CH 4 - Derivatives

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Velocity

🐳 **Info — Average Velocity and Instantaneous Velocity**

$$v_{avg} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

$$v_{inst} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Definition of Derivatives

🐳 **Info — Average Rate of Change and Instantaneous Rate of Change (Derivative)**

$$f_{avg} = \frac{f(b) - f(a)}{b - a}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

If $f'(x)$ exists at $x = a$, then $f(x)$ is **differentiable** at $x = a$

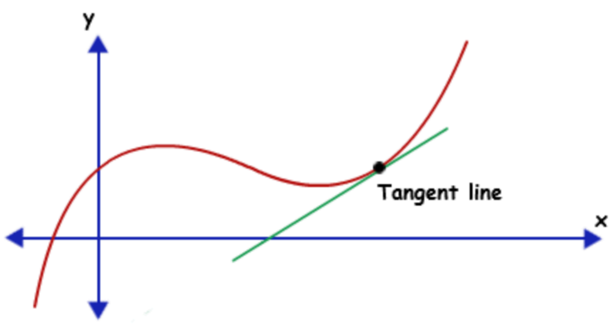
🐳 **Info — Tangent Line**

If $f(x)$ is differentiable at $x = a$, then the **tangent line** to $f(x)$ at $x = a$ is the line passing through $(a, f(a))$ with slope $f'(a)$

The equation of the tangent line

$$y = f'(a)(x - a) + f(a)$$

$(a, f(a))$ is the **point of tangency**



Examples:

Find the tangent line to $f(x) = \frac{1}{x+5}$ at $x = 3$
$f(3) = \frac{1}{8}$
$f'(3) = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h+5} - \frac{1}{a+5}}{h} =$

$\lim_{h\to 0} \frac{1}{h} \frac{a+5-(a+h+5)}{(a+5)(a+h+5)} = \lim_{h\to 0} -\frac{1}{(a+5)(a+h+5)} = -\frac{1}{(a+5)^2} = -\frac{1}{64}$

$y = -\frac{1}{64}(x-3) + \frac{1}{8}$

🐳 **Info — Differentiability Implies Continuity**
If a function f is differentiable at $x = a$, then f is continuous at $x = a$

Proof

f is differentiable at $x = a$ then, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists
$\lim_{h\to 0}[f(a+h) - f(a)] = 0 \Longrightarrow \lim_{h\to 0}[f(a+h) - f(a) + f(a)] = \lim_{h\to 0} f(a) \Longrightarrow$
$\lim_{h\to 0} f(a) = f(a)$

⚠️ **Warning — Continuity Not Implies Differentiability**
$f(x) = |x| = \begin{cases} -x \text{ if } x<0 \\ x \text{ if } x \geq 0 \end{cases}$

$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \frac{h-0}{h} = 1$

$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \frac{-h-0}{h} = -1$

Thus $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} =$ DNE but continuous.

$\therefore$ continuity does not impliy differentiability

🐳 **Info — Differentiability of Funciton**
We say that f is **differentiable** on an interval I if $f'(a)$ exists $\forall a \in I$.

We define the derivative funciton $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

We sometimes also write $f'(x)$ as $\frac{d}{dx} f(x)$, and $f'(a) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)\big|_a$

🐳 **Info** — Constant Function
$f(x) = c$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

🐳 **Info** — Linear Function
$f(x) = mx + b$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(m(x+h)+b)-f(mx+b)}{h} = \lim_{h\to 0} m\frac{h}{h} = m$

🐳 **Info** – Quadratic Function
$f(x) = px^2 + sx + c$
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[p(x+h)^2+s(x+h)+c]-[px^2+sx+c]}{h} = \lim_{h\to 0} \frac{2xph+xh^2+sh}{h} = \lim_{h\to 0} 2xp + xh + s = 2xp + s$

🐳 **Info** – Basic Trig
$f(x) = \sin x$
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$

$\lim_{h\to 0} \frac{[\sin x(\cos h - 1)]}{h} + \lim_{h\to 0} \cos x \frac{\sin h}{h} = \sin x \cdot \lim_{h\to 0} \frac{\cos^2 h - 1}{h\cdot(\cos h + 1)} + \cos x =$

