

CH 1 – Vectors in Euclidean Space

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Vector Addition and Scalar Multiplication

Info – Vector

The set \mathbb{R}^n is defined as $\left\{ \vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$

A **vector** is an element $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$ of \mathbb{R}^n

The row notation of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ is $\vec{v} = [v_1 \ v_2 \ v_3]^T$

Info – Equality

We say that vectors $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} \in \mathbb{R}^m$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ are **equal**

if $n = m$ and $u_i = v_i \forall i = 1, 2, \dots, n$.

We denote it: $\vec{w} = \vec{v}$

Info – Addition Properties

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} \in \mathbb{R}^n$.

Then $\vec{w} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \dots \\ u_n + v_n \end{bmatrix}$

1. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$
2. $\vec{w} + \vec{v} + \vec{w} = \vec{w} + (\vec{v} + \vec{w})$
3. There is a zero **vector**, $\vec{0} = [0 \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{R}^n$
4. $\vec{v} + \vec{0} = \vec{v}$
5. $\vec{v} + (-\vec{v}) = 0$

Info – Additive Inverse

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \in \mathbb{R}^n$. The additive inverse of \vec{w} denoted $-\vec{w}$ is defined as

$$-\vec{w} = \begin{bmatrix} -u_1 \\ -u_2 \\ \dots \\ -u_n \end{bmatrix}$$

$$\vec{w} - \vec{w} = \vec{w} + (-\vec{w}) = \vec{0}$$

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \\ \dots \\ v_n - u_n \end{bmatrix}$$

Info – Scalar Multiplication

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$. Then the scalar product $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \dots \\ cv_n \end{bmatrix}$

1. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
2. $c(\vec{w} + \vec{v}) = c\vec{w} + c\vec{v}$
3. $0\vec{w} = \vec{0}$
4. If $c\vec{v} = \vec{0}$ then $c = 0 \vee \vec{v} = 0$
5. $c(d\vec{v}) = (cd)\vec{v}$

Info – Linear Combination

For $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, and $c_1, \dots, c_k \in \mathbb{R}$ we call the expression

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$$

a **linear combination** of $\vec{v}_1, \dots, \vec{v}_k$.

Examples:

1. Let $\vec{u} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ then $2\vec{u} - 3\vec{v} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$
2. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Is $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ a linear combination of \vec{u} and \vec{v} ?

We set $\vec{x} = c_1\vec{u} + c_2\vec{v}$ and try to solve for c_1, c_2

That is $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 \end{bmatrix}$, we obtain $c_1 = 2, c_2 = \frac{1}{2}$. So \vec{x} is a linear combination of \vec{u}, \vec{v}

Bases

Info – Span

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in \mathbb{R}^n . We define the **span** of \mathcal{B} by

$$\text{Span } \mathcal{B} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \mid c_1, c_2, \dots, c_l \in \mathbb{R}\}$$

We say that the set $\text{Span } \mathcal{B}$ is spanned by \mathcal{B} adn that \mathcal{B} is a spanning set for $\text{Span } \mathcal{B}$

Span might not cover the entire plane if

- Vectors are linear dependent to each other
- One of them is $\vec{0}$

Theorem

Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$. Some vector $\vec{v}_i, 1 \leq i \leq k$, can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}, \dots, \vec{v}_k$ if and only if $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \dots, \vec{v}_k\}$

Proof

Example:

Consider $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Describe $\text{Span } \vec{v}_1, \vec{v}_2$ geometrically.

$$\text{Span } \{\vec{v}_1, \vec{v}_2\} = \text{Span } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \mathbb{R}^2$$

Info – Linear Dependence/Independence

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\} \in \mathbb{R}^n$ is said to be **linearly dependent** if there exist coefficients c_1, \dots, c_k not all zero such that

$$\vec{0} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$$

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\} \in \mathbb{R}^n$ is said to be **linearly Independent** if the only solution to c_1, \dots, c_k not all zero such that

$$\vec{0} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$$

is $c_1 = c_2 = \dots = c_k = 0$ (called **trivial solution**)

Examples:

1. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linearly dependent \iff at least one of \vec{u}, \vec{v} is a scalar multiple of the other.

