

## CH 2 — Sequence and Limits

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### Triangle Inequality

$$|x - y| \leq |x - z| + |z - y|$$

for  $x, y, z \in \mathbb{R}$

Idea: the straight-line distance is shortest.

Without loss of generality assume  $x \leq y$ ; swapping  $x, y$  preserves the statement.

Number-line proof by cases:

- Case 1  $z \leq x \leq y$ :  $|x - y| \leq |z - y| \leq |x - z| + |z - y|$
- Case 2  $x \leq z \leq y$ :  $|x - y| = |x - z| + |z - y|$
- Case 3  $x \leq y \leq z$ :  $|x - y| \leq |x - z| + |z - y|$

### Triangle Inequality 2

For all  $a, b \in \mathbb{R}$

$$|a + b| \leq |a| + |b|$$

Proof:

apply the triangle inequality to  $x = a, y = -b, z = 0$ .

### Quick check

Is  $|a - b| \leq |a| - |b|$  for all  $a, b$ ?

No

Example:

$a = 10, b = -9$  gives  $|10 - (-9)| = 19$ , while  $|10| - |-9| = 1$

Hence this statement is false.

### Interval translations

1.  $|x - a| < \delta \Rightarrow x \in (a - \delta, a + \delta)$
2.  $|x - a| \leq \delta \Rightarrow x \in [a - \delta, a + \delta]$
3.  $0 \leq |x - a| \leq \delta \Rightarrow x \in (a - \delta, a) \cup (a, a + \delta)$

### Practice

1) Solve  $|2x - 5| < 3$

$$-3 < 2x - 5 < 3 \Rightarrow 1 < x < 4$$

**Answer:**  $x \in (1, 4)$

2) Solve  $2 < |x + 7| \leq 3$

Split into  $|x + 7| > 2$  and  $|x + 7| \leq 3$

**Answer:**  $x \in [-10, -9] \cup (-5, -4]$

3) Solve  $\frac{|x+2|}{|x-2|} > 5$

Consider regions  $(-\infty, -2)$ ,  $(-2, 2)$ ,  $(2, \infty)$  and track signs of  $x + 2$  and  $x - 2$

**Answer:**  $x \in (\frac{4}{3}, 2) \cup (2, 3)$

## Infinite Sequences

A sequence is an ordered list  $a_1, a_2, a_3, \dots$ ; write  $\{a_n\}_{n=1}^{\infty}$

A subsequence chooses indices  $n_1 < n_2 < \dots$ , yielding  $a_{n_1}, a_{n_2}, \dots$

The tail with cutoff  $k$  is  $a_k, a_{k+1}, a_{k+2}, \dots$

## Convergence (definition)

### IMPORTANT

We say  $\lim_{n \rightarrow \infty} a_n = L$  if for every  $\varepsilon > 0$  there exists  $N$  such that  $n > N \Rightarrow |a_n - L| < \varepsilon$

### Examples

1) Show  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{n}} = 0$

Choose  $N = \frac{1}{\varepsilon^3}$

Then  $n > N \Rightarrow \left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon$

2) Show  $\lim_{n \rightarrow \infty} \frac{3n^2 + 2n}{4n^2 + n + 1} = \frac{3}{4}$

Estimate  $\left| \frac{3n^2 + 2n}{4n^2 + n + 1} - \frac{3}{4} \right| \leq \frac{5}{16n + 4}$

Pick  $N > \frac{5}{16\varepsilon} - \frac{1}{4}$

## Theorem (Equivalent definitions of the limit of a sequence)

### IMPORTANT

For a sequence  $(a_n)$  and a number  $L$ , the following are equivalent

- 1)  $\lim_{n \rightarrow \infty} a_n = L$
- 2) For every  $\varepsilon > 0$ , the interval  $(L - \varepsilon, L + \varepsilon)$  contains a tail of  $\{a_n\}$
- 3) For every  $\varepsilon > 0$ , only finitely many  $n$  satisfy  $|a_n - L| \geq \varepsilon$
- 4) Every interval  $(a, b)$  containing  $L$  contains a tail of  $\{a_n\}$
- 5) Given any interval  $(a, b)$  containing  $L$ , only finitely many terms of  $\{a_n\}$  lie outside  $(a, b)$

