

# CH 5- Set Theory

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## Empty Set

$\emptyset = \{\}$  but  $\{\emptyset\} \neq \emptyset$

## Cardinality

The number of elements in a finite set is  $S$  called the cardinality of  $S$ , denoted by  $|S|$

## Set Notation

### Set Builder Notation - Type 1

The notation  $\{x \in \mathcal{U} : P(x)\}$

Describes the set consisting of all objects  $x$  such that  $x$  is an element of  $\mathcal{U}$ , and  $P(x)$  is true

Example:  $A = \{n \in \mathbb{N} : n \mid 12\} = \{1, 2, 3, 4, 6, 12\}$

### Set Builder Notation - Type 2

The notation  $\{f(x) : x \in \mathcal{U}\}$

Describes the set consisting of all objects of the form  $f(x)$  such that  $x$  is an element of  $\mathcal{U}$

Example:  $B = \{2k : k \in \mathbb{Z}\}$  = all even numbers

### Set Builder Notation - Type 3

The notation  $\{f(x) : x \in \mathcal{U}, P(x)\}$  or  $\{f(x) : P(x), x \in \mathcal{U}\}$  Both describes the set consisting of all objects of the form  $f(x)$  such that  $x$  is an element  $\mathcal{U}$  and  $P(x)$  is true

Example:  $C = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$

Practices:

1. All multiple of 7:  $\{x \in \mathbb{N} : 7 \mid x\}$
2. All odd perfect square:  $\{(2x+1)^2 : x \in \mathbb{Z}\}$
3. All points on a circle of radius 8 centered at origin:  $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 64\}$
4. All sets of three integers which are the side lengths of a triangle:

$\{(x, y, z) : x, y, z \in \mathbb{N}, x < y < z, x + y < z\}$

## Union and Intersection

The **union** of two sets  $S$  and  $T$ , denoted,  $S \cup T$ , is the set of all elements belonging to either set  $S$  or set  $T$ .

The **intersection** of two sets  $S$  and  $T$ , denoted,  $S \cap T$ , is the set of all elements belonging to either set  $S$  and set  $T$ .

Practice:

Let  $C = \{3, 5, 7, 10\}$ ,  $D = \{1, 3, 6, 7, 8\}$

1.  $C \cup D = \{1, 3, 5, 6, 7, 8, 10\}$
2.  $C \cap D = \{3, 7\}$

Let  $A = \{m \in \mathbb{Z} : 2 \mid m\}$ ,  $B = \{2k + 1 : k \in \mathbb{Z}\}$

1.  $A \cup B = \mathbb{Z}$
2.  $A \cap B = \emptyset$

For non empty sets  $A$  and  $B$

1. If  $|A| = 12$ ,  $|B| = 4$ ,  $|A \cap B| = 2$ ,  $|A \cup B| = 14$
2. If  $|A| = 10$ ,  $|B| = 20$ ,  $|A \cup B| = 25$ ,  $|A \cap B| = 5$

## Set Difference

The **set difference** of two sets  $S$  and  $T$ , written  $S - T$  or  $S \setminus T$  is the set of all elements belonging to  $S$  but not  $T$ .

Symbolically:  $S - T = \{x : (x \in S) \wedge x \notin T\}$

## Complement

The **complement** of a set  $S$  whose elements belong to  $\mathcal{U}$ , written  $S^c$ , is the set of all elements in  $\mathcal{U}$  but not in  $S$ .