$\sin x \cdot \lim_{h\to 0} \frac{\sin^2 h}{h\cdot(\cos h+1)} = \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{\sin h}{\cos h + 1} + \cos x = \cos x$

We define e to be the unique base of an exponential function with slope 1 through (0, 1)

🐳 **Info** – **Derivative Rules**

Let $f(x)$ and $g(x)$ be differentiable at $x = a$

1. $w(x) = cf(x) \implies w'(x) = cf'(x)$
2. $w(x) = f(x) \pm g(x) \implies w'(x) = f'(x) \pm g'(x)$
3. $w(x) = f(x)g(x) \implies w'(x) = f'(x)g(x) + f(x)g'(x)$
4. If $g(x) \neq 0, w'(x) = \frac{f(x)}{g(x)} \implies w'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$
5. If $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R} \setminus \{0\} \implies f'(x) = \alpha x^{\alpha-1}$
6. $w(x) = (g \circ f)(x) = g(f(x)) \implies w'(x) = g'(f(x)) \cdot f'(x) \sim \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$

⚠️ **Warning** – Power Rule
If $x = 0, x^-1$ does not make sense so that is why $\alpha \in \mathbb{R} \setminus \{0\}$

Proof

We suppose that $f(x), g(x)$ are differentiable, so that the limits:

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ exists

1. Product rule:

$\lim_{h\to 0} \frac{w(x+h)-w(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$

$\lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0} \frac{(f(x+h)-f(x))g(x+h)}{h} + \frac{(g(x+h)-g(x))f(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \cdot \lim_{h\to 0} f(x)$

$= f'(x)g(x) + f(x)g'(x)$

2. Quotient rule

$\lim_{h\to 0} \frac{w(x+h)-w(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)g(x)}{g(x+h)g(x)} - \frac{f(x)g(x+h)}{g(x+h)g(x)}}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x)g(x+h)}}{h}$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} = \lim_{h \to 0} \frac{g(x)(f(x+h) - f(x))}{\frac{h}{g(x)g(x+h)}} - \frac{f(x)(g(x+h) - g(x))}{\frac{h}{g(x)g(x+h)}}$$

$$= \frac{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to} g(x) - \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}{\lim_{h \to 0} g(x+h)g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$

$\square$

Basic Derivatives

🐳 Info — Basic Trig Derivatives

$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin x}{\cos} x = \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x} \csc x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\sin x} = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\csc x \cot x$

$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\cos x} = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$

$\frac{\mathrm{d}}{\mathrm{d}x} \cot x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\cos x}{\sin x} = \frac{\cos' x \sin x - \cos x \sin' x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\csc^2 x$

🐳 Info — Exponential/Logarithmic Derivatives

For $a^x, x > 0$:

$\frac{\mathrm{d}}{\mathrm{d}x} a^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x \ln(a)} = e^{x \ln a} \cdot \ln a = a^x \ln a$

$\frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln(x)}{\ln(a)} = \frac{1}{x} \cdot \frac{1}{\ln a} = \frac{1}{x \ln a}$

Example:

1. $\frac{\mathrm{d}}{\mathrm{d}x} x^3 e^{2x} \cos x = 3x^2 e^{2x} \cos x + 2x^3 e^{2x} \cos x - x^3 e^{2x} \sin x$
2. $\frac{\mathrm{d}}{\mathrm{d}x} 3^{\csc x} = 3^{\csc x} \ln 3 \cdot -\csc x \cot x = -3^{\csc x} \csc x \cot x \ln 3$
3. $\frac{\mathrm{d}^{67}}{\mathrm{d}x^{67}} \sin x$. Note that $\sin' x = \cos x, \sin'' x = -\sin x, \sin''' - \cos x, \frac{\mathrm{d}^4}{\mathrm{d}x^4} \sin x = \sin x /$

$67 \bmod 4 \equiv 3$, that is $\frac{\mathrm{d}^{67}}{\mathrm{d}x^{67}} \sin x = -\cos x$

