

# Classical Related Rates Problems

This note summarizes the most common and instructive examples of **related rates** problems encountered in calculus. Each scenario is described with a short setup, diagram description, and key relations.

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## 1. Ladder Sliding Down a Wall

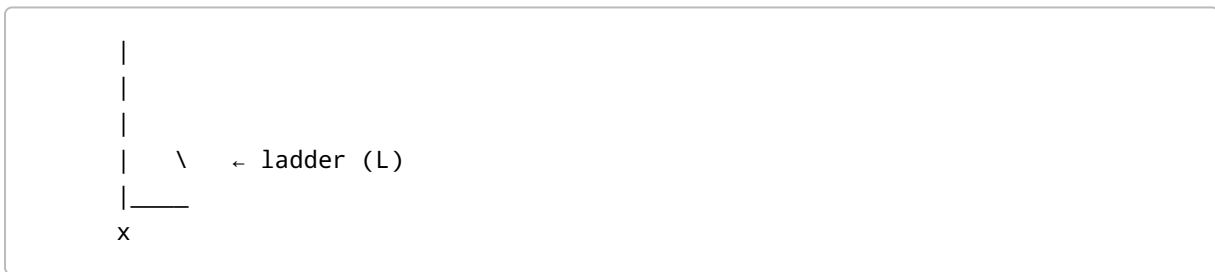
**Setup:** A ladder of fixed length  $L$  leans against a vertical wall. The bottom slides away at a rate  $dx/dt$ . Find how fast the top slides down ( $dy/dt$ ).

**Relation:**  $x^2 + y^2 = L^2$

**Differentiate:**  $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$

**Illustration:**



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## 2. Shadow of a Person under a Lamp

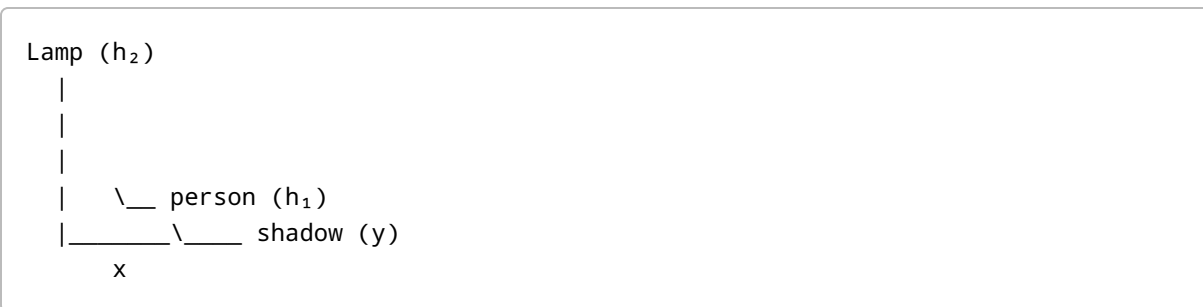
**Setup:** A person of height  $h_1$  walks away/toward a lamp of height  $h_2$ . The shadow length is  $y$ , and the distance from lamp to person is  $x$ . The person walks at  $dx/dt$ .

**Relation:**  $\frac{h_2}{x+y} = \frac{h_1}{y}$

**Differentiate:**  $h_2 y' (x+y) - h_1 (x+y) y' = h_1 y x'$

Simplify to:  $\frac{dy}{dt} = \frac{h_1}{h_2 - h_1} \frac{dx}{dt}.$

**Illustration:**



### 3. Water Pouring into a Conical Tank

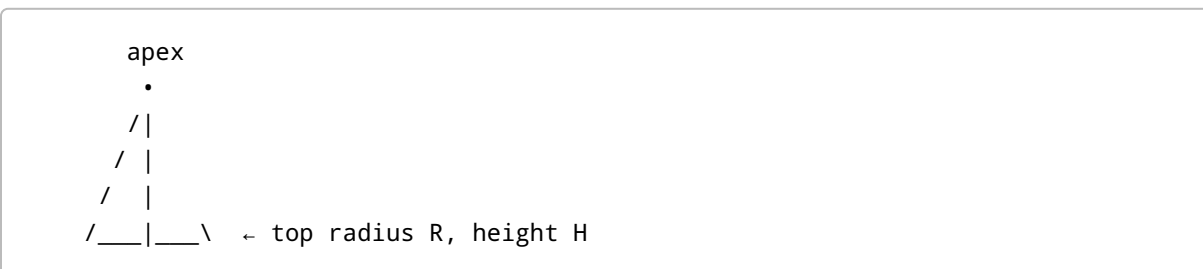
**Setup:** Water is poured into an inverted cone. Given  $dh/dt$ , find  $dV/dt$  or vice versa.

**Relation:** Volume of a cone  $V = \frac{1}{3}\pi r^2 h$ . For similar triangles,  $r/h = R/H$ , so  $r = (R/H)h$ .

**Differentiate:** Substitute for  $r$  and differentiate  $V$  with respect to  $t$ :

$$\frac{dV}{dt} = \frac{\pi R^2}{H^2} h^2 \frac{dh}{dt}.$$

**Illustration:**



### 4. Expanding Circle (Area & Radius)

**Setup:** The radius  $r$  of a circle increases at  $dr/dt$ . Find how fast the area  $A$  or circumference  $C$  increases.

**Relations:** -  $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$  . -  $C = 2\pi r \Rightarrow \frac{dC}{dt} = 2\pi \frac{dr}{dt}$ .

