CH 5 - Applications of Derivatives

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Related Rates

> 💡 **Tip** — Steps for Related Rates Questions
>
> 1. Draw diagram
> 2. Identitfy **changing** quantities
> 3. Find **constant** quantities (if possible)
> 4. Derive equations relating the quantities that are changing
> 5. **Implicitly differentiate** the key equations
> 6. Solve for the desired rate of change, substituting in known quantities.
> 7. **Concluding statement** (and also check units)

Example:

1. Laindon is taking a hot air ballon ride. A giant fan is blowing hot air into the ballon in a rate of $200\frac{\text{m}^3}{\text{min}}$. Assuming that at any given point in time the balloon sphere, find the rate at which the radius of the balloon is changing when the diameter is 12 m.

   ANS:

   1. Picture: The problem is trivial so the graph is omitted

   2. Changing variable: Volume(m^3), Radius(m), time(t)

   3. Constant quantities: $\frac{\mathrm{d}V}{\mathrm{d}t} = 200\frac{\text{m}^3}{\text{min}}$

   4. Key Equation: $V = \frac{4}{3}\pi r^3(t)$

   5. Implicit Differentiation: $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2(t) \cdot \frac{\mathrm{d}r}{\mathrm{d}t}$

   6. $\frac{\mathrm{d}r}{\mathrm{d}t}\big|_{r=6} = \frac{1}{4\pi(6)^2} \cdot 200 = \frac{200}{144\pi} = \frac{25}{18\pi}\frac{\text{m}}{\text{min}}$

   7. Concluding statement: When the diameter of the balloon is 12m, the rate of change of the radius is expanding by $\frac{25}{18\pi}\frac{\text{m}}{\text{min}}$

2. The construction workers building M4 accidentally left a 20 foot ladder propped up against a concrete wall that is 80 feet in height. The base of the ladder begins to slide away from the wall at a rate of 2ft/sec, and the top begins to move down as a result. When the base of the ladder is 14 ft from the wall, how fast is the top of the ladder sliding down the wall?

   ANS:

   1. Picture is omitted and left as an exercise for the reader

   2. Changing variable: Distance from wall of base of ladder (m), Heing where ladder touches the wall (m)

   3. Constant quantities : $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$

   4. Key Equation: $x^2 + y^2 = 20^2$

5. Implicit Differentiation: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

6. $\frac{dy}{dt} = -\frac{14}{\sqrt{400-14^2}} \cdot 2 = -\frac{14}{\sqrt{51}}\frac{\text{ft}}{\text{sec}}$

7. Concluding statement: When the base of ladder is 14cm, the top of the ladder is falling at a speed of $\frac{14}{\sqrt{51}}\frac{\text{ft}}{\text{sec}}$

Extrema

> 🐋 **Info − Extrema**
>
> Let $f(x)$ be a funciton defined on an interval I, and let $c \in I$. We say f has
>
> 1. A **global minimum** on I at $x = c$ if $f(c) \leq f(x) \forall x \in I$
> 2. A **global maximum** on I at $x = c$ if $f(c) \geq f(x) \forall x \in I$
> 3. A **global extremum** on I at $x = c$ if f has either a global minimum or global maximum.
>
> - Every point on a constant function is both a global minimum and global maximum
> - Every global extremum can be a local extremum in some interval

Examples:

1. Find all global extrema of $f(x) = x^2$ on $[0, 1)$
   - The global minimum be $x = 0$ because $f(0) \leq f(x) \; \forall x \in [0, 1)$
   - The global maximum DNE as the end point is missing. That is infinitely numbers lie on the interval $[0, 1]$

2. Find all global extrema of $f(x) = \frac{1}{x}$ on $[-1, 1]$

   - The global extrema DNE as $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

> 🐋 **Info − Extreme Value Theorem (Existence Thm)**
>
> Assume that $f(x)$ is continuous on the closed interval $[a, b]$. Then **there exist** two numbers $c_1, c_2 \in [a, b]$ s.t. $f(c_1) \leq f(x) \leq f(c_2) \; \forall x \in [a, b]$.
>
> In other words, there is a global minimum at $x = c_1$ and a global maximum at x_{c_2}

> 🐋 **Info − Local Extrema**
>
> Let f be a function. We say that f has
>
> 1. a **local minimum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \leq f(x) \; \forall x \in (a, b)$
> 2. a **local maximum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \geq f(x) \; \forall x \in (a, b)$
> 3. a **local extremum** at $x = c$ if there is either a local minimum or a loacl maximum

> ⚠️ **Warning −** Local Extrema
>
> If c is an endpoint of the domain of f, c can never be a local extremum, even if it is a global extremum

> 🐳 **Info − Fermat's Theorem**
>
> If there is a local extremum for $f(x)$ at $x = c$ and $f'(c)$ exists, hence $f'(c) = 0$. That is we cannot put an open interval around the point.

