CH 3 — Function Limits and Continuity

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Definitions

If $f: \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}$, $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

Examples:

1)Prove using the $\varepsilon - \delta$ definition that $\lim_{x\to 0} f(x)$ DNE where

$$f(x) = \begin{cases} -2 & \text{if } x < 0\\ 3 & \text{if } x > 0 \end{cases}$$

Domain: $\mathbb{R} \setminus \{0\}$

Take $\varepsilon = 1$. Consider some $\delta > 0$. Whitin $(0 - \delta, 0 + \delta)$

We have both $(-\delta,0)$ where f(x)=-2 and $(0,\delta)$ where f(x)=3. If this δ exists for $\varepsilon=1$ then the limit L would need to be distance 1 or both -2 and 3, where is impossible.

$$\therefore \lim_{x \to 0} f(x) = \text{DNE}$$

2)
$$\lim_{x\to 7} 8x - 3 = 53$$

Let $\varepsilon > 0$ be arbitrary.

We want find δ s.t. if $0<|x-7|<\delta$ then $|8x-3-53|<\varepsilon\to\delta=\frac{\varepsilon}{8}$

Pick $\delta = \frac{\varepsilon}{8}$.

Then if
$$0 < |x-7| < \frac{\varepsilon}{8}, |(8x-3)-53| = |8x-56| = 8|x-7| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$$

3)
$$\lim_{x\to 1} x^2 + 3x + 4 = 8$$

We want for any $\varepsilon>0$ and $\delta>0:|x-1|<\delta,$ then $|f(x)-L|<\varepsilon$

$$\leftrightarrow |x^2 + 3x - 4| < \varepsilon \leftrightarrow |(x+4)(x-1)| < \varepsilon \leftrightarrow |x+4| - |x-1| < \varepsilon$$

I can always make δ smaller if I need to.

take
$$\delta < 1$$
, then $|x-1| < 1 \Longrightarrow 0 < x < 2$ $|x+4| < 6 \to |x+4| x - 1| < 6\delta$, but $6\delta < \varepsilon \leftrightarrow \delta < \frac{\varepsilon}{4}$. Say $\frac{\varepsilon}{7} < \frac{\varepsilon}{6}$ for all epsilon. Take $\delta < \min(1, \frac{\varepsilon}{6})$

Proof

Let $\varepsilon>0$ be given. Take $\delta=\min\left(\frac{1}{2},\frac{\varepsilon}{7}\right)$. Then, if $|x-1|<\delta,|x^2+3x+4-8|=|x^2+3x-4|=|(x+4)(x-1)|=|(x+4)(x-1)|<6\cdot\frac{\varepsilon}{7}<\varepsilon$

🔪 Info — Sequential Characterization of Limits Theorem

Let $a \in \mathbb{R}$. let the function f(x) be defined on an open interval containing a, expect possibly at x = a itself. Then the following are equivalent:

- 1. $\lim_{x\to a} f(x) = L$
- 2. For all sequences $\{x_n\}$ satisfying $\lim_{n\to\infty}x_n=a$ and $x_n\neq a, \forall n\in\mathbb{N}$, we have that $\lim_{n\to\infty}f(x_n)=L$

▼ Tip — Usage of Sequential Characterization of Limits

- 1. Find a sequence $\{x_n\}$ with $x_n \to a$
- 2. Find two sequences $\{x_n\}, \{y_n\}$ with $x_n, y_n \to a$ and $x_n, y_n \neq a, \forall n \in \mathbb{N}$ but which $\{f(x_n)\}, \{f(y_n)\}$ converge to different values

Proof

$$\Longrightarrow : \lim_{x \to a} f(x) = L \text{ means } \forall \varepsilon > 0, \exists \delta > 0 : |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$
 Let $\{x_n\}$ be s.t. $x_n \to a \text{(meaning that } \forall \varepsilon > 0, \exists N \in \mathbb{R} : \forall n > \mathbb{N}, |x_n - a| < \varepsilon_2) \text{ and } x_n \neq a \text{ for any } n.$

In particular, let
$$\varepsilon$$
 for $x_n \to a$ be δ . Then $\forall n > N$, $|x_n - a| < \delta$, and so $|f(x)_n| < \varepsilon_1$. Then $\forall n > N$, $|x_n - a| < \delta$ and so $|f(x_n) - L| < \varepsilon_1$. So by definition, $\lim_{n \to \infty} f(x_n) = L$

Side Question: We saw the limit of a sequence is unique. Is the same true for limits of functions?

