CH 2 — Sequence and Limits

Luke Lu • 2025-09-15

Triangle Inequality

$$|x - y| \leq |x - z| + |z - y|$$

for $x, y, z \in \mathbb{R}$

Idea: the straight-line distance is shortest.
Without loss of generality assume $x \leq y$; swapping x, y preserves the statement.

Number-line proof by cases:

- Case 1 $z \leq x \leq y$: $|x - y| \leq |z - y| \leq |x - z| + |z - y|$
- Case 2 $x \leq z \leq y$: $|x - y| = |x - z| + |z - y|$
- Case 3 $x \leq y \leq z$: $|x - y| \leq |x - z| + |z - y|$

Triangle Inequality 2
For all $a, b \in \mathbb{R}$

$$|a + b| \leq |a| + |b|$$

Proof:
apply the triangle inequality to $x = a$, $y = -b$, $z = 0$.

Quick check
Is $|a - b| \leq |a| - |b|$ for all a, b ?
No
Example:
$a = 10$, $b = -9$ gives $|10 - (-9)| = 19$, while $|10| - |-9| = 1$
Hence this statement is false.

Interval translations

1. $|x - a| < \delta \Rightarrow x \in (a - \delta, a + \delta)$
2. $|x - a| \leq \delta \Rightarrow x \in [a - \delta, a + \delta]$
3. $0 \leq |x - a| \leq \delta \Rightarrow x \in (a - \delta, a) \cup (a, a + \delta)$

Practice

1) Solve $|2x - 5| < 3$

$-3 < 2x - 5 < 3 \Rightarrow 1 < x < 4$

Answer: $x \in (1, 4)$

2) Solve $2 < |x + 7| \leq 3$

Split into $|x + 7| > 2$ and $|x + 7| \leq 3$

Answer: $x \in [-10, -9) \cup (-5, -4]$

3) Solve $\frac{|x+2|}{|x-2|} > 5$

Consider regions $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$ and track signs of $x + 2$ and $x - 2$

Answer: $x \in \left(\frac{4}{3}, 2\right) \cup (2, 3)$

Infinite Sequences

A sequence is an ordered list $a_1, a_2, a_3, \ldots$; write $(a_n)_{\{n=1\}}^{\{\infty\}}$

A subsequence chooses indices $n_1 < n_2 < \ldots$, yielding $a_{\{n_1\}}, a_{\{n_2\}}, \ldots$

The tail with cutoff k is $a_k, a_{\{k+1\}}, a_{\{k+2\}}, \ldots$

Convergence (definition)

IMPORTANT

We say $\lim_{\{n\to\infty\}} a_n = L$ if for every $\varepsilon > 0$ there exists N such that $n > N \Rightarrow |a_n - L| < \varepsilon$

Equivalent formulations:

- Every interval $(L - \varepsilon, L + \varepsilon)$ contains a tail of (a_n)
- Only finitely many terms lie outside $(L - \varepsilon, L + \varepsilon)$
- More generally, any open interval (a, b) containing L contains a tail

Examples

1) Show $\lim_{\{n\to\infty\}} \frac{1}{\sqrt[3]{n}} = 0$

Choose $N = \frac{1}{\varepsilon^3}$

Then $n > N \Rightarrow \left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon$

2) Show $\lim_{\{n\to\infty\}} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$

Estimate $\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| \leq \frac{5}{16n+4}$

Pick $N > \frac{5}{16\varepsilon} - \frac{1}{4}$

Theorem (Equivalent definitions of the limit of a sequence)

IMPORTANT

For a sequence (a_n) and a number L, the following are equivalent

1) $\lim_{\{n\to\infty\}} a_n = L$

2) For every $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains a tail of (a_n)

3) For every $\varepsilon > 0$, only finitely many n satisfy $|a_n - L| \geq \varepsilon$

4) Every interval (a, b) containing L contains a tail of (a_n)

5) Given any interval (a, b) containing L, only finitely many terms of (a_n) lie outside (a, b)

Example 1

Show $\lim_{\{n\to\infty\}} \frac{1}{\sqrt[3]{n}} = 0$

Side work:

$\left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon \Rightarrow \frac{1}{\sqrt[3]{n}} < \varepsilon \Rightarrow \sqrt[3]{n} > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon^3}$