Proof

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$

1. Assume $\{\vec{u}, \vec{v}\}$ is linearly dependent. Then $\exists c_1, c_2 \in \mathbb{R}$, not both zero s.t.

$$c_1 \vec{u} + c_2 \vec{v} = \vec{0}$$

WLOG, assume $c_1 \neq 0$ Then, $c_1 \vec{u} = -c_2 \vec{v} \implies \vec{u} = -\frac{c_2}{c_1} \vec{v}$. Thus \vec{u} is a scalar multiple of \vec{v}

2. Assume WLOG \vec{u} is a scalar multiple of \vec{v} . Then $\exists a \in \mathbb{R}$ s.t.

$$\vec{u} = a\vec{v} \implies 1\vec{u} - a\vec{v} = \vec{0}$$

Since $1 \neq 0$, $\{\vec{u}, \vec{v}\}$ is linearly dependent.

□

2. Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Consider the equation $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Which gives $\begin{cases} c_1 + c_2 + 2c_3 = 0 \\ 2c_1 - c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$

$\implies c_2 = c_1 = -c_3$. Thus we get a solution for any c_3 . Pick $c_3 = -1 \implies c_2 = c_1 = 1$.

Thus the set is linearly dependednt.

Info – Linear Dependence Theorem

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n\}$ is linearly dependent if and only if

$$\vec{v}_i \in \text{Span } \{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\} \text{ for some } i, 1 \leq i \leq k$$

Info – Zero Vector and Linear Dependence

If a set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ contains the zero vector, then it is linearly dependent.

Proof:

Let $\vec{v}_i = \vec{0}$

$$0\vec{v}_1 + \dots + 0\vec{v}_{i-1} + \vec{v}_i + 0\vec{v}_{i+1} + 0\vec{v}_k = \vec{0}$$

□

Info – Basis and Standard Basis

Basis

Let S be a subset of \mathbb{R}^n . If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set of vectors in \mathbb{R}^n s.t. $S = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then the set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is called a **basis** for S .

We define a basis for the set $\{\vec{0}\}$ to be the empty set

Standard Basis

In \mathbb{R}^n , let \vec{e}_i be the vector whose i^{th} component is 1 with all other components 0. The set $E = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called the **standard basis for \mathbb{R}^n**

$$(\text{i.e. } \mathbb{R}^3 \text{ is } \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\})$$

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n$ then we call v_1, v_2, \dots, v_n the **components of \vec{v}**

Examples:

Is B is a basis for \mathbb{R}^2

1. $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. This set of vectors is linearly independent, thus is a standard basis for \mathbb{R}^2 .

That is $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \mathbb{R}^2$

2. $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$. Note that this set of vectors is linearly dependent as one is a scalar multiple of another, thus cannot be considered as a basis for \mathbb{R}^2

3. $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ is linearly independent since neither scalar multiple of another. We need to prove:

- $\text{Span } B \subseteq \mathbb{R}^2$, which is obvious, since the vectors in B are in \mathbb{R}^2 the linear combinatin of these will be \mathbb{R}^2

• $\mathbb{R}^2 \subseteq \text{Span } B$, consider an arbitrary $\vec{x} \in \mathbb{R}^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ that is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ 2c_1 + 4c_2 \end{bmatrix}$.

We obtain $\begin{cases} c_1 + 3c_2 = x_1 \\ 2c_1 + 4c_2 = x_2 \end{cases} \Rightarrow c_1 = x_1 - 3c_2 \Rightarrow 2(x_1 - 3c_2) + 4c_2 = x_2 \Rightarrow c_2 = -\frac{1}{2}x_2 + x_1 \Rightarrow$

$$c_1 = x_1 - 3(-\frac{1}{2}x_2 + x_1) = -2x_1 + \frac{3}{2}x_2$$

Therefore $\mathbb{R}^2 \subseteq \text{Span } B$

That is B is a standard basis for \mathbb{R}^2

Info – Theorem

If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subset S of \mathbb{R}^n , then every vector $\vec{x} \in S$ can be written as a unique linear combination of the vectors in \mathcal{B}

Proof

Let $\vec{x} \in S$ and assume $\exists c_1, \dots, c_k, d_1, \dots, d_k \in \mathbb{R}$ s.t. $\vec{x} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ and $\vec{x} = d_1\vec{v}_1 + \dots + d_k\vec{v}_k$.