ANS: NO, it is like saying $\lim_{x\to a} f(x) = L$ and = M and $L \neq M$ Suppose true. By Sequential Characterization of Limits, $\forall \{x_n\} \to a$ but $x_n \neq a \forall n, f(x_n) \to L$ and $f(x_n) \to M$ but $L \neq M$ Since the limits of sequences are unique, thus there is a contradiction.

Examples:

Prove that $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist

We take sequences of peak points of $\cos(\frac{1}{x})$, that is -1, 1. Then will converge to -1, 1 repeatedly, so by Sequential Characterization, $\lim_{x\to 0}\cos(\frac{1}{x})$ will not exist.

$$\cos\left(\frac{1}{x}\right)=1 \text{ if } x=\frac{1}{2k\pi}, k\in\mathbb{Z} \text{, and } \cos\left(\frac{1}{x}\right)=-1 \text{ if } x=\frac{1}{(2k+1)\pi}, k\in\mathbb{Z}.$$

Let $x_n=\frac{1}{2}n\pi$ and $y_n=\frac{1}{2n+1}\pi$. Then $x_n,y_n\to 0, x_n,y_n\neq 0 \forall n$. It converges to both -1 and 1. By Sequential Characterization, the limit DNE.

Limit Laws

info — Let f,g be functions with $\lim_{x\to a}f(x)=L$, $\lim_{x\to a}g(x)=M$ for some $L,M\in\mathbb{R}$ then:

- 1. For any $c \in \mathbb{R}$, if f(x) = c for all n then L = c
- 2. For any $c \in \mathbb{R}$, if $\lim_{x \to a} cf(x) = cL$
- 3. $\lim_{x\to a}(f(x)+g(x))=L+M$
- 4. $\lim_{x \to 0} f(x) \cdot g(x) = LM$
- 5. $\lim_{n\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
- 6. If $\alpha > 0$ and L > 0, then $\lim_{x>a} f(x)^{\alpha} = L^{\alpha}$

Proof

We assume functions f,g are defined on a punctured neighborhood of a and $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$. In the quotient law we also assume $M\neq 0$.

1. Product law

Claim. $\lim_{x\to a} (f(x)g(x)) = LM$.

Proof. Let $\varepsilon > 0$. Then $|f(x)g(x) - LM| = |f(x)(g(x) - M) + M(f(x) - L)| \le |f(x)| \cdot |g(x) - M| + |M| \cdot |f(x) - L|$.

Since $f(x) \to L$, choose $\delta_0 > 0$ with $|x - a| < \delta_0 \Rightarrow |f(x) - L| < 1$, hence $|f(x)| \le |L| + 1$ there.

Choose $\delta_1, \delta_2 > 0$ so that $|x-a| < \delta_1 \Rightarrow |g(x)-M| < \frac{\varepsilon}{2(|L|+1)}$ and $|x-a| < \delta_2 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2(|M|+1)}$.

Let $\delta = \min(\delta_0, \delta_1, \delta_2)$. For $0 < |x-a| < \delta$, $|f(x)g(x) - LM| \le (|L|+1) \cdot \frac{\varepsilon}{2(|L|+1)} + |M| \cdot \frac{\varepsilon}{2(|M|+1)} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus $\lim_{x\to a} (fg) = LM$.

2. Quotient law (with $M \neq 0$)

Claim. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Proof. Let $\varepsilon>0$. Because $g(x)\to M\neq 0$, there exists $\delta_0>0$ such that $|x-a|<\delta_0\Rightarrow |g(x)-M|<|M_{\frac{1}{2}}$, hence $|g(x)|\geq |M_{\frac{1}{2}}$.

