

CH 10 - Complex Numbers

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Standard Form

Info – Definition of Complex Numbers

A **complex number** z in **standard form** is an expression of the form $z = x + yi$ where $x, y \in \mathbb{R}$.

The real number x is called the **real part** of z , and is written $\Re(x)$.

The real number y is called the **imaginary part** of z , and is written $\Im(z)$.

The set of complex numbers is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$$

The complex number $z = x + yi$ and $w = u + vi$ are equal ($z = w$) if and only if $x = u, y = v$

A complex number z is said to be purely real if $y = 0$ (i.e. $1 = 1 + 0i$)

A complex number z is said to be purely imaginary if $x = 0$ (i.e. $i = 0 + 1i$)

0 is purely real and purely imaginary (i.e. $0 = 0 + 0i$)

Info – Complex Arithmetics

Let $z = a + bi$ and $w = c + di$ be complex numbers. Then the

Addition is defined as

$$z + w = (a + c) + (b + d)i$$

Multiplication is defined as

$$zw = (ac - bd) + (ad + bc)i$$

Examples:

Let $z = 2 + 3i, w = -1 + 7i$

1. $z + w = (2 - 1) + (3 + 7)i = 1 + 10i$
2. $zw = (-2 - 21) + (14 - 3)i = -23 + 11i$
3. $i^2 = ii = (0 + 1i) \cdot (0 + 1i) = (0 - 1) + (0 + 0)i = -1$
 $\therefore i^2 = -1$

From (3), we can derive a easier way of multiplication

 **Tip** – Multiplication trick

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Properties of Complex Arithmetics

1. $z + 0 = 0 + z = z$
2. $z0 = 0z = 0$
3. $z + (-1)z = (-1)z + z = 0$
4. $z1 = 1z = z$


 **Info** – Multiplicative Inverse

For all complex numbers z , the multiplicative inverse of z exists if and only if $z \neq 0$. Moreover, for $a + bi \neq 0$, the multiplicative inverse is unique given by

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i = \frac{a - bi}{a^2 + b^2}$$

Examples:

1. $\frac{(1-2i)-(3+4i)}{(5-6i)} = [(1-2i) - (3+4i)] \cdot (5-6i)^{-1} = (-2-6i) \cdot \frac{1}{25+36} \cdot (5+6i)$
 $= \frac{1}{61} \cdot (-10 - 30i - 12i + 36) = \frac{26-42i}{61} = \frac{26}{61} - \frac{42}{61}i$
2. $i^{2025} = (i^4)^{506} \cdot i = (-1)^{506} \cdot i = i$

 **Tip** – i^n Solutions

$$i^n = \begin{cases} 1, n \equiv 0 \pmod{4} \\ i, n \equiv 1 \pmod{4} \\ -1, n \equiv 2 \pmod{4} \\ -i, n \equiv 3 \pmod{4} \end{cases}$$

 **Info** – Properties of Complex Arithmetics

In complex arithmetic, the following properties are valid for $u, v, z \in \mathbb{C}$

1. Associativity of addition: $(u + v) + z = u + (v + z)$
2. Commutativity of addition: $u + v = v + u$
3. Additive identity: $0 = 0 + 0i$ has the property that $z + 0 = z$
4. Additive inverses: If $z = a + bi$, then there exists an additive inverse of z , written $-z$, with the property that $z + (-z) = 0$. The additive inverse of $z = a + bi$ is $-z = -a - bi$
5. Associativity of multiplication: $(uv)z = u(vz)$
6. Commutativity of multiplication: $uv = vu$
7. Multiplicative identity: $1 = 1 + 0i$ has property that $z1 = z$
8. Multiplicative inverses: If $z = a + bi \neq 0$, then there exists a multiplicative inverse of z , written z^{-1} , with the property that $zz^{-1} = 1$. The multiplicative inverse of $z = a + bi \neq 0$ is $z^{-1} = \frac{a-bi}{a^2+b^2}$
9. Distributivity: $z(u + v) = zu + zv$

Example:

