CH 1 — Vectors in Euclidean Space

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Vector Addition and Scalar Multiplication

🐳 **Info — Vector**

The set $\mathbb{R}^n$ is defined as $\left\{ \vec{x} = \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} \mid x_1, ..., x_n \in \mathbb{R} \right\}$

A **vector** is an element $\vec{x} = \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}$ of $\mathbb{R}^n$

The row notation of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$ is $\vec{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$

🐳 **Info — Equality**

We say that vectors $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix} \in \mathbb{R}^m$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ are **equal**

if $n = m$ and $u_i = v_i \forall\ i = 1, 2, ..., n$.

We denote it: $\vec{w} = \vec{v}$

🐳 **Info — Addition Properties**

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ ... \\ w_n \end{bmatrix} \in \mathbb{R}^n.$

Then $\vec{w} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ ... \\ u_n+v_n \end{bmatrix}$

1. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$
2. $\vec{w} + \vec{v} + \vec{w} = \vec{w} + (\vec{v} + \vec{w})$
3. There is a zero **vector**, $\vec{0} = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \end{bmatrix}^T \in \mathbb{R}^n$
4. $\vec{v} + \vec{0} = \vec{v}$
5. $\vec{v} + (-\vec{v}) = 0$

🐳 **Info − Additive Inverse**

Let $\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix} \in \mathbb{R}^n$. The additive inverse of $\vec{w}$ denoted $-\vec{w}$ is defined as

$$-\vec{w} = \begin{bmatrix} -u_1 \\ -u_2 \\ ... \\ -u_n \end{bmatrix}$$

$$\vec{w} - \vec{w} = \vec{w} + (-\vec{w}) = \vec{0}$$

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \\ ... \\ v_n - u_n \end{bmatrix}$$

🐳 **Info − Scalar Multiplication**

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}, \vec{w} \in \mathbb{R}^n, c, d \in \mathbb{R}$. Then the scalar product $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ ... \\ cv_n \end{bmatrix}$

1. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
2. $c(\vec{w} + \vec{v}) = c\vec{w} + c\vec{v}$
3. $0\vec{w} = \vec{0}$
4. If $c\vec{v} = \vec{0}$ then $c = 0 \vee \vec{v} = 0$
5. $c(d\vec{v}) = (cd)\vec{v}$

🐳 **Info − Linear Combination**

For $\vec{v_1}, ..., \vec{v_k} \in \mathbb{R}^n$, and $c_1, ..., c_k \in \mathbb{R}$ we call the expression

$$c_1\vec{v_1} + c_2\vec{v_2} + ... + c_k\vec{v_k}$$

a **linear combination** of $\vec{v_1}, ..., \vec{v_k}$.

Examples:

1. Let $\vec{u} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ then $2\vec{u} - 3\vec{v} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} - + \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

2. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Is $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ a linear combination of $\vec{u}$ and $\vec{v}$?

   We set $\vec{x} = c_1\vec{u} + c_2\vec{v}$ and try to solve for c_1, c_2

   That is $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 \end{bmatrix}$, we obtain $c_1 = 2, c_2 = \frac{1}{2}$. So $\vec{x}$ is a linear combination of $\vec{u}, \vec{v}$

Bases

> 🐳 **Info − Span**
>
> Let $\mathcal{B} = \{\vec{v_1}, ..., \vec{v_k}\}$ be a set of vectors in $\mathbb{R}^n$. We define the **span** of $\mathcal{B}$ by
>
> $$\text{Span } \mathcal{B} = \{c_1\vec{v_1} + c_2\vec{v_2} + ... + c_k\vec{v_k} \mid c_1, c_2, ...c_l \in \mathbb{R}\}$$
>
> We say that the set Span $\mathcal{B}$ is spanned by $\mathcal{B}$ adn that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$
>
> Span might not cover the entire plane if
>
> - Vectors are linear dependent to each other
> - One of them is $\vec{0}$
>
> **Theorem**
>
> Let $\vec{v_1}, ..., \vec{v_k} \in \mathbb{R}^n$. Some vector $\vec{v_i}, 1 \le i \le k$, can be written as a linear combination of $\vec{v_1}, ..., \vec{v_{i-1}}, ..., \vec{v_k}$ if and only if $\text{Span}\{\vec{v_1}, ..., \vec{v_k}\} = \text{Span}\{\vec{v_1}, ..., \vec{v_{i-1}}, ..., \vec{v_k}\}$

Proof

Example:

Consider $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Describe Span $\vec{v_1}, \vec{v_2}$ geometrically.