$$\begin{array}{l} \operatorname{Now}\mid \frac{f(x)}{g(x)} - \frac{L}{M}\mid = \mid Mf(x) - Lg(x) \frac{\mid}{\mid M\mid \cdot \mid g(x)\mid} \leq \frac{\mid M\mid \cdot \mid f(x) - L\mid + \mid L\mid \cdot \mid g(x) - M\mid}{\mid M\mid \cdot \mid g(x)\mid} \leq \left(\frac{2}{\mid}M\mid\right) \cdot \mid f(x) - L\mid + \left(2\mid L\mid M\mid^2\right) \cdot \mid g(x) - M\mid. \end{array}$$

 $\text{Choose } \delta_1, \delta_2 > 0 \text{ with } |x-a| < \delta_1 \Rightarrow |f(x)-L| < \left(|M_{\frac{1}{4}}\right) \cdot \varepsilon \text{ and } |x-a| < \delta_2 \Rightarrow |g(x)-M| < \left(|M_{\frac{|^2}{4(|L|+1)}}\right) \cdot \varepsilon.$

Let
$$\delta = \min(\delta_0, \delta_1, \delta_2)$$
. Then for $0 < |x - a| < \delta$, $|\frac{f(x)}{g(x)} - \frac{L}{M}| \le \left(\frac{2}{|M|}\right) \cdot \left(|M_{\frac{1}{4}}^{\perp}\right) \cdot \varepsilon + \left(2 \ |L_{\frac{1}{2}}^{\perp}M|^2\right) \cdot \left(|M_{\frac{1}{4(|L|+1)}}^{\perp}\right) \cdot \varepsilon \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Therefore $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Then $\lim_{x\to a} p(x) = p(a)$

Proof

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} x^i = a^i$$

$$\lim_{x\to a}a_ix^i=a_ia^i$$

$$\lim_{x\to a}\sum_{i=0}^n a_ix^i=\sum_{i=0}^n a_ia^i$$

≥ Info – Limit of Rational Functions

Let $f(x) = \frac{p(x)}{q(x)}$ when p,q be polynomial functions and $a \in \mathbb{R}$

- 1. If $q(a) \neq 0$ then $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$
- 2. If $\lim_{x\to a} q(a) = 0$ but then $\lim_{x\to a} p(x) \neq 0$ then $\lim_{x\to a} \frac{p(x)}{q(x)}$ is DNE. If $x\to a, x<0$, then the limit diverges to $-\infty$. If $x\to a, x>0$, then the limit diverges to ∞ .
- 3. Otherwise, p(a)=0=q(a), so both p(x) and q(x) have (x-a) as a factor. Divide it out and then repeat the process.

Examples:

$$\begin{array}{l} \text{1. } \lim_{x \to -3} \frac{x^3 + 10x^2 + 13x - 24}{x^2 - 4x - 21} \\ \Rightarrow \stackrel{\left[\begin{array}{c} 0 \\ \end{array} \right]}{=} \lim_{x \to -3} \frac{(x + 3)(x - 1)(x + 8)}{(x + 3)(x - 7)} = \lim_{x \to -3} \frac{(x - 1)(x + 8)}{(x - 7)} = \frac{(-3 - 1)(-3 + 8)}{(-3 - 7)} = \frac{-20}{-10} = 2 \end{array}$$

№ Info — Squeeze Theorem(Functions):

If $g(x) \leq f(x) \leq h(x)$ be functions defined in an open interval I around a except possibly at a.

If $\forall a \in I \setminus \{a\}$ we have $g(x) < f(x) \le h(x)$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$

- ho **Tip** When to apply Squeeze Theorem
- 1. Trigonometeric functions with clear bounds and polynomial terms before
- 2. Exponential Functions with constants terms or by defining a certain inverval

2. Evaluate $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2 \cos \left(\frac{1}{x}\right) \le x^2$$

Notice that x^2 are polynomial function that is defined in $x \in \mathbb{R}$.

$$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$$

By Squeeze Theorem, $\lim_{x\to 0} x^2 \cos(\frac{1}{x}) = 0$

One Sided Limits and the Fundamental Trig Limit

- 1. We say that L is the **right side limit** of f at a, and write $\lim_{x\to a^+} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x-a| < \delta$ and x > a then $|f(x) L| < \varepsilon$
- 2. We say that L is the **left side limit** of f at a, and write $\lim_{x\to a^-} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x-a| < \delta$ and x < a then $|f(x)-L| < \varepsilon$