Subtracting these two equations: $\vec{0} = (c_1 - d_1)\vec{v}_1 + \dots + (c_k - d_k)\vec{v}_k$, with $\{\vec{v}_1, \dots, \vec{v}_k\}$ is basis, thus linearly independent, so there is $(c_1 - d_1) = \dots = (c_k - d_k) = 0$, thus \vec{x} can be written as a unique linear combination.

□

Subspace

Info – Definition

A subset S of \mathbb{R}^n is called a **subspace** of \mathbb{R}^n if for every $\vec{x}, \vec{y}, \vec{w} \in S$ and $c, d \in \mathbb{R}$ we have

1. $\vec{x} + \vec{y} \in S$
2. $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
4. $\exists \vec{0} \in S$ s.t. $\vec{x} + \vec{0} = \vec{x} \quad \forall \vec{x} \in S$
5. For every $\vec{x} \in S$, $\exists (-\vec{x} \in S)$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$
6. $c\vec{x} \in S$
7. $c(d\vec{x}) = (cd)\vec{x}$
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
9. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
10. $1\vec{x} = \vec{x}$

Info – Subspace Test

Let S be a **non-empty** subset of \mathbb{R}^n . If $\vec{x} + \vec{y} \in S$ and $c\vec{x} \in S \quad \forall \vec{x}, \vec{y} \in S$ and $c \in \mathbb{R}$, then S is a subspace of \mathbb{R}^n

Be aware that $\vec{0} \in S$

Examples:

1. Let $\vec{u}, \vec{w} \in \mathbb{R}^n$ and let $S = \text{Span}\{\vec{u}, \vec{w}\}$. Is S a subspace of \mathbb{R}^n ?

Proof

- Note that $\vec{0} \in S$ since $\vec{0} = 0\vec{u} + 0\vec{w} \quad \forall \vec{u}, \vec{w} \in \mathbb{R}^n$.
- Let $\vec{x}, \vec{y} \in S$, $\exists c_1, c_2, d_1, d_2 \in \mathbb{R}$ s.t. $\vec{x} = c_1\vec{u} + c_2\vec{w}$, $\vec{y} = d_1\vec{u} + d_2\vec{w}$. Then $\vec{x} + \vec{y} = c_1\vec{u} + c_2\vec{w} + d_1\vec{u} + d_2\vec{w} = (c_1 + d_1)\vec{u} + (c_2 + d_2)\vec{w}$.

Therefore $\vec{x} + \vec{y} \in S$

- Let $c \in \mathbb{R}$, then $c\vec{x} = c(c_1\vec{u} + c_2\vec{w}) = (cc_1)\vec{u} + (cc_2)\vec{w} \in S$

By Subspace Test, $S \in \mathbb{R}^n$

□

2. Let $\mathbb{S} = \left\{ \vec{x} \in \mathbb{R}^2 \mid \vec{x} = r \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}, r \in \mathbb{R} \right\}$. Is \mathbb{S} a subspace of \mathbb{R}^2 ?

False

Proof

For contradiction, assume $\vec{0} \in \mathbb{S}$, then $\vec{0} = r \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

That is $\begin{cases} r+3=0 \\ 2r+4=0 \end{cases} \implies r = -3 \wedge r = -2 \implies \text{contradiction. Since } -3 \neq -2, \text{ therefore } \vec{0} \notin \mathbb{S}$

Hence \mathbb{S} is not a subspace of \mathbb{R}^2

□

3. Let $\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + 2x_2 = 0 \right\}$. Is \mathbb{S} a subspace of \mathbb{R}^3 ?