### Example 1

Show  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{n}} = 0$

Side work:

$$\left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon \Rightarrow \frac{1}{\sqrt[3]{n}} < \varepsilon \Rightarrow \sqrt[3]{n} > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon^3}$$

### Proof

Let  $\varepsilon > 0$  and choose  $N = \frac{1}{\varepsilon^3}$

If  $n > N$  then  $\left| \frac{1}{\sqrt[3]{n}} \right| < \frac{1}{\sqrt[3]{N}} = \frac{1}{\sqrt[3]{\frac{1}{\varepsilon^3}}} = \varepsilon$

### Example 2

Prove  $\lim_{n \rightarrow \infty} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$

Rough work

$$\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| = \frac{|5n-3|}{16n^2+4n+4} \leq \frac{5n}{16n^2+4n} = \frac{5}{16n+4}$$

### Proof

Given  $\varepsilon > 0$ , pick  $N = \frac{5}{16\varepsilon} - \frac{1}{4}$

Then for  $n > N$

$$\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| < \frac{5}{16n+4} \leq \frac{5}{16N+4} < \varepsilon$$

### Limits

Thinking question:

Can a sequence converge to two different limits  $L \neq M$ ?

No, we are saying then  $\varepsilon < \frac{|L-M|}{2}$

If  $a_n \rightarrow L$  a tail of the sequence lies in  $(L - \varepsilon, L + \varepsilon)$  so only finite many terms can lie in the interval  $(M - \varepsilon, M + \varepsilon)$ , that is  $a_n \nrightarrow M$

### IMPORTANT

#### Theorem (Uniqueness of Limits):

Let  $\{a_n\}$  be a sequence. If  $\{a_n\}$  has limit  $L$ , then the value  $L$  is unique.

We say that a sequence **diverges to  $\infty$**  if for every  $m > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $a_n > m$ .

We say that a sequence **diverges to  $\infty$**  if any interval of the form  $(m, \infty)$  for some  $m > 0$  contains a tail of  $\{a_n\}$ . We write that  $\lim_{n \rightarrow \infty} a_n = \infty$

We say that a sequence **diverges to  $-\infty$**  if for every  $m < 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $a_n < m$

We say that a sequence **diverges to  $-\infty$**  if any interval of the form  $(-\infty, m)$  for some  $m < 0$  contains a tail of  $\{a_n\}$ . We write that  $\lim_{n \rightarrow \infty} a_n = -\infty$

Thinking questions:

1. If a sequence consists of non-negative terms, is the limit non-negative?

ANS: YES Suppose not, then  $a_n \rightarrow L$ ,  $a_n > 0, \forall n$ . Consider  $\varepsilon < \frac{|L|}{2}$ . Then  $(L - \varepsilon, L + \varepsilon)$  only contains negative numbers, so it can't include a tail of  $a_n$ , contradiction.

2. If a sequence consists of positive terms, is the limit positive?

ANS: NO, consider the sequence  $\{\frac{1}{n}\}$ ,  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Examples: Prove that  $\lim_{n \rightarrow \infty} \frac{1}{n^3} = 0$ . Let  $m > 0$  and consider the interval  $(0, m)$ . If  $n > \sqrt[3]{m}$  then  $n^3 > m$  and so  $\frac{1}{n^3} < \frac{1}{m}$ . So choose  $k = \lceil \sqrt[3]{m} \rceil + 1$ , then the tails lies in  $(0, m)$

## Limit Laws

### IMPORTANT

Let  $\{a_n\}, \{b_n\}$  be sequences with  $\lim_{n \rightarrow \infty} a_n = a, \lim_{n \rightarrow \infty} b_n = b$  for some  $a, b \in \mathbb{R}$  then:

1. For any  $c \in \mathbb{R}$ , if  $a_n = c$  for all  $n$  then  $c = a$
2. For any  $c \in \mathbb{R}$ , if  $\lim_{n \rightarrow \infty} ca_n = ca$
3.  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$
4.  $\lim_{n \rightarrow \infty} a_n b_n = ab$
5.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$  if  $b \neq 0$
6. If  $a_n \geq 0$  for all  $n$  and  $\alpha > 0$ , then  $\lim_{n \rightarrow \infty} a_n^\alpha = a^\alpha$
7. For any  $k \in \mathbb{N}$ ,  $\lim_{n \rightarrow \infty} a_{n+k} = a$

