CH 4 - Derivatives

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Velocity

> 🐳 **Info** — **Average Velocity and Instantaneous Velocity**
>
> $$v_{avg} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$
>
> $$v_{inst} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Definition of Derivatives

> 🐳 **Info** — **Average Rate of Change and Instantaneous Rate of Change (Derivative)**
>
> $$f_{avg} = \frac{f(b) - f(a)}{b - a}$$
>
> $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
>
> If $f'(x)$ exists at $x = a$, then $f(x)$ is **differentiable** at $x = a$
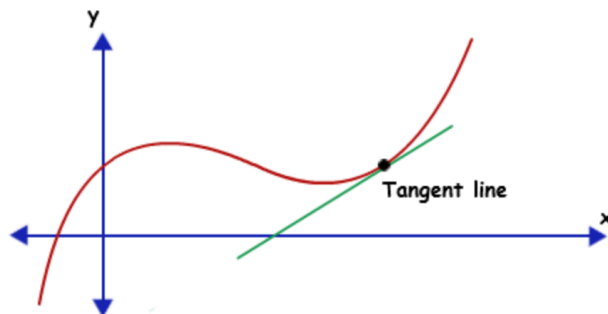
> 🐳 **Info** — **Tangent Line**
> If $f(x)$ is differentiable at $x = a$, then the **tangent line** to $f(x)$ at $x = a$ is the line passing through $(a, f(a))$ with slope $f'(a)$
> The equation of the tangent line
>
> $$y = f'(a)(x - a) + f(a)$$
>
> $(a, f(a))$ is the **point of tangency**

Examples:

Find the tangent line to $f(x) = \frac{1}{x+5}$ at $x = 3$

$f(3) = \frac{1}{8}$

$f'(3) = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h+5} - \frac{1}{a+5}}{h} =$

$\lim_{h\to 0} \frac{1}{h} \frac{a+5-(a+h+5)}{(a+5)(a+h+5)} = \lim_{h\to 0} -\frac{1}{(a+5)(a+h+5)} = -\frac{1}{(a+5)^2} = -\frac{1}{64}$

$y = -\frac{1}{64}(x-3) + \frac{1}{8}$

> 🐳 **Info** $-$ **Differentiability Implies Continuity**
>
> If a function f is differentiable at $x = a$, then f is continuous at $x = a$

Proof

f is differentiable at $x = a$ then, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists

$\lim_{h\to 0}[f(a+h) - f(a)] = 0 \implies \lim_{h\to 0}[f(a+h) - f(a) + f(a)] = \lim_{h\to 0} f(a) \implies$

$\lim_{h\to 0} f(a) = f(a)$

> ⚠️ **Warning** $-$ **Continuity Not Implies Differentiability**
>
> $f(x) = |x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x\geq 0 \end{cases}$
>
> $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \frac{h-0}{h} = 1$
>
> $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \frac{-h-0}{h} = -1$
>
> Thus $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} =$ DNE but continuous.
>
> $\therefore$ continuity does not impliy differentiability

> 🐳 **Info** $-$ **Differentiability of Funciton**
>
> We say that f is **differentiable** on an interval I if $f'(a)$ exists $\forall a \in I$.
>
> We define the derivative funciton $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
>
> We sometimes also write $f'(x)$ as $\frac{d}{dx}f(x)$, and $f'(a) = \frac{d}{dx}f(x)\big|_a$

> 🐳 **Info** $-$ Constant Function
>
> $f(x) = c$
>
> $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

> 🐳 **Info** $-$ Linear Function
>
> $f(x) = mx + b$
>
> $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(m(x+h)+b)-f(mx+b)}{h} = \lim_{h\to 0} m\frac{h}{h} = m$

> 🐋 **Info** − Quadratic Function
>
> $f(x) = px^2 + sx + c$
>
> $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{[p(x+h)^2+s(x+h)+c]-[px^2+sx+c]}{h} = \lim_{h \to 0} \frac{2xph+xh^2+sh}{h} =$
>
> $\lim_{h \to 0} 2xp + xh + s = 2xp + s$

