

CH 3 — Function Limits and Continuity

Luke Lu • 2025-09-26

Definitions

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function and $a \in \mathbb{R}$, $\lim_{x \rightarrow a} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

Examples:

1) Prove using the $\varepsilon - \delta$ definition that $\lim_{x \rightarrow 0} f(x)$ DNE where

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 3 & \text{if } x > 0 \end{cases}$$

Domain: $\mathbb{R} \setminus \{0\}$

Take $\varepsilon = 1$. Consider some $\delta > 0$. Within $(0 - \delta, 0 + \delta)$

We have both $(-\delta, 0)$ where $f(x) = -2$ and $(0, \delta)$ where $f(x) = 3$. If this δ exists for $\varepsilon = 1$ then the limit L would need to be distance 1 or both -2 and 3, where is impossible.

$$\therefore \lim_{x \rightarrow 0} f(x) = \text{DNE}$$

2) $\lim_{x \rightarrow 7} 8x - 3 = 53$

Let $\varepsilon > 0$ be arbitrary.

We want find δ s.t. if $0 < |x - 7| < \delta$ then $|8x - 3 - 53| < \varepsilon \rightarrow \delta = \frac{\varepsilon}{8}$

Pick $\delta = \frac{\varepsilon}{8}$.

Then if $0 < |x - 7| < \frac{\varepsilon}{8}$, $|(8x - 3) - 53| = |8x - 56| = 8|x - 7| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$

3) $\lim_{x \rightarrow 1} x^2 + 3x + 4 = 8$

We want for any $\varepsilon > 0$ and $\delta > 0$: $|x - 1| < \delta$, then $|f(x) - L| < \varepsilon$

$$\Leftrightarrow |x^2 + 3x - 4| < \varepsilon \Leftrightarrow |(x + 4)(x - 1)| < \varepsilon \Leftrightarrow |x + 4| - |x - 1| < \varepsilon$$

I can always make δ smaller if I need to.

take $\delta < 1$, then $|x - 1| < 1 \Rightarrow 0 < x < 2$

$|x + 4| < 6 \rightarrow |x + 4||x - 1| < 6\delta$, but $6\delta < \varepsilon \Leftrightarrow \delta < \frac{\varepsilon}{6}$. Say $\frac{\varepsilon}{7} < \frac{\varepsilon}{6}$ for all epsilon.

Take $\delta < \min(1, \frac{\varepsilon}{6})$

Proof

Let $\varepsilon > 0$ be given. Take $\delta = \min(\frac{1}{2}, \frac{\varepsilon}{7})$. Then, if $|x - 1| < \delta$, $|x^2 + 3x + 4 - 8| = |x^2 + 3x - 4| = |(x + 4)(x - 1)| = |x + 4||x - 1| < 6 \cdot \frac{\varepsilon}{7} < \varepsilon$

□

Info – Sequential Characterization of Limits Theorem

Let $a \in \mathbb{R}$. let the function $f(x)$ be defined on an open interval containing a , except possibly at $x = a$ itself. Then the following are equivalent:

1. $\lim_{x \rightarrow a} f(x) = L$
2. For all sequences $\{x_n\}$ satisfying $\lim_{n \rightarrow \infty} x_n = a$ and $x_n \neq a \forall n \in \mathbb{N}$, we have that $\lim_{n \rightarrow \infty} f(x_n) = L$

💡 Tip – Usage of Sequential Characterization of Limits

1. Find a sequence $\{x_n\}$ with $x_n \rightarrow a$
2. Find two sequences $\{x_n\}, \{y_n\}$ with $x_n, y_n \rightarrow a$ and $x_n, y_n \neq a \forall n \in \mathbb{N}$ but which $\{f(x_n)\}, \{f(y_n)\}$ converge to different values

Proof

\Rightarrow : $\lim_{x \rightarrow a} f(x) = L$ means $\forall \varepsilon > 0, \exists \delta > 0 : |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Let $\{x_n\}$ be s.t. $x_n \rightarrow a$ (meaning that $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, |x_n - a| < \varepsilon$) and $x_n \neq a$ for any n .

In particular, let ε for $x_n \rightarrow a$ be δ . Then $\forall n > N, |x_n - a| < \delta$, and so $|f(x_n) - L| < \varepsilon$. Then $\forall n > N, |x_n - a| < \delta$ and so $|f(x_n) - L| < \varepsilon$. So by definition, $\lim_{n \rightarrow \infty} f(x_n) = L$

Side Question: We saw the limit of a sequence is unique. Is the same true for limits of functions?