Prove the Sum of Sequences Rule

### Proof

$$a_n \rightarrow a, b_n \rightarrow b$$

$$\forall \varepsilon > 0, \exists M, N \in \mathbb{R}, \forall n > M, n > N, |a_n - a| < \varepsilon, |b_n - b| < \varepsilon$$

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

### IMPORTANT

Tandem Convergence Theorem:

If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists and  $\lim_{n \rightarrow \infty} b_n = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$

Examples:

Evaluate the following limits

$$\begin{aligned} 1) \lim_{n \rightarrow \infty} \frac{3n^2 + 2n}{4n^2 + n + 1} \\ = \lim_{n \rightarrow \infty} \frac{n^2(3 + \frac{2}{n})}{n^2(4 + \frac{1}{n} + \frac{1}{n^2})} = \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{4 + \frac{1}{n} + \frac{1}{n^2}} = \frac{\lim_{n \rightarrow \infty} 3 + \lim_{n \rightarrow \infty} \frac{2}{n}}{\lim_{n \rightarrow \infty} 4 + \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n^2}} = \frac{3+0}{4+0+0} = \frac{3}{4} \end{aligned}$$

$$\begin{aligned} 2) \lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n, \text{ We have indeterminate form } [\infty - \infty] \\ = \lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n \cdot \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} = \lim_{n \rightarrow \infty} \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \lim_{n \rightarrow \infty} \frac{n}{n(\sqrt{1 + \frac{1}{n}} + 1)} = \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{\sqrt{1 + \lim_{n \rightarrow \infty} \frac{1}{n}} + 1} = \frac{1}{1+0+1} = \frac{1}{2} \end{aligned}$$

3) Let the sequence  $\{a_n\}$  be defined recursively by  $a_1 = 16$  and for all  $n > 1, a_n = \frac{1}{2} \left( a_{n-1} + \frac{260}{a_{n-1}} \right)$ . Given that  $\lim_{n \rightarrow \infty} a_n$  exists, compute its value

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{1}{2} \left( a_{n-1} + \frac{260}{a_{n-1}} \right) = \frac{1}{2} \left( \lim_{n \rightarrow \infty} a_{n-1} + \frac{260}{\lim_{n \rightarrow \infty} a_{n-1}} \right) \\ &= \frac{1}{2} \left( \lim_{n \rightarrow \infty} a_n + \frac{260}{\lim_{n \rightarrow \infty} a_n} \right) \end{aligned}$$

$$\text{Let } L = \lim_{n \rightarrow \infty} a_n, \text{ then } L = \frac{1}{2} \left( L + \frac{260}{L} \right) \Leftrightarrow L^2 = \frac{1}{2} L^2 + 260 \Leftrightarrow L^2 = 520 \Leftrightarrow L = \pm \sqrt{520}$$

Since  $a_n$  consists of non-negative terms, thus its limit converges to a value that is non-negative.

Thus,  $\lim_{n \rightarrow \infty} a_n = \sqrt{520}$

## IMPORTANT

### Squeeze Theorem:

If  $a_n \geq b_n \geq c_n$  for all  $n \in \mathbb{N}$  with  $n \geq M$  for some  $M \in \mathbb{R}$  and  $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$  for some  $L \in \mathbb{R}$ , then  $\lim_{n \rightarrow \infty} b_n = L$

### Proof

Since  $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$  for any  $\varepsilon > 0$ ,  $\exists N_a, N_c \in \mathbb{R} : n > N_a, n > N_c \cdot |a_n - L| < \varepsilon, |c_n - L| < \varepsilon$ . Let  $N = \max(N_a, N_c)$  but  $a_n \geq b_n \geq c_n$ , so  $a_n \in (L - \varepsilon, L + \varepsilon), b_n \in (L - \varepsilon, L + \varepsilon), c_n \in (L - \varepsilon, L + \varepsilon)$   
 $\therefore \lim_{n \rightarrow \infty} b_n = L$

$$4) \lim_{n \rightarrow \infty} \frac{\sin(n)}{n}$$

$-1 \leq \sin(n) \leq 1$  for any  $n \in \mathbb{N}$ , so  $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}, \forall n \in \mathbb{N}$   
 $\lim_{n \rightarrow \infty} -\frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

