CH 2 — Inequalities and Limits

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Triangle Inequality

|x - y| <= |x - z| + |z - y| for $x, y, z \in \mathbb{R}$

Sketch: the straight-line distance is shortest. Without loss of generality assume $x \leq y$; swapping x, y preserves the statement.

Number-line proof by cases:

- Case 1 $z \leq x \leq y$: $|x - y| \leq |z - y| \leq |x - z| + |z - y|$
- Case 2 $x \leq z \leq y$: $|x - y| = |x - z| + |z - y|$
- Case 3 $x \leq y \leq z$: $|x - y| \leq |x - z| + |z - y|$

Triangle Inequality 2

For all $a, b \in \mathbb{R}$

|a + b| <= |a| + |b|

Proof: apply the triangle inequality to $x = a$, $y = -b$, $z = 0$.

Quick check

Is $|a - b| \leq |a| - |b|$ for all a, b No

Example: $a = 10$, $b = -9$ gives $|10 - (-9)| = 19$ while $|10| - |-9| = 1$

Interval translations

1. $|x - a| < \delta \Rightarrow x \in (a - \delta, a + \delta)$
2. $|x - a| \leq \delta \Rightarrow x \in [a - \delta, a + \delta]$
3. $0 \leq |x - a| \leq \delta \Rightarrow x \in (a - \delta, a) \cup (a, a + \delta)$

Practice

1) Solve $|2x - 5| < 3$

$-3 < 2x - 5 < 3 \Rightarrow 1 < x < 4$

Answer: $x \in (1, 4)$

2) Solve $2 < |x + 7| \leq 3$

Split into $|x + 7| > 2$ and $|x + 7| \leq 3$

Answer: $x \in [-10, -9) \cup (-5, -4]$

3) Solve $\frac{|x+2|}{|x-2|} > 5$

Consider regions $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$ and track signs of $x + 2$ and $x - 2$

Answer: $x \in \left(\frac{4}{3}, 2\right) \cup (2, 3)$

Infinite Sequences

A sequence is an ordered list $a_1, a_2, a_3, \ldots$; write $(a_n)_{\{n=1\}}^{\{\infty\}}$

A subsequence chooses indices $n_1 < n_2 < \ldots$, yielding $a_{\{n_1\}}, a_{\{n_2\}}, \ldots$

The tail with cutoff k is $a_k, a_{\{k+1\}}, a_{\{k+2\}}, \ldots$

Convergence (definition)
We say $\lim_{\{n\to\infty\}} a_n = L$ if for every $\varepsilon > 0$ there exists N such that

$n > N \Rightarrow |a_n - L| < \varepsilon$

Equivalent formulations:

- Every interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of (a_n)
- Only finitely many terms lie outside $(L-\varepsilon, L+\varepsilon)$
- More generally, any open interval (a, b) containing L contains a tail

Examples
1) Show $\lim_{\{n\to\infty\}} \frac{1}{\sqrt[3]{n}} = 0$

Choose $N = \frac{1}{\varepsilon^3}$

Then $n > N \Rightarrow \left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon$

2) Show $\lim_{\{n\to\infty\}} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$

Estimate $\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| \leq \frac{5}{16n+4}$

Pick $N > \frac{5}{16\varepsilon} - \frac{1}{4}$

Theorem (Equivalent definitions of the limit of a sequence)
For a sequence (a_n) and a number L, the following are equivalent

1) $\lim_{\{n\to\infty\}} a_n = L$

2) For every $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of (a_n)

3) For every $\varepsilon > 0$, only finitely many n satisfy $|a_n - L| \geq \varepsilon$

4) Every interval (a, b) containing L contains a tail of (a_n)

5) Given any interval (a, b) containing L, only finitely many terms of (a_n) lie outside (a, b)

Example 1 (worked)

Show $\lim_{\{n\to\infty\}} \frac{1}{\sqrt[3]{n}} = 0$

Side work:

$\left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon \Rightarrow \frac{1}{\sqrt[3]{n}} < \varepsilon$

$$\Rightarrow \sqrt[3]{n} > \frac{1}{\varepsilon}$$

$\Rightarrow n > \frac{1}{\varepsilon^3}$

Formal proof:

Let $\varepsilon > 0$ and choose $N = \frac{1}{\varepsilon^3}$

If $n > N$ then $\left| \frac{1}{\sqrt[3]{n}} \right| < \frac{1}{\sqrt[3]{N}} = \frac{1}{\sqrt[3]{\frac{1}{\varepsilon^3}}} = \varepsilon$

Example 2 (worked)

Prove $\lim_{\{n\to\infty\}} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$

Estimate:

$$\mid \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \mid = \frac{|5n-3|}{16n^2+4n+4} \leq \frac{5n}{16n^2+4n} = \frac{5}{16n+4}$$

Given $\varepsilon > 0$, pick $N > \frac{5}{16\varepsilon} - \frac{1}{4}$

Then for $n > N$

$$\mid \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \mid < \frac{5}{16n+4} \leq \frac{5}{16N+4} < \varepsilon$$