CH 1 — Sets, Quantifiers, and Statements

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Set Notation

$\mathbb{N}$ is the set of natural numbers, 0 excluded

$\mathbb{Z}$ is the set of integers

$\mathbb{Q}$ is the set of all numbers of the form $\frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $b \neq 0$

$\mathbb{R}$ is the set of all real numbers

Parity: whether an integer is even or odd

Statements

A statement is a sentence that has a definite truth value (true or false)

Examples that are not statements:

- questions
- orders
- sentences without a definite truth value This statement is false Let x be ... $x > 3$ $x = x$

Negation

Suppose A is a statement

The negation of A, written $\neg A$, asserts the opposite truth value to A

$\neg(\neg A)$ is equivalent to A

Quantifiers

- Universal $\forall$ means for all
- Existential $\exists$ means there exists

Example: $\exists x \in \mathbb{R}, x > 3$ means there exists a real number x with $x > 3$

Negating quantified statements

Basic rules

- $\neg(\forall x \in D, P(x))$ is equivalent to $\exists x \in D, \neg P(x)$
- $\neg(\exists x \in D, P(x))$ is equivalent to $\forall x \in D, \neg P(x)$

Nested rules

- $\neg(\forall x \in X, \exists y \in Y, P(x, y))$ is equivalent to $\exists x \in X, \forall y \in Y, \neg P(x, y)$
- $\neg(\exists x \in X, \forall y \in Y, P(x, y))$ is equivalent to $\forall x \in X, \exists y \in Y, \neg P(x, y)$

Useful patterns

- $\neg(P \land Q)$ is equivalent to $\neg P \lor \neg Q$
- $\neg(P \lor Q)$ is equivalent to $\neg P \land \neg Q$
- $\neg(P \Rightarrow Q)$ is equivalent to $P \land \neg Q$

Examples

- negate $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x > y$ $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x \leq y$
- negate $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x > y$ $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x \leq y$

Terminology

Variable, (e.g. x)

A domain is any set, (e.g. $\mathbb{R}$)

An open sentence $P(x)$ is an expression involving a variable that is true or false once a value from the domain is specified, (e.g. $x > 3$)

Examples

The phrase $x + 1 > x$ is not a statement until a domain for x is specified

The universally quantified sentence $\forall x \in \mathbb{R}, x + 1 > x$ is a statement and it is true

Nested Quantifier

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x > y$ is true

Every real x has a smaller real y with $x > y$

Caution: $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x > y$ is false

This would claim there is a smallest real number, which is impossible

Two Nested Quantifiers

If the domain and quantifier are the same for both variables, abbreviate as

$\forall x, y \in \mathbb{R}, P(x, y)$

$\exists x, y \in \mathbb{R}, P(x, y)$

Assignment 1

Q1

a) S

b) N

c) N

d) S

Q2

a) F

b) T

c) T

d) F

Q3

a) Solution

We know $\neg(\forall x \in X, P(x))$ is equivalent to $\exists x \in X, \neg P(x)$

We have $\neg(\forall a \in \mathbb{Z}, (a - 5)^2 \geq 0)$ is equivalent to $\exists a \in \mathbb{Z}, (a - 5)^2 < 0$

Thus the answer is $\exists a \in \mathbb{Z}, (a-5)^2 < 0$

b) Solution

We know $\neg(\exists x \in X, \forall y \in Y, Q(x,y))$ is equivalent to $\forall x \in X, \exists y \in Y, \neg Q(x,y)$

We have $\neg(\exists \theta \in \mathbb{R}, \forall \alpha \in \mathbb{R}, \sin(\theta) = \cos(\alpha))$ is equivalent to $\forall \theta \in \mathbb{R}, \exists \alpha \in \mathbb{R}, \sin(\theta) \neq \cos(\alpha)$

Thus the answer is $\forall \theta \in \mathbb{R}, \exists \alpha \in \mathbb{R}, \sin(\theta) \neq \cos(\alpha)$

Q4

a) $f(x) = \sin(x)$ and $f(x) \in [-1,1]$

b) $\forall r \in \mathbb{R}, \exists a, b \in \mathbb{R}$ such that ...

Q5

a) Statement is False

b) Statement is True

c) Statement is True

d) Open sentence depending on w

e) Statement is True

Q6

a) $\forall x, y \in S, P(x,y)$ with domain $\mathbb{Q}_{\{>0\}}$, where $P_4 : x * y \in S$

b) Let $T = \{-1, 0, 1\}$ $\forall x \in S, \exists y \in S, P(x,y)$ with domain T, where $P_3 : \sin(\frac{\pi y}{2}) = |x|$

c) $\exists x \in S, \forall y \in S, P(x,y)$ with domain $\mathbb{R}$, where $P_2 : y^2 \geq x$

d) Let $U = \left\{3, \frac{1}{3}\right\}$ $\exists x, y \in S$ with domain U, where $P_1(x,y) : x = 27^y$

Q7

a) True

b) Free response Seeking help is a tip that resonates with me. Whenever I encounter a problem, I tend to try solving it alone, which can go poorly. In CEGEP I often sought help for tough questions. Professors and classmates were open to discussion and gave tips that led me to solutions. It reminds me there is a community around me with resources and kindness, and it is never too late to reach out