

CH 1 — Vectors in Euclidean Space

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Vector Addition and Scalar Multiplication

Info — Vector

The set \mathbb{R}^n is defined as $\left\{ \vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$

A **vector** is an element $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$ of \mathbb{R}^n

The row notation of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ is $\vec{v} = [v_1 \ v_2 \ v_3]^T$

Info — Equality

We say that vectors $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} \in \mathbb{R}^m$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in \mathbb{R}^n are **equal**

if $n = m$ and $u_i = v_i \forall i = 1, 2, \dots, n$.

We denote it: $\vec{w} = \vec{v}$

Info — Addition and Properties

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} \in \mathbb{R}^n$.

Then $\vec{w} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \dots \\ u_n + v_n \end{bmatrix}$

1. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$
2. $\vec{w} + \vec{v} + \vec{w} = \vec{w} + (\vec{v} + \vec{w})$
3. There is a zero **vector**, $\vec{0} = [0 \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{R}^n$
4. $\vec{v} + \vec{0} = \vec{v}$
5. $\vec{v} + (-\vec{v}) = \vec{0}$

Info – Additive Inverse

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \in \mathbb{R}^n$. The additive inverse of \vec{w} denoted $-\vec{w}$ is defined as

$$-\vec{w} = \begin{bmatrix} -u_1 \\ -u_2 \\ \dots \\ -u_n \end{bmatrix}$$

$$\vec{w} - \vec{w} = \vec{w} + (-\vec{w}) = \vec{0}$$

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \\ \dots \\ v_n - u_n \end{bmatrix}$$

Info – Scalar Multiplication

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$. Then the scalar product $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \dots \\ cv_n \end{bmatrix}$

1. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
2. $c(\vec{w} + \vec{v}) = c\vec{w} + c\vec{v}$
3. $0\vec{w} = \vec{0}$
4. If $c\vec{v} = \vec{0}$ then $c = 0 \vee \vec{v} = \vec{0}$
5. $c(d\vec{v}) = (cd)\vec{v}$

Info – Linear Combination

For $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, and $c_1, \dots, c_k \in \mathbb{R}$ we call the expression

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$$

a **linear combination** of $\vec{v}_1, \dots, \vec{v}_k$.

Examples:

1. Let $\vec{u} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ then $2\vec{u} - 3\vec{v} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$
2. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Is $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ a linear combination of \vec{u} and \vec{v} ?

We set $\vec{x} = c_1\vec{u} + c_2\vec{v}$ and try to solve for c_1, c_2

That is $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 \end{bmatrix}$, we obtain $c_1 = 2$, $c_2 = \frac{1}{2}$. So \vec{x} is a linear combination of \vec{u}, \vec{v}

Bases

Info – Span

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in \mathbb{R}^n . We define the **span** of \mathcal{B} by

$$\text{Span } \mathcal{B} = \{c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}$$

We say that the set $\text{Span } \mathcal{B}$ is spanned by \mathcal{B} and that \mathcal{B} is a spanning set for $\text{Span } \mathcal{B}$

Span might not cover the entire plane if

- Vectors are linear dependent to each other
- One of them is $\vec{0}$

Theorem

Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$. Some vector $\vec{v}_i, 1 \leq i \leq k$, can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}, \dots, \vec{v}_k$ if and only if $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \dots, \vec{v}_k\}$

Proof

Example:

Consider $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Describe $\text{Span } \vec{v}_1, \vec{v}_2$ geometrically.

$$\text{Span } \{\vec{v}_1, \vec{v}_2\} = \text{Span } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \mathbb{R}^2$$

Info – Linear Dependence/Independence

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\} \in \mathbb{R}^n$ is said to be **linearly dependent** if there exist coefficients c_1, \dots, c_k not all zero such that

$$\vec{0} = c_1 \vec{v}_1 + \dots + c_k \vec{v}_k$$

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\} \in \mathbb{R}^n$ is said to be **linearly Independent** if the only solution to c_1, \dots, c_k not all zero such that

$$\vec{0} = c_1 \vec{v}_1 + \dots + c_k \vec{v}_k$$

is $c_1 = c_2 = \dots = c_k = 0$ (called **trivial solution**)

Examples:

1. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linearly dependent \iff at least one of \vec{u}, \vec{v} is a scalar multiple of the other.

