# CH 8 - Modular Arithmetics

Luke Lu • 2025-10-29

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## Basic Modular Arithmetics

🐳 **Info — Congruence and Modular Expression**

Let $m$ be a fixed positive integer. For integers $a$ and $b$, we say that $a$ is **congruent** to $b$ **modulo** $m$, and write

$$a \equiv b \pmod{m}$$

if and only if $m \mid (a - b)$. For integers $a$ and $b$ such that $m \nmid (a - b)$, we write $a \not\equiv b \pmod{m}$. We refer to $\equiv$ as **congruence**, and $m$ as its **modulus**.

$$a \equiv b \pmod{m} \Longleftrightarrow m \mid (a - b) \Longleftrightarrow \exists k \in \mathbb{Z}, a - b = km \Longleftrightarrow \exists k \in \mathbb{Z}, a = km + b$$

Examples:

1. $6 \equiv 18 \pmod{12} : 6 - 18 = -12, 12 \mid -12$
2. $73 \equiv 1 \pmod{2} : 13 - 1 = 72, 2 \mid 72$
3. $5 \equiv 1 \pmod{4} : 5 - 1 = 4, 4 \mid 4$
4. $24 \equiv 0 \pmod{24} : 24 - 0 = 24, 24 \mid 24$
5. $-5 \equiv 7 \pmod{12} : -5 - 7 = -12, 12 \mid -12$

🐳 **Info — Equality Properties**

1. Reflexivity: $\forall a \in \mathbb{Z}, a = a$
2. Symmetry: $\forall a, b \in \mathbb{Z}, a = b \Longrightarrow b = a$
3. Transitivity: $\forall a, b, c \in \mathbb{Z}, a = b \land b = c \Longrightarrow a = c$

🐳 **Info — Congruence Relations**

$\forall a, b, c \in \mathbb{Z}$

1. $a \equiv a \pmod{m}$
2. $a \equiv b \pmod{m} \Longrightarrow b \equiv a \pmod{m}$
3. $a \equiv b \pmod{m} \land b \equiv c \pmod{m} \Longrightarrow a \equiv c \pmod{m}$

**Info — Modular Arithmetics**

$\forall a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $\forall n \in \mathbb{N}$, if $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$ then

1. $a_1 + a_2 \equiv b_1 + b_2 \pmod{m}$
2. $a_1 - a_2 \equiv b_1 - b_2 \pmod{m}$
3. $a_1 a_2 \equiv b_1 b_2 \pmod{m}$
4. $a_1 + a_2 + ... + a_n \equiv b_1 + b_2 + ... + b_n \pmod{m}$
5. $a_i \equiv b_i \implies a_1 a_2 ... a_n \equiv b_1 b_2 ... b_n \pmod{m}$
6. $\forall a, b \in \mathbb{Z}$ if $a \equiv b \pmod{m}$ then $a^n \equiv b^n \pmod{m}$
7. $\forall a, b, c \in \mathbb{Z}$, if $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$

**Proof**

$\forall a_1, a_2, b_1, b_2 \in \mathbb{Z}$ where $a_1 \equiv b_1 \pmod{m}$ and$a_2 \equiv b_2 \pmod{m}$

1. $a_1 + a_2 - b_1 - b_2 = a_1 - b_1 + a_2 - b_2 \pmod{m}$.

   Since $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2$, therefore $m \mid (a_1 - b_1)$ and $m \mid (a_2 - b_2)$.

   By DIC $m \mid (a_1 - b_1 + a_2 - b_2) \equiv m \mid (a_1 + a_2 - (b_1 + b_2))$.

   By definition of Congruence, $a_1 + a_2 \equiv b_1 + b_2 \pmod{m}$

2. $a_1 - a_2 - b_1 + b_2 = a_1 - b_1 + a_2 - b_2 \pmod{m}$.

   Since $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$, therefore $m \mid (a_1 - b_1)$ and $m \mid (a_2 - b_2)$.

   By DIC $m \mid (a_1 - b_1 - a_2 + b_2) \equiv m \mid (a_1 - a_2 - (b_1 - b_2))$.

   By definition of Congruence, $a_1 - a_2 \equiv b_1 - b_2 \pmod{m}$

3. Since $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$,

   therefore $\exists k, l \in \mathbb{Z}$ s.t. $a_1 = km + b_1; a_2 = lm + b_2$.

   $a_1 b_1 - b_1 b_2 = (km + b_1)(lm + b^2) - b_1 b_2 = klm^2 + kmb_2 + b_1 lm + b_1 lm + b_1 b_2$

   $(klm + kb_2 + b_1 l) \cdot m \implies m \mid (klm + kb_2 + b_1 l)$.

   Hence, $a_1 a_2 \equiv b_1 b_2 \pmod{m}$

$\square$

Examples:

1. Is $5^9 + 62^{2000} - 14$ divisible by 7

   $5^9 + 62^{2000} - 14 \equiv 0 \pmod{7}$

   $5^9 + 62^{2000} \equiv 0 \pmod{7}$ since $14 \equiv 0 \pmod{7}$

   $(5^2)^4 \cdot 5 + (-1)^{2000} \equiv 0 \pmod{7}$ since $62 \equiv -1 \pmod{7}$ because $62 - (-1) = 63, 7 \mid 63$

   $4^4 \cdot 5 + 1 \equiv 0 \pmod{7}$ since $25 \equiv 4 \pmod{7}$

   $2^2 \cdot 5 + 1 \equiv 0 \pmod{7}$ since $7 \mid (16 - 2)$

   $21 \equiv 0 \pmod{7}$ since $7 \mid 21$

   $\therefore 5^9 + 62^{2000} - 14 \equiv 0 \pmod{7}$ since $7 \mid 5^9 + 62^{2000} - 14$, meaning, $5^9 + 62^{2000} - 14$ is divisible by 7.

2. Illustration of Congruence Divide
   $3 \equiv 27 \pmod 6$
   $3 \cdot 1 \equiv 3 \cdot 9 \pmod 6, 1 \not\equiv 9 \pmod 6$ since $\gcd(3, 6) \neq 1$

## Congruence and Remaidners

> 🐳 **Info — Congruent Iff Same Remainder**
>
> $\forall a, b \in \mathbb{Z}, a \equiv b \pmod m$ if and only if $a$ and $b$ have the same remainder when divided by $m$

> 🐳 **Info — Congruent to Remainder**
> $\forall a, b \in \mathbb{Z}$ with $0 \leq b < m, a \equiv b \pmod m$ if and only if $a$ has a remainder $b$ when divided by $m$

Examples:

1. What is the remaidner when $77^{100} \cdot 999 - 6^{83}$ divided by 4?
   $77 \equiv 1 \pmod 4$
   $999 \equiv -1 \pmod 4$
   $6 \equiv 2 \pmod 4$
   $\equiv 1^{100} \cdot -1 - 2^{83} \pmod 7$
   $\equiv -1 - 2^{82} \cdot 2 \equiv -1 - 2(4)^{41} \equiv -1 - 2(0) \equiv -1 \pmod 4$

   By CTR $3 \equiv -1 \pmod 4$, the remainder is 3

   > 💡 **Tip — Divisibility by 3**
   > For all non-negative integers $a$, $a$ is divisible by 3 if and only if the sum of the digits in the decimal representation of $a$ is divisible by 3