Examples:

1. Does the converse of Fermat's Theorem hold? That is if $f'(0) = 0$, then is a local extremum at $x = c$.

   This is false. Let $f(x) = x^3$, $f'(x) = 3x^2$, $f'(0) = 0$ but is not a local extremum on any interval containing $x = 0$

2. Why is it worth mentioning $f'(c)$ has to exist?
   It is important because it is like saying $f(x)$ is differentiable at $x = c$. If not, let $f(x) = |x|$. $f(x)$ is continuous. It has a local minimum at $x = 0$ but $f'(0)$ DNE as it is not differentiable.

> 🐳 **Info − Critical Points**
>
> We say that a function f has a **critical point** at $x = c$ if $f'(c) = 0$ or $f'(c) =$ DNE for $c \in$ the domain of f. These are our candidates for local extrema.

> 💡 **Tip − Closed Interval Method**
>
> Let $f(x)$ be continuous function on $[a, b]$.
> 1. Calculate $f(a)$ and $f(b)$
> 2. Find $f'(x)$
> 3. Find all the critical points of f on $[a, b]$
> 4. Calculate $f(c)$

Example:

$f(x) = \frac{1}{3}x^3 - 3\sqrt[3]{x}$ on $[-8, 1]$

$f(-8) = -\frac{512}{3} - 3(-2) = -\frac{496}{3}$
$f(1) = \frac{1}{3} - 3 = -\frac{8}{3}$

$f'(x) = x^2 - x^{-\frac{2}{3}}$

$f'(c) = 0 \implies c^2 - c^{-\frac{2}{3}} = 0 \implies c^{\frac{8}{3}} = 1 \implies c = -1, 1$. $f'(c) =$ DNE $\implies c = 0$

$f(0) = 0$

$f(-1) = -\frac{1}{3} + 3 = \frac{8}{3}$ Global maximum at $x = -1$, global minimum at $f(-8)$

> 🐳 **Info − Rolle's Theorem (Existence Thm)**
>
> If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = k \in \mathbb{R}$, then there exists $c \in (a, b)$ s.t. $f'(c) = 0$

Proof

If $f(x) = k \; \forall x \in [a, b]$, any value of c works.

Otherwise, $\exists x_0 \in [a, b]$ s.t. $f(x_0) \neq k$. Since f is continuous on $[a, b]$, it attains a maximum/minimum on $[a, b]$.

Since $f(x_0) \neq k \implies f(x_0) > k \leftrightarrow f(a), f(b)$ are not maximum, or $f(x_0) < k \leftrightarrow f(a), f(b)$ are not minimum. So one of maximum or minimum is in (a, b), thus differentiable at some c.

By Fermat's Theorem, $f'(c) = 0$ or $f'(c) = $ DNE. But f is differentiable on $(a, b) \implies f'(c)$ exists.

$\therefore f'(c) = 0$

> 🐳 **Info — Mean Value Theorem (Existence Thm)**
>
> If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t.
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

Let $h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]$

$h(a) = f(a) - \left[f(a) + \frac{f(b) - f(a)}{b - a} \cdot (a - a) \right] = 0$

$h(b) = f(b) - \left[f(a) + \frac{f(b) - f(a)}{b - a} \cdot (b - a) \right] = 0$

Since $h(b) = h(a) \overset{\text{Rolle's Theorem}}{\implies} \exists c \in (a, b)$ s.t. $h'(c) = 0$

That is $h'(x) = f'(x) \frac{f(b) - f(a)}{b - a} \implies f'(x) = \frac{f(b) - f(a)}{b - a}$

Finally, $h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \leftrightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

Tip: the construction of $\left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]$ is the linear approximation of $f(x)$ near a

Antiderivative

> 🐳 **Info — Antiderivative**
>
> Given a function $f(x)$, an **antiderivative** is a function $F(x)$ s.t. $F'(x) = f(x)$. If $F'(x) = f(x)$ for all $x \in I$ for some interval I, then $F(x)$ is an antiderivative of $f(x)$ on I
>
> e.g. $\frac{d}{dx} - \ln(\cos x)) = -\frac{1}{\cos}x \sin x = -\frac{-\sin x}{\cos x} = \tan x$
>
> Note: one function can have infinitely many antiderivatives, that is why we insist **an antiderivative** of $f(x)$

> 🐳 **Info — Constant Function Theorem**
>
> Suppose that $f'(x) = 0 \; \forall x \in I$ for some interval I. Then $\exists \alpha \in \mathbb{R}$ s.t. $f(x) = \alpha \; \forall x \in I$

Proof

Let $x_1 < x_2 \in I$.

Since f is differentiable on I, it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

By MVT, $\exists c \in (x_1, x_2)$ s.t. $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ But $f'(c) = 0$ since $f'(x) = 0$ on I.

Thus, $0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \wedge x_2 - x_1 \neq 0 \implies f(x_2) - f(x_1) = 0 \iff f(x_2) = f(x_1)$.