$\text{Span } \{\vec{v_1}, \vec{v_2}\} = \text{Span } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \mathbb{R}^2$

> 🐳 **Info − Linear Dependence/Independence**
>
> A set of vectors $\{\vec{v_1}, ..., \vec{v_k}\} \in \mathbb{R}^n$ is said to be **linearly dependent** if there exist coefficients $c_1, ..., c_k$ not all zero such that
>
> $$\vec{0} = c_1\vec{v_1} + ... + c_k\vec{v_k}$$
>
> A set of vectors $\{\vec{v_1}, ..., \vec{v_k}\} \in \mathbb{R}^n$ is said to be **linearly Independent** if the only solution to $c_1, ..., c_k$ not all zero such that
>
> $$\vec{0} = c_1\vec{v_1} + ... + c_k\vec{v_k}$$
>
> is $c_1 = c_2 = ... = c_k = 0$ (called **trivial solution**)

Examples:

1. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linearly dependent $\iff$ at least one of $\vec{u}, \vec{v}$ is a scalar multiple of the other.

Proof

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$

1. Assume $\{\vec{u}, \vec{v}\}$ is linearly dependent. Then $\exists c_1, c_2 \in \mathbb{R}$, not both zero s.t.

$$c_1 \vec{u} + c_2 \vec{v} = \vec{0}$$

WLOG, assume $c_1 \neq 0$ Then, $c_1 \vec{u} = -c_2 \vec{v} \implies \vec{u} = -\frac{c_2}{c_2}\vec{v}$. Thus $\vec{u}$ is a scalar multiple of $\vec{v}$

2. Assume WLOG $\vec{u}$ is a scalar multiple of $\vec{v}$. Then $\exists a \in \mathbb{R}$ s.t.

$$\vec{u} = a\vec{v} \implies 1\vec{u} - a\vec{v} = \vec{0}$$

Since $1 \neq 0$, $\{\vec{u}, \vec{v}\}$ is linearly dependent.

$\square$

2. Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Consider the equation $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Which gives $\begin{cases} c_1 + c_2 + 2c_3 = 0 \\ 2c_1 - c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$

$\implies c_2 = c_1 = -c_3$. Thus we get a solution for any c_3. Pick $c_3 = -1 \implies c_2 = c_1 = 1$.

Thus the set is linearly dependednt.

> 🐳 **Info — Linear Dependence Theorem**
> A set of vectors $\{\vec{v_1}, ..., \vec{v_k} \in \mathbb{R}^n\}$ is linearly dependent if and only if
>
> $$\vec{v_i} \in \text{Span} \{\vec{v_1}, ..., \vec{v_{i-1}}, \vec{v_{i+1}}, ..., \vec{v_k}\} \text{ for some } i, 1 \leq i \leq k$$

> 🐳 **Info — Zero Vector and Linear Dependence**
> If a set of vectors $\{\vec{v_1}, ..., \vec{v_k}\}$ contains the zero vector, then it is linearly dependent.
>
> Proof:
>
> Let $\vec{v_i} = \vec{0}$
>
> $$0\vec{v_1} + ... + 0\vec{v_{i-1}} + \vec{v_i} + 0\vec{v_{i+1}} + 0\vec{v_k} = \vec{0}$$
>
> $\square$

Basis

Let S be a subset of $\mathbb{R}^n$. If $\{\vec{v_1}, ..., \vec{v_k}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ s.t. $S = \text{Span}\{\vec{v_1}, ..., \vec{v_k}\}$, then the set $\{\vec{v_1}, ..., \vec{v_k}\}$ is called a **basis** for S.

