

CH 5- Set Theory

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Empty Set

$\emptyset = \{\}$ but $\{\emptyset\} \neq \emptyset$

Cardinality

The number of elements in a finite set is S called the cardinality of S , denoted by $|S|$

Set Notation

Set Builder Notation - Type 1

The notation $\{x \in \mathcal{U} : P(x)\}$

Describes the set consisting of all objects x such that x is an element of \mathcal{U} , and $P(x)$ is true

Example: $A = \{n \in \mathbb{N} : n \mid 12\} = \{1, 2, 3, 4, 6, 12\}$

Set Builder Notation - Type 2

The notation $\{f(x) : x \in \mathcal{U}\}$

Describes the set consisting of all objects of the form $f(x)$ such that x is an element of \mathcal{U}

Example: $B = \{2k : k \in \mathbb{Z}\} = \text{all even numbers}$

Set Builder Notation - Type 3

The notation $\{f(x) : x \in \mathcal{U}, P(x)\}$ or $\{f(x) : P(x), x \in \mathcal{U}\}$ Both describes the set consisting of all objects of the form $f(x)$ such that x is an element \mathcal{U} and $P(x)$ is true

Example: $C = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$

Practices:

1. All multiple of 7: $\{x \in \mathbb{N} : 7 \mid x\}$
2. All odd perfect square: $\{(2x + 1)^2 : x \in \mathbb{Z}\}$
3. All points on a circle of radius 8 centered at origin: $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 64 : \}$
4. All sets of three integers which are the side lengths of a triangle:

$\{(x, y, z) : x, y, z \in \mathbb{N}, x < y < z, x + y < z\}$

Union and Intersection

The **union** of two sets S and T , denoted, $S \cup T$, is the set of all elements belonging to either set S or set T .

The **intersection** of two sets S and T , denoted, $S \cap T$, is the set of all elements belonging to either set S and set T .

Practice:

Let $C = \{3, 5, 7, 10\}$, $D = \{1, 3, 6, 7, 8\}$

1. $C \cup D = \{1, 3, 5, 6, 7, 8, 10\}$
2. $C \cap D = \{3, 7\}$

Let $A = \{m \in \mathbb{Z} : 2 \mid m\}$, $B = \{2k + 1 : k \in \mathbb{Z}\}$

1. $A \cup B = \mathbb{Z}$
2. $A \cap B = \emptyset$

For non empty sets A and B

1. If $|A| = 12$, $|B| = 4$, $|A \cap B| = 2$, $|A \cup B| = 14$
2. If $|A| = 10$, $|B| = 20$, $|A \cup B| = 25$, $|A \cap B| = 5$

Set Difference

The **set difference** of two sets S and T , written $S - T$ or $S \setminus T$ is the set of all elements belonging to S but not T .

Symbolically: $S - T = \{x : (x \in S) \wedge x \notin T\}$

Complement

The **complement** of a set S whose elements belong to \mathcal{U} , written S^c , is the set of all elements in \mathcal{U} but not in S .

Symbolically: $\{x \in \mathcal{U} \mid x \notin S\}$

Disjoin Set

Two sets S and T are said to be disjoint when $S \cap T = \emptyset$

Practice:

1. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $C = \{3, 5, 7, 10\}$, $D = \{1, 3, 6, 7, 8\}$

Find $|C - D^c| = 2$

2. Let $A = \{x : x \in \mathbb{N}, x \text{ is even}\}$, $B = \{x : x \in \mathbb{N}, x \text{ is not a prime}\}$

$A \cup B = \{1, 2, 4, 6, 8, 9, \dots\}$ $A \cap B = \{2\}$

3. If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$: False
4. If $|A \cap B| = |A|$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$: True
5. If $|A \cap B| = |A|$ and $|A \cap C| = |A|$, then $B \cap C = \emptyset$: False

Subsets

A set S is called a **subset** of a set T , denoted $S \subseteq T$ when every element of S belongs to T . T is **superset** of S

A set is called a **proper subset**, denoted $S \subsetneq T$, meaning S is a subset of T and there exists an element in T which does not belong to S . T is a **proper superset** of S

Examples:

1)

$$\{5, 15, 25\} \subseteq \{5, 10, 15, 20, 25\}$$

$$\{5, 15, 25\} \subsetneq \{5, 10, 15, 20, 25\}$$

2)

$$\{2, 4, 6\} \not\subseteq \{2025\}$$

$$\{2, 4, 6\} \not\subseteq \{1, 2, 3, 4, 5\}$$

3)

$$\mathbb{N} \subseteq \mathbb{Z}, \mathbb{Z} \subseteq \mathbb{Q}, \mathbb{Q} \subseteq \mathbb{R}$$

4)

$$\emptyset \subseteq S \text{ for all sets } S$$

$S \subseteq S$ for all sets S , but S is never a proper subset of S

5)

$$\text{For all sets } S \text{ and } T, S \cap T \subseteq T \text{ and } S \subseteq S \cup T$$

IMPORTANT

Subset can be expressed as an implication

To prove $S \subseteq T$, we need to prove the universally quantified implication

$$\forall x \in \mathcal{U}, (x \in S) \implies (x \in T)$$

Equal notation:

$$S \subseteq T \text{ and } T \subseteq S \iff S = T$$

Example:

1. Let $A = \{n \in \mathbb{N} : 4 \mid (n - 3)\}$ and let $B = \{2k + 1 : k \in \mathbb{Z}\}$, prove that $A \subseteq B$

Let $n \in \mathbb{Z}, (n \in A) \implies (n \in B)$

$A \equiv$ a set of natural numbers in form $4q + 3$

$B \equiv$ a set of integers that are odd

Since $4q + 3 = 2(2q + 1) + 1$, which is always odd.

$\therefore A \subseteq B$ and also $A \subsetneq B$

2. Prove $S = T \iff S \cap T = S \cup T$

$$(S = T \implies S \cup T = S \cap T) \wedge (S \cup T = S \cap T \implies S = T)$$

(\implies)

Suppose $S = T$, we need to show $S \cap T \subseteq S \cup T$ and $S \cup T \subseteq S \cap T$

If $x \in S \cap T$, then $x \in S, x \in S \cup T$.

Assume $x \in S \cup T$, then without loss of generality, we may suppose that $x \in S$.

Then since $S = T$, it follows that $x \in T$.

$\therefore x \in S \wedge x \in T \implies x \in S \cap T$

(\Leftarrow)

Suppose $S \cap T = S \cup T$, we must show $S \subseteq T \wedge T \subseteq S$

If $x \in S$ then $x \in S \cup T$.

Furthermore, $S \cap T = S \cup T \implies x \in S \cap T \implies x \in S \wedge x \in T$

$x \in T \implies S \subseteq T$. The proof of $T \subseteq S$ is similar and will be omitted

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