Since x_1, x_2 are arbitrary, therefore f is constant on I

🐳 **Info − Antiderivative Theorem**

Suppose that $F'(x) = G'(x) \forall x \in I$ for some interval I. Then there exists $\alpha \in \mathbb{R}$ s.t. $F(x) = G(x) + \alpha \forall x \in I$

Proof

Let $h(x) = F(x) - G(x)$. Then $h'(x) = F'(x) - G'(x) = 0$ on I.

By the CFT, $h(x) = \alpha$ for some $\alpha \in \mathbb{R}$, so $F(x) - G(x) = \alpha \implies F(x) = G(x) + \alpha \quad (\forall x \in I)$

First Derivatives

🐳 **Info − Definition of Increasing/Decreasing**

Let I be in interval and $x_1, x_2 \in I$, then f(x)
- **increasing** on I if $f(x_1) \leq f(x_2) \ \forall x_1 < x_2$
- **decreasing** on I if $f(x_1) \geq f(x_2) \ \forall x_1 < x_2$
- **strictly increasing** on I if $f(x_1) < f(x_2) \ \forall x_1 < x_2$
- **strictly decreasing** on I if $f(x_1) > f(x_2) \ \forall x_1 < x_2$

Note: a constant function is both increasing and decreasing but not strictly

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🐳 **Info − Increasing/Decreasing Function Theorem**

Let I be an interval

1. If $f'(x) \geq 0 \ \forall x \in I$, then $f(x)$ is increasing on I
2. If $f'(x) > 0 \ \forall x \in I$, then $f(x)$ is strictly increasing on I
3. If $f'(x) \leq 0 \ \forall x \in I$, then $f(x)$ is decreasing on I
4. If $f'(x) < 0 \ \forall x \in I$, then $f(x)$ is strictly decreasing on I

Proof

Let $x_1 < x_2 \in I$. $f'(x) > 0$ on I, so it exists on $I \implies f$ is differentiable on (x_1, x_2), continuous on $[x_1, x_2]$.

By MVT, $\exists c \in (x_1, x_2)$ s.t. $f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Given $f'(x) > 0 \ \forall x \in I$ and $x_1 < x_2$ and $f(x_1) < f(x_2)$

Since x_1, x_2 are arbitrary, thus f is strictly increasing on I

Proof for increasing, strictly decreasing and decreasing is similar thus be omitted.

Question: If f is strictly increasing on $I \implies f'(x) > 0 \ \forall x \in I$?

ANS: No, counterexample $f(x) = x^3$

Question: If f is strictly decreasing on $I \implies f'(x) < 0 \ \forall x \in I$?

ANS: No, counterexample $f(x) = -\sqrt[3]{x}$

> 🐋 **Info − Bounded Derivative Theorem**
>
> Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).
>
> Suppose that $m \leq f'(x) \leq M \ \forall x \in (a, b)$. Then
>
> $$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$
>
> $\forall x \in [a, b]$

Proof

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).

It also applies to $[a, x_1]$.

Case1:

For $x_1 \in (a, b]$

By MVT, $\exists \ c \in (a, x_1)$ s.t. $f'(c) = \frac{f(x_1) - f(a)}{x_1 - a}$

Since $m \leq f'(c) \leq M$ so $m(x_1 - a) \leq f(x_1) - f(a) \leq M(x_1 - a)$.

Then $m(x_1 - a) + f(a) \leq f(x_1) \leq M(x_1 - a) + f(a)$

Case 2:

When $x = a, m(x - a) + f(a) = f(a)$ and similar applies to $M(x - a)$.

Resulting to $f(a) \leq f(a) \leq f(a)$

Example:

Prove that $\sqrt{51} \in \left[7 + \frac{1}{8}, 7 + \frac{1}{7}\right]$.

Let $f(x) = \sqrt{x}$ and let $[a, b]$ be [49,64]

Since $f(x)$ is continuous on $[49, 64]$ and differentiable on $[49, 64]$

By Bounded Derivative Theorem, $f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$

where $m \leq f'(x) \leq M$ on $[49, 64]$ $\forall x$.

$f'(x) = \frac{1}{2\sqrt{x}}, \frac{1}{\sqrt{64}} \leq \frac{1}{2\sqrt{x}} \leq \frac{1}{2\sqrt{49}} \implies 7 + \frac{1}{2} \cdot \frac{1}{8}(51 - 49) \leq \sqrt{51} \leq \sqrt{49} + \frac{1}{2} \cdot \frac{1}{7}(51 - 49)$

$\implies 7 + \frac{1}{8} \leq \sqrt{51} \leq 7 + \frac{1}{7}$

> 🐋 **Info − Comparison via First Derivative Theorem**
>
> Assume $f(x)$ and $g(x)$ are continuous at $x = a$ with $f(a) = g(a)$. Then
>
> - f and g are differentiable for $x > a$ and $f'(x) \leq g'(x)$ $\forall x > a \implies f(x) \leq g(x)$ $\forall x > a$
>
> - f and g are differentiable for $x < a$ and $f'(x) \leq g'(x)$ $\forall x < a \implies f(x) \geq g(x)$ $\forall x < a$

Example:

Show that $x - \frac{1}{2}x^2 < \ln(1 + x)$ $\forall x > 0$

Let $f(x) = x - \frac{1}{2}x^2, g(x) = \ln(1 + x)$

At $x = 0, f(0) = 0, g(0) = 0$, that is $f(0) = g(0)$

All functions are differentiable on $(0, \infty)$

$f'(x) = 1 - x, g'(x) = \frac{1}{1+x}$.