We define a basis for the set $\{\vec{0}\}$ to be the empty set

Standard Basis

In $\mathbb{R}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set $\text{E} = \{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$ is called the **standard basis for** $\mathbb{R}^n$

(i.e. $\mathbb{R}^3$ is $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$)

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix} = v_1\vec{e_1} + v_2\vec{e_2} + ... + v_n\vec{e_n}$ then we call $v_1, v_2, ..., v_n$ the **components of** $\vec{v}$

Examples:

Is B is a basis for $\mathbb{R}^2$

1. $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. This set of vectors is linearly independent, thus is a standard basis for $\mathbb{R}^2$.

   That is $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \left\{ c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2, \in \mathbb{R} \right\} = \mathbb{R}^2$

2. $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$. Note that this set of vectors is linearly dependent as one is a scalar multiple of another, thus cannot be considered as a basis for $\mathbb{R}^2$

3. $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ is linearly independent since neither scalar multiple of another. We need to prove:

   - Span $B \subseteq \mathbb{R}^2$, which is obvious, since the vectors in B are in $\mathbb{R}^2$ the linear combination of these will be $\mathbb{R}^2$
   - $\mathbb{R}^2 \subseteq$ Span B, consider an arbitrary $\vec{x} \in \mathbb{R}^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ that is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ 2c_1 4c_2 \end{bmatrix}$.

   We obtain $\begin{cases} c_1 + 3c_2 = x_1 \\ 2c_1 - 4c_2 = x_2 \end{cases} \Longrightarrow c_1 = x_1 - 3c_2 \Longrightarrow 2(x_1 - 3c_2) + 4c_2 = x_2 \Longrightarrow c_2 = -\frac{1}{2}x_2 + x_1 \Longrightarrow$

   $c_1 = x - 3\left(-\frac{1}{2}x_2 + x_1 = -2x_1 + \frac{3}{2}x_2\right)$

   Therefore $\mathbb{R}^2 \subseteq \text{Span}B$

   That is B is a standard basis for $\mathbb{R}^2$

If $\mathcal{B} = \{\vec{v_1}, ..., \vec{v_k}\}$ is a basis for a subset S of $\mathbb{R}^n$, then every vector $\vec{x} \in S$ can be written as a unique linear combination of the vectors in $\mathcal{B}$

Proof

Let $\vec{x} \in S$ and assume $\exists c_1, ..., c_k, d_1, ..., d_k \in \mathbb{R}$ s.t. $\vec{x} = c_1 \vec{v_1} + ... + c_k \vec{v_k}$ and $\vec{x} = d_1 \vec{v_1} + ... + d_k \vec{v_k}$.

Subtracting these two equations: $\vec{0} = (c_1 - d_1)\vec{v_1} + ... + (c_k - d_k)\vec{v_k}$, with $\{\vec{v_1}, ..., \vec{v_k}\}$ is basis, thus linearly independent, so there is $(c_1 - d_1) = ... = (c_k - d_k) = 0$, thus $\vec{x}$ can be written as a uniue linear combination.

$\square$

Subspace

> 🐋 **Info − Definition**
>
> A subset $\mathbb{S}$ of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if for every $\vec{x}, \vec{y}, \vec{w} \in \mathbb{S}$ and $c, d \in \mathbb{R}$ we have
>
> 1. $\vec{x} + \vec{y} \in \mathbb{S}$
> 2. $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
> 3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
> 4. $\exists \vec{0} \in \mathbb{S}$ s.t. $\vec{x} + \vec{0} = \vec{x} \; \forall \vec{x} \in \mathbb{S}$
> 5. For every $\vec{x} \in \mathbb{S}, \exists(-\vec{x} \in \mathbb{S})$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$
> 6. $c\vec{x} \in \mathbb{S}$
> 7. $c(d\vec{x}) = (cd)\vec{x}$
> 8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
> 9. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
> 10. $1\vec{x} = \vec{x}$