Proof

Let $\varepsilon > 0$ and choose $N = \frac{1}{\varepsilon^3}$

If $n > N$ then $| \frac{1}{\sqrt[3]{n}} | < \frac{1}{\sqrt[3]{N}} = \frac{1}{\sqrt[3]{\frac{1}{\varepsilon^3}}} = \varepsilon$

Example 2)

Prove $\lim_{\{n \to \infty\}} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$

Rough work

$$| \frac{3n^2 + 2n}{4n^2 + n + 1} - \frac{3}{4} | = \frac{|5n - 3|}{16n^2 + 4n + 4} \leq \frac{5n}{16n^2 + 4n} = \frac{5}{16n + 4}$$

Proof

Given $\varepsilon > 0$, pick $N > \frac{5}{16\varepsilon} - \frac{1}{4}$

Then for $n > N$

$$| \frac{3n^2 + 2n}{4n^2 + n + 1} - \frac{3}{4} | < \frac{5}{16n + 4} \leq \frac{5}{16N + 4} < \varepsilon$$

Limits

Thinking quesiton: can a sequence converge to two different limits $L \neq M$?
No, we are saying then $\varepsilon < \frac{|L-M|}{2}$
If $a_n \to L$ a tail of the sequence lies in $(L - \varepsilon, L + \varepsilon)$ so only finite many terms can lie in the interval $(M - \varepsilon, M + \varepsilon)$, that is $a_n \nrightarrow M$

IMPORTANT

Theorem (Uniqueness of Limits):
Let $\{a_n\}$ be a sequence. If $\{a_n\}$ has limit L, then the value L is unique.

We say that a sequence **diverges to ∞** if for every $m > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N, a_n > m$.
We say that a sequence **diverges to ∞** if any interval of the form (m, ε) for some $m > 0$ contains a tail of $\{a_n\}$. We write that $\lim_{n \to \infty} a_n = \infty$

We say that a sequence **diverges to −∞** if for every $m < 0$, there exists $N \in \mathbb{N}$ such that for all $n > N, a_n < m$

We say that a sequence **diverges to −∞** if any interval of the form (m, ε) for some $m < 0$ contains a tail of $\{a_n\}$. We write that $\lim_{n \to \infty} a_n = -\infty$

Thinking questions:

1. If a seuqnce consists of non-negative terms, is the limit non-negative?
   ANS: YES Suppose not, then $a_n \to L, a_n > 0, \forall n$. Consider $\varepsilon < \frac{|L|}{2}$. Then $(L - \varepsilon, L + \varepsilon)$ only contains negative numbers, so it can't include a tail of a_n, contradiction.

2. If a sequence consists of positive terms, is the limit positive?
   ANS: NO, consider the sequence $\left\{\frac{1}{n}\right\}$, $\lim_{n \to \infty} \frac{1}{n} = 0$

3. $\lim_{n \to \infty} a_n + b_n = a + b$

4. $\lim_{n \to \infty} a_n b_n = ab$

5. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}, b \neq 0$

6. If $a_n \geq 0 \forall n, \alpha > 0$, then $\lim_{\{n \to \infty\}} a_n^\alpha = a^\alpha$

7. For any $k \in \mathbb{N}, \lim_{n \to \infty} = a_n$

Examples: Prove that $\lim_{\{n \to \infty}$ Let $m > 0$ and consider the interval m, ∞. If $n > \sqrt[3]{m}$ then $n^3 > m$ and ao $n^3 \in (m, \infty)$. So choose $k = \lceil \sqrt[3]{m} \rceil + 1$, then the tails lies in (m, ∞)

Limit Laws

IMPORTANT

Let $\{a_n\}, \{b_n\}$ be sequences with $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} b_n = b$ for some $a, b \in \mathbb{R}$ then:
1. For any $c \in \mathbb{R}$, if $a_n = c$ for all n then $c = a$
2. For any $c \in \mathbb{R}$, if $\lim_{n \to \infty} ca_n = ca$

Prove the SUm of Sequences Rule

Proof

$a_n \to a, b_n \to b$
$\forall \varepsilon > 0, \exists M, N \in \mathbb{R}, \forall n > M, n > N, |a_n - a| < \varepsilon, |b_n - b| < \varepsilon$
$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n + b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

IMPORTANT

Tandem Convergence Theorem:
If $\lim_{n \to \infty} \frac{a_n}{b_n}$ exists and $\lim_{n \to \infty} b_n = 0$, then $\lim_{n \to \infty} a_n = 0$