Since $x > 0, (1 - x)(1 + x) < 1 \implies 1 - x^2 < 1 \implies -x^2 < 0$

Thus by CFDT, $x - \frac{1}{2}x^2 < \ln(1 + x)$ $\forall x > 0$

Second Derivatives

> 🐋 **Info − Concavity and Second Derivative Theorem**
>
> 1. We say f is **concave up** on an interval I if for all $a, b \in I$, the secant line between $(a, f(a))$ and $(b, f(b))$ lies **above** the graph of $f(x)$.
>
> 2. We say f is **concave down** on an interval I if for all $a, b \in I$, the secant line between $(a, f(a))$ and $(b, f(b))$ lies **below** the graph of $f(x)$.
>
> 3. If $f''(x) > 0$ $\forall x$ in an inverval I then $f(x)$ is **concave up** on I
>
> 4. If $f''(x) < 0$ $\forall x$ in an inverval I then $f(x)$ is **concave down** on I

Example:

$f(x) = |x|$ is neither concave up nor concave down as the secant line lies on the graph if a and b are same side of the absolute function.

> 🐳 **Info — Point of Inflection**
>
> A point $(c, f(x))$ is called a **point of inflection** of $f(x)$ if $f(x)$ is continuous at $x = c$ and the concavity of $f(x)$ changes at $x = c$
>
> If $f''(x)$ is continuous at c and $(c, f(c))$ is a point of inflection, then $f''(x) = 0$ or DNE

However, the converse is false: $f(x) = x^4$, $f''(0) = 0$, but the concavity does not change

Examples:

1. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$

$f''(x) < 0 \; \forall x \in (-\infty, 0)$, concave down

$f''(x) > - \; \forall x \in (0, \infty)$, concave up

$f''(x) = 0 \Longrightarrow 6x = 0 \Longrightarrow x = 0.$

2. $g(x) = \frac{1}{x}$, $g'(x) = -\frac{1}{x^2}$, $g''(x) = \frac{2}{x^3}$.

$g''(x) < 0 \; \forall x \in (-\infty, 0)$, concave down

$g''(x) > - \; \forall x \in (0, \infty)$, concave up

$g''(x)$ is discontinuous at $x = 0$, therefore, $x = 0$ is not an infleciton point.

> ⚠️ **Warning —** If a function is concave down before $x = c$, and concave up after $x = c$, it is not necessary that there exists an infleciton point. Notably: $f(x) = \frac{1}{x}$

Derivative Tests

> 🐳 **Info — First Derivative Test**
>
> Let $f(x)$ has a critical point at $x = c$ and suppose that $f(x)$ is continuous at c. If there is an interval (a, b) containing c s.t.
>
> 1. $f'(x) \geq 0$ on (a, c) and $f'(x) \leq 0$ on (c, b), then f has a local maximum at c
> 2. $f'(x) \leq 0$ on (a, c) and $f'(x) \geq 0$ on (c, b), then f has a local minimum at c
>
> Otherwise c is neither a local maximum nor a local minimum

> 🐳 **Info — Second Derivative Test**
>
> Suppose that $f'(c) = 0$ and $f''(x)$ is continuous at c, then
>
> 1. if $f''(c) < 0$, then there is a local maximum for f a c
> 2. if $f''(c) > 0$, then there is a local minimum for f at c
> 3. if $f''(x) = 0$, then it is inconclusive, that is, there might be a local maximum, a local minimum, or neither.

Comparison:

FDT:
- Requires an interval, that is points around c
- Requires lesser steps of differentiation
- Conclusive as long as the constraints are satisfied

SDT:
- Requires an interval, that is the point at c
- Requires more steps of differentiation
- Inconclusive in some certain cases $f''(x) = 0$

Examples:

1. Classify all critical points of $f(x) = x^3 - 13x + 12$

   $f'(x) = 3x^2 - 13, f''(x) = 6x$

   $f'(x) = 3x^2 - 13 \implies f'(x) \geq 0 \ \forall x \in \left(-\infty, -\sqrt{\frac{13}{3}}\right] \cup \left[\sqrt{\frac{13}{3}}, \infty\right)$

   and $f'(x) \leq 0 \ \forall x \in [-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}}$

   $f'(0) < 0, f'(3) > 0, f'(-3) > 0$

   That is, by FDT, we have a local maximum at $x = -\sqrt{\frac{13}{3}}$ and a local minimum at $x = \sqrt{\frac{13}{3}}$

   $f''(x) = 6x, f''(x) < 0 \ \forall x \in (-\infty, 0), f''(x) > - \ \forall x \in (0, \infty)$.

