CH 10 - Complex Numbers

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Standard Form

🐳 Info — Definition of Complex Numbers

A **complex number** z in **standard form** is an expression of the form $z = x + yi$ where $x, y \in \mathbb{R}$.

The real number x is called the **real part** of z, and is written $\Re(x)$.

The real number y is called the **imaginary part** of z, and is written $\Im(z)$.

The set of complex numbers is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$$

The complex number $z = x + yi$ and $w = u + vi$ are equal $(z = w)$ if and only if $x = u, y = v$

A complex number z is said to be purely real if $y = 0$ (i.e. $1 = 1 + 0i$)

A complex number z is said to be purely imaginary if $x = 0$ (i.e. $i = 0 + 1i$)

0 is purely real and purely imaginary (i.e. $0 = 0 + 0i$)

🐳 Info — Complex Arithmetics

Let $z = a + bi$ and $w = c + di$ be complex numbers. Then the

Addition is defined as

$$z + w = (a + c) + (b + d)i$$

Multiplication is defined as

$$zw = (ac - bd) + (ad + bc)i$$

Examples:

Let $z = 2 + 3i, w = -1 + 7i$

1. $z + w = (2 - 1) + (3 + 7)i = 1 + 10i$
2. $zw = (-2 - 21) + (14 - 3)i = -23 + 11i$
3. $i^2 = ii = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + (0 + 0)i = -1$

$\therefore i^2 = -1$

From (3), we can derive a easier way of multiplication

💡 **Tip** — Multiplication trick

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Properties of Complex Arithmetics

1. $z + 0 = 0 + z = z$
2. $z0 = 0z = 0$
3. $z + (-1)z = (-1)z + z = 0$
4. $z1 = 1z = z$

🐳 **Info — Multiplicative Inverse**

For all complex numbers z, the multiplicative inverse of z exists if and only if $z \neq 0$. Moreover, for $a + bi \neq 0$, the multiplicative inverse is unique given by

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i = \frac{a - bi}{a^2 + b^2}$$