ANS: NO, it is like saying $\lim_{x \rightarrow a} f(x) = L$ and $= M$ and $L \neq M$ Suppose true. By Sequential Characterization of Limits, $\forall \{x_n\} \rightarrow a$ but $x_n \neq a \forall n, f(x_n) \rightarrow L$ and $f(x_n) \rightarrow M$ but $L \neq M$ Since the limits of sequences are unique, thus there is a contradiction.

Examples:

Prove that $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$ does not exist

We take sequences of peak points of $\cos\left(\frac{1}{x}\right)$, that is $-1, 1$. Then will converge to $-1, 1$ repeatedly, so by Sequential Characterization, $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$ will not exist.

$$\cos\left(\frac{1}{x}\right) = 1 \text{ if } x = \frac{1}{2k\pi}, k \in \mathbb{Z}, \text{ and } \cos\left(\frac{1}{x}\right) = -1 \text{ if } x = \frac{1}{(2k+1)\pi}, k \in \mathbb{Z}.$$

Let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+1)\pi}$. Then $x_n, y_n \rightarrow 0, x_n, y_n \neq 0 \forall n$. It converges to both -1 and 1 . By Sequential Characterization, the limit DNE.

Limit Laws

Info – Let f, g be functions with $\lim_{x \rightarrow a} f(x) = L, \lim_{x \rightarrow a} g(x) = M$ for some $L, M \in \mathbb{R}$ then:

1. For any $c \in \mathbb{R}$, if $f(x) = c$ for all x then $L = c$
2. For any $c \in \mathbb{R}$, if $\lim_{x \rightarrow a} cf(x) = cL$
3. $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$
4. $\lim_{x \rightarrow a} f(x) \cdot g(x) = LM$
5. $\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
6. If $\alpha > 0$ and $L > 0$, then $\lim_{x \rightarrow a} f(x)^\alpha = L^\alpha$

Info – Limit of Polynomial Functions Let $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ be a polynomial.

Then $\lim_{x \rightarrow a} p(x) = p(a)$

Proof

$$\lim_{x \rightarrow a} x = a$$

$$\lim_{x \rightarrow a} x^i = a^i$$

$$\lim_{x \rightarrow a} a_i x^i = a_i a^i$$

$$\lim_{x \rightarrow a} \sum_{i=0}^n a_i x^i = \sum_{i=0}^n a_i a^i$$

Info – Limit of Rational Functions

Let $f(x) = \frac{p(x)}{q(x)}$ when p, q be polynomial functions and $a \in \mathbb{R}$

1. If $q(a) \neq 0$ then $\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$
2. If $\lim_{x \rightarrow a} q(x) = 0$ but then $\lim_{x \rightarrow a} p(x) \neq 0$ then $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$ is DNE.
If $x \rightarrow a, x < 0$, then the limit diverges to $-\infty$.
If $x \rightarrow a, x > 0$, then the limit diverges to ∞ .
3. Otherwise, $p(a) = 0 = q(a)$, so both $p(x)$ and $q(x)$ have $(x - a)$ as a factor. Divide it out and then repeat the process.

Examples:

$$1. \lim_{x \rightarrow -3} \frac{x^3 + 10x^2 + 13x - 24}{x^2 - 4x - 21}$$

$$\Rightarrow \stackrel{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]}{=} \lim_{x \rightarrow -3} \frac{(x+3)(x-1)(x+8)}{(x+3)(x-7)} = \lim_{x \rightarrow -3} \frac{(x-1)(x+8)}{(x-7)} = \frac{(-3-1)(-3+8)}{(-3-7)} = \frac{-20}{-10} = 2$$

Info – Squeeze Theorem(Functions):

If $g(x) \leq f(x) \leq h(x)$ be functions defined in an open interval I around a except possibly at a .

If $\forall a \in I \setminus \{a\}$ we have $g(x) < f(x) \leq h(x)$ and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$

 **Tip** – When to apply Squeeze Theorem

1. Trigonometric functions with clear bounds and polynomial terms before
2. Exponential Functions with constants terms or by defining a certain interval

2. Evaluate $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right)$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$$

Notice that x^2 are polynomial function that is defined in $x \in \mathbb{R}$.

$$\lim_{x \rightarrow 0} -x^2 = \lim_{x \rightarrow 0} x^2 = 0$$

By Squeeze Theorem, $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0$