CH 3 - Matrices and Linear Mapping

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🐳 **Info — Matrix Definition**

A $m \times n$ **matrix** A is a rectangular array with m rows and n columns. We denote the entry in the i-th row and j-th column by a_{ij} or $(A)_{ij}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1}j & \dots & a_{1}n \\ a_{21} & a_{22} & \dots & a_{2}j & \dots & a_{2}n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{jn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Two $m \times n$ matrices A, B are equal if $a_{ij} = b_{ij}, \ \forall 1 \leq i \leq m, 1 \leq j \leq n$.

The set of all $m \times n$ matrices with real entries is denoted by $M_{m \times n}(\mathbb{R})$

Let $A, B \in M_{m \times n}(\mathbb{R}), c \in \mathbb{R}$

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$$
$$(cA)_{ij} = c(A)_{ij}$$

Example:

$$\begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 6 & -3 \end{bmatrix} + 7 \begin{bmatrix} \pi & -1 \\ 6 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2+7\pi & -6 \\ 42 & 26 \\ 6 & -3 \end{bmatrix}$$

🐳 **Info — Matrix Properties**

If $A, B, C \in M_{m \times n}(\mathbb{R}), c, d \in \mathbb{R}$ then

1. $A + B \in M_{m \times n}(\mathbb{R})$
2. $(A + B) + C = A + (B + C)$
3. $A + B = B + A$
4. There exists a matrix $O_{m,n} \in M_{m \times n}(\mathbb{R})$, s.t. $A + O_{m,n} = A \ \forall A$
5. For every $A \in M_{m \times n}(\mathbb{R}), \exists (-A) \in M_{m \times n}(\mathbb{R})$ s.t. $A + (-A) = O_{m,n}$
6. $cA = M_{m \times n}(\mathbb{R})$
7. $c(dA) = (cd)A$
8. $(c + d)A = cA + dA$
9. $c(A + B) = cA + cB$
10. $1A = A$

🐳 **Info — Transpose**

The **transpose** of a $m \times n$ matrix A is the $n \times m$ matrix A^T whose ij-th entry is the ji-th entry of A. $\left(A^T\right)_{ij} = (A)_{ji}$

If $A, B \in M_{m \times n}(\mathbb{R}), c \in \mathbb{R}$

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = cA^T$

Proof

$\left((cA)^T\right)_{ij} = (cA)_{ji} = c(A)_{ji} = c\left(A^T\right)_{ij} = \left(cA^T\right)_{ij}$

□

Note: we can regard vectors in $\mathbb{R}^n$ as matrices in $M_{n \times 1}(\mathbb{R})$

Examples:

1. $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 6 & -3 \end{bmatrix}$

   $A^T = \begin{bmatrix} 2 & 0 & 6 \\ 1 & 5 & -3 \end{bmatrix}$

2. $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

   $\vec{x}^T = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$

Matrix Multiplications

Note that

- $A = \begin{bmatrix} \overrightarrow{a_1}^T \\ \overrightarrow{a_2}^T \\ \dots \\ \overrightarrow{a_m}^T \end{bmatrix}$
- $\vec{a_\imath} = \vec{x} = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \dots \\ 1_{in} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$

Thus we can the linear system with augmented matrix $\left[A \mid \vec{b}\right]$ as $\begin{bmatrix} \overrightarrow{a_1} \cdot \vec{x} \\ \overrightarrow{a_2} \cdot \vec{x} \\ \dots \\ \overrightarrow{a_m} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$

🐳 Info — Matrix-Vector Multiplication

1. Let A be an $m \times n$ matrix whose rows are denoted $\vec{a}_i^T$ for $1 \leq i \leq m$. For any $\vec{x} \in \mathbb{R}^n$

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \dots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$$

2. Let $A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$ be $m \times n$ matrix , $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$A\vec{x} = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \dots + \vec{a}_n x_n$$

Note that $\vec{a}_i$ refers to the i −th column of the matrix A

Properties

If $A \in M_{m \times n}(\mathbb{R}), \vec{x}, \vec{y} \in \mathbb{R}^n, c \in \mathbb{R}$

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
2. $A(c\vec{x}) = c(A\vec{x}$

Examples:

1. Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \\ 0 & -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