> 🐋 **Info − Subspace Test**
>
> Let $\mathbb{S}$ be a **non-empty** subset of $\mathbb{R}^n$. If $\vec{x} + \vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S} \; \forall \vec{x}, \vec{y} \in \mathbb{S}$ and $c \in \mathbb{R}$, then $\mathbb{S}$ is a subspace of $\mathbb{R}^n$
>
> Be aware that $\vec{0} \in \mathbb{S}$

Examples:

1. Let $\vec{u}, \vec{w} \in \mathbb{R}^n$ and let $\mathbb{S} = \text{Span}\{\vec{u}, \vec{w}\}$. Is $\mathbb{S}$ a subspace of $\mathbb{R}^n$?

   Proof

   - Note that $\vec{0} \in \mathbb{S}$ since $\vec{0} = 0\vec{u} + 0\vec{w} \; \forall \vec{u}, \vec{w} \in \mathbb{R}^n$.
   - Let $\vec{x}, \vec{y} \in \mathbb{S}, \exists c_1, c_2, d_1, d_2 \in \mathbb{R}$ s.t. $\vec{x} = c_1 \vec{u} + c_2 \vec{w}, \vec{y} = d_1 \vec{u} + d_2 \vec{w}$. Then $\vec{x} + \vec{y} = c_1 \vec{u} + c_2 \vec{w} + d_1 \vec{u} + d_2 \vec{w} = (c_1 + d_1)\vec{u} + (c_2 + d_2)\vec{w}$.

     Therefore $\vec{x} + \vec{y} \in \mathbb{S}$

   - Let $c \in \mathbb{R}$, then $c\vec{x} = c(c_1 \vec{u}) + (c_2 \vec{w}) = (cc_1)\vec{u} + (cc_2)\vec{u} \in \mathbb{S}$

   By Subspace Test, $\mathbb{S} \in \mathbb{R}^n$

   $\square$

2. Let $\mathbb{S} = \left\{ \vec{x} \in \mathbb{R}^2 \mid \vec{x} = r\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}, r \in \mathbb{R} \right\}$. Is $\mathbb{S}$ a subsapce of $\mathbb{R}^2$?

False

Proof

For contradiction, assume $\vec{0} \in \mathbb{S}$, then $\vec{0} = r\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

That is $\begin{cases} r+3=0 \\ 2r+4=0 \end{cases} \implies r = -3 \wedge r = -2 \implies$ contradiction. Since $-3 \neq -2$, therefore $\vec{0} \notin \mathbb{S}$

Hence $\mathbb{S}$ is not a subspace of $\mathbb{R}^2$

$\square$

3. Let $\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + 2x_2 = 0 \right\}$. Is $\mathbb{S}$ a subspace of $\mathbb{R}^3$?

True

Proof

- Note that $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $0 + 2(0) = 0$, so $\vec{0} \in \mathbb{S}$

- Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{S}, c \in \mathbb{R}$, then $x_1 + 2x_2 = 0$ and $y_1 + 2y_2 = 0$. Then $\vec{x} + \vec{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$ and $\vec{x_1} + \vec{y_1} + 2(\vec{x_1} + \vec{y_1}) = \vec{x_1} + \overrightarrow{2x_2} + \vec{y_1} + \overrightarrow{2y_2} = 0 \implies \vec{x} + \vec{y} \in \mathbb{S}$

- $c\vec{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} \implies c(x_1 + 2x_2) = c \cdot 0 = 0 \implies c\vec{x} \in \mathbb{S}$

By Subspace Test, $\mathbb{S} \in \mathbb{R}^3$

$\square$

> 🐳 **Info − Theorem**
>
> If $\vec{v_1}, ..., \vec{v_k} \in \mathbb{R}^n$, then $\mathbb{S} = \text{Span}\{\vec{v_1}, ..., \vec{v_k}\}$ is a subspace of $\mathbb{R}^n$
>
> Note that the converse is also true. The proof will be given in CH5

Example:

Find $\mathbb{S} = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a - b + c = c \right\}$

1. An arbitrary element $\vec{x} \in \mathbb{S}$ has form

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ a-b+c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \\ -b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore $\mathbb{S} = \text{Span } B$ where $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