True

Proof

- Note that $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $0 + 2(0) = 0$, so $\vec{0} \in \mathbb{S}$
- Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{S}$, $c \in \mathbb{R}$, then $x_1 + 2x_2 = 0$ and $y_1 + 2y_2 = 0$. Then $\vec{x} + \vec{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$ and $\vec{x}_1 + \vec{y}_1 + 2(\vec{x}_1 + \vec{y}_1) = \vec{x}_1 + \vec{2x}_2 + \vec{y}_1 + \vec{2y}_2 = 0 \implies \vec{x} + \vec{y} \in \mathbb{S}$
- $c\vec{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} \implies c(x_1 + 2x_2) = c \cdot 0 = 0 \implies c\vec{x} \in \mathbb{S}$

By Subspace Test, $\mathbb{S} \in \mathbb{R}^3$

□

Info – Theorem

If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then $\mathbb{S} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of \mathbb{R}^n

Note that the converse is also true. The proof will be given in CH5

Example:

$$\text{Find } \mathbb{S} = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a - b + c = d \right\}$$

1. An arbitrary element $\vec{x} \in \mathbb{S}$ has form

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ a-b+c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \\ -b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Therefore } \mathbb{S} = \text{Span } B \text{ where } B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2. We set $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\implies c_1 = c_2 = c_3 = 0$$

Thus B is linearly independent

Therefore B is a basis for \mathbb{S} , that is $\dim(B) = 3$ (since there are 3 vectors)

Dot Product

Info – Dot Product

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$ be vectors in \mathbb{R}^n . We defined their **dot product** by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^n u_i v_i$$

1. $\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$
2. $(\vec{w} + \vec{v}) \cdot \vec{w} = \vec{w} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(c\vec{w}) \cdot \vec{v} = c(\vec{w} \cdot \vec{v})$
4. $\vec{w} \cdot \vec{w} \geq 0$, with $\vec{w} \cdot \vec{w} = 0 \iff \vec{w} = 0$

Example:

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, then the dot product $\vec{x} \cdot \vec{y} = (1)(-1) + (2)(0) + (3)(1) + (4)(2) = 10$

Info – Norms

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$

1. The **length** of vector \vec{w} is $\|\vec{w}\| = \sqrt{\vec{w} \cdot \vec{w}} = \sqrt{w_1^2 + \dots + w_n^2}$
2. If $c \in \mathbb{R}$, $\vec{w} \in \mathbb{R}^n$, then $\|c\vec{w}\| = |c| \|\vec{w}\|$
3. \vec{v} is a **unit vector** if $\|\vec{v}\| = 1$
4. Properties of norm
 1. $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \iff \vec{v} = 0$
 2. $\|c\vec{v}\| = |c| \|\vec{v}\|$
 3. $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \|\vec{w}\|$
 4. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

Examples:

1. Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, then $\|\vec{x}\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$

2. Let $\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then $\|\vec{e}_i\| = 1$

This is an unit vector since all standard basis vectors are unit vectors

Info – Orthogonality/Angle

1. **Normalization** is when some \vec{v} is a non-zero vector,

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

in the direction of \vec{v} by scaling \vec{v}

2. With \vec{v}, \vec{w} non-zero vectors. The angle $\theta, 0 \leq \theta \leq \pi$ between \vec{v} is such that

$$\vec{w} \cdot \vec{v} = \|\vec{w}\| \|\vec{v}\| \cos \theta,$$

$$\theta = \arccos\left(\frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\| \|\vec{v}\|}\right)$$

3. $\vec{w}, \vec{v} \in \mathbb{R}^n$ are **orthogonal/perpendicular** if $\vec{w} \cdot \vec{v} = 0$

- Note that $\vec{0}$ is orthogonal to any vector

4. A set of vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ is called an orthogonal set if $\vec{v}_i \cdot \vec{v}_j = 0 \forall i \neq j$

- Any set of standard basis is an **orthonormal** set because the elements are orthogonal and their norm are all 1

The Cross Product in \mathbb{R}^3

Info – Cross Products

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{w} = \dots \in \mathbb{R}^3$.