Proof

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$

1. Assume $\{\vec{u}, \vec{v}\}$ is linearly dependent. Then $\exists c_1, c_2 \in \mathbb{R}$, not both zero s.t.

$$c_1 \vec{u} + c_2 \vec{v} = \vec{0}$$

WLOG, assume $c_1 \neq 0$. Then, $c_1 \vec{u} = -c_2 \vec{v} \implies \vec{u} = -\frac{c_2}{c_1} \vec{v}$. Thus \vec{u} is a scalar multiple of \vec{v} .

2. Assume WLOG \vec{u} is a scalar multiple of \vec{v} . Then $\exists a \in \mathbb{R}$ s.t.

$$\vec{u} = a\vec{v} \implies 1\vec{u} - a\vec{v} = \vec{0}$$

Since $1 \neq 0$, $\{\vec{u}, \vec{v}\}$ is linearly dependent.

□

2. Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Consider the equation $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Which gives $\begin{cases} c_1 + c_2 + 2c_3 = 0 \\ 2c_1 - c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$

$\implies c_2 = c_1 = -c_3$. Thus we get a solution for any c_3 . Pick $c_3 = -1 \implies c_2 = c_1 = 1$.

Thus the set is linearly dependent.



Info – Linear Dependence Theorem

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n\}$ is linearly dependent if and only if

$$\vec{v}_i \in \text{Span} \{ \vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k \} \text{ for some } i, 1 \leq i \leq k$$



Info – Zero Vector and Linear Dependence

If a set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ contains the zero vector, then it is linearly dependent.

Proof:

Let $\vec{v}_i = \vec{0}$

$$0\vec{v}_1 + \dots + 0\vec{v}_{i-1} + \vec{v}_i + 0\vec{v}_{i+1} + \dots + 0\vec{v}_k = \vec{0}$$

□

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Info — Standard Basis

In \mathbb{R}^n , let \vec{e}_i be the vector whose i^{th} component is 1 with all other components 0. The set $E = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called the **standard basis for \mathbb{R}^n**

(i.e. \mathbb{R}^3 is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$)

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_n \vec{e}_n$ then we call v_1, v_2, \dots, v_n the **components of \vec{v}**

Dot Product

Info — Dot Product

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$ be vectors in \mathbb{R}^n . We defined their **dot product** by

$$\vec{w} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

1. $\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$
2. $(\vec{w} + \vec{v}) \cdot \vec{w} = \vec{w} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(c\vec{w}) \cdot \vec{v} = c(\vec{w} \cdot \vec{v})$
4. $\vec{w} \cdot \vec{w} \geq 0$, with $\vec{w} \cdot \vec{w} = 0 \iff \vec{w} = 0$

Info — Vector Unit Basics

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$

1. The **length** of vector \vec{w} is $\|\vec{w}\| = \sqrt{\vec{w} \cdot \vec{w}}$
2. If $c \in \mathbb{R}$, $\vec{w} \in \mathbb{R}^n$, then $\|c\vec{w}\| = |c| \|\vec{w}\|$
3. \vec{v} is a **unit vector** if $\|\vec{v}\| = 1$
4. **Normalization** is when some \vec{v} is a non-zero vector,

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

in the direction of \vec{v} by scaling \vec{v}

5. With \vec{w}, \vec{v} non-zero vectors. The angle θ , $0 \leq \theta \leq \pi$ between \vec{v} is such that

$$\vec{w} \cdot \vec{v} = \|\vec{w}\| \|\vec{v}\| \cos \theta \text{ that is } \theta = \arccos \left(\frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\| \|\vec{v}\|} \right)$$

6. \vec{w}, \vec{v} are **orthogonal/perpendicular** if $\vec{w} \cdot \vec{v} = 0$