> 🐋 **Info** − Basic Trig
>
> $f(x) = \sin x$
>
> $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$
>
> $\lim_{h \to 0} \frac{[\sin x(\cos h - 1)]}{h} + \lim_{h \to 0} \cos x \frac{\sin h}{h} = \sin x \cdot \lim_{h \to 0} \frac{\cos^2 h - 1}{h \cdot (\cos h + 1)} + \cos x =$
>
> $\sin x \cdot \lim_{h \to 0} \frac{\sin^2 h}{h \cdot (\cos h + 1)} = \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{\sin h}{\cos h + 1} + \cos x = \cos x$

We define e to be the unique base of an exponential function with slope 1 through $(0, 1)$

> 🐋 **Info** − **Derivative Rules**
>
> Let $f(x)$ and $g(x)$ be differentiable at $x = a$
>
> 1. $w(x) = cf(x) \implies w'(x) = cf'(x)$
>
> 2. $w(x) = f(x) \pm g(x) \implies w'(x) = f'(x) \pm g'(x)$
>
> 3. $w(x) = f(x)g(x) \implies w'(x) = f'(x)g(x) + f(x)g'(x)$
>
> 4. If $g(x) \neq 0, w'(x) = \frac{f(x)}{g(x)} \implies w'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$
>
> 5. If $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R} \setminus \{0\} \implies f'(x) = \alpha x^{\alpha-1}$
>
> 6. $w(x) = (g \circ f)(x) = g(f(x)) \implies w'(x) = g'(f(x)) \cdot f'(x) \sim \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

> ⚠️ **Warning** − Power Rule
>
> If $x = 0, x^{-1}$ does not make sense so that is why $\alpha \in \mathbb{R} \setminus \{0\}$

Proof

We suppose that $f(x), g(x)$ are differentiable, so that the limits:

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}, \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$ exists

1. Product rule:

$\lim_{h \to 0} \frac{w(x+h)-w(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$

$\lim_{h \to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} = \lim_{h \to 0} \frac{(f(x+h)-f(x))g(x+h)}{h} + \frac{(g(x+h)-g(x))f(x)}{h}$

$= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \cdot \lim_{h \to 0} f(x)$

$= f'(x)g(x) + f(x)g'(x)$

2. Quotient rule

$\lim_{h \to 0} \frac{w(x+h)-w(x)}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)g(x)}{g(x+h)g(x)} - \frac{f(x)g(x+h)}{g(x+h)g(x)}}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x)g(x+h)}}{h}$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} = \lim_{h \to 0} \frac{g(x)(f(x+h) - f(x))}{\frac{h}{g(x)g(x+h)}} - \frac{f(x)(g(x+h) - g(x))}{\frac{h}{g(x)g(x+h)}}$$

$$= \frac{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to} g(x) - \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}{\lim_{h \to 0} g(x+h)g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

$\square$

Basic Derivatives

> 🐳 **Info − Basic Trig Derivatives**
>
> $\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin x}{\cos} x = \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
>
> $\frac{\mathrm{d}}{\mathrm{d}x} \csc x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\sin x} = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\csc x \cot x$
>
> $\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\cos x} = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$
>
> $\frac{\mathrm{d}}{\mathrm{d}x} \cot x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\cos x}{\sin x} = \frac{\cos' x \sin x - \cos x \sin' x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\csc^2 x$

> 🐳 **Info − Exponential/Logarithmic Derivatives**
>
> For $a^x, x > 0$:
>
> $\frac{\mathrm{d}}{\mathrm{d}x} a^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x \ln(a)} = e^{x \ln a} \cdot \ln a = a^x \ln a$
>
> $\frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln(x)}{\ln(a)} = \frac{1}{x} \cdot \frac{1}{\ln a} = \frac{1}{x \ln a}$

Example:

1. $\frac{\mathrm{d}}{\mathrm{d}x} x^3 e^{2x} \cos x = 3x^2 e^{2x} \cos x + 2x^3 e^{2x} \cos x - x^3 e^{2x} \sin x$

2. $\frac{\mathrm{d}}{\mathrm{d}x} 3^{\csc x} = 3^{\csc x} \ln 3 \cdot - \csc x \cot x = -3^{\csc x} \csc x \cot x \ln 3$

3. $\frac{\mathrm{d}^{67}}{\mathrm{d}x^{67}} \sin x$. Note that $\sin' x = \cos x, \sin'' x = -\sin x, \sin''' - \cos x, \frac{\mathrm{d}^4}{\mathrm{d}x^4} \sin x = \sin x /$

