CH 5- Set Theory

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Empty Set

$\emptyset = \{\}$ but $\{\emptyset\} \neq \emptyset$

Cardinality

The number of elements in a finite set is S called the cardinality of S, denoted by $|S|$

Set Notation

Set Builder Noataion - Type 1

The notation $\boldsymbol{\{x \in \mathcal{U} : P(x)\}}$

Describes the set consisting of all objects x such that x is an element of $\mathcal{U}$, and $P(x)$ is true

Example: $A = \{n \in \mathbb{N} : n \mid 12\} = \{1, 2, 3, 4, 6, 12\}$

Set Builder Notation -Type 2

The notation $\boldsymbol{\{f(x) : x \in \mathcal{U}\}}$

Describes the set consisting of all objects of the form $f(x)$ such that x is an element of $\mathcal{U}$

Example: $B = \{2k : k \in \mathbb{Z}\} =$ all even numbers

Set Builder Notation - Type 3

The notation $\boldsymbol{\{f(x) : x \in \mathcal{U}, P(x)\}}$ or $\boldsymbol{\{f(x) : P(x), x \in \mathcal{U}\}}$ Both describes the set consisting of all objects of the form $f(x)$ such that x is an element $\mathcal{U}$ and $P(x)$ is true

Example: $C = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$

Practices:

1. All multiple of 7: $\{x \in \mathbb{N} : 7 \mid x\}$
2. All odd perfect square: $\{(2x+1)^2 : x \in \mathbb{Z}\}$
3. All points on a circle of radius 8 centered at origin: $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 64 :\}$
4. All sets of three integers which are the side lengths of a triangle:

$\{(x, y, z) : x, y, z \in \mathbb{N}, x < y < z, x + y < z\}$

Union and Intersection

The **union** of two sets S and T, denoted, $S \cup T$, is the set of all elements belonging to either set S or set T.

The **intersection** of two sets S and T, denoted, $S \cap T$, is the set of all elements belonging to either set S and set T.

Practice:

Let $C = \{3, 5, 7, 10\}, D = \{1, 3, 6, 7, 8\}$

1. $C \cup D = \{1, 3, 5, 6, 7, 8, 10\}$
2. $C \cap D = \{3, 7\}$

Let $A = \{m \in \mathbb{Z} : 2 \mid m\}, B = \{2k + 1 : k \in \mathbb{Z}\}$
1. $A \cup B = \mathbb{Z}$
2. $A \cap B = \emptyset$

For non empty sets A and B
1. If $|A| = 12, |B| = 4, |A \cap B\ | = 2, |A \cup B| = 14$
2. If $|A| = 10, |B| = 20, |A \cup B| = 25, |A \cap B| = 5$

Set Difference

The **set difference** of two sets S and T, written $S - T$ or $S \setminus T$ is the set of all elements belonging to S but not T.
Symbolically: $S - T = \{x : (x \in S) \wedge x(\notin \mathbb{T})\}$

Complement

The **complement** of a set S whose elements belong to $\mathcal{U}$, written S, is the set of all elements in $\mathcal{U}$ but not in S.
Symbolically: $\{x \in \mathcal{U} \notin S\}$

Disjoin Set

Two sets S and T are said to be disjoint when $S \cap T = \emptyset$

Practice:
1. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, C = \{3, 5, 7, 10\}, D = \{1, 3, 6, 7, 8\}$

Find $|C - D^{\mathrm{C}}| = 2$

2. Let $A = \{x : x \in \mathbb{N}, x \text{ is even }\}, B = \{x : x \in \mathbb{N}, x \text{ is not a prime}\}$

$A \cup B = \{1, 2, 4, 6, 8, 9, ...\}$ $A \cap B = \{2\}$

3. If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$: False
4. If $|A \cap B| = |A|$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$: True
5. If $|A \cap B| = |A|$ and $|A \cap C| = |A|$, then $B \cap C = \emptyset$: False

Subsets

A set S is called a **subset** of a set T, denoted $S \subseteq T$ when every element of S belongs to T. T is **superset** of S

A set is called a **proper subset**, denoted $S \subsetneq T$, meaning S is a subset of T and there exists an element in T which does no belong to S. T is a **proper superset** of S

Examples:

1)

$$\{5, 15, 25\} \subseteq \{5, 10, 15, 20, 25\}$$

$$\{5, 15, 25\} \subsetneq \{5, 10, 15, 20, 25\}$$

2)

$$\{2, 4, 6\} \not\subseteq \{2025\}$$

$$\{2, 4, 6\} \not\subseteq \{1, 2, 3, 4, 5\}$$

3)

$$\mathbb{N} \subseteq \mathbb{Z}, \mathbb{Z} \subseteq \mathbb{Q}, \mathbb{Q} \subseteq \mathbb{R}$$

4)

$$\emptyset \subseteq S \text{ for all sets } S$$

$$S \subseteq S \text{ for all sets } S, \text{but } S \text{ is never a proper subset of } S$$

5)

$$\text{For all sets } S \text{ and } T, S \cap T \subseteq T \text{ and } S \subseteq S \cup T$$

IMPORTANT

Subset can be expressed as an implication
To prove $S \subseteq T$, we need to prove the universally quantified implication

$$\forall x \in \mathcal{U}, (x \in S) \Longrightarrow (x \in T)$$

Equal notation:

$$S \subseteq T \text{ and } T \subseteq S \Longleftrightarrow S = T$$

Example:

1. Let $A = \{n \in \mathbb{N} : 4 \mid (n - 3)\}$ and let $B = \{2k + 1 : k \in \mathbb{Z}\}$, prove that $A \subseteq B$

Let $n \in \mathbb{Z}, (n \in A) \Longrightarrow (n \in B)$
$A \equiv \text{a set of natural numbers in form } 4q + 3$
$B \equiv \text{a set of integers that are odd}$
$\text{Since } 4q + 3 = 2(2q + 1) + 1, \text{which is always odd.}$
$\therefore A \subseteq B \text{ and also } A \subsetneq B$

2. Prove $S = T \Longleftrightarrow S \cap T = S \cup T$

$$(S = T \Longrightarrow S \cup T = S \cap T) \land (S \cup T = S \cap T \Longrightarrow S = T)$$

$(\Longrightarrow)$

$\text{Suppose } S = T, \text{we need to show } S \cap T \subseteq S \cup T \text{ and } S \cup T \subseteq S \cap T$
If $x \in S \cap T, \text{then } x \in S, x \in S \cup T.$
Assume $x \in S \cup T$, then without loss of generality, we may suppose that $x \in S$.
Then since $S = T$, it follows that $x \in T$.
$\therefore x \in S \land x \in T \Longrightarrow x \in S \cap T$

($\Longleftarrow$)

Suppose $S \cap T = S \cup T$, we must show $S \subseteq T \wedge T \subseteq S$
If $x \in S$ then $x \in S \cup T$.
Furthermore, $S \cap T = S \cup T \implies x \in S \cap T \implies x \in S \wedge x \in T$
$x \in T \implies S \subseteq T$. The proof of $T \subseteq S$ is similar and will be omitted

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