2. Compute E1 in another way:

$$3 \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 6 & 3 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} -9 \\ -1 \\ 7 \\ -3 \end{bmatrix}$$

🐳 Info — Column Extraction Theorem

If $\vec{e}_i$ is the i-th standard basis and $A = [\vec{a_1} \dots \vec{a_n}]$ then

$$A\vec{e}_i = \vec{a}_i$$

Example:

Let A be the coefficient matrix of the homogeneous linear system $\left[A \mid \vec{0}\right]$ or $A\vec{x} = \vec{0}$

Let S be the solution set. Prove that S is a subspace of $\mathbb{R}^n$

Proof

1. $\vec{x} = \vec{0}$ is a solution to $\left[A \mid \vec{0}\right], \vec{0} \in S$
2. For $\vec{x}, \vec{y} \in S$, we have $A\vec{x} = \vec{0}, A\vec{y} = \vec{0}$. Then $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0} \in S$
3. For $\vec{x} \in S, c \in \mathbb{R}$, $A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0} \in S$

By subspace test, S is a subspace of $\mathbb{R}^n$

□

Info — Transpose - Dot Product Theorem
If $\vec{x}, \vec{y} \in \mathbb{R}^n$

$$\vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$$

Info — Matrix-Marix Multiplication

Let A be a $m \times n$ matrix, B be $n \times p$ matrix. Then AB is the $m \times p$ matrix defined by

$$AB = \left[A\vec{b}_1 \ A\vec{b}_2 \ \dots \ A\vec{b}_p\right]$$

where $B = \left[\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_p\right], A = \left[\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_p\right]$

Consider the ijth entry of AB, that is the entry of AB in the i-th row and j-th column, that is, $(AB)_{ij}$

1. The j-th column of AB is $A\vec{b}_j$, and thus its i-th entry will be the ij-th entry of AB
2. If we write $A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \dots \\ \vec{a}_m^T \end{bmatrix}$, then $A\vec{b}_j = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \dots \\ \vec{a}_m^T \end{bmatrix} \vec{b}_j = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 \\ \vec{a}_2 \cdot \vec{b}_2 \\ \dots \\ \vec{a}_m \cdot \vec{b}_j \end{bmatrix}$

   Then the i-th entry of this vector is $(AB)_{ij} = \vec{a}_i \cdot \vec{b}_j$

Properties

If A, B, C be matrices of the correct size so that the required products are defined, $t \in \mathbb{R}$,

1. $A(B + C) = AB + AC$
2. $(A + B)C = AC + BC$
3. $t(AB) = (tA)(B) = A(tB)$
4. $A(BC) = (ABC)$
5. $(AB)^T = B^T A^T$

Note that AB is not BA generally speaking. There might be special cases.

Examples:

1. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 5 & 6 \\ 0 & 1 & -1 \end{bmatrix}$

   Then $AB = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 & 6 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} (2)(3)+(1)(0) & (2)(5)+(1)(1) & (2)(6)+(1)(-1) \\ (1)(3)+(-2)(0) & (1)(5)+(-2)(1) & (1)(6)+(-2)(-1) \\ (3)(3)+(4)(0) & (3)(5)+(4)(1) & (3)(5)+(4)(-1) \end{bmatrix} = \begin{bmatrix} 6 & 11 & 11 \\ 3 & 4 & 8 \\ 9 & 19 & 14 \end{bmatrix}$

   $BA = \begin{bmatrix} 3 & 5 & 6 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (3)(2)+(5)(1)+(6)(3) & (3)(1)+(5)(-2)+(6)(4) \\ (0)(2)+(1)(1)+(-1)(3) & (0)(1)+(-1)(-2)+(1)(4) \end{bmatrix} = \begin{bmatrix} 29 & 17 \\ -2 & -6 \end{bmatrix}$

2. Let$A = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$

   Then $AB = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ -6 & 20 \end{bmatrix}$

   $BA = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 17 \end{bmatrix}$

🐳 **Info — Matrix Equality Theorem**

If A and B are $m \times n$ matrices s.t. $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$, then $A = B$

Proof

If $A\vec{x} = B\vec{x}, \ \forall \vec{x} \in \mathbb{R}^n$, then $A\vec{e}_i = B\vec{e}_i, \ \forall 1 \leq i \leq n$.