The **cross product** of \vec{u}, \vec{v} is defined to be the vector in \mathbb{R}^3 given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Tip – Tricks for the Cross Product

- 1st coordinate: $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ Ignore row 1 \implies compute the difference of the diagonals $v_2w_3 - v_3w_2$
- 2nd coordinate: $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ Ignore row 2 \implies compute the difference of the diagonals $v_1w_3 - v_3w_1$
- 3rd coordinate: $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ Ignore row 3 \implies compute the difference of the diagonals $v_1w_2 - v_2w_1$

Warning – Cross Product

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$

$$\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$$

Example:

Given vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ we want to find a vector $\vec{n} \in \mathbb{R}^3$ which is orthogonal to both.

What is, we want $\vec{v} \cdot \vec{n} = \vec{w} \cdot \vec{n} = 0 \implies \vec{n} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$

Info – Linearity of the Cross Product

Let $c \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, then

1. If $\vec{n} = \vec{v} \times \vec{u}$ then for any $\vec{y} \in \text{Span } \{\vec{v}, \vec{w}\}$ we have $\vec{y} \cdot \vec{n} = 0$
2. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$
3. $\vec{v} \times \vec{v} = \vec{0}$
4. $\vec{v} \times \vec{w} = \vec{0}$ if and only if either $\vec{v} = \vec{0}$ or \vec{w} is a scalar multiple of \vec{v}
5. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
6. $\vec{u} \times c(\vec{v}) = c(\vec{u} \times \vec{v})$
7. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| |\sin \theta|$ where θ is the angle between \vec{u} and \vec{v}

CUTOFF

Scalar Equation of a plane

 **Info – Definition of Scalar Equation of a Plane** Let $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^3$ with $\{\vec{v}, \vec{w}\}$ being linearly independent and let P be a plane in \mathbb{R}^3 with vector equation $\vec{x} = s\vec{v} + t\vec{w} + \vec{b}$, $s, t \in \mathbb{R}$. If $\vec{n} = \vec{v} \times \vec{w}$, then an equation for the plane is

$$(\vec{x} - \vec{b}) \cdot \vec{n} = 0$$

 **Info – Normal Vector Scalar Equation**

Let P be a plane in \mathbb{R}^3 that passes through the point $B(b_1, b_2, b_3)$. If $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \in \mathbb{R}^3$ is a vector s.t.

$$n_1x_1 + n_2x_2 + n_3x_3 = b_1n_1 + b_2n_2 + b_3n_3$$

is an equation for P , then \vec{n} is called a **normal vector** for P . We can the equation a **scalar equation**

Example:

1. Find a scalar equation of the plane in \mathbb{R}^3 given by vector equation $\vec{x} = s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$

Let $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$

Therefore, by definition, the scalar equation for this plane is $x_1 - 5x_2 + 3x_3 = 5 - 5 + 6 \Rightarrow x_1 - 5x_2 + 3x_3 = 6$

2. Let P be the plane in \mathbb{R}^3 given by the scalar equation $2x_1 - 3x_2 + 407x_3 = \pi$. Find a normal vector to P .

By observation, we obtain that $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ 407 \end{bmatrix}$

Projections

Info – Projection

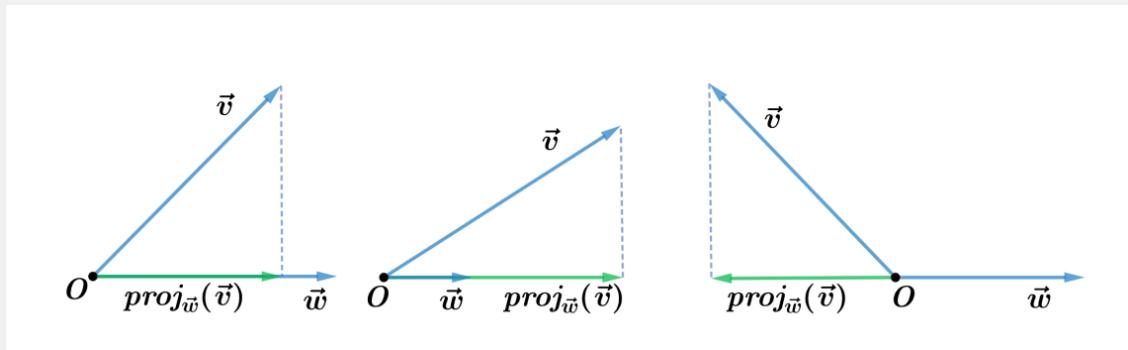
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq 0$.

