CH 1 — Integration

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Definite Integrals

🐳 **Info — Riemann Sums**

Given $f(x)$ that is defined over $[a, b]$ with $a < b$, the area under function $f(x)$ can be found by

1. **Left-Endpoint Riemann Sum**

$$L_n = \sum_{i=0}^{n-1} f(x_i^*)\Delta x$$

- Underestimates Increasing Funcitons

2. **Right-Endpoint Riemann Sum**

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- Overestimates Increasing Functions

where

- $\Delta x = \frac{b-a}{n}$ under regular partition
- $x_i^* = a + i\Delta x = a + i\frac{b-a}{n}$

$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$ for $f(x)$ on interval $[a, b]$

Regular Parition means interval $[a, b]$ is equally partitionined into n rectangles with identical width

Example:

Estimate area under the curve for $f(x) = x^2$ on $x \in [0, 1]$

$R_n = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6n}$
$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n} = \frac{1}{3}$

🐳 **Info — Definite Integral**

$f(x)$ defined on $x \in [a, b]$ with regular partition with n subintervals

The definite integral of $f(x)$ on $[a, b]$ is defined

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

A function is integrable on $x \in [a, b]$ provided that the limit of Riemann Sum exists and has the same value regardless of the choice of x_i^*

🐳 Info — Integrability Theorem for Continuous Functions

Integrability: $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$

1. If f is continuous of $[a, b]$ then f is integrable on $[a, b]$
2. f is bounded on $[a, b]$ and has a **finite** number of discontinuities, then f is integrable on $[a, b]$

That is continuity implies integrability and the other way is false

Examples:

1. $f(x) = x^2$
2. $f(x) = \begin{cases} 2 \text{ if } x \leq 3 \\ 1 \text{ if } x > 3 \end{cases}$, note that $f(x)$ is discontinuous
3. $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ on $[0, 1]$
   - x_i^* is rational

     $\int_0^1 f(x) \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 1$
   - x_i^* is irrational $\int_0^1 f(x) \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x^* + i)\Delta x = 0$

   Thus not integrable

For geometric interpretation, Riemann Sums and Definite Integrals measures the "signed" area where there is no more than 1 inflection point

- A positive result of w implies the area under the curve above x-axis is w
- A negative result of w implies the area under the curve under x-axis is w

🐳 Info — Parity of Functions and Definite Integrals

Let $f(x)$ be bounded and integrable on $[-a, a]$

1. If $f(x)$ is odd function, then

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 0$$

2. If $f(x)$ is even function where $\int_0^a f(x) \,\mathrm{d}x = w$

$$\int_{-a}^{a} f(x) \,\mathrm{d}x = 2w$$

Examples:

1. $\int_1^3 x^2 - 3x \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n} = \mathrm{im}_{n\to\infty} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^3 - 3\left(1 + \frac{2i}{n}\right)\right] \cdot \frac{2}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^n \left(-\frac{4i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right) = -\frac{10}{3}$

2. $\int_0^5 x^3 - 2 \,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(i\frac{5}{n}\right)\frac{5}{n} = \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{5}{n}\right)\left(\left(i\frac{5}{n}\right)^3 - 2\right)\right] = \frac{583}{4}$

🐳 **Info — Basic Property of Definite Integral**

Let $f(x), g(x)$ be integrable on $[a, b]$

1. For any $c \in \mathbb{R}$, the function $cf(x)$ is integrable and
$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$
2. The function $f + g$ is integrable and
$$\int_a^b (f+g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$
3. If $m, M \in \mathbb{R}$ and $m \leq f(x) \leq M \forall x \in [a, b]$, then
$$m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$$
4. If $f(x) \geq 0 \forall x$, then
$$\int_a^b f(x)\,\mathrm{d}x \geq 0$$
5. If $f(x) \leq g(x) \forall x \in [a, b]$, then
$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$
6. The function $|f|$ is integrable on $[a, b]$ and
$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$$
7. Bound fliping
$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$
8. $$\int_a^a f(x)\,\mathrm{d}x = 0$$

🐳 **Info — Separation of Domain of Definite Integral**

If $f(x)$ is also integrable on an interval containing a, b, c, then
$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

🐳 **Info — Average Value of Function**

Let f be a function that is continuous on an interval $[a, b]$ with $a < b$. The **average calue of f on $[a, b]$** is defined as

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x$$

Examples:

1. Determine the average value of $f(x) = 1 - x^2$ on [−1, 1]

   $f_{\text{avg}} = \frac{1}{1-(-1)} \int_{-1}^{1} f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x = \lim_{x\to\infty} \sum_{i=1}^{n} \frac{1-\left(\frac{i}{n}\right)^2}{n}$
   $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1 - \frac{i^2}{n^2} = \lim_{n\to\infty} \frac{1}{n}\left(n - \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right) = 1 - \frac{1}{3} = \frac{2}{3}$
2. Suppose that f, g are integrable on $[-1, 1]$, $\int_1^{-1} f(t)\,\mathrm{d}t = 5$, and g is an evern function with $\int_0^1 g(t)\,\mathrm{d}t = 2$.

   $\int_{-1}^{1} 3f(x) - g(x)\,\mathrm{d}x = 3\int_{-1}^{1} f(x)\,\mathrm{d}x - \int_{-1}^{1} g(x)\,\mathrm{d}x = -3\int_1^{-1} f(x)\,\mathrm{d}x - 2\int_0^1 g(x)\,\mathrm{d}x = -19$

🐳 **Info — Fundamental Theorem of Calculus (FTC - 1)**

Let $a \in \mathbb{R}$. If f is continuous on an open interval I containing a, then the funciton

$$G(x) = \int_a^x f(t)\,\mathrm{d}t$$

is differentiable $\forall x \in I$ and $G'(x) = f(x)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

Proof

Given $x \in I$, from the definition of the derivative, we have
$G'(x) = \lim_{h\to\infty} \frac{G(x+h)-G(x)}{h}$

$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{h} = \lim_{h\to 0} \frac{\int_a^x f(t)\,\mathrm{d}t + \int_x^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t.$

For all $h \neq 0$, sufficiently close to 0, and $h > 0$ f is continuous on $[x, x+h]$.

$\forall h, \exists c = c(h)$ in $[x, x+h]$ s.t.

$$f(c(h)) = \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t$$

Since $x \leq c(h) \leq x + h$, by Squeeze Theorem, $\lim_{h\to 0} c_h = x$, thus

$$G'(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t = \lim_{h\to 0}(c_h) = f(x)$$

□