≥ Info – Theorem

$$\lim\nolimits_{x\to a}f(x)=L\Longleftrightarrow\lim\nolimits_{x\to a^{-}}f(x)=\lim\nolimits_{x\to a^{+}}f(x)=L$$

Example:

Show that $\lim_{x\to 0}\sin(x)=0$, $\lim_{x\to 0}\cos(x)=1$, and $\lim_{x\to 0}\tan(x)=0$

1. $\lim_{x\to 0} \sin(x)$:

Construct a unit circle with a triangle that lies in the first quadrant and an arbitrary point, say P(x,y). Then $P(x,y) = P(\cos(x),\sin(y))$. The area of the triangle can be represented as $\frac{1}{2}\sin(x)$.

Contruct another unit circle and draw P(x,y) at the same location as the previous triangle, however, contruct an sector. The area of this new sector is $\frac{1}{2}x$.

Notice that the area bounded by the sector is bigger than the triangle.

We then have $0 \le \frac{1}{2}\sin(x) \le \frac{1}{2}x \Longrightarrow 0 \le \sin(x) \le x$. Since $\lim_{x\to 0^+} 0 = \lim_{x\to 0^+} x = 0$, by Squeeze Theorem, $\lim_{x\to 0^+}\sin(x) = 0$

 $\lim_{x\to 0^-}\sin(x)=0$ can be achieved similarly to the prove of right side limit and will be omitted.

Thus $\lim_{x\to 0} \sin(x) = 0$

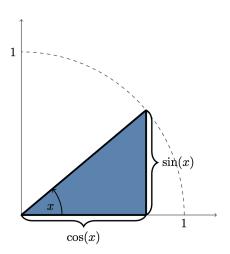
2. $\lim_{x\to 0} \cos(x) = 1$:

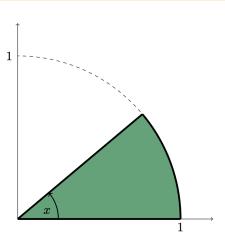
$$\lim_{x\to 0}\cos(x)=\lim_{x\to 0}\sqrt{1-\sin^2(x)}=1$$

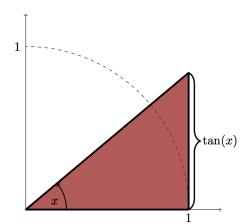
3.
$$\lim_{x\to 0} \tan(x) = \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = 1$$

🔥 Warning — The Fundamental Trig Limit:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$







We have that $\frac{1}{2}\cos(x)\sin(x) \le \frac{1}{2}x \le \frac{1}{2}\tan(x) \Longrightarrow \cos(x) \le \frac{\sin(x)}{x} \le \frac{1}{\cos(x)}$.

By Squeeze Theorem, $\lim_{x\to 0^+}\frac{\sin(x)}{x}=1.$

Since $\sin(x)$ is a even function, then $\frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x}$ so $\lim_{x\to 0^-} \frac{\sin(x)}{x} = 1$

$$\therefore \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Examples:

1.
$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(s)} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{\cos(x)} = 1 \cdot 1 = 1$$

$$2. \ \lim_{x \to 0} \frac{\sin(72x)}{\tan(9x)} = \lim_{x \to 0} \frac{\sin(72x)}{72x} \cdot \frac{9x}{\tan(9x)} \cdot 8 = \lim_{x \to 0} \frac{\sin(72x)}{72x} \cdot \lim_{x \to 0} \frac{9x}{\tan(9x)} \cdot 8 = 1 \cdot 1 \cdot 8$$

$$2. \lim_{x \to 0} \frac{\sin(72x)}{\tan(9x)} = \lim_{x \to 0} \frac{\sin(72x)}{72x} \cdot \frac{9x}{\tan(9x)} \cdot 8 = \lim_{x \to 0} \frac{\sin(72x)}{72x} \cdot \lim_{x \to 0} \frac{9x}{\tan(9x)} \cdot 8 = 1 \cdot 1 \cdot 8$$

$$3. \lim_{x \to 0} \frac{\sin(x^2 - 1)}{\sin(x - 1)} = \lim_{x \to 0} \frac{\sin(x^2 - 1)}{\sin(x - 1)} \cdot \frac{x^2 - 1}{x^2 - 1} \cdot \frac{x - 1}{x - 1} = \lim_{x \to 0} \frac{\sin(x^2 - 1)}{x^2 - 1} \cdot \lim_{x \to 0} \frac{x - 1}{\sin(x - 1)} \cdot \lim_{x \to 0} \frac{$$

