CH 3 - Matrices and Linear Mapping

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🐳 **Info — Matrix Definition**

A $m \times n$ **matrix** A is a rectangular array with m rows and n columns. We denote the entry in the i-th row and j-th column by a_{ij} or $(A)_{ij}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_1 j & \dots & a_1 n \\ a_{21} & a_{22} & \dots & a_2 j & \dots & a_2 n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{jn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Two $m \times n$ matrices A, B are equal if $a_{ij} = b_{ij}, \ \forall 1 \leq i \leq m, 1 \leq j \leq n$.

The set of all $m \times n$ matrices with real entries is denoted by $M_{m \times n}(\mathbb{R})$

Let $A, B \in M_{m \times n}(\mathbb{R}), c \in \mathbb{R}$

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$$
$$(cA)_{ij} = c(A)_{ij}$$

Example:

$$\begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 6 & -3 \end{bmatrix} + 7 \begin{bmatrix} \pi & -1 \\ 6 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2+7\pi & -6 \\ 42 & 26 \\ 6 & -3 \end{bmatrix}$$

🐳 **Info — Matrix Properties**

If $A, B, C \in M_{m \times n}(\mathbb{R}), c, d \in \mathbb{R}$ then

1. $A + B \in M_{m \times n}(\mathbb{R})$
2. $(A + B) + C = A + (B + C)$
3. $A + B = B + A$
4. There exists a matrix $O_{m,n} \in M_{m \times n}(\mathbb{R})$, s.t. $A + O_{m,n} = A \ \forall A$
5. For every $A \in M_{m \times n}(\mathbb{R}), \exists (-A) \in M_{m \times n}(\mathbb{R})$ s.t. $A + (-A) = O_{m,n}$
6. $cA = M_{m \times n}(\mathbb{R})$
7. $c(dA) = (cd)A$
8. $(c + d)A = cA + dA$
9. $c(A + B) = cA + cB$
10. $1A = A$

🐳 **Info — Transpose**

The **transpose** of a $m \times n$ matrix A is the $n \times m$ matrix A^T whose ij-th entry is the ji-th entry of A. $\left(A^T\right)_{ij} = (A)_{ji}$

If $A, B \in M_{m \times n}(\mathbb{R}), c \in \mathbb{R}$

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = cA^T$

Proof

$\left((cA)^T\right)_{ij} = (cA)_{ji} = c(A)_{ji} = c\left(A^T\right)_{ij} = \left(cA^T\right)_{ij}$

$\square$

Note: we can regard vectors in $\mathbb{R}^n$ as matrices in $M_{n \times 1}(\mathbb{R})$

Examples:

1. $$A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 6 & -3 \end{bmatrix}$$
   $$A^T = \begin{bmatrix} 2 & 0 & 6 \\ 1 & 5 & -3 \end{bmatrix}$$
2. $$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$
   $$\vec{x}^T = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$$

Matrix-Vector Multiplication

Note that

- $$A = \begin{bmatrix} \vec{a_1}^T \\ \vec{a_2}^T \\ \dots \\ \vec{a_m}^T \end{bmatrix}$$
- $$\vec{a_\imath} = \vec{x} = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \dots \\ 1_{in} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

Thus we can the linear system with augmented matrix $\left[A \mid \vec{b}\right]$ as $\begin{bmatrix} \vec{a_1} \cdot \vec{x} \\ \vec{a_2} \cdot \vec{x} \\ \dots \\ \vec{a_m} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$