

CH 4 - Derivatives

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Velocity

🐼 Info — Average Velocity and Instantaneous Velocity

$$v_{avg} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

$$v_{inst} = \lim_{t \rightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \rightarrow 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Definition of Derivatives

🐼 Info — Average Rate of Change and Instantaneous Rate of Change (Derivative)

$$f_{avg} = \frac{f(b) - f(a)}{b - a}$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

If $f'(x)$ exists at $x = a$, then $f(x)$ is **differentiable** at $x = a$

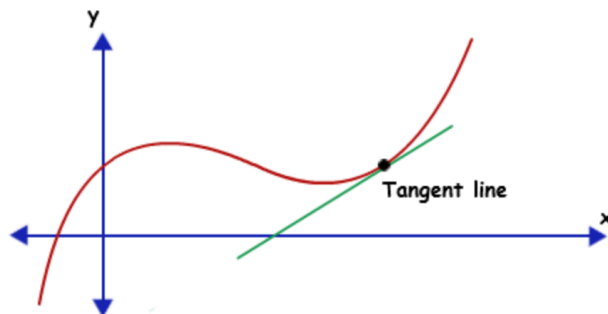
🐼 Info — Tangent Line

If $f(x)$ is differentiable at $x = a$, then the **tangent line** to $f(x)$ at $x = a$ is the line passing through $(a, f(a))$ with slope $f'(a)$

The equation of the tangent line

$$y = f'(a)(x - a) + f(a)$$

$(a, f(a))$ is the **point of tangency**



Examples:

Find the tangent line to $f(x) = \frac{1}{x+5}$ at $x = 3$

$$f(3) = \frac{1}{8}$$

$$f'(3) = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{a+h+5} - \frac{1}{a+5}}{h} =$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \frac{a+5 - (a+h+5)}{(a+5)(a+h+5)} = \lim_{h \rightarrow 0} -\frac{1}{(a+5)(a+h+5)} = -\frac{1}{(a+5)^2} = -\frac{1}{64}$$

$$y = -\frac{1}{64}(x - 3) + \frac{1}{8}$$

Info – Differentiability Implies Continuity

If a function f is differentiable at $x = a$, then f is continuous at $x = a$

Proof

f is differentiable at $x = a$ then, $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists

$$\lim_{h \rightarrow 0} [f(a+h) - f(a)] = 0 \implies \lim_{h \rightarrow 0} [f(a+h) - f(a) + f(a)] = \lim_{h \rightarrow 0} f(a) \implies$$

$$\lim_{h \rightarrow 0} f(a) = f(a)$$

Warning – Continuity Not Implies Differentiability

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h-0}{h} = 1$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h-0}{h} = -1$$

Thus $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \text{DNE}$ but continuous.

\therefore continuity does not imply differentiability

Info – Differentiability of Function

We say that f is **differentiable** on an interval I if $f'(a)$ exists $\forall a \in I$.

We define the derivative function $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

We sometimes also write $f'(x)$ as $\frac{d}{dx} f(x)$, and $f'(a) = \frac{d}{dx} f(x) \big|_a$

Info – Constant Function

$$f(x) = c$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c-c}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

Info – Linear Function

$$f(x) = mx + b$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(mx+h)+b - f(mx+b)}{h} = \lim_{h \rightarrow 0} m \frac{h}{h} = m$$

Info – Quadratic Function

$$f(x) = px^2 + sx + c$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[p(x+h)^2 + s(x+h) + c] - [px^2 + sx + c]}{h} = \lim_{h \rightarrow 0} \frac{2xph + xh^2 + sh}{h} = \lim_{h \rightarrow 0} 2xp + xh + s = 2xp + s$$

Info – Basic Trig

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$$

$$\lim_{h \rightarrow 0} \frac{[\sin x (\cos h - 1)]}{h} + \lim_{h \rightarrow 0} \cos x \frac{\sin h}{h} = \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos^2 h - 1}{h \cdot (\cos h + 1)} + \cos x =$$

$$\sin x \cdot \lim_{h \rightarrow 0} \frac{\sin^2 h}{h \cdot (\cos h + 1)} = \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \lim_{h \rightarrow 0} \frac{\sin h}{\cos h + 1} + \cos x = \cos x$$

We define e to be the unique base of an exponential function with slope 1 through $(0, 1)$

Info – Derivative Rules

Let $f(x)$ and $g(x)$ be differentiable at $x = a$

1. $w(x) = cf(x) \implies w'(x) = cf'(x)$
2. $w(x) = f(x) \pm g(x) \implies w'(x) = f'(x) \pm g'(x)$
3. $w(x) = f(x)g(x) \implies w'(x) = f'(x)g(x) + f(x)g'(x)$
4. If $g(x) \neq 0$, $w(x) = \frac{f(x)}{g(x)} \implies w'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
5. If $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R} \setminus \{0\} \implies f'(x) = \alpha x^{\alpha-1}$
6. $w(x) = (g \circ f)(x) = g(f(x)) \implies w'(x) = g'(f(x)) \cdot f'(x) \sim \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

Warning – Power Rule

If $x = 0$, x^{-1} does not make sense so that is why $\alpha \in \mathbb{R} \setminus \{0\}$

Proof

We suppose that $f(x), g(x)$ are differentiable, so that the limits:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \text{ exists}$$

1. Product rule:

$$\lim_{h \rightarrow 0} \frac{w(x+h) - w(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} = \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{(g(x+h) - g(x))f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \rightarrow 0} g(x+h) + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h \rightarrow 0} f(x)$$

$$= f'(x)g(x) + f(x)g'(x)$$

2. Quotient rule

$$\lim_{h \rightarrow 0} \frac{w(x+h) - w(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} = \lim_{h \rightarrow 0} \frac{g(x)(f(x+h) - f(x))}{\frac{h}{g(x)g(x+h)}} - \frac{f(x)(g(x+h) - g(x))}{\frac{h}{g(x)g(x+h)}} \\
&= \frac{\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \rightarrow 0} g(x) - \lim_{h \rightarrow 0} f(x) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}}{\lim_{h \rightarrow 0} g(x+h)g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}
\end{aligned}$$