Symbolically:  $\{x \in \mathcal{U} : x \notin S\}$

## Disjoin Set

Two sets  $S$  and  $T$  are said to be disjoint when  $S \cap T = \emptyset$

Practice:

1. Let  $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ ,  $C = \{3, 5, 7, 10\}$ ,  $D = \{1, 3, 6, 7, 8\}$

Find  $|C - D^c| = 2$

2. Let  $A = \{x : x \in \mathbb{N}, x \text{ is even}\}$ ,  $B = \{x : x \in \mathbb{N}, x \text{ is not a prime}\}$

$$A \cup B = \{1, 2, 4, 6, 8, 9, \dots\} \quad A \cap B = \{2\}$$

3. If  $A \cap B = \emptyset$  and  $B \cap C = \emptyset$ , then  $A \cap C = \emptyset$ : False

4. If  $|A \cap B| = |A|$  and  $B \cap C = \emptyset$ , then  $A \cap C = \emptyset$ : True

5. If  $|A \cap B| = |A|$  and  $|A \cap C| = |A|$ , then  $B \cap C = \emptyset$ : False

## Subsets

A set  $S$  is called a **subset** of a set  $T$ , denoted  $S \subseteq T$  when every element of  $S$  belongs to  $T$ .  $T$  is a **superset** of  $S$

A set is called a **proper subset**, denoted  $S \subsetneq T$ , meaning  $S$  is a subset of  $T$  and there exists an element in  $T$  which does not belong to  $S$ .  $T$  is a **proper superset** of  $S$

Examples:

1)

$$\{5, 15, 25\} \subseteq \{5, 10, 15, 20, 25\}$$

$$\{5, 15, 25\} \subsetneq \{5, 10, 15, 20, 25\}$$

2)

$$\{2, 4, 6\} \not\subseteq \{2025\}$$

$$\{2, 4, 6\} \not\subseteq \{1, 2, 3, 4, 5\}$$

3)

$$\mathbb{N} \subseteq \mathbb{Z}, \mathbb{Z} \subseteq \mathbb{Q}, \mathbb{Q} \subseteq \mathbb{R}$$

4)

$$\emptyset \subseteq S \text{ for all sets } S$$

$$S \subseteq S \text{ for all sets } S, \text{ but } S \text{ is never a proper subset of } S$$

5)

$$\text{For all sets } S \text{ and } T, S \cap T \subseteq T \text{ and } S \subseteq S \cup T$$

## IMPORTANT

Subset can be expressed as an implication

To prove  $S \subseteq T$ , we need to prove the universally quantified implication

$$\forall x \in \mathcal{U}, (x \in S) \implies (x \in T)$$

Equal notation:

$$S \subseteq T \text{ and } T \subseteq S \iff S = T$$

Example:

1. Let  $A = \{n \in \mathbb{N} : 4 \mid (n - 3)\}$  and let  $B = \{2k + 1 : k \in \mathbb{Z}\}$ , prove that  $A \subseteq B$

Let  $n \in \mathbb{Z}, (n \in A) \implies (n \in B)$

$A \equiv$  a set of natural numbers in form  $4q + 3$

$B \equiv$  a set of integers that are odd

Since  $4q + 3 = 2(2q + 1) + 1$ , which is always odd.

$\therefore A \subseteq B$  and also  $A \subsetneq B$

2. Prove  $S = T \iff S \cap T = S \cup T$

$$(S = T \implies S \cup T = S \cap T) \wedge (S \cup T = S \cap T \implies S = T)$$

$(\implies)$

Suppose  $S = T$ , we need to show  $S \cap T \subseteq S \cup T$  and  $S \cup T \subseteq S \cap T$

If  $x \in S \cap T$ , then  $x \in S, x \in T$ .

Assume  $x \in S \cup T$ , then without loss of generality, we may suppose that  $x \in S$ .

Then since  $S = T$ , it follows that  $x \in T$ .

$\therefore x \in S \wedge x \in T \implies x \in S \cap T$

( $\Leftarrow$ )

Suppose  $S \cap T = S \cup T$ , we must show  $S \subseteq T \wedge T \subseteq S$

If  $x \in S$  then  $x \in S \cup T$ .

Furthermore,  $S \cap T = S \cup T \implies x \in S \cap T \implies x \in S \wedge x \in T$

$x \in T \implies S \subseteq T$ . The proof of  $T \subseteq S$  is similar and will be omitted

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