Proof of PCA Part 5: $(uv)z = u(vz) \quad \forall u, v, z \in \mathbb{C}$

Let $u = a + bi, v = c + di, z = x + yi$ where $a, b, c, d, x, y \in \mathbb{R}$ so that $u, v, z \in \mathbb{C}$

$$(uv)z = [(a + bi)(c + di)](x + yi) = (ac - bd + (ad + bc)i)(x + yi) =$$

$$= ((ac - bd)x - (ad + bc)y) + ((ac - bd)y + (ad + bc)x)i$$

$$= (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$$

$$u(vz) = (a + bi)[(c + di)(x + yi)] = (a + bi)(cx - dy) + (cy + dx)i$$

$$= (a(cx - dy) - b(cy + dx)) + (a(cy + dx) + b(cx - dy))i$$

$$= (acx - ady - bcy - bdx) + (acy + adx + bcx - bdy)i$$

$$\text{Thus } (uv)z = u(vz) = (axc - bdx - ady - bcy) + (acy - bdy + adx + bcx)i$$

□

Info – Other Arithmetic of Complex Numbers

For $z \in \mathbb{C}$

1. $z^0 = 1$
2. $z^1 = z$
3. $z^{k+1} = z^k z \quad \forall k \in \mathbb{N}$
4. $(z^n)^m = z^{nm}$ and $z^n z^m = z^{n+m} \quad \forall n, m \in \mathbb{N} \cup \{0\}$

For $n \notin \mathbb{N} \cup \{0\}$ will be discussed in later lectures

Examples:

$$\text{Find a real solution to } 6z^3 + (1 + 3\sqrt{2}i)z^2 - (11 - 2\sqrt{2})i - 6 = 0$$

Let $z = x \in \mathbb{R}$, that is $z = x + 0i$

$$\implies 6x^3 + (1 + 3\sqrt{2}i)x^2 - (11 - \sqrt{2}i)x - 6 = 0$$

$$\implies (6x^3 + x^2 - 11x - 6) + (3\sqrt{2}x^2 + 2\sqrt{2}x)i = 0 + 0i$$

$$\implies (6x^3 + x^2 - 11x - 6) = 0 \text{ and } 3\sqrt{2}x^2 + 2\sqrt{2}x = 0$$

$x = 0$ or $x = -\frac{2}{3}$ for the imaginary part.

However, $x = 0$ does not satisfy the real part.

$$\therefore x = -\frac{2}{3} \implies z = -\frac{2}{3} + 0i$$

Conjugate and Modulus

Info – Complex Conjugate

The complex **conjugate** of a complex number $z = x + yi$ written \bar{z} is the complex number

$$\bar{z} = x - yi$$



Info – Properties of Conjugate

For the complex conjugate, the following properties hold $\forall z, w \in \mathbb{C}$

1. $\overline{(\bar{z})} = z$
2. $\overline{z + w} = \bar{z} + \bar{w}$
3. $z + \bar{z} = 2\Re(z)$ and $z - \bar{z} = 2\Im(z)i$
4. $\overline{zw} = \bar{z} \cdot \bar{w}$
5. If $z \neq 0$, $\overline{(z^{-1})} = (\bar{z})^{-1}$
6. If $w \neq 0$, $\overline{(\frac{z}{w})} = \frac{\bar{z}}{\bar{w}}$

Proof

Part 3

$$z + \bar{z} = 2\Re(z) \text{ and } z - \bar{z} = 2\Im(z)i$$

Let $z = x + yi$

$$z + \bar{z} = (x + yi) + (x - yi) = 2x + 0i = 2\Re(z)$$

$$z - \bar{z} = (x + yi) - (x - yi) = 0x + 2i = 2\Im(z)i$$

Part 4

$$\overline{zw} = \bar{z} \cdot \bar{w}$$

Let $z = x + yi, w = a + bi$

$$\overline{zw} = \overline{(x + yi)(a + bi)} = \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb + ya)i$$

$$\begin{aligned} \overline{zw} &= \overline{x + yia + bi} = (x - yi)(a - bi) = (xa - (-y)(-b)) + (x(-b) + (-y)a)i \\ &= (xa - yb) - (xb + ya)i \end{aligned}$$

Thus $\overline{zw} = \bar{z} \cdot \bar{w}$

□

Examples:

1. Prove $z \in \mathbb{R} \iff z = \bar{z}$

Let $z = x + yi \quad \forall x, y \in \mathbb{R}$

\implies

Suppose $z \in \mathbb{R}$, then $y = 0$, so that $z = x + 0i = x \in \mathbb{R}$.

