

CH 2 — Differential Equations

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Introduction to Differential Equations

Info — Differential Equations

A differential equation (DE), is an equation involving an unknown function and its derivatives. The term ordinary differential equation (ODE) refers to a differential equation involving single-variable functions, whereas the term partial differential equation (PDE) refers to a differential equation involving multivariable functions (i.e., functions with multiple inputs).

An ODE is expressed

$$F(x, y, y', y'', \dots, y^n) = 0$$

for some $n \in \mathbb{N}$

The order of a DE is the order of the **highest derivative** that appears in the equation.

A function $y = \varphi(x)$ is a solution to the differential equation $F(x, y, y', y'', \dots, y^n) = 0$ if

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), \dots, \varphi^n(x)) = 0$$

The graph of a solution to a DE is called a **solution curve**

The complete collection of solutions to a DE, including any arbitrary constants, is called its general solution. A particular solution to a DE is one in which all arbitrary constants have been specified.

A differential equation together with one or more initial conditions is known as an initial value problem (IVP)

Examples:

1. First order differential equation $\frac{dy}{dx} = x + y$
2. Second order differential equation $\frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = \ln x + y$
3. Solve $\frac{dy}{dx} = \sin x \implies y = -\cos x + C, c \in \mathbb{R}$
4. $\frac{dy}{dx} = \sin x, y(0) = 2 \implies -\cos x + C \implies -\cos x + 3$
5. $y' = x + y$

$$y = -1 - x + 2e^x \implies y' = -1 + 2e^x \implies (x + y) = x + (-1 - x + 2e^x) = -1 + 2e^x = y'$$

6. $y = -1 - x - 5e^x$

$$y' = -1 - 5e^x \implies x + (-1 - x - 5e^x) = -1 - 5e^x = y'$$

7. $y = -5 - x + 2e^x$

$$y' = -1 + 2e^x \implies x + (-5 - x + 2e^x) = -5 + 2e^x \neq y'$$

Thus $y = -1 - x + Ce^x$, for $C \in \mathbb{R}$ is always a solution

8. Determine all real numbers k s.t. $x(t) = \sin(kt)$ is a solution to the second-order differential equation $\frac{d^2y}{dx^2} = -2x$

$$\begin{cases} x'(t) = k \cos(kt) \\ x''(t) = -k^2 \sin(kt) \end{cases} \implies -k^2 \sin kt = 0 \sin kt$$

$$(k^2 - 2) \sin(kt) = 0$$

$$k = \pm\sqrt{2}, k = 0$$

Direction Fields



Info – Direction Field

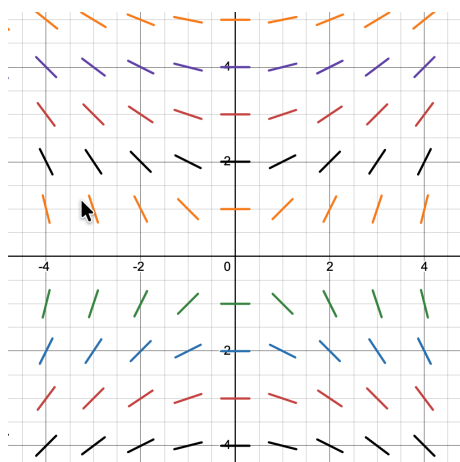
A direction field for the differential equation $y' = F(x)$ displays short line segments of slope $F(x, y)$ at various points in the Cartesian plane



Tip – Direction field plotter for DE: <https://www.desmos.com/calculator/p7vd3cdmei>

Examples:

1. $\frac{dy}{dx} = \frac{x}{y}$

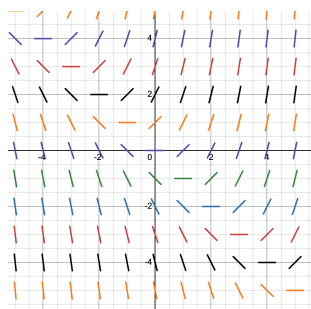


We can obtain this when plugging-in numbers.