4. $\frac{\mathrm{d}}{\mathrm{d}x} \frac{x}{\left(1+e^{x^2}\right)^3} = \frac{\mathrm{d}}{\mathrm{d}x} x \cdot \left(1 + e^{x^2}\right)^{-3} = \frac{1}{\left(1+e^{x^2}\right)^3} - 3(\left(1 + e^{x^2}\right)^{-4} \cdot x^2 e^{x^2} \cdot 2x = \frac{1}{\left(1+e^{x^2}\right)^3} - \frac{6x^3 e^{x^2}}{(\left(1+e^{x^2}\right)^4}$
5. $\frac{\mathrm{d}}{\mathrm{d}x} x^{x^x}$

$\frac{\mathrm{d}}{\mathrm{d}x} x^x = x^x \ln x \cdot (\ln x + 1)$

$\frac{\mathrm{d}}{\mathrm{d}x} x^{f(x)} = x^{f(x)} \cdot (\ln x \cdot f'(x) + x) = x^{x^x} \cdot x^x (\ln^2(x) + \ln(x) + x^{x-1})$

Linear Approximation

With the assumption of $f(x)$ is continuous ay $x = a$, we can derive

$$f'(a) = \lim_{x \to a \frac{f(x) - f(a)}{x - a}} \leftrightarrow f'(a)(x - a) = f(x) - f(a) \leftrightarrow f(x) = f'(a)(x - a) + f(a)$$

Which is the linear approximation of $f(x)$ near $x = a$

🐳 Info — Linear Approximation

Let $f(x)$ be differentiable at $x = a$. The **linear approximation** to $f(x)$ at $x = a$ is given by

$$L_a^f(x) = f'(a)(x-a) + f(a)$$

If it is clear what function f we are talking about, we sometimes denote $L_a(x)$ instead.

🐳 Info — Upper Bound Error of Linear Approximation

The error of linear approximation is defined as:

$$\text{error} = |f(x) - L_a^f(x)|$$

Assume that $f(x)$ is such that $|f''(x)| \leq M$ for each x in an interval I containing $x = a$. Then,

$$\text{error} = |f(x) - L_a^f(x)| \leq \frac{M}{2}(x-a)^2$$

forr each $x \in I$

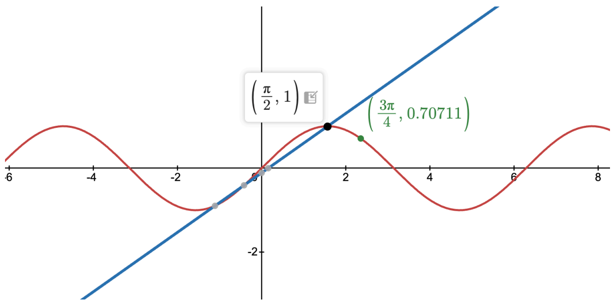
Examples:

1. Find the linearization of $\sqrt{x}$ at $x = 4$ and use it ti estimate $\sqrt{4.01}$

   $L_a^f(x) = f'(a)(x-a) + f(a)$ where $f(x) = \sqrt{x} \Longrightarrow f'(x) = \frac{1}{2\sqrt{x}}$

   $L_a^f(x) = \frac{1}{2\sqrt{x}}(x-4) + 2 \Longrightarrow L_a^f(0.01) = \frac{1}{2\sqrt{0.01}}(0.1) + 2 \approx 2.0024984$

2. What factors could affect the error in linear approximation?
   - The distance from x to a. (e.g. $f(x) = \sin(x), x = 3\frac{\pi}{2}, a = \frac{\pi}{4}$.)



   - The curvature (e.g. $f(x) = e^x, g(x) = e^{\frac{x}{10}}$)

3. Find a upper bound on the error in using L_9 to appriximate $f(x) = \sqrt{x}$ on $[5, 13]$

   If $|f''(x)| \leq M$ on I then: error $\leq \frac{M}{2}(x-a)^2 \forall x \in I$

   $f(x) = \sqrt{x}; f'(x) = \frac{1}{2\sqrt{x}}; f''(x) = -\frac{1}{4x^{\frac{3}{2}}}$

   $|f''(x)| = \frac{1}{4x^{\frac{3}{2}}} \leq= \frac{1}{4(5)^{\frac{3}{2}}} = \frac{1}{20\sqrt{5}}$

   So the error $\leq \frac{1}{40\sqrt{5}}(x-9)^2 = \frac{1}{40\sqrt{5}}(13-9)^2 = \frac{2}{5\sqrt{5}}$