2.
We set $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies c_1 = c_2 = c_3 = 0$

Thus B is linearly independent

Therefore B is a basis for $\mathbb{S}$, that is $\dim(B) = 3$ (since there are 3 vectors)

Dot Product

> 🐳 **Info − Dot Product**
>
> Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ ... \\ w_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. We defined their **dot product** by
>
> $$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + ... + u_n v_n = \sum_{i=1}^{n} u_i v_i$$
>
> 1. $\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$
>
> 2. $(\vec{w} + \vec{v}) \cdot \vec{w} = \vec{w} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
>
> 3. $(c\vec{w}) \cdot \vec{v} = c(\vec{w} \cdot \vec{v})$
>
> 4. $\vec{w} \cdot \vec{w} \geq 0$, with $\vec{w} \cdot \vec{w} = 0 \iff \vec{w} = 0$

Example:

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, y = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, then the dot product $\vec{x} \cdot \vec{y} = (1)(-1) + (2)(0) + (3)(1) + (4)(2) = 10$

> 🐳 **Info − Norms**
>
> Let $\vec{v}, \vec{w} \in \mathbb{R}^n$
>
> 1. The **length** of vector $\vec{w}$ is $\|\vec{w}\| = \sqrt{\vec{w} \cdot \vec{w}} = \sqrt{w_1^2 + ... + w_n^2}$
>
> 2. If $c \in \mathbb{R}, \vec{w} \in \mathbb{R}^n$, then $\|\vec{w}\| = |c| \, \|\vec{w}\|$
>
> 3. $\vec{v}$ is a **unit vector** if $\|\vec{v}\| = 1$
>
> 4. Properties of norm
>
>   1. $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \iff \vec{v} = 0$
>
>   2. $\|c\vec{v}\| = |c| \, \|\vec{v}\|$
>
>   3. $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$
>
>   4. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

Examples:

1.
Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, then $\|\vec{x}\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$

2.

$$\text{Let } \vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \text{ then } \|e_i\| = 1$$

This is an unit vector since all standard basis vectors are unit vectors

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🐳 **Info** — **Orthogonality/Angle**

1. **Normalization** is when some $\vec{v}$ is a non-zero vector,

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

in the direction of $\vec{v}$ by scaling $\vec{v}$

2. With $\vec{v}, \vec{w}$ non-zero vectors. The angle $\theta, 0 \leq \theta \leq \pi$ between $\vec{v}$ is such that

$$\vec{w} \cdot \vec{v} = \|\vec{w}\|\|\vec{v}\| \cos \theta,$$

$$\theta = \arccos\left(\frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\|\|\vec{v}\|}\right)$$

3. $\vec{w}, \vec{v} \in \mathbb{R}^n$ are **orthogonal/perpendicular** if $\vec{w} \cdot \vec{v} = 0$

   - Note that $\vec{0}$ is orthogonal to any vector

4. A set of vectors $\vec{v_1}, ..., \vec{v_k} \in \mathbb{R}^n$ is called an orthogonal set if $\vec{v}_i \cdot \vec{v}_j = 0 \ \forall \ i \neq j$

   - Any set of standard basis is an **orthonormal** set because the elements are orthogonal and their norm are all 1

The Cross Product in $\mathbb{R}^3$

🐳 **Info** — **Cross Products**

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{w} = ... \in \mathbb{R}^3.$

The **cross product** of $\vec{u}, \vec{v}$ is defined to be the vector in $\mathbb{R}^3$ given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

> 💡 **Tip** — Tricks for the Cross Product
>
> - 1st coordinate: $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ _{Ignore row 1} $\implies$ compute the difference of the diagonals $v_2 w_3 - v_3 w_2$
>
> - 2nd coordinate: $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ _{Ignore row 2} $\implies$ compute the difference of the negative of the diagonals
> $-(v_1 w_3 - v_3 w_1)$
>
> - 3rd coordinate: $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ _{Ignore row 3} $\implies$ compute the difference of the diagonals $v_1 w_2 - v_2 w_1$

> ⚠️ **Warning** — **Cross Product**
>
> Let $\vec{u}, \vec{v} \in \mathbb{R}^3$
>
> $$\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$$

Example:

Given vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ we want to find a vector $\vec{n} \in \mathbb{R}^3$ which is orthogonal to both.