   By SDT, we have a local maximum at $x = -\sqrt{\frac{13}{3}}$ and a local minimum at $x = \sqrt{\frac{13}{3}}$

2. Find all extrema of $f(x) = x\sqrt[3]{x - 4}$ on interval $[0, 5]$

   $f'(x) = \sqrt[3]{x - 4} + \frac{1}{3}x(x - 4)^{-\frac{2}{3}}$

   $f''(x) = \frac{1}{3}(x - 4)^{-\frac{2}{3}} + \frac{1}{3}(x - 4)^{-\frac{2}{3}} - \frac{2}{9}(x - 4)^{-\frac{5}{3}} = \frac{2}{3}(x - 4)^{-\frac{2}{3}}\left(1 - \frac{x}{3(x-4)}\right)$

   $f'(4) = \text{DNE}, f'(x) = 0 \iff 1 + \frac{x}{3(x-4)} \iff x = 3$

   $f'(0) < 0, f'(5) > 0, f'(3.5) > 0$

   By FDT, $x = 3$ is a local minimum

   SDT is inconclusive as $x = 4$ as $f''(4) = \text{DNE}$

   $f(0) = 0, f(3) = -3, f(4) = 0, f(5) = 5$

   The global maximum is at $x = 5$ and global minimum at $x = 3$ on the interval $[0, 5]$

L'Hôpital Rule

> 🐳 **Info — Indeterminate Forms**
>
> 1. $\frac{0}{0}$
>
> 2. $\frac{\pm\infty}{\pm\infty}$
>
> 3. $0 \cdot \pm\infty$
>
> 4. $\infty - \infty$
>
> 5. 1^∞
>
> 6. ∞^0
>
> 7. 0^0
>
> These forms signal to apply L'Hôpital.
>
> 1 and 2 are classic form, 3 to 7 need to be manipulated into classical form to apply L'Hôpital

> 🐳 **Info — L'Hôpital Rule**
>
> Suppose that $f'(x), g'(x)$ exists near a, except possibly at $x = a$, and that $g'(x) \neq 0$ near a, except possibly at $x = a$.
>
> Suppose that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then
>
> $$\lim(x \to a)\frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$
>
> whenever the right side exists or equals $\pm\infty$
>
> You need to write $\overset{\text{LHR}}{=}$ just for the sake of this course

> ⚠️ **Warning — Case do not use LHR**
>
> $\lim_{x \to 0} \frac{\sin x}{x} \overset{\text{LHR}}{=} \lim_{x \to 0} \cos x = 0$ **WRONG, CIRCULAR LOGIC, REMEMBER FUNDAMENTAL TRIG LIMIT**

> ⚠️ **Warning — Determinate Form**
>
> $0^\infty, \frac{0}{\infty}$, and ∞^∞ are determinate forms

Examples:

1. $\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x} \overset{\text{LHR}}{=} \lim_{x \to \pi} = 2\frac{x}{\cos x} = 2\frac{\pi}{-1} = -2\pi$

2. $\lim_{x \to \infty} \frac{\ln x}{x} \overset{\text{LHR}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \frac{1}{x} = \frac{1}{\infty} = 0$

3. $\lim_{x \to 0^+} \frac{\frac{1}{x^2}}{\ln x} \overset{\text{LHR}}{=} \lim_{x \to 0^+} \frac{-\frac{2}{x^3}}{\frac{1}{x}} = \lim_{x \to 0^+} = -\frac{2}{x^2} = -\frac{2}{0} = -\infty$

> 🐳 **Info** $- 0 \cdot \pm\infty$
>
> $$\lim_{x\to a} f(x)g(x) \overset{0\cdot\pm\infty}{=} \lim_{x\to a} \frac{\frac{1}{f(x)}}{g(x)} \overset{\text{LHR}}{\underset{\pm\infty}{\pm\infty}} \dots$$

Examples:

1. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} = \frac{\ln x}{\frac{1}{x}} \overset{\text{LHR}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\frac{x^2}{x} = 0$

2. $\lim_{x\to -\infty} x^{\frac{5}{3}} \cdot e^x = \lim_{x\to -\infty} \frac{x^{\frac{5}{3}}}{e^{-x}} \overset{\text{LHR}}{=} \lim_{x\to -\infty} \frac{5\frac{x^{\frac{2}{3}}}{3}}{-e^{-x}} \overset{\text{LHR}}{=}$

$\lim_{x\to -\infty} \frac{10\frac{x^{-\frac{1}{3}}}{9}}{e^{-x}} = \frac{10}{9} \lim_{x\to -\infty} \frac{e^x}{x^{\frac{1}{3}}} = \frac{0}{-\infty} = 0$