1. The **projection** of \vec{v} onto \vec{w} is defined by

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

We also refer to this as the **projection of \vec{v} in the \vec{w} direction**

Illustration of $\text{proj}_{\vec{w}}(\vec{v})$:



2. We refer to the quantity

$$\|\vec{v}\| \cos \theta = \vec{v} \cdot \hat{w}$$

as the **component** (or scalar component) of \vec{v} along \vec{w}

3. The **perpendicular** of \vec{v} onto \vec{w} is defined by $\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$
4. The projection and the perpendicular of a vector \vec{v} onto \vec{w} are orthogonal; that is

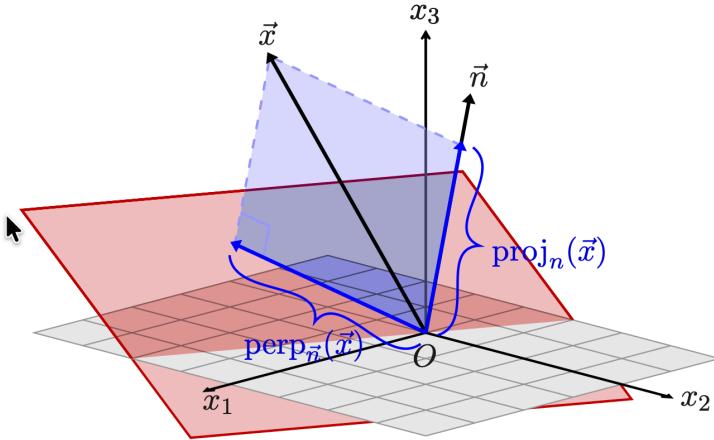
$$\text{perp}_{\vec{w}}(\vec{v}) \cdot \text{proj}_{\vec{w}}(\vec{v}) = 0$$

Example:

$$\text{Let } \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{Find } \text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$



We project instead on a plane, we project onto the normal vector \vec{n} . Then we use the fact that a normal vector is orthogonal to the plane P

Info – Projection onto Plane

Let P be a plane in \mathbb{R}^3 that passes through the origin and has normal \vec{n} .

The **projection** of $\vec{x} \in \mathbb{R}^3$ onto P is defined by

$$\text{proj}_{P(\vec{x})} = \text{perp}_{\vec{n}}(\vec{x})$$

The **perpendicular** of \vec{x} onto P is defined by

$$\text{perp}_{P(\vec{x})} = \text{proj}_{\vec{n}}(\vec{x})$$

Example:

Let P be the plane $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$, find $\text{proj}_P(\vec{u})$

$$\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{proj}_P(\vec{u}) = \text{perp}_{\vec{n}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{n}}(\vec{u}) = \vec{u} - \frac{\vec{u} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} - 7 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Complex Euclidean Space (TBD at the end of term)

Info – Vectors in \mathbb{C}^n

The set \mathbb{C}^n is defined as $\left\{ \vec{z} = \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix} : z_1, \dots, z_n \in \mathbb{C} \right\}$

The **vector** is an element $\vec{z} = \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix}$ of \mathbb{C}^n

In \mathbb{C}^n , let \vec{e}_i be the vector whose i^{th} component is 1 with all other components 0. The set $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called the **standard basis for \mathbb{C}^n**

Standard Inner Product in \mathbb{C}^n

Info – Standard inner product

Let $c \in \mathbb{C}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$

The **standard inner product** of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$ is

$$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w_1} + v_2 \overline{w_2} + \dots + v_n \overline{w_n}$$

1. $\langle \vec{u}, \vec{w} \rangle = \overline{\langle \vec{v}, \vec{w} \rangle}$
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3. $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$, with $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$
5. The length: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$
6. \vec{w}, \vec{v} are **orthogonal/perpendicular** if $\langle \vec{w}, \vec{v} \rangle = 0$
7. With $\vec{w} \neq 0$. The **projection of \vec{v} onto \vec{w}** is defined by

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \hat{w}$$