**Illustration:**



## 5. Two Cars Moving at Right Angles

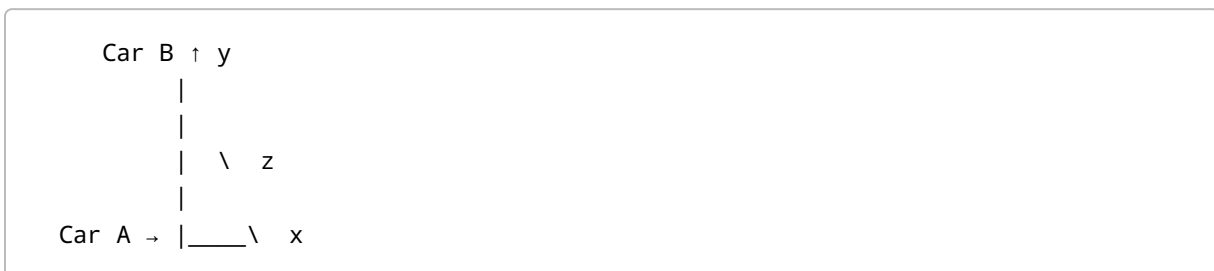
**Setup:** Car A moves east at  $dx/dt$ , Car B north at  $dy/dt$ . Distance between them is  $z$ .

**Relation:**  $z^2 = x^2 + y^2$

**Differentiate:**  $2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$

$\Rightarrow \frac{dz}{dt} = \frac{xx' + yy'}{z}$ .

**Illustration:**



## 6. Rotating Beacon on a Shoreline

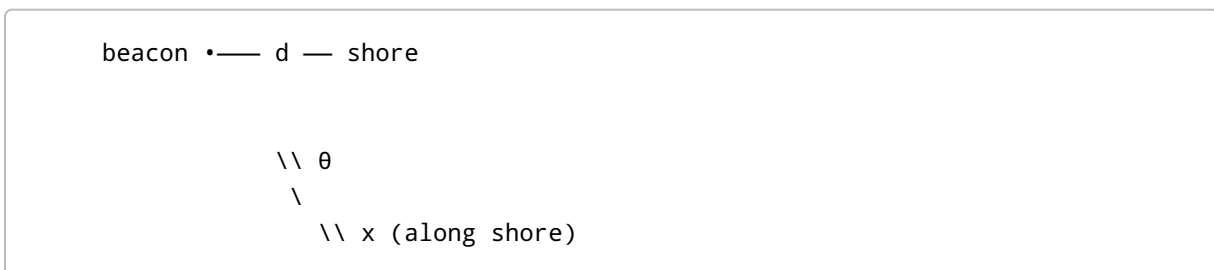
**Setup:** A beacon located a fixed distance  $d$  from the shore rotates, sweeping a beam along the shore. The angle is  $\theta$ , and the distance along the shore is  $x$ .

**Relation:**  $\tan \theta = x/d$

**Differentiate:**  $\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{d} \frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = d \sec^2 \theta \frac{d\theta}{dt}$ .

**Illustration:**



## 7. Draining Cylindrical Tank

**Setup:** A cylindrical tank drains so that  $h$  decreases at a known rate. Find  $dV/dt$ .

**Relation:**  $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

If  $r$  also changes, use product rule.

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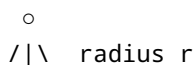
## 8. Inflating a Sphere

**Setup:** A balloon's radius  $r$  increases at  $dr/dt$ . Find  $dV/dt$ .

**Relation:**  $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

**Illustration:**



## 9. Airplane Flying Over Radar Station

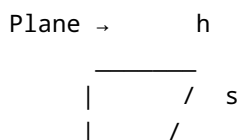
**Setup:** A plane flying horizontally at constant altitude  $h$  and horizontal speed  $dx/dt$ . The radar measures distance  $s$  from station to plane.

**Relation:**  $s^2 = x^2 + h^2$

**Differentiate:**  $2s \frac{ds}{dt} = 2x \frac{dx}{dt}$

$$\Rightarrow \frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt}.$$

**Illustration:**



| \_\_\_\_ / x  
Radar

## 10. Boats Pulling In or Out (Rope Problem)

**Setup:** A boat is pulled toward a dock by a rope passing through a pulley located  $h$  meters above the boat's bow. The rope is pulled at  $dr/dt$ , and the distance from the boat to dock is  $x$ .

**Relation:**  $r^2 = x^2 + h^2$

**Differentiate:**  $2r \frac{dr}{dt} = 2x \frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{r}{x} \frac{dr}{dt}$ .

## Summary Table

Scenario	Key Equation	Goal
Ladder	$x^2 + y^2 = L^2$	$dy/dt$ given $dx/dt$
Shadow	$\frac{h_2}{x+y} = \frac{h_1}{y}$	$dy/dt, d(x+y)/dt$
Cone	$V = \frac{1}{3}\pi r^2 h$	$dV/dt$ or $dh/dt$
Circle	$A = \pi r^2$	$dA/dt$
Cars	$z^2 = x^2 + y^2$	$dz/dt$
Beacon	$\tan \theta = x/d$	$dx/dt$
Cylinder	$V = \pi r^2 h$	$dV/dt$
Sphere	$V = \frac{4}{3}\pi r^3$	$dV/dt$
Airplane	$s^2 = x^2 + h^2$	$ds/dt$
Boat	$r^2 = x^2 + h^2$	$dx/dt$