> 🐳 **Info** $- \infty - \infty$
>
> $\lim_{x\to a} f(x) - g(x)$ needed to be alegraically manipulated using one of two
>
> 1. Using ln
>
> 2. Using conjugate/fractions

Examples:

1. $\lim_{x\to \infty} \ln x + \ln\left(\frac{67}{x+1}\right) = \lim_{x\to \infty} \ln(67\frac{x}{x+1} = \ln\left(\lim_{x\to \infty} 67\frac{x}{x+1}\right) = \ln 67$

2. $\lim_{x\to 0^+} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0^+} \frac{x - \sin x}{x \sin x} \overset{\text{LHR}}{=} \lim_{x\to 0^+} \frac{1-\cos x}{\sin x + x \cos x} \overset{\text{LHR}}{=} \lim_{x\to 0^+} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0$

> 🐳 **Info** $- 1^\infty, \infty^0, 0^0$
>
> $\lim_{x\to a} f^{g(x)}(x) = e^{\lim_{x\to \infty} g(x)\ln(f(x))}$
>
> Then evaluate $\lim_{x\to \infty} g(x)\ln(f(x)) = L$ the final answer of the entire limit is e^L

Examples:

1. $\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to \infty} e^{x \ln(1+\frac{1}{x})} = e^{\lim_{x\to \infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} \overset{\text{LHR}}{=}$

$e^{\lim_{x\to \infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot -\frac{1}{x^2}}{-\frac{1}{x^2}}} = e^{\lim_{x\to \infty} \frac{1}{1+\frac{1}{x}}} = e^1 = e$

2. $\lim_{x\to 0^+} x^{\frac{1}{\ln(-\ln x)}} = e^{\lim_{x\to 0^+} \frac{\ln x}{\ln(-\ln x)}} \overset{\text{LHR}}{=} e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{-\ln x}\left(-\frac{1}{x}\right)}} = e^{\lim_{x\to 0^+} \ln x} = e^{-\infty} = 0$

Curve Sketching

> 🐳 **Info — Graph Sketching Procedure**
>
> 1. Determine the **domain** of the functions and the values at the endpoints
>
> 2. Find the x, y **intercepts**
>
> 3. Find the **horizontal asymptotes** by checking $\lim_{x \to \pm\infty} f(x)$
>
> 4. Find the **vertical asymptotes** by computing $\lim_{x \to a^{\pm} f(x)}$
>
> 5. Find all the **critical points** $f'(x) = 0$ or $f'(x) = \text{DNE}$
>
> 6. Find all candidates for **points of inflection** $f''(x) = 0$ or $f''(x) = \text{DNE}$
>
> 7. Find the shape of the function via intervals of increase/decrease and concavity between the points form steps 5-6 plus discontinuous
>
> 8. Find local **extrema** and points of inflection using the information from step 7
>
> 9. **Plot** the x-intercept, y-intercepts, point of inflection, critical points, and extrema
>
> 10. Connect everything together

Examples:

1. $f(x) = \frac{x^2-1}{x^2+3x}$, $f'(x) = \frac{3x^2+2x+3}{x^2(x+3)}$, $f''(x) = \frac{-6(x+1)(x^2+3)}{x^3(x+3)^3}$

   1. Domain is $x \in (-\infty, -3) \cup (-3, 0) \cup (0, \infty)$

   2. • The y-intercept DNE as 0 is not in domain

      • x-intercept at $x = \pm 1$

   3. • $\lim_{x \to \infty} \frac{x^2-1}{x^2+3x} = 1$

      • $\lim_{x \to -\infty} \frac{x^2-1}{x^2+3x} = 1$

   4. • $\lim_{x \to 0^+} \frac{x^2-1}{x^2+3x} = -\frac{1}{0} = -\infty$

      • $\lim_{x \to 0^-} \frac{x^2-1}{x^2+3x} = -\frac{1}{-0} = \infty$

      • $\lim_{x \to 3^+} \frac{x^2-1}{x^2+3x} = -\frac{1}{0} = -\infty$

      • $\lim_{x \to 3^-} \frac{x^2-1}{x^2+3x} = -\frac{1}{-0} = \infty$

   5. $f'(x) = \frac{3x^2+2x+3}{x^2(x+3)} = 0$

      • $3x^2 + 2x + 3 = 0$, no real solution

      • $x^2(x+3) = 0$, $x = 0, -3$ but not in domain

      • no critical points

   6. $f''(x) = \frac{-6(x+1)(x^2+3)}{x^3(x+3)^3} = 0$

      • $-6(x+1)(x^2+3) = 0$, $x = -1$

      • $x^3(x+3)^3 = 0$, $x = 0, -3$

      • $x = -1$ is inflection point

7. Increasing/Decreasing

$f'(1) > 0 \implies$ increasing

$f'(-1) > 0 \implies$ increasing

$f'(-4) > 0 \implies$ increasing

Concavity

$f''(1) =< 0 \implies$ concave down

$f''\left(-\frac{1}{2}\right) > 0 \implies$ concave up

$f''(-2) < 0 \implies$ concave down

$f''(-4) > 0 \implies$ concave up

8.