Horizontal Asymptotes and the Fundamental Log Limit

\longrightarrow Info – Limit at $\pm \infty$

Let $L \in \mathbb{R}$. We say that $\lim_{x \to \infty} \text{ if } \forall \varepsilon > 0, \exists N \in \mathbb{R} \text{ s.t. if } x > N, \text{ then } |f(x) - L| < \varepsilon$.

Similarly, $\lim_{x \to -\infty}$ if $\forall \varepsilon > 0, \exists N \in \mathbb{R} \text{ s.t. if } x < N, \text{then } |f(x) - L| < \varepsilon.$

≥ Info — Horizontal Asymptotes

If $\lim_{x\to\infty}f(x)=L \text{ or } \lim_{x\to-\infty}f(x)=L \text{ for some } L\in\mathbb{R}$ then we way y=L is a

Horizontal Asymptote of f

Note: you can cross horizontal asymptotes multiple times

≥ Info — Divergence of Limits

- 1. We say that $\lim_{x\to\infty} f(x) = \infty$ if, $\forall M>0, \exists N\in\mathbb{R}$ s.t. if x>N we have f(x)>M.
- 2. We say that $\lim_{x \to -\infty} f(x) = \infty$ if, $\forall M > 0, \exists N \in \mathbb{R}$ s.t. if x < N we have f(x) > M.
- 3. We say that $\lim_{x \to \infty} f(x) = -\infty$ if, $\forall M < 0, \exists N \in \mathbb{R}$ s.t. if x > N we have f(x) < M.
- 4. We say that $\lim_{x \to -\infty} f(x) = -\infty$ if, $\forall M < 0, \exists N \in \mathbb{R}$ s.t. if x < N we have f(x) < M.

$\stackrel{\triangleright}{\mathbf{W}}$ Info − Squeeze Theorem at $\pm \infty$

If $g(x) \le f(x) \le h(x) \forall x \ge N$ for some $N \in \mathbb{R}$, and if

 $\lim_{x\to\infty} g(x) = L = \lim_{x\to\infty} h(x)$, then $\lim_{x\to\infty} f(x) = L$

⚠ Warning — The Fundamental Log Limit

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = 0$$

Proof

 $0 \le \frac{\ln(x)}{x}$ true whenever $x \ge 1$. Since $x \to \infty$, assume $x \ge 1$.

$$\frac{\ln(x)}{x} = \frac{\ln(\sqrt{x} \cdot \sqrt{x})}{\sqrt{x} \cdot \sqrt{x}} = \frac{2\ln(\sqrt{x})}{\sqrt{x} \cdot \sqrt{x}} = \frac{2}{\sqrt{x}} \cdot \frac{\ln(\sqrt{x})}{\sqrt{x}} \le 1 \le \frac{2}{\sqrt{x}} \text{ (since } \ln(z) \le z, \forall z \text{ arbitrarily large)}$$

$$0 \leq rac{\ln(x)}{x} \leq rac{2}{\sqrt{x}}$$
. By Squeeze Theorem $\lim_{x o \infty} rac{\ln(x)}{x} = 0$

Examples:

1. Show that
$$\lim_{x \to \infty} \frac{\ln(x)}{x^p} = 0, \forall p > 0$$

$$\lim\nolimits_{x\to\infty} \frac{\ln(x)}{x^p} = \lim\nolimits_{x\to\infty} \frac{\ln(x^p)}{x^p} \cdot \frac{1}{p} \qquad \text{ Let } u = x^p, x\to\infty, u\to\infty$$

$$\lim_{u \to \infty} \frac{\ln(u)}{u} \cdot \frac{1}{p} = 0 \cdot \frac{1}{p} = 0$$