Projection

Info – Projection

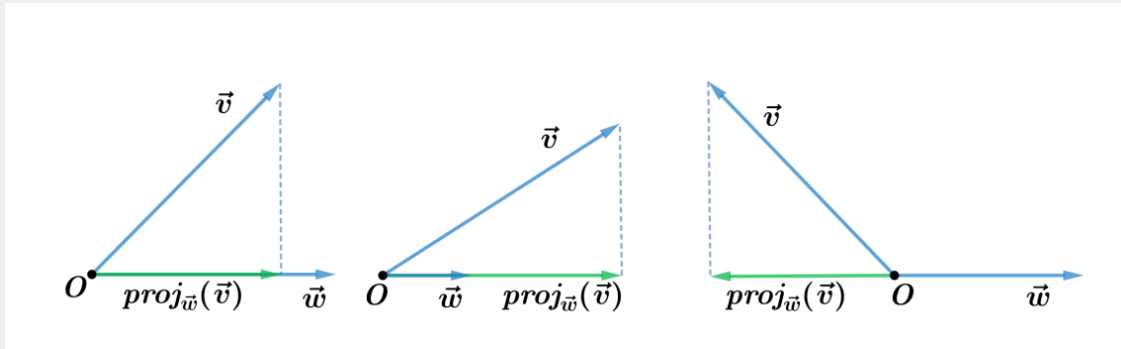
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq 0$.

1. The **projection** of \vec{v} onto \vec{w} is defined by

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

We also refer to this as the **projection of \vec{v} in the \vec{w} direction**

Illustration of $\text{proj}_{\vec{w}}(\vec{v})$:



2. We refer to the quantity

$$\|\vec{v}\| \cos \theta = \vec{v} \cdot \hat{w}$$

as the **component** (or scalar component) of \vec{v} along \vec{w}

3. The **perpendicular** of \vec{v} onto \vec{w} is defined by $\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$
4. The projection and the perpendicular of a vector \vec{v} onto \vec{w} are orthogonal; that is

$$\text{perp}_{\vec{w}}(\vec{v}) \cdot \text{proj}_{\vec{w}}(\vec{v}) = 0$$

Vectors in \mathbb{C}^n

Info – Vectors in \mathbb{C}^n

The set \mathbb{C}^n is defined as $\left\{ \vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_1, \dots, z_n \in \mathbb{C} \right\}$

The **vector** is an element $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ of \mathbb{C}^n

In \mathbb{C}^n , let \vec{e}_i be the vector whose i^{th} component is 1 with all other components 0. The set $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called the **standard basis** for \mathbb{C}^n

Standard Inner Product in \mathbb{C}^n

Info — Standard inner product

Let $c \in \mathbb{C}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$

The **standard inner product** of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$ is

$$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w_1} + v_2 \overline{w_2} + \dots + v_n \overline{w_n}$$

1. $\langle \vec{u}, \vec{w} \rangle = \overline{\langle \vec{v}, \vec{w} \rangle}$
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3. $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$, with $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$
5. The length: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$
6. \vec{w}, \vec{v} are **orthogonal/perpendicular** if $\langle \vec{w}, \vec{v} \rangle = 0$
7. With $\vec{w} \neq 0$. The **projection of \vec{v} onto \vec{w}** is defined by

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w}$$

The Cross Product in \mathbb{R}^3

 Info — Cross Products Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$.

The **cross product** of \vec{u}, \vec{v} is defined to be the vector in \mathbb{R}^3 given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Let $\vec{z} = \vec{u} \times \vec{v}$

1. $\vec{z} \cdot \vec{u} = \vec{z} \cdot \vec{v} = 0$
2. $\vec{v} \times \vec{u} = -\vec{z} = -\vec{u} \times \vec{v}$
3. If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, then $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$ where θ is the angle between \vec{u} and \vec{v}

Info – Linearity of the Cross Product

Let $c \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, then

1. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. $\vec{u} \times c(\vec{v}) = c(\vec{u} \times \vec{v})$