$67 \bmod 4 \equiv 3$, that is $\frac{\mathrm{d}^{67}}{\mathrm{d}x^{67}} \sin x = -\cos x$

4. $\frac{\mathrm{d}}{\mathrm{d}x} \frac{x}{\left(1 + e^{x^2}\right)^3} = \frac{\mathrm{d}}{\mathrm{d}x} x \cdot \left(1 + e^{x^2}\right)^{-3} = \frac{1}{\left(1 + e^{x^2}\right)^3} - 3\left(\left(1 + e^{x^2}\right)^{-4} \cdot x^2 e^{x^2} \cdot 2x\right) = \frac{1}{\left(1 + e^{x^2}\right)^3} - \frac{6x^3 e^{x^2}}{\left(\left(1 + e^{x^2}\right)^4\right)}$

5. $\frac{\mathrm{d}}{\mathrm{d}x} x^{x^x}$

$\frac{\mathrm{d}}{\mathrm{d}x} x^x = x^x \ln x \cdot (\ln x + 1)$

$\frac{\mathrm{d}}{\mathrm{d}x} x^{f(x)} = x^{f(x)} \cdot (\ln x \cdot f'(x) + x) = x^{x^x} \cdot x^x (\ln^2(x) + \ln(x) + x^{x-1})$

Linear Approximation

With the assumption of $f(x)$ is continuous ay $x = a$, we can derive

$$f'(a) = \lim_{x \to a \frac{f(x) - f(a)}{x - a}} \leftrightarrow f'(a)(x - a) = f(x) - f(a) \leftrightarrow f(x) = f'(a)(x - a) + f(a)$$

Which is the linear approximation of $f(x)$ near $x = a$

> 🐳 **Info − Linear Approximation**
>
> Let $f(x)$ be differentiable at $x = a$. The **linear approximation** to $f(x)$ at $x = a$ is given by
>
> $$L_a^f(x) = f'(a)(x - a) + f(a)$$
>
> If it is clear what function f we are talking about, we sometimes denote $L_a(x)$ instead.

> 🐳 **Info − Upper Bound Error of Linear Approximation**
>
> The error of linear approximation is defined as:
>
> $$\text{error} = |f(x) - L_a^f(x)|$$
>
> Assume that $f(x)$ is such that $|f''(x)| \leq M$ for each x in an interval I containing $x = a$. Then,
>
> $$\text{error} = |f(x) - L_a^f(x)| \leq \frac{M}{2}(x - a)^2$$
>
> forr each $x \in I$

Examples:

1. Find the linearization of $\sqrt{x}$ at $x = 4$ and use it ti estimate $\sqrt{4.01}$

   $L_a^f(x) = f'(a)(x - a) + f(a)$ where $f(x) = \sqrt{x} \Longrightarrow f'(x) = \frac{1}{2\sqrt{x}}$

   $L_a^f(x) = \frac{1}{2\sqrt{x}}(x - 4) + 2 \Longrightarrow L_a^f(0.01) = \frac{1}{2\sqrt{0.01}}(0.1) + 2 \approx 2.0024984$

2. What factors could affect the error in linear approximation?
   - The distance from x to a. (e.g. $f(x) = \sin(x), x = 3\frac{\pi}{2}, a = \frac{\pi}{4}$.)
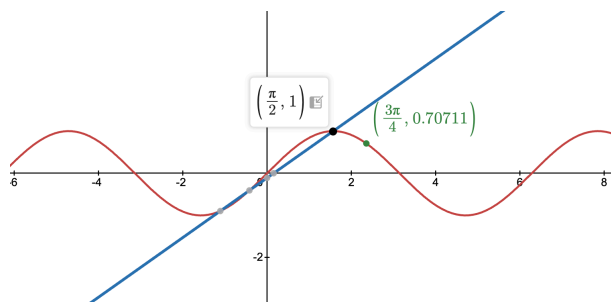
   - The curvature (e.g. $f(x) = e^x, g(x) = e^{\frac{x}{10}}$)

3. Find a upper bound on the error in using L_9 to appriximate $f(x) = \sqrt{x}$ on $[5, 13]$
   If $|f''(x)| \leq M$ on I then: error $\leq \frac{M}{2}(x - a)^2 \forall x \in I$

   $f(x) = \sqrt{x}; f'(x) = \frac{1}{2\sqrt{x}}; f''(x) = -\frac{1}{4x^{\frac{3}{2}}}$