By Squeeze Theorem,  $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$

$$5) \lim_{n \rightarrow \infty} \frac{4+(-1)^n}{n^3+n^2-1}$$

$$\frac{3}{n^3+n^2-1} \leq \frac{4+(-1)^n}{n^3+n^2-1} \leq \frac{5}{n^3+n^2-1}$$

$$\lim_{n \rightarrow \infty} \frac{3}{n^3+n^2-1} = \lim_{n \rightarrow \infty} \frac{5}{n^3+n^2-1} = 0$$

By Squeeze Theorem,  $\lim_{n \rightarrow \infty} \frac{4+(-1)^n}{n^3+n^2-1} = 0$

$\lim_{n \rightarrow \infty} \frac{4+(-1)^n+(-1)^{n^2+n+2}}{n^3+n^2+100}$  can be solved similarly

## Definitions

A sequence  $\{a_n\}$  is

1. increasing if  $a_n \leq a_{n+1} \forall n \in \mathbb{N}$
2. decreasing if  $a_n \geq a_{n+1} \forall n \in \mathbb{N}$
3. monotonic if it is increasing or decreasing

A set  $S \subset \mathbb{R}$  is

1. bounded above if there exists some  $\alpha \in \mathbb{R}$  with  $a \leq \alpha \forall x \in S$ , and we call such an  $\alpha$  an upper bound for  $S$ . The least upper bound is the smallest such  $\alpha$
2. bounded below if there exists some  $\beta \in \mathbb{R}$  with  $a \geq \beta \forall x \in S$ , and we call such an  $\beta$  an lower bound for  $S$ . The greatest lower bound is the largest such  $\beta$
3. bounded if it is both bounded above and bounded below

If a set  $S \subset \mathbb{R}$  is bounded above, it has a least upper bound. If it is bounded below, it has a greatest lower bound.

Greatest lower bound and least upper bound do not have to be in part of the set

## IMPORTANT

**Theorem(Monotone Convergence Theorem):** Let  $\{a_n\}$  be an increasing sequence. If  $\{a_n\}$  is bounded above, it converges to its least upper bound, otherwise to  $\infty$

### Proof

Let  $\{a_n\}$  be increasing, bounded above. Then it has a lowest upper bound say  $L$ . Suppose  $\lim_{n \rightarrow \infty} a_n \neq L$ . So there is some bad  $\varepsilon$  s.t. no tail of  $\{a_n\}$  lies in  $(L - \varepsilon, L + \varepsilon)$ . But then no term from  $a_n$  lies in  $(L - \varepsilon, L + \varepsilon)$  since  $a_n$  is increasing. Hence  $L - \varepsilon$  is an upper bound for  $\{a_n\}$ , but  $L - \varepsilon < L$  and  $L$  is the least upper bound of  $\{a_n\}$  is a contradiction. The assumption of  $\lim_{n \rightarrow \infty} a_n \neq L$  is false.  $\therefore \lim_{n \rightarrow \infty} a_n = L$

Let  $\{a_n\}$  be a decreasing sequence. If  $\{a_n\}$  is bounded below, it converges to its greatest lower bound, otherwise it diverges to  $-\infty$

### Proof

Let  $L =$  greatest lower bound of  $\{a_n\}$  since  $\{a_n\}$  is decreasing,  $\{-a_n\}$  is increasing with lowest upper bound is  $-L$ . By the Monotone Convergence Theorem, it is true.

## Proof by Induction

### IMPORTANT

Let  $P(n)$  be a statement over the natural numbers  $\mathbb{N}$

- 1) Prove the basic case  $P(1)$  is true
- 2) Prove that if  $P(n)$  is true, then  $P(n + 1)$  is true  $\forall n \in \mathbb{N}$
- 3) Apply 2) repeatedly starting at  $P(1)$

Prove a recursive sequence  $(a_n)$  converges:

- 1) Show that  $(a_n)$  is monotone
- 2) Show that  $(a_n)$  is bounded above if increasing or bounded below if decreasing.
- 3) By the Monotone Convergence Theorem,  $\lim_{n \rightarrow \infty} a_n$  exists. Use limit laws to solve for it, keeping in mind that the initial term and whether  $\{a_n\}$  is increasing or decreasing will tell you which solution is admissible if there are multiple.