What is, we want $\vec{v} \cdot \vec{n} = \vec{w} \cdot \vec{n} = 0 \implies \vec{n} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$

> 🐳 **Info** — **Linearity of the Cross Product**
>
> Let $c \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, then
>
> 1. If $\vec{n} = \vec{v} \times \vec{u}$ then for any $\vec{y} \in \text{Span} \{\vec{v}, \vec{w}\}$ we have $\vec{y} \cdot \vec{n} = 0$
>
> 2. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$
>
> 3. $\vec{v} \times \vec{v} = \vec{0}$
>
> 4. $\vec{v} \times \vec{w} = \vec{0}$ if and only if either $\vec{v} = \vec{0}$ or $\vec{w}$ is a scalar multiple of $\vec{v}$
>
> 5. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
>
> 6. $\vec{u} \times c(\vec{v}) = c(\vec{u} \times \vec{v})$
>
> 7. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \, |\sin \theta|$ where θ is the angle between $\vec{u}$ and $\vec{v}$

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Scalar Equation of a plane

Example:

1. Find a scalar equation of the plane in $\mathbb{R}^3$ given by vector equation $\vec{x} = s\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$

   Let $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$

   Therefore, by definition, the scalar equation for this plane is $x_1 - 5x_2 + 3x_3 = 5 - 5 + 6 \Longrightarrow$ $x_1 - 5x_2 + 3x_3 = 6$

2. Let P be the plane in $\mathbb{R}^3$ given by the scalar equation $2x_1 - 3x_2 + 407x_3 = \pi$. Find a normal vector to P.

   By observation, we obtain that $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ 407 \end{bmatrix}$
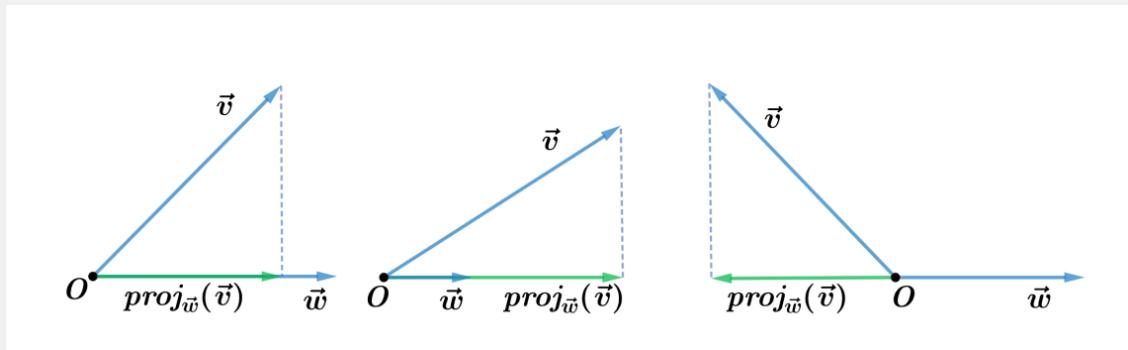
Projections