2. Show that $\lim_{x \to \infty} \frac{\ln(x^p)}{x} = 0, \forall p > 0$

$$\lim\nolimits_{x\to\infty}\frac{\ln(x^p)}{x}=\lim\nolimits_{x\to\infty}p\cdot\lim\nolimits_{x\to\infty}\frac{\ln(x)}{x}=p\cdot0=0$$

3. Show that $\lim_{x\to\infty} \frac{x^p}{e^x} = 0, \forall p > 0$

$$\begin{array}{ll} \lim_{x\to\infty}\frac{x^p}{e^x} & \text{Let } x=\ln u \Leftrightarrow u=e^x, x\to\infty, u\to\infty \\ \lim_{u\to\infty}\frac{\ln(u)^p}{u}=\lim_{u\to\infty}\left(\frac{\ln(u)}{u^\frac{1}{p}}\right)^p=0^p=0 \end{array}$$

4. Show that $\lim_{x\to 0^+} \frac{x^p}{\ln(x)} = 0, \forall p > 0$

$$\begin{array}{l} \lim_{x\to 0^+} \frac{x^p}{\ln(x)} \left(\text{Let } u = \frac{1}{x} \right) \\ \lim_{u\to \infty} \frac{1}{u^p} \cdot \ln \left(\frac{1}{u} \right) = 0 \end{array}$$

≥ Info − Vertical Asymptotes

- 1. $\lim_{x \to a^+} f(x) = \infty$ if for every $m > 0, \exists \delta > 0$ s.t. if $a < x < a + d + \delta$ then f(x) > m
- 2. $\lim_{x \to a^-} f(x) = \infty$ if for every $m < 0, \exists \delta > 0$ s.t. if $a < x < a + d + \delta$ then f(x) < m
- 3. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \infty \Longrightarrow \lim_{x\to a} f(x) = \infty$
- 4. If $\lim_{x\to a^\pm} f(x) = \pm \infty$ then there is a vertical asymptote at x=a

Info — Continuity

We say that a function f is **continuous** at a if f is defined at a and

- 1. $\lim_{x\to a} f(x) = f(a)$
- 2. For every $\varepsilon > 0, \exists \delta > 0$ s.t. if $|x a| < \delta$, then $|f(x) f(a)| < \varepsilon$
- 3. for every sequence $\{x_n\}$ with $x_n \to a$, we have $f(x_n) \to f(a)$
- 4. If f is not continuous at a, then it is called **discountinous at** a

Example:

1. f(x) = |x| is continuous at a

$$f(x) = \begin{cases} x \text{ if } x > 0 \\ -x \text{ if } x \le 0 \end{cases}$$

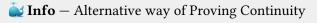
 $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

Thus $\lim_{x\to 0} |x| = 0$

igcap Tip — Polynomials are continuous at all points

$$\lim_{x\to a} p(x) = p(a)$$



$$\lim_{x\to a} f(x) = f(a) \equiv \lim_{h\to 0} f(a+h) = f(a)$$

Example:

1. Prove $\sin(x)$ is continuous at $x \in \mathbb{R}$

$$\sin(a+h) = \sin(a)\cos(h) + \sin(h)\cos(a)$$

$$\lim_{x \to a} \sin(x) = \lim_{h \to 0} \sin(a+h)$$

$$= \lim\nolimits_{h \to 0} \sin(a) \cdot \lim\nolimits_{h \to 0} \cos(h) + \lim\nolimits_{h \to 0} \sin(h) \cdot \lim\nolimits_{h \to 0} \cos(a)$$

$$= \sin(a) \cdot 1 + 0 \cdot \cos(a) = \sin(a)$$

2. Given that e^x is continuous at x=0, prove that it is continuous on all of $\mathbb R$

$$\lim_{x\to a} e^x = \lim_{h\to 0} e^{a+h} = e^a \cdot 1 = \lim_{h\to 0}$$

? Tip − Continuity of Inverse

Let f(x) be invertible at x = a with f(a) = b. Then $f^{-1}(y)$ is continuous at y = b

Let f, g be continuous at x = a.

