CH 1 — Vectors

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Introduction

🐳 **Info — Vector**

The set $\mathbb{R}^n$ is defined as $\left\{ \vec{x} = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}$

A **vector** is an element $\vec{x} = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$ of $\mathbb{R}^n$

The row notation of $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}$ is $\vec{v} = [v_1 \ v_2 \ v_3]^T$

Operations

🐳 **Info — Equality**

We say that vectors $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$ are **equal** if $u_i = v_i \forall \ i = 1, 2, \dots, n$.

Denoted $\vec{u} = \vec{v}$

🐳 **Info — Addition and Properties**

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix} \in \mathbb{R}^n$.

Then $\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \dots \\ u_n+v_n \end{pmatrix}$

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $\vec{u} + \vec{v} + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. There is a zero **vector**, $\vec{0} = [0 \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{R}^n$
4. $\vec{v} + \vec{0} = \vec{v}$

🐳 Info — Additive Inverse

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} \in \mathbb{R}^n$. The additive inverse of $\vec{u}$ denoted $-\vec{u}$ is defined as

$$-\vec{u} = \begin{pmatrix} -u_1 \\ -u_2 \\ \dots \\ -u_n \end{pmatrix}$$

$$\vec{u} - \vec{u} = \vec{u} + (-\vec{u}) = \vec{0}$$

$$\vec{v} - \vec{u} = \vec{v} + (-\vec{u}) = \begin{pmatrix} v_1 - u_1 \\ v_2 - u_2 \\ \dots \\ v_n - u_n \end{pmatrix}$$

🐳 Info — Scalar Multiplication

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}, \vec{u} \in \mathbb{R}^n, c, d \in \mathbb{R}$. Then the scalar product $c\vec{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ \dots \\ cv_n \end{pmatrix}$

1. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
2. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
3. $0\vec{u} = \vec{0}$
4. If $c\vec{v} = \vec{0}$ then $c = 0 \vee \vec{v} = 0$

🐳 Info — Standard Basis

In $\mathbb{R}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set $\mathrm{E} = \{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}$ is called the **standard basis for $\mathbb{R}^n$**

(i.e. $\mathbb{R}^3$ is $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$)

If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} = v_1\vec{e_1} + v_2\vec{e_2} + \dots + v_n\vec{e_n}$ then we call $v_1, v_2, \dots, v_n$ the **components of $\vec{v}$**

Vectors in $\mathbb{C}^n$

🐳 **Info — Vectors in $\mathbb{C}^n$**

The set $\mathbb{C}^n$ is defined as $\left\{ \vec{z} = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix} : z_1, \dots, z_n \in \mathbb{C} \right\}$

The **vector** is an element $\vec{z} = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$ of $\mathbb{C}^n$

In $\mathbb{C}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set $\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}$ is called the **standard basis for $\mathbb{C}^n$**

Dot Product

🐳 **Info — Dot Product**

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$. We defined their **dot product** by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
4. $\vec{u} \cdot \vec{u} \geq 0$, with $\vec{u} \cdot \vec{u} = 0 \Longleftrightarrow \vec{u} = 0$
5. The length of vector $\vec{u} \in \mathbb{R}^n$ is $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$
6. If $c \in \mathbb{R}, \vec{u} \in \mathbb{R}^n$, then $\|\vec{u}\| = |c| \ \|\vec{u}\|$