Then $\bar{z} = x - 0i = x$

$$\therefore z = \bar{z}$$

\Longleftarrow

Suppose $z = \bar{z}$, then $x + yi = x - yi$

This implies $x = x, y = -y$. Thus $y = 0$

$$\therefore z = x + 0i = x \in \mathbb{R}$$

2. Prove that z is purely imaginary $\Leftrightarrow z = -\bar{z}$

3. Solve $z^2 = i\bar{z}$

Let $z = x + yi$. Then $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$

and $i\bar{z} = i(x - yi) = ix - i^2y = xi + y$

Then the equation becomes $x^2 - y^2 + 2xyi = xi + y$

$$\Rightarrow \begin{cases} x^2 - y^2 = y \\ 2xy = x \end{cases} \Rightarrow x = 0 \wedge y = \frac{1}{2} \Rightarrow \begin{cases} x^2 - \frac{1}{4} = \frac{1}{2} \\ -y^2 = 0 \end{cases} \Rightarrow z = \left\{ 0, -i, \pm \left(\frac{\sqrt{3}}{2} \right) + \frac{1}{2}i \right\}$$

Modulus of Complex Numbers

Info – Modulus of Complex Number

The **modulus** of the complex number $z = x + yi$, written $|z|$, is the non-negative real number

$$|z| = \sqrt{x^2 + y^2}$$

Info – Properties of Modulus

For the modulus, the following properties $\forall z, w \in \mathbb{C}$:

1. $|z| = 0 \Leftrightarrow z = 0$

2. $|\bar{z}| = |z|$

3. $\bar{z}z = |z|^2$

4. $|zw| = |z||w|$

5. If $z \neq 0$ then $|z^{-1}| = |z|^{-1}$

Side note: for $z \neq 0$, $z^{-1} = \frac{\bar{z}}{|z|^2}$

6. If $w \neq 0$, $\frac{z}{w} = z \cdot \frac{\bar{w}}{|w|^2}$

7. $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$

8. $\overline{\sum z_i} = \sum \overline{(z_i)}$

9. $\overline{\prod z_i} = \prod \overline{(z_i)}$

10. $|\prod z_i| = \prod (|z_i|)$

Proof

Part 3

$$\bar{z}z = |z|^2$$

Let $z = a + bi$, $\forall a, b \in \mathbb{R}$

$$\text{then } \bar{z}z = (a - bi)(a + bi) = (a^2 - (-b)(b)) + (ab + (-b)a)i$$

$$= a^2 + b^2 + 0i = \left(\sqrt{a^2 + b^2} \right)^2 = |z|^2$$

□

The Complex Plane and Polar Form

Complex Plane

💡 Tip – Geometric Interpretation and Graphical Properties

- z and \bar{z} are reflection of each other over real axis
- Modulus is the distance from the point z to origin
- For addition, it is similar to vector addition that is the parallelogram rule, $z + w$
- For subtraction, consider $z + w - w = z$ and the rest is same for addition

📘 Info – Triangle Inequality

For all $z, w \in \mathbb{C}$, we have

$$|z + w| \leq |z| + |w|$$

Note that $|z|$ is the modulus of z , not absolute value

Proof

Let $z = x + yi, w = u + vi$, where $x, y, u, v \in \mathbb{R}$

$$\begin{aligned} |z + w| &= |(x + u) + (y + v)i| = \sqrt{(x + u)^2 + (y + v)^2} \\ &= \sqrt{(x - (-u))^2 + (y - (-v))^2} \end{aligned}$$

(The Euclidean distance formula between (x, y) and $(-u, -v)$)

Consider the triangle ABC constructed from points

$$A : (0, 0); \quad B(x, y) = (z = x + yi); \quad C : (-u, -v) = (-w = -u - vi)$$