- 1. For any constant $c \in \mathbb{R}$, cf(x) is continuous at x = a
- 2. $f(x) \pm g(x)$ remain continuous
- 3. f(x)g(x) remain continuous
- 4. If $g(x) \neq 0$, $\frac{f(x)}{g(x)}$ remain continuous

≥ Info − Composition of Continuous Function

If f is continuous at x = a, and g is continuous at x = f(a), then

 $h = g \circ f$ is continuous at x = a

Proof:

Given f is continuous at a, then for any $\{x_n\}$, $x_n \to a$, $f(x_n) \to f(a)$ But g is continuous at f(a), so for any $\{y_n\}$ with $y_n \to f(a)$, $g(y_n) \to g(f(a))$. Let $y_n = f(x_n)$. Then $g(f(x_n)) \to g(f(a))$ since x_n is aribitrary, $g \circ f$ is continuous at x = a

ho Tip — Continuity on Invervals

- 1. We say that a function f is continuous on (a, b) if it is continuous at each $x \in (a, b)$
- 2. We say that a function f is continuous on [a,b] if it is continuous on (a,b) and we have $\lim_{x\to a^-}f(x)=f(a)$ and $\lim_{x\to b^+}f(x)=f(b)$

Example:

Find the largest interval on which $f(x) = x^{\frac{1}{4}}$ is continuous.

For $(0, \infty)$ given $a \in (0, \infty)$, f is continuous at a^- because it is the inverse of x^4 , which is continuous at \mathbb{R} .

For $a=0,\lim_{x\to 0^+}x^{\frac{1}{4}}=0$, so f is continuous at 0 as well.

f is continuous on $[0, \infty)$

Info — Types of Discountinuity

- 1. A discontinuity is a removable discontinuity if $\lim_{x\to a} f(x)$ exists but does not equal f(a). All discontinuities not of this type are non-removable.
- 2. A discontinuity is a jump discontinuity if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but do not agree.
- 3. A discontinuity is infinite if either $\lim_{x\to a^-} f(x) = \pm \infty$ or $\lim_{x\to a^+} f(x) = \pm \infty$.
- 4. A discontinuity is oscillatory if $\lim_{x\to a} f(x)$ does not exist, but f is bounded and oscillates infinitely often near x = a.

🚉 Info — Intermediate Value Theorem (IVT)

If f is continuous on a closed interval [a, b] and $\alpha \in \mathbb{R}$ is such that either $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$, then $\exists c \in (a, b)$ s.t. $f(c) = \alpha$

Examples:

1. Show that $\sin(x)$ and $1-x^2$ intersects on (0,1)

Since $\sin(x)$ and $1-x^2$ are continuous on $\mathbb R$ therefore continuous on [0,1]

For
$$x = 1$$
, $\sin(1) - 1 + (1)^2 = 1^2 - 1 > 1$

For
$$x = 0$$
, $\sin(0) - 1 + (0)^2 = -1 < 1$

By IVT, $\exists c \in (0,1), f(c) = 0$, meaning the given function intersect

2. Show that $x^3 + 3x^2 - x - 3$ has a root in the interval (-5, -2)

Since $x^3 + 3x^2 - x - 3$ is continuous on $\mathbb R$ therefore continuous on [-5, -2]

For
$$x = -5$$
, $(-5)^3 + 3(-5)^2 - (-5) - 3 = -48 < 0$

For
$$x = -2$$
, $(-2)^3 + 3(-2)^2 - (-2) - 3 = 3 > 0$

$$-48 < 0 < 3$$

By IVT, $\exists c \in (-5, -2), f(c) = 0$, meaning there is a point where the given function has a root

🔪 Info — Bisection Method

Let F be a continuous function.

To approximate a value c s.t. F(c) = 0 with an error of at most ε

- 1. We need to find a, b s.t. F(a) and F(b) have opposite signs
- 2. IVT gives that $c \in (a, b)$
- 3. Look at the mid-point $a + \frac{b-a}{2}$. Is $F(a + \frac{b-a}{2})$ positive or negative?
- $\int 1$. If it has the same sigh as F(a), then it has the opposite sign with $F(b) \Longrightarrow \text{IVT}$ gives $c \in (a + \frac{b-a}{2}, b)$
- 2. If it has the opposite sigh as F(a), then it has the same sign with $F(b) \Longrightarrow IVT$ gives $c \in (a, a + \frac{b-a}{2})$
- 4. Repeat Step 3 until $|b-a| < \varepsilon$

CHAPTER ENDS