Example:

- 1) Find the limit of the sequence  $(a_n)$  given by

$$a_1 = 1, \quad a_n = \sqrt{3 + 2a_{n-1}} \\ \text{for} \\ n \geq 2.$$

### Proof

Let  $P(n)$  be the statement that  $a_n \leq a_{n+1}$ .

$$a_1 = 1,$$

**Base Case:**  $P(1)$ ,  $a_2 = \sqrt{5}$ , so  $a_1 < a_2$ .

**Inductive Hypothesis:**  $(P(a) \rightarrow P(a + 1))$  suppose  $P(n)$  is true for some  $n$ .

Then  $a_n < a_{n+1}$ ; we want to show that  $a_{n+1} \leq a_{n+2}$ . From  $a_n \leq a_{n+1}$ :

$$2a_n \leq 2a_{n+1}$$

$$3 + 2a_n \leq 3 + 2a_{n+1}$$

$$\sqrt{3 + 2a_n} \leq \sqrt{3 + 2a_{n+1}}$$

$$a_{n+1} \leq a_{n+2}$$

By induction,  $P(n)$  is true for all  $n \in \mathbb{N}$ , so  $(a_n)$  is increasing.

**Step2:** Choose upper bound to be big to make proof easier.

Let  $P(n)$  be the statement that  $a_n \leq 100$ .  $P(1)$  is true since  $a_1 = 1 < 100$ .

Suppose  $P(n)$  is true for some  $n$ . Then  $a_n \leq 100$ .

We want to show  $a_{n+1} \leq 100$ :

$$a_{n+1} = \sqrt{3 + 2a_n} \leq \sqrt{3 + 2 * 100} < \sqrt{10000} = 100.$$

By induction  $P(n)$  ...

Since  $(a_n)$  is increasing and bounded above. By MCT,  $a$  converges to least upper bound. So, let  $a_n \rightarrow L$ ,

$$a_{n+1} = \sqrt{3 + 2a_n}.$$

$$\lim_{n \rightarrow \infty} a_{n+1} = \sqrt{3 + 2 \lim_{n \rightarrow \infty} a_n} \rightarrow L = \sqrt{3 + 2L}.$$

So  $L = -1, 3 = \lim_{n \rightarrow \infty} a_n$ . Since the sequence is increasing, we choose  $L = 3$

□

2) Find the limit of the sequence  $(b_n)$  given by

$$b_1 = 4, \quad b_n = \frac{7 + b_{n-1}}{22}$$

for  
 $n \geq 2$ .

Notice that the sequence is a decreasing sequence.

Let  $P(n)$  be the statement that  $b_n > b_{n+1} \forall n \in \mathbb{N}$  and  $b_n \geq \frac{1}{3} \forall n$ .

$$n = 1,$$

Base case:  $b_1 = 4 \geq \frac{1}{3}$  and  $b_2 = \frac{1}{2} < 4 = b_1$ .

Inductive hypothesis: suppose  $P(n)$  is true for some  $n = k$ . Then

$$b_{n+1} = \frac{7 + b_k}{22} \geq \frac{7 + \frac{1}{3}}{22} > \frac{1}{3}.$$

We have to show  $b_k \geq b_{k+1} \leftrightarrow 22b_k \geq 7 + b_k \leftrightarrow 21b_k \geq 7 \leftrightarrow b_k > \frac{1}{3}$ , so  $P(k + 1)$  holds. By Induction,  $P(n)$  holds for all  $n$ .

By MCT,  $(b_n)$  converges to  $L$ , since  $b_n = \frac{7 + b_{n-1}}{22}$ , so

$$\lim_{n \rightarrow \infty} b_n = \frac{7 + \lim_{n \rightarrow \infty} b_{n-1}}{22} \rightarrow L = \frac{7 + L}{22} \rightarrow 22L = 7 + L \rightarrow L = \frac{1}{3}.$$