CH 1 — Vectors in Euclidean Space

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Vector Addition and Scalar Multiplication

🐳 **Info — Vector**

The set $\mathbb{R}^n$ is defined as $\left\{ \vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$

A **vector** is an element $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$ of $\mathbb{R}^n$

The row notation of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ is $\vec{v} = [v_1 \ v_2 \ v_3]^T$

🐳 **Info — Equality**

We say that vectors $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} \in \mathbb{R}^m$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ are **equal**

if $n = m$ and $u_i = v_i \forall \ i = 1, 2, \dots, n$.

We denote it: $\vec{w} = \vec{v}$

🐳 **Info — Addition and Properties**

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} \in \mathbb{R}^n$.

Then $\vec{w} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \dots \\ u_n+v_n \end{bmatrix}$

1. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$
2. $\vec{w} + \vec{v} + \vec{w} = \vec{w} + (\vec{v} + \vec{w})$
3. There is a zero **vector**, $\vec{0} = [0 \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{R}^n$
4. $\vec{v} + \vec{0} = \vec{v}$
5. $\vec{v} + (-\vec{v}) = 0$

🐳 Info — Additive Inverse

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \in \mathbb{R}^n$. The additive inverse of $\vec{w}$ denoted $-\vec{w}$ is defined as

$$-\vec{w} = \begin{bmatrix} -u_1 \\ -u_2 \\ \dots \\ -u_n \end{bmatrix}$$

$$\vec{w} - \vec{w} = \vec{w} + (-\vec{w}) = \vec{0}$$

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \\ \dots \\ v_n - u_n \end{bmatrix}$$

🐳 Info — Scalar Multiplication

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}, \vec{w} \in \mathbb{R}^n, c, d \in \mathbb{R}$. Then the scalar product $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \dots \\ cv_n \end{bmatrix}$

1. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
2. $c(\vec{w} + \vec{v}) = c\vec{w} + c\vec{v}$
3. $0\vec{w} = \vec{0}$
4. If $c\vec{v} = \vec{0}$ then $c = 0 \vee \vec{v} = 0$
5. $c(d\vec{v}) = (cd)\vec{v}$

🐳 Info — Linear Combination

For $\vec{v_1}, \dots, \vec{v_k} \in \mathbb{R}^n$, and $c_1, \dots, c_k \in \mathbb{R}$ we call the expression

$$c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k}$$

a **linear combination** of $\vec{v_1}, \dots, \vec{v_k}$.

Examples:

1. Let $\vec{u} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ then $2\vec{u} - 3\vec{v} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} - + \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$
2. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Is $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ a linear combination of $\vec{u}$ and $\vec{v}$?

   We set $\vec{x} = c_1\vec{u} + c_2\vec{v}$ and try to solve for c_1, c_2

   That is $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 \end{bmatrix}$, we obtain $c_1 = 2, c_2 = \frac{1}{2}$. So $\vec{x}$ is a linear combination of $\vec{u}, \vec{v}$

Bases

🐳 Info — Span

Let $\mathcal{B} = \{\vec{v_1}, ..., \vec{v_k}\}$ be a set of vectors in $\mathbb{R}^n$. We define the **span** of $\mathcal{B}$ by

$$\text{Span } \mathcal{B} = \{c_1\vec{v_1} + c_2\vec{v_2} + ... + c_k\vec{v_k} \mid c_1, c_2, ...c_l \in \mathbb{R}\}$$

We say that the set Span $\mathcal{B}$ is spanned by $\mathcal{B}$ adn that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$

Span might not cover the entire plane if

- Vectors are linear dependent to each other
- One of them is $\vec{0}$

Let $\vec{v_1}, ..., \vec{v_k} \in \mathbb{R}^n$. Some vector $\vec{v_i}, 1 \leq i \leq k$, can be written as a linear combination of $\vec{v_1}, ..., \overrightarrow{v_{i-1}}, ..., \vec{v_k}$ if and only if $\text{Span}\{\vec{v_1}, ..., \vec{v_k}\} = \text{Span}\{\vec{v_1}, ..., \overrightarrow{v_{i-1}}, ..., \vec{v_k}\}$

Example:

Consider $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Describe Span $\vec{v_1}, \vec{v_2}$ geometrically.

Span $\{\vec{v_1}, \vec{v_2}\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \mathbb{R}^2$

🐳 Info — Standard Basis

In $\mathbb{R}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set $\text{E} = \{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$ is called the **standard basis for** $\mathbb{R}^n$

(i.e. $\mathbb{R}^3$ is $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$)

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix} = v_1\vec{e_1} + v_2\vec{e_2} + ... + v_n\vec{e_n}$ then we call $v_1, v_2, ..., v_n$ the **components of** $\vec{v}$

Dot Product

🐳 Info — Dot Product

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ ... \\ w_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. We defined their **dot product** by

$$\vec{w} \cdot \vec{v} = u_1v_1 + u_2v_2 + ... + u_nv_n$$

1. $\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$
2. $(\vec{w} + \vec{v}) \cdot \vec{w} = \vec{w} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(c\vec{w}) \cdot \vec{v} = c(\vec{w} \cdot \vec{v})$
4. $\vec{w} \cdot \vec{w} \geq 0$, with $\vec{w} \cdot \vec{w} = 0 \Longleftrightarrow \vec{w} = 0$

