CH 1 — Integration

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Definite Integrals

🐳 **Info — Riemann Sums**

Given $f(x)$ that is defined over $[a, b]$ with $a < b$, the area under function $f(x)$ can be found by

1. **Left-Endpoint Riemann Sum**

$$L_n = \sum_{i=0}^{n-1} f(x_i^*)\Delta x$$

- Underestimates Increasing Funcitons

2. **Right-Endpoint Riemann Sum**

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- Overestimates Increasing Functions

where

- $\Delta x = \frac{b-a}{n}$ under regular partition
- $x_i^* = a + i\Delta x = a + i\frac{b-a}{n}$

$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$ for $f(x)$ on interval $[a, b]$

Regular Parition means interval $[a, b]$ is equally partitionined into n rectangles with identical width

Example:

Estimate area under the curve for $f(x) = x^2$ on $x \in [0, 1]$

$R_n = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6n}$

$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n} = \frac{1}{3}$

🐳 **Info — Definite Integral**

$f(x)$ defined on $x \in [a, b]$ with regular partition with n subintervals

The definite integral of $f(x)$ on $[a, b]$ is defined

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

A function is integrable on $x \in [a, b]$ provided that the limit of Riemann Sum exists and has the same value regardless of the choice of x_i^*

🐳 **Info — Integrability Theorem for Continuous Functions**

Integrability: $\lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n$

1. If f is continuous of $[a, b]$ then f is integrable on $[a, b]$
2. f is bounded on $[a, b]$ and has a **finite** number of discontinuities, then f is integrable on $[a, b]$

That is continuity implies integrability and the other way is false

Examples:

1. $f(x) = x^2$
2. $f(x) = \begin{cases} 2 \text{ if } x \leq 3 \\ 1 \text{ if } x > 3 \end{cases}$, note that $f(x)$ is discontinuous
3. $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ on $[0, 1]$
   - x_i^* is rational

     $\int_0^1 f(x) \, \mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^n f(x^* + i) \Delta x = 1$
   - x_i^* is irrational $\int_0^1 f(x) \, \mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^n f(x^* + i) \Delta x = 0$

   Thus not integrable

For geometric interpretation, Riemann Sums and Definite Integrals measures the "signed" area where there is no more than 1 inflection point

- A positive result of w implies the area under the curve above x-axis is w
- A negative result of w implies the area under the curve under x-axis is w

🐳 **Info — Properties of Definite Integrals**

Let $f(x)$ be integrable on $[a, b]$

1. $\int_{-a}^a f(x) \, \mathrm{d}x = 0$ if $f(x)$ is odd function
2. $\int_{-a}^a f(x) \, \mathrm{d}x = 2w$ where $\int_0^a f(x) \, \mathrm{d}x = w$ if $f(x)$ is even function

Examples:

1. $\int_1^3 x^2 - 3x \, \mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n} = \mathrm{im}_{n \to \infty} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^3 - 3\left(1 + \frac{2i}{n}\right)\right] \cdot \frac{2}{n}$

$= \lim_{n \to \infty} \sum_{i=1}^n \left(-\frac{4i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right) = -\frac{10}{3}$

2. $\int_0^5 x^3 - 2 \, \mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^n f\left(i \frac{5}{n}\right) \frac{5}{n} = \lim_{n \to \infty} \sum_{i=1}^n \left[\left(\frac{5}{n}\right)\left(\left(i \frac{5}{n}\right)^3 - 2\right)\right] = \frac{583}{4}$