By Column Extraction Theorem, $A\vec{e}_i = \vec{a}_i, B\vec{e}_i = \vec{b}_i$ where $\vec{a}_i = \vec{b}_i$, are also the i-th column of A and B respectively. Hence $A = B$

□

🐳 **Info — Identity Matrix**

The $n \times n$ **identity matrix**, denoted by I or I_n, is the matrix s.t. $(I)_{ii} = 1$ for $1 \leq i \leq n$, $(I)_{ij} = 0, \forall i \neq j$.

$$I = [\vec{e}_1 \ ... \ \vec{e}_n] = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

If $I = [\vec{e}_1 \ ... \ \vec{e}_n]$, thenfor any $n \times n$ matrix A:

$$AI = A = IA$$

Example:

$\begin{bmatrix} 6 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 1 & 2 \end{bmatrix}$

Linear Mappngs

🐳 **Info — Matrix Mappings**

If A is an $m \times n$ matrix, then we define a function $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(\vec{x}) = A\vec{x}$ called a matrix mapping.

Note that we can write $f(x_1, ..., x_n) = (y_1, ..., y_m)$

Example:

Let $A = \begin{bmatrix} 6 & 0 \\ 1 & 2 \end{bmatrix}$. Then the matrix mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, given by $f(\vec{x}) = A\vec{x}$ can be expressed as $f(\vec{x}) = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 6 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 \\ x_1 + 2x_2 \end{bmatrix}$

Info — Distrbutive Matrix Property

If A is an $m \times n$ matrix and $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $f(\vec{x}) = A\vec{x}$, then $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$ we have

$$f(b\vec{x} + c\vec{y}) = bf(\vec{x}) + cf(\vec{y})$$

Info — Linear Mapping

A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **linear mapping** if for every $\vec{x}, \vec{y} \in \mathbb{R}^n; b, c \in \mathbb{R}$ we have

$$L(b\vec{x}) + c\vec{y} = bL(\vec{x}) + cL(\vec{y})$$

Two linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ are said to be **equal** if $L(\vec{x}) = M(\vec{x}),\ \forall \vec{x} \in \mathbb{R}^n$, that is$L = M$.

A linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is somtimes called a **linear operator**. This is often done when we wish to stress the fact that the domain and codomain of the linear mapping are thhe same.

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $L\left(\vec{0}\right) = \vec{0}$

Tip — Linear Mapping and Matrix Mapping

Every linear mapping can be represented as matrix mapping

Every matrix mapping is a linear mapping

Observe: for an arbitrary $\vec{x} \in \mathbb{R}^n$, we can write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$.

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping, then

$$\begin{aligned} L(\vec{x}) &= x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n \\ &= x_1L(\vec{e}_1) + x_2L(\vec{e}_2) + \dots + x_nL(\vec{e}_n) \\ &= [L(\vec{e}_1)\ L(\vec{e}_2)\ \dots\ L(\vec{e}_n)] \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \\ &= [L]\vec{x} \end{aligned}$$

Example:

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$. Is L linear.

Let $\vec{x}, \vec{y} \in \mathbb{R}^n; b, c \in \mathbb{R}$. Then $L(b\vec{x} + c(\vec{y})) = L\left(\begin{bmatrix} bx_1 + cy_1 \\ bx_2 + cy_2 \end{bmatrix}\right) = \begin{bmatrix} bx_2 + cy_2 \\ -bx_1 - cy_1 \end{bmatrix} = b\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} + c\begin{bmatrix} y_2 \\ -y_1 \end{bmatrix} = bL(\vec{x}) + cL(\vec{y})$.

Hence L is linear.

🐳 **Info — Standard Matrix**

Every linear mapping $L : R^n \to \mathbb{R}^m$ can be represented as a matrix mapping with matrix whose i-th column is the image of the i-th standard basis vector of $\mathbb{R}^n$ under L for all $1 \leq i \leq n$. That is, $L(\vec{x}) = [L]\vec{x}$ where

$$[L] = [L(\vec{e}_1) \ \dots \ L(\vec{e}_n)]$$

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ e a linear mapping. The amtrix $[L] = [L(\vec{e}_1) \ \dots \ L(\vec{e}_n)]$ is called the **standard matrix** of L. It satisfies $L(\vec{x}) = [L]\vec{x}$