🐳 **Info — Vector Unit Basics**

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$

1. The **length** of vector $\vec{w}$ is $\|\vec{w}\| = \sqrt{\vec{w} \cdot \vec{w}}$
2. If $c \in \mathbb{R}, \vec{w} \in \mathbb{R}^n$, then $\|\vec{w}\| = |c| \ \|\vec{w}\|$
3. $\vec{v}$ is a **unit vector** if $\|\vec{v}\| = 1$
4. **Normalization** is when some $\vec{v}$ is a non-zero vector,

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

   in the direction of $\vec{v}$ by scaling $\vec{v}$

5. With $\vec{v}, \vec{v}$ non-zero vectors. The angle $\theta, 0 \leq \theta \leq \pi$ between $\vec{v}$ is such that

$$\vec{w} \cdot \vec{v} = \|\vec{w}\|\|\vec{v}\| \cos\theta \text{ that is } \theta = \arccos\left(\frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\|\|\vec{v}\|}\right)$$

6. $\vec{w}, \vec{v}$ are **orthogonal/perpendicular** if $\vec{w} \cdot \vec{v} = 0$

Projection

🐳 **Info — Projection**

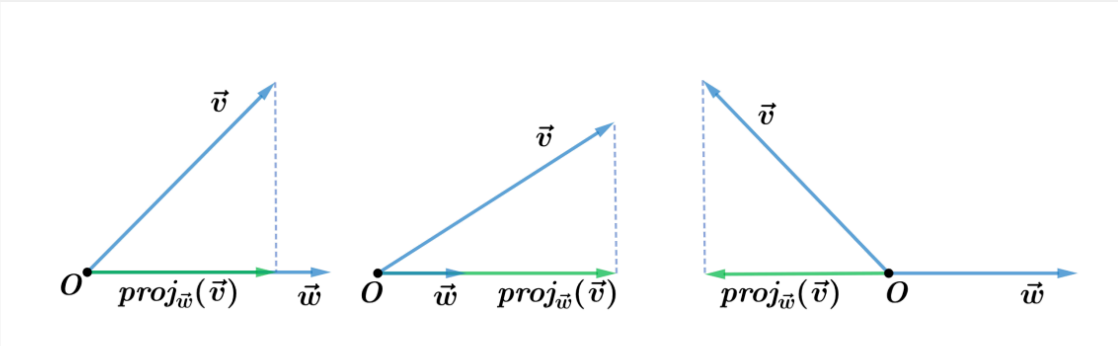
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq 0$.

1. The **projection** of $\vec{v}$ onto $\vec{w}$ is defined by

$$\operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w}$$

   We also refer to this as the **projection of $\vec{v}$ in the $\vec{w}$ direction**

   Illustration of $\operatorname{proj}_{\vec{w}}(\vec{v})$:



2. We refer to the quantity

$$\|\vec{v}\| \cos\theta = \vec{v} \cdot \hat{w}$$

   as the **component** (or scalar component) **of $\vec{v}$ along $\vec{w}$**

3. The **perpendicular** of $\vec{v}$ onto $\vec{w}$ is defined by $\operatorname{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \operatorname{proj}_{\vec{w}}(\vec{v})$
4. The projection and the perpendicular of a vector $\vec{v}$ onto $\vec{w}$ are orthogonal; that is

$$\operatorname{perp}_{\vec{w}}(\vec{v}) \cdot \operatorname{proj}_{\vec{w}}(\vec{v}) = 0$$

Vectors in $\mathbb{C}^n$

🐳 **Info — Vectors in $\mathbb{C}^n$**

The set $\mathbb{C}^n$ is defined as $\left\{ \vec{z} = \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix} : z_1, \dots, z_n \in \mathbb{C} \right\}$

The **vector** is an element $\vec{z} = \begin{bmatrix} z_1 \\ \dots \\ z_n \end{bmatrix}$ of $\mathbb{C}^n$

In $\mathbb{C}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set $\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}$ is called the **standard basis for** $\mathbb{C}^n$

Standard Inner Product in $\mathbb{C}^n$

🐳 **Info — Standard inner product**

Let $c \in \mathbb{C}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$

The **standard inner product** of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$ is

$$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w_1} + v_2 \overline{w_2} + \dots + v_n \overline{w_n}$$

1. $\langle \vec{u}, \vec{w} \rangle = \overline{\langle \vec{v}, \vec{w} \rangle}$
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3. $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$, with $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$
5. The length: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$
6. $\vec{w}, \vec{v}$ are **orthogonal/perpendicular** if $\langle \vec{w}, \vec{v} \rangle = 0$
7. With $\vec{w} \neq 0$. The **projection of $\vec{v}$ onto $\vec{w}$** is defined by

$$\operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} = \langle \vec{v}, \vec{w} \rangle \hat{w}$$

The Cross Product in $\mathbb{R}^3$

🐳 **Info — Cross Products** Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$.

The **cross product** of $\vec{u}, \vec{v}$ is defined to be the vector in $\mathbb{R}^3$ given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1; v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Let $\vec{z} = \vec{u} \times \vec{v}$

1. $\vec{z} \cdot \vec{u} = \vec{z} \cdot \vec{v} = 0$
2. $\vec{v} \times \vec{u} = -\vec{z} = -\vec{u} \times \vec{v}$
3. If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, then $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$

Info — Linearity of the Cross Product

Let $c \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, then

1. $(\vec{u} + \vec{v})\vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. $\vec{u} \times c(\vec{v}) = c(\vec{u} \times \vec{v})$