + v_2 \overline{w_2} + \dots + v_n \overline{w_n}$$

1.  $\langle \vec{u}, \vec{w} \rangle = \overline{\langle \vec{v}, \vec{w} \rangle}$
2.  $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3.  $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
4.  $\langle \vec{v}, \vec{v} \rangle \geq 0$ , with  $\langle \vec{v}, \vec{v} \rangle = 0$  if and only if  $\vec{v} = \vec{0}$
5. The length:  $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$
6.  $\vec{w}, \vec{v}$  are **orthogonal/perpendicular** if  $\langle \vec{w}, \vec{v} \rangle = 0$
7. With  $\vec{w} \neq 0$ . The **projection of  $\vec{v}$  onto  $\vec{w}$**  is defined by

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w} = \langle \vec{v}, \vec{w} \rangle \hat{w}$$

## The Cross Product in $\mathbb{R}^3$

 Info — Cross Products Let  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ .

The **cross product** of  $\vec{u}, \vec{v}$  is defined to be the vector in  $\mathbb{R}^3$  given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Let  $\vec{z} = \vec{u} \times \vec{v}$

1.  $\vec{z} \cdot \vec{u} = \vec{z} \cdot \vec{v} = 0$
2.  $\vec{v} \times \vec{u} = -\vec{z} = -\vec{u} \times \vec{v}$
3. If  $\vec{u} \neq \vec{0}$  and  $\vec{v} \neq \vec{0}$ , then  $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$  where  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$

### Info – Linearity of the Cross Product

Let  $c \in \mathbb{R}$  and  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ , then

1.  $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
2.  $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
3.  $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4.  $\vec{u} \times c(\vec{v}) = c(\vec{u} \times \vec{v})$