No local extrema

Point of inflection at $x == 1$
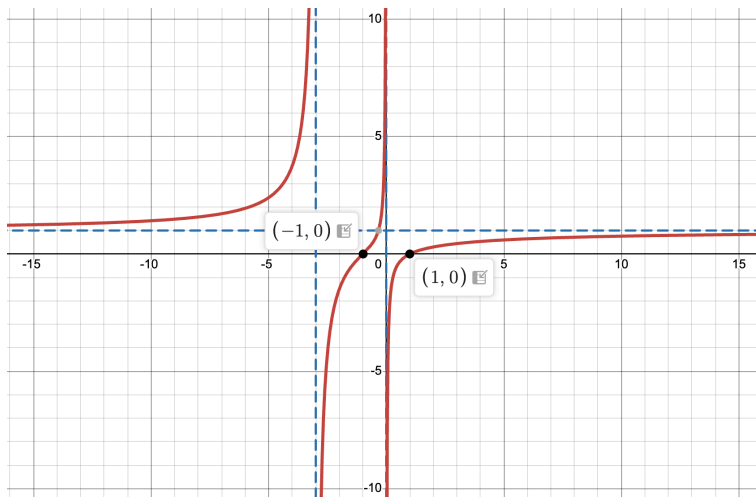
9. Vertical asymptotes at $x = -3, x = 0$

Horizontal asymptote at $y = 1$

Increasing on $(-\infty, -3) \cup (-3, 0) \cup (0, \infty)$

Concave up on $(-\infty, -3) \cup [-1, 0)$

Concave down on $(-3, -1] \cup (0, \infty)$

10.

2. $f(x) = \frac{e^x(x-2)}{x^2-2x}, f'(x) = \frac{e^{x(x-1)}(x-2)}{x^3-2x^2}, f''(x) = \frac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3}$

1. Domain $x \in (-\infty, 0) \cup (0, 2) \cup (2, \infty)$

2.

   - The y-intercept DNE as 0 is not in domain
   - The x-intecerpt DNE as $f(x)$ has no solution

3. Horizontal asymptote

   - $\lim_{x \to \infty} \frac{e^x(x-2)}{x^2-2x} = \frac{e^x}{x} = \infty$

   - $\lim_{x \to -\infty} \frac{e^x(x-2)}{x^2-2x} = \frac{e^x}{x} = \frac{0}{\infty} = 0$

4. Vertical asymptotes

   - $\lim_{x \to 2} \frac{e^x(x-2)}{x^2-2x} = \frac{e^2}{2}$ is a removable discontinuity

   - $\lim_{x \to 0^+} \frac{e^x(x-2)}{x^2-2x} \infty$

   - $\lim_{x \to 0^-} \frac{e^x(x-2)}{x^2-2x} - \infty$

5. $f'(x) = \frac{e^x(x-1)(x-2)}{x^3-2x^2} = 0$

   $x = 0, 1, 2$

   Critical point at $x = 1$

   DNE at $x = 0, 2$

6. $f''(x) = \frac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3} = 0$

   $x^2 - 2x + 2$ has no solutions

   No point of inflection as $x = 0, 2$ are not in domain

7. Increasing/Decreasing

   $f'(-1) =< 0 \implies$ decreasing

   $f'\left(\frac{1}{2}\right) < 0 \implies$ decreasing

   $f'\left(\frac{3}{2}\right) > 0 \implies$ increasing

   Concavity

   $f''(-1) = \frac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3} < 0 \implies$ concave down

   $f''(1) = \frac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3} > 0 \implies$ concave up

8. Local minimum at $x = 1$

   No point of inflection

9. Decreasing on $(-\infty, 0), (0, 1]$ as an arbitrary point on left inverval and right interval will be increasing
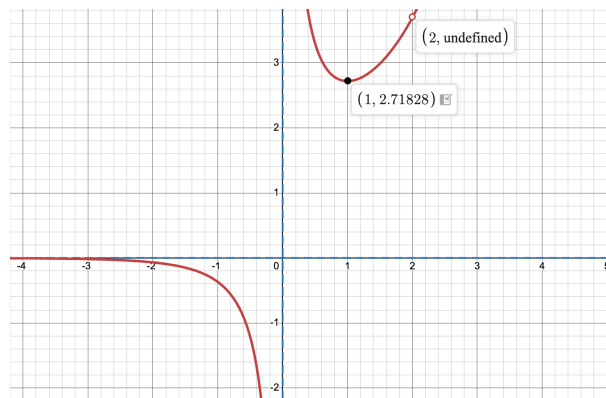
   Increasing on $[1, 2) \cup (2, \infty)$

   Concave up on $(0, 2) \cup (0, \infty)$

   Concave down on $(-\infty, 0)$

   Vertical asymptotes at $x = 0$

   Horizontal asymptote at $y = 0$

10.