Let l_{AB} = length of side AB, l_{BC} and l_{AC} have the similar constructed

From geometric perspective, $l_{BC} \leq l_{AB} + l_{AC}$

$$\text{Note that } l_{AB} = \sqrt{x^2 + y^2} = |z|, l_{AC} = \sqrt{(-u)^2 + (-v)^2} = |w|$$

$$l_{BC} = \sqrt{(x - (-u))^2 + (y - (-v))^2} = |z + w|$$

$$\text{Therefore } |z + w| \leq |z| + |w|$$

□

Exercise:

Let $z \neq \pm i$. Prove that $\frac{z}{1+z^2}$ is real $\iff z \in \mathbb{R}$ or $|z| = 1$

Polar Form

Info – Polar Form

A **polar form** of a complex number z is denoted

$$z = r(\cos \theta + i \sin \theta)$$

where $r \geq 0$, being the modulus of z and angle $\theta \in \mathbb{R}$ be an **argument** of z

- θ is not unique unless given restriction of $\theta \in [0, 2\pi)$

Notice that in standard form $z = x + yi$

- $x = r \cos \theta$
- $y = r \sin \theta$
- $r = |z| = \sqrt{x^2 + y^2}$
- $\theta = \arctan\left(\frac{y}{x}\right)$

Examples:

Convert polar form to standard form

1. $z = 3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

2. $z = \cos \frac{15}{6}\pi + i \sin \frac{15}{6}\pi = 0 + 1i = i$

Convert from standard form to polar form

3. $z = \sqrt{6} + \sqrt{2}i$

$$r = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = 2\sqrt{2}$$

$$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \text{ is one possibility as the angle is not unique}$$

$$z = 2\sqrt{2}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$$

4. $z = -3\sqrt{2} + 3\sqrt{6}i = 6\sqrt{2}\left(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi\right)$

Info – Polar Multiplication for \mathbb{C}

For all complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, then their product is

$$z_1 z_2 = r_1 r_2 (\cos \theta(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

Proof

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, $z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \in \mathbb{C}$

$$z_1 z_2 = r_1(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

$$= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + 2i \sin \theta_1 \cos \theta_2)$$

$$= r_1 r_2 (\cos \theta(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

□

Examples:

1. Compute $(i + 1)i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \cdot (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \stackrel{\text{By PMC}}{=} \sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$
 $= \sqrt{2}(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = -1 + i$
2. Find $(\sqrt{6} + \sqrt{2}i)(-3\sqrt{2} + 3\sqrt{6}i)$
 $= 2\sqrt{2}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \cdot 6\sqrt{2}(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) = 24(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}) = 24(-\frac{\sqrt{3}}{2} + \frac{i}{2})$
 $= -12\sqrt{3} + 12i$

De Moivre's Theorem

Info – De Moivre's Theorem (DMT)

$\forall \theta, n \in \mathbb{R},$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Info – Corollary of DMT

For all complex number $z = r(\cos \theta + i \sin \theta)$ and integer n , except $|z| = r = 0$ and $n < 0$ we have

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

Examples:

1. $(\sqrt{3} - i)^{10} = [2(\cos -\frac{\pi}{6} + i \sin -\frac{\pi}{6})]^{10} = 1024(\cos -\frac{10\pi}{6} + i \sin -\frac{10\pi}{6}) = 512 + 512\sqrt{3}i$
2. Prove that $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$ (Hint: Consider $(\cos \theta + i \sin \theta)^4$ expand normally and apply DMT then compare real parts)
3. Prove that $\forall n \in \mathbb{Z}$ if $w \in \mathbb{C}, |w| = 1$ and θ is an argument of w , then $-\frac{i}{2}(w^n - w^{-n}) = \sin(n\theta)$

Info – Additional Information

For notation, we write $\text{cis}(\theta) = \cos \theta + i \sin \theta$ notation wise

Also, $\text{cis}(\theta) = e^{i\theta}$ that is having the similar properties of exponentials

$e^{i\pi} = \text{cis}(\pi) = -1 \implies e^{i\pi} + 1 = 0$ which is the Euler's Formula