> 🐳 **Info − Projection**
>
> Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq 0$.
>
> 1. The **projection** of $\vec{v}$ onto $\vec{w}$ is defined by
>
> $$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w}$$
>
> We also refer to this as the **projection of $\vec{v}$ in the $\vec{w}$ direction**
>
> Illustration of $\text{proj}_{\vec{w}}(\vec{v})$:
>
>
>
> 2. We refer to the quantity
>
> $$\|\vec{v}\| \cos \theta = \vec{v} \cdot \hat{w}$$
>
> as the **component** (or scalar component) **of $\vec{v}$ along $\vec{w}$**
>
> 3. The **perpendicular** of $\vec{v}$ onto $\vec{w}$ is defined by $\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$
>
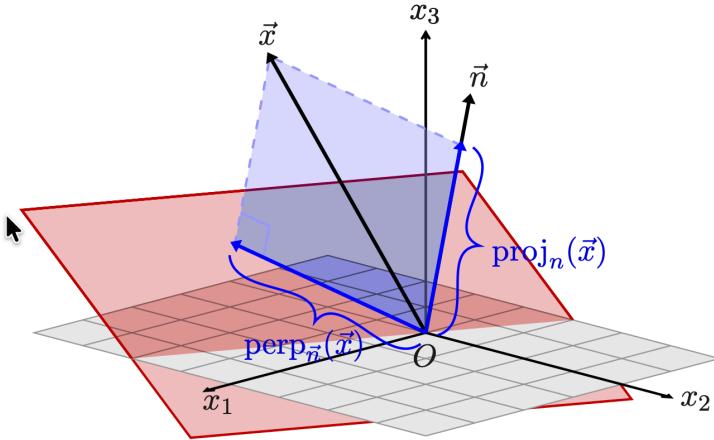> 4. The projection and the perpendicular of a vector $\vec{v}$ onto $\vec{w}$ are orthogonal; that is
>
> $$\text{perp}_{\vec{w}}(\vec{v}) \cdot \text{proj}_{\vec{w}}(\vec{v}) = 0$$

Example:

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Find $\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$

We project instead on a plane, we project onto the normal vector $\vec{n}$. Then we use the fact that a normal vector is orthogonal to the plane P

🐳 **Info** − **Projection onto Plane**

Let P be a plane in $\mathbb{R}^3$ that passes through the origin and has normal $\vec{n}$.

The **projection** of $\vec{x} \in \mathbb{R}^3$ onto P is defined by

$$\text{proj}_{P(\vec{v})} = \text{perp}_{\vec{n}}(\vec{x})$$

The **perpendicular** of $\vec{x}$ onto P is defined by

$$\text{perp}_{P(\vec{x})} = \text{proj}_{\vec{n}}(\vec{x})$$

Example:

Let P be the plane $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \vec{u} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$, find $\text{proj}_P(\vec{u})$

$$\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{proj}_P(\vec{u}) = \text{perp}_{\vec{n}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{n}}(\vec{u}) = \vec{u} - \frac{\vec{u} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} - 7 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Complex Euclidean Space (TBD at the end of term)

🐳 **Info** − **Vectors in $\mathbb{C}^n$**

The set $\mathbb{C}^n$ is defined as $\left\{ \vec{z} = \begin{bmatrix} z_1 \\ ... \\ z_n \end{bmatrix} : z_1, ..., z_n \in \mathbb{C} \right\}$

The **vector** is an element $\vec{z} = \begin{bmatrix} z_1 \\ ... \\ z_n \end{bmatrix}$ of $\mathbb{C}^n$

In $\mathbb{C}^n$, let $\vec{e}_i$ be the vector whose i^{th} component is 1 with all other components 0. The set $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ is called the **standard basis for** $\mathbb{C}^n$

Standard Inner Product in $\mathbb{C}^n$

> 🐳 **Info — Standard inner product**
>
> Let $c \in \mathbb{C}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$
>
> The **standard inner product** of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ ... \\ w_n \end{bmatrix}$ is
>
> $$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w_1} + v_2 \overline{w_2} + ... + v_n \overline{w_n}$$
>
> 1. $\langle \vec{u}, \vec{w} \rangle = \overline{\langle \vec{v}, \vec{w} \rangle}$
>
> 2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
>
> 3. $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
>
> 4. $\langle \vec{v}, \vec{v} \rangle \geq 0$, with $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$
>
> 5. The length: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$
>
> 6. $\vec{w}, \vec{v}$ are **orthogonal/perpendicular** if $\langle \vec{w}, \vec{v} \rangle = 0$
>
> 7. With $\vec{w} \neq 0$. The **projection of $\vec{v}$ onto $\vec{w}$** is defined by
>
> $$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} = \langle \vec{v}, \vec{w} \rangle \hat{w}$$