CH 3 — Function Limits and Continuity

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Definitions

If $f : \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}, \lim_{x \to a} f(x) = L$ if
for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

Examples:

1)Prove using the $\varepsilon - \delta$ definition that $\lim_{x \to 0} f(x)$ DNE where

$$f(x) = \begin{cases} -2 \text{ if } x < 0 \\ 3 \text{ if } x > 0 \end{cases}$$

Domain: $\mathbb{R} \setminus \{0\}$

Take $\varepsilon = 1$. Consider some $\delta > 0$. Whitin $(0 - \delta, 0 + \delta)$
We have both $(-\delta, 0)$ where $f(x) = -2$ and $(0, \delta)$ where $f(x) = 3$. If this δ exists for $\varepsilon = 1$ then the limit L would need to be distance 1 or both −2 and 3, where is impossible.

$$\therefore \lim_{x \to 0} f(x) = \text{DNE}$$

2) $\lim_{x \to 7} 8x - 3 = 53$

Let $\varepsilon > 0$ be arbitrary.
We want find δ s.t. if $0 < |x - 7| < \delta$ then $|8x - 3 - 53| < \varepsilon \to \delta = \frac{\varepsilon}{8}$
Pick $\delta = \frac{\varepsilon}{8}$.
Then if $0 < |x - 7| < \frac{\varepsilon}{8}, |(8x - 3) - 53| = |8x - 56| = 8|x - 7| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$