(2, undefined)

(1, 2.71828)

Optimization

Example:

Minimize $x^2 + y^2$ given $x - y = 10$

ANS: $x^2 + y^2, y = x - 10 \implies x^2 + (x - 10)^2 = x^2 + y^2$

$f(x) = x^2 + (x - 10)^2 = 10, f'(x) = 2x + 2(x - 10) = 0 \implies x = 5$

$f'(4) < 0, f'(6) > 0$, by FDT, $x = 5$ is a local minimum

$5 - y = 10, y = -5 \implies 5^2 + (-5)^2 = 50$

> 🐳 **Info − Optimization**
> 1. Determine maximizing or minimizing
> 2. Identitfy constraints
> 3. Draw diagram
> 4. Create objective function
> 5. Rewrite objective function using constraing in single variable
> 6. Differentiation with FDT or SDT
> 7. Ending statement
>
> Acronym: divorce

Examples:

1. A rectangular garden is to be set up with a river on one side and fencing on the other three sides. If the area of the garden must be 200 m^2, determine the dimensions that will minimize the fencing required.

   1. Minimize perimeter of fence

   2. $xy = 200 \implies y = \frac{200}{x}$

   3. Picture is a three sided rectangle with one side is next to rive

   4. $2x + y$

   5. $2x + \frac{200}{x}$

   6. $f'(x) = 2 - \frac{200}{x^2}$

      $f'(x) = 0 \implies x = 10$

      $f'(5) = 2 - \frac{200}{25} = 2 - 8 = -6$

      $f'(20) = 2 - \frac{200}{400} = 2 - \frac{1}{2} = \frac{3}{2}$

      By FDT, $x = 10$ is a global minimum

   7. $y = \frac{200}{10} = 20$

   The dimension of the fence is $10m \cdot 20m$

2. Peter's Packaging is working on designing a new, cutting-edge box. Peter takes a piece of cardboard, which has dimensions 25 cm by 10 cm, and cuts out squares from each corner. He then folds the net he has made to create an open-top box. What side length of square should Peter cut from each corner to maximize the volume of the resulting box?

   1. Maxmize volume of the box

   2. 25 cm and 10 cm

   3. s is the side length of small squares and height of volume

   4. lws

   5. $(10 - 2s)(25 - 2s)s = (250 - 70s + 4s^2)s = 250s - 70s^2 + 4s^3$

   6. $f'(x) = 12s^2 - 140s + 250$
      $f'(x) = 2(6s^2 - 70s + 125) = 0 \implies x = \frac{35 \pm 5\sqrt{19}}{6}$
      $x = \frac{35 - 5\sqrt{19}}{6}$ is the only working point as $10 - 2s > 0$
      $f'(1) = 61 > 0$
      $f'(3) = -31 < 0$
      By FDT, $x = s = \frac{35 - 5\sqrt{19}}{6}$ is local maximum

   7. The side length of the square is $\frac{35 - 5\sqrt{19}}{6}$ cm

3. An island is 2 km due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 km west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 km/h and swims at a rate of 3 km/h. How far should the visitor run before swimming to minimize the time it takes to reach the island?

   1. Minimize travel time

   2. 2km north, 6km east, run: 8km/h, swim: 3km/h

   3. Right triangle with 2km height and $6 - r$ km base
      r is the distance of running and s is the distance of swimming

   4. $\frac{r}{8} + \frac{s}{3}$

   5. $\frac{r}{8} + \frac{\sqrt{(6-r)^2 + 4}}{3}$

   6. $f'(r) = \frac{1}{8} + \frac{r-6}{3(40 - 12r + r^2)^{\frac{1}{2}}} \implies r = 6 \pm \frac{6}{\sqrt{55}} \implies 6 - \frac{6}{\sqrt{55}}$ is a valid point

      Note that there is no DNE point for $f'(x)$

      $f'(4) < 0$

      $f'(7) > 0$

      By FDT, $r = 6 - \frac{6}{\sqrt{55}}$ is a local minimum

   7. The person should run $6 - \frac{6}{\sqrt{55}}$ km before start swimming

4. In information theory, the self-information of an event x_i which occurs with probability p_i is given by $-\log_2(p_i)$. The entropy H of an distribution X with options x_i is thus given by the weighted sum $H(X) = \sum_i -p_i \log_2 \pi$. Find the probability with which a single event ξ contributes the minimum possible value to this entropy.

Minimize $-p_i \log_2 p_i$

Let $x = \frac{1}{p_i} \; \forall x \in [1, \infty)$

$-p_i \log_2 p_i = -\frac{1}{x} \log_2\left(\frac{1}{x}\right) = \frac{\ln x}{x \ln 2}$

$h'(x) = \frac{1 - \ln x}{x^2} = 0 \Longrightarrow x = e$

p_i is minimized at $p_i = \frac{1}{e}$