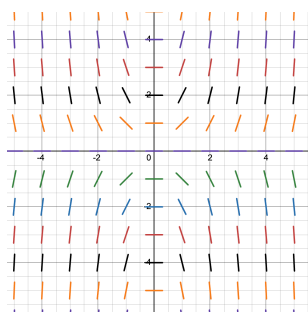
$$\text{At } (-2, 1), \left. \frac{dy}{dx} \right|_{(x,y)=(-2,1)} = \left. \frac{x}{y} \right|_{(x,y)=(-2,1)} \implies -\frac{2}{1} = -2$$

Hence a slope of -2 at $(-2, 1)$ and other points have the same method

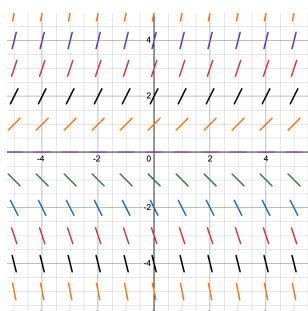
2. $y'x = x + y$



3. $y' = xy$



4. $y' = y$



Separable Differential Equations



Info — Separable Differential Equation

A first order differential equation is said to be separable if it is written in format

$$\frac{dy}{dx} = g(x)h(y)$$

Examples:

1. $y' = \sin x$

2. $y' = \frac{x}{y}$

3. $y' = 5y$

4. $y^2 y' = 2y + xy$

5. $y' = x + y$ is not separable but linear, see in Linear First Order DEs

 **Tip – Solving Separable DE** We have two cases

1. Determine any solution y with $h(y) = 0$
2. Find the solutions y where $h(y) \neq 0$ by evaluating

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

If possible, isolate y as a function of x in the resulting equation.

The general solution is the collection of all solutions obtained in Case 1 together with all solutions obtained in Case 2.

Examples:

1. $y' = y$

Case 1: $y = 0$

Case 2:

$$\begin{aligned} \int \frac{1}{y} dy &= \int 1 dx \\ \ln|y| &= x + C_1 \\ |y| &= e^{x+C_1} y = \pm e^{C_1} e^x = \pm C e^x, \quad C \in \mathbb{R} \end{aligned}$$

The general solution is $y = Ae^x$, $A \in \mathbb{R}$

2. $y' = \frac{x}{y}$

Case 1: $h(y) = \frac{1}{y} \neq 0$

Case 2:

$$\begin{aligned} \int y dy &= \int x dx \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ y^2 &= x^2 + 2C \\ y &= \pm \sqrt{x^2 + 2C} \end{aligned}$$

The general solution $y = \pm \sqrt{x^2 + D}$, $D \in \mathbb{R}$

3. $y' = xy^2, y(0) = 3$

Case 1: $h(y) = 0 \implies y \equiv 0$

Case 2:

$$\int \frac{1}{y^2} dy = \int x dx \implies -\frac{1}{y} = \frac{x^2}{2} + C \implies y = -2 \frac{2}{x^2 + 2C}$$

The general solution is $y \equiv 0$ or $y = \frac{-2}{x^2 + D}$, $D \in \mathbb{R}$

For $y(0) = 3 \implies -\frac{2}{D} = 3 \implies D = -\frac{2}{3} \implies$ the unique solution is $y = \frac{-2}{x^2 - \frac{2}{3}}$

$$4. \quad y' - x^2 - y^2 - 2xy + 1 = 0 \implies y' = (x + y)^2 - 1 \xrightarrow{v=x+y} v' = v^2$$

$$\text{Case 1: } h(v) = v^2 \equiv 0 \implies v \equiv 0$$

$$\text{Case 2: } \int \frac{1}{v^2} dv = x \implies -\frac{1}{v} = x + C \implies v = -\frac{1}{x+C}, C \in \mathbb{R}$$

$$\text{The general solution } v \equiv 0 \text{ or } v = -\frac{1}{x+C}, C \in \mathbb{R}$$

$$y = -x \text{ or } y = -x - \frac{1}{x+C}, C \in \mathbb{R}$$

Linear First-Order Differential Equations

Info – First-Order Linear Differential Equation

First order Linear DE has the form

$$A_0(x)y + A_1(x)y' = B(x) \quad A_1(x) \neq 0$$

Such an equation can be written in the form $y' + P(x)y = Q(x)$ called the standard form.

Info – Integrating Factor

Given a DE of the form $y' + P(x)y = Q(x)$,

$$\mu = e^{\int P(x) dx}$$

is called the integrating factor for the DE.

Tip – Solving First-Order Linear DE

Given $A_1(x)y' + A_0(x)y = B(x)$

1. Divide by $A_1(x)$ to rewrite the DE in standard form
2. Multiply the equation by the integrating factor.
3. Rewrite LHS as $(\mu(x)y)'$
4. Integrate in respect to x
5. Isolate y

CUTOFF Q5

Applications

Tip – Formula - Mixing Problem

$$\frac{dA}{dt} = (\text{rate of substance in}) - (\text{rate of substance out})$$