   $|f''(x)| = \frac{1}{4x^{\frac{3}{2}}} <= \frac{1}{4(5)^{\frac{3}{2}}} = \frac{1}{20\sqrt{5}}$

   So the error $\leq \frac{1}{40\sqrt{5}}(x - 9)^2 = \frac{1}{40\sqrt{5}}(13 - 9)^2 = \frac{2}{5\sqrt{5}}$

Example:

A poor group of students is writing a midterm. A nefarious professor decides to make things harder by cranking up the thermostat to uncomfortable levels. To make things as uncomfortable as possible, the professor increases the temperature according to an equation of the form $T(t) = 2t + c$ for some constant $c \in R$, where t is the time elapsed in minutes. Initially, at t = 0, the room is a cool 21° C. Estimate the change in the room's temperature 90 seconds after the professor alters the thermostat.

ANS:
$T'(t) = 2^t \ln 2$
$\triangle T = T'(0) \cdot \triangle t = \ln 2 \cdot 1.5 = \left(3 \frac{\ln 2}{2}\right)°\text{C}$

Newton's Method

> 🐳 **Info — Newton's Method**
> 1. Make an initial guess, x_1, of where the root of $f(x)$. IVT can be helpful.
> 2. Take the linear appoximation, $L^f_{x_1}(x)$ and find its root and call it x_2
> 3. Repeat Step 3 at x_2 to find x_3
>
> Note that NM converges faster than Bisection Method

The Visualization of the procedure
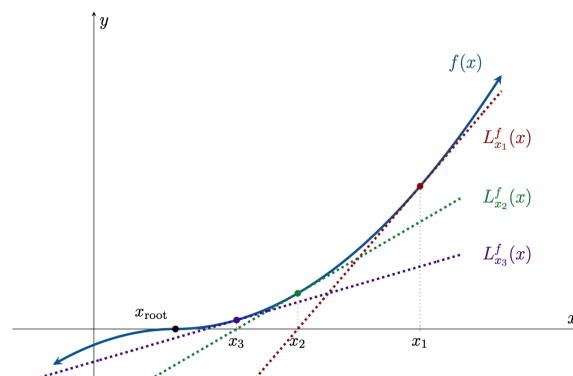
> 💡 **Tip — Root Finding Formula for NM**
>
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

$x^2 - 4x - 7$ at $x = 5$ to nearest thousandth

ANS:

Let $x_1 = 5$, given $f'(x) = 2x - 4$

$x_2 = x_1 - \frac{f(5)}{f'(5)} = 5 - \left(-\frac{1}{3}\right) = \frac{16}{3}$

$x_3 = x_2 - \frac{f\left(\frac{16}{3}\right)}{f'\left(\frac{16}{3}\right)} = \frac{16}{3} - \frac{1}{60}$

Derivatives of Inverse Functions

Example:

$f(x) = 5\sqrt{x}$

$f(x)$ is continous and differentiable in $(0, \infty)$

$b = f(a) \iff 5 = f(a) \iff a = 1 \ f'(x) = \frac{5}{2}x^{-\frac{1}{2}}$

$(f^{-1})'(5) = \frac{1}{f'(1)} = \frac{1}{\frac{5}{2}} = \frac{2}{5}$

> 🐳 **Info** $- \ln x$
>
> $$f(x) = e^x, \ f^{-1}(x) = \ln x$$
>
> $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$
>
> $$(\ln(x))' = \frac{1}{f'(\ln(x))}$$
>
> $$= e^{\frac{1}{\ln(x)}}$$
>
> $$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

Inverse Trigonometric Functions

1. $\frac{\mathrm{d}}{\mathrm{d}x} \arccos x = \frac{1}{\cos'(\arccos x)} = -\frac{1}{\sin(\arccos x)} = -\frac{1}{\sqrt{1-x^2}}$

   $\arccos(x) = \theta \implies \cos(\theta) = \frac{x}{1}$

   $\sin(\arccos(x)) = \sin(\theta) = \frac{\sqrt{1-x^2}}{1}$

   -Domain $\arccos x \in (-1, 1)$ -Range $\arccos x \in (0, \pi)$

2. $\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

   -Domain $\arcsin x \in (-1, 1)$ -Range $\arcsin x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

3. $\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{x^2+1}$

   -Domain $\arctan x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ -Range $\arctan x \in (-\infty, \infty)$