

## CH 4 - Derivatives

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### Velocity

🐡 Info — Average Velocity and Instantaneous Velocity

$$v_{avg} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

$$v_{inst} = \lim_{t \rightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \rightarrow 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

### Definition of Derivatives

🐡 Info — Average Rate of Change and Instantaneous Rate of Change (Derivative)

$$f_{avg} = \frac{f(b) - f(a)}{b - a}$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

If  $f'(x)$  exists at  $x = a$ , then  $f(x)$  is **differentiable** at  $x = a$

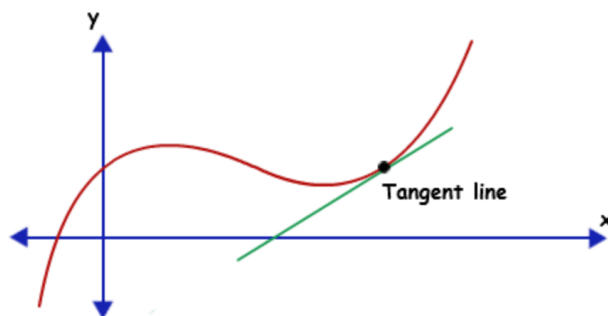
🐡 Info — Tangent Line

If  $f(x)$  is differentiable at  $x = a$ , then the **tangent line** to  $f(x)$  at  $x = a$  is the line passing through  $(a, f(a))$  with slope  $f'(a)$

The equation of the tangent line

$$y = f'(a)(x - a) + f(a)$$

$(a, f(a))$  is the **point of tangency**



Examples:

Find the tangent line to  $f(x) = \frac{1}{x+5}$  at  $x = 3$

$$f(3) = \frac{1}{8}$$

$$f'(3) = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{a+h+5} - \frac{1}{a+5}}{h} =$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \frac{a+5 - (a+h+5)}{(a+5)(a+h+5)} = \lim_{h \rightarrow 0} -\frac{1}{(a+5)(a+h+5)} = -\frac{1}{(a+5)^2} = -\frac{1}{64}$$

$$y = -\frac{1}{64}(x - 3) + \frac{1}{8}$$

### **Info – Differentiability Implies Continuity**

If a function  $f$  is differentiable at  $x = a$ , then  $f$  is continuous at  $x = a$

### **Proof**

$f$  is differentiable at  $x = a$  then,  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$  exists

$$\lim_{h \rightarrow 0} [f(a+h) - f(a)] = 0 \implies \lim_{h \rightarrow 0} [f(a+h) - f(a) + f(a)] = \lim_{h \rightarrow 0} f(a) \implies$$

$$\lim_{h \rightarrow 0} f(a) = f(a)$$

### **Warning – Continuity Not Implies Differentiability**

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h-0}{h} = 1$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h-0}{h} = -1$$

Thus  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \text{DNE}$  but continuous.

$\therefore$  continuity does not imply differentiability

### **Info – Differentiability of Function**

We say that  $f$  is **differentiable** on an interval  $I$  if  $f'(a)$  exists  $\forall a \in I$ .

We define the derivative function  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

We sometimes also write  $f'(x)$  as  $\frac{d}{dx} f(x)$ , and  $f'(a) = \frac{d}{dx} f(x) \big|_a$

### **Info – Constant Function**

$$f(x) = c$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c-c}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

### **Info – Linear Function**

$$f(x) = mx + b$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(mx+h)+b - f(mx+b)}{h} = \lim_{h \rightarrow 0} m \frac{h}{h} = m$$

### Info – Quadratic Function

$$f(x) = px^2 + sx + c$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[p(x+h)^2 + s(x+h) + c] - [px^2 + sx + c]}{h} = \lim_{h \rightarrow 0} \frac{2xph + xh^2 + sh}{h} = \lim_{h \rightarrow 0} 2xp + xh + s = 2xp + s$$

### Info – Basic Trig

$$f(x) = \sin x$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \\ &= \lim_{h \rightarrow 0} \frac{[\sin x (\cos h - 1)]}{h} + \lim_{h \rightarrow 0} \cos x \frac{\sin h}{h} = \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos^2 h - 1}{h \cdot (\cos h + 1)} + \cos x = \\ &= \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin^2 h}{h \cdot (\cos h + 1)} = \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \lim_{h \rightarrow 0} \frac{\sin h}{\cos h + 1} + \cos x = \cos x \end{aligned}$$

We define  $e$  to be the unique base of an exponential function with slope 1 through  $(0, 1)$

### Info – Derivative Rules

Let  $f(x)$  and  $g(x)$  be differentiable at  $x = a$

1.  $w(x) = cf(x) \implies w'(x) = cf'(x)$
2.  $w(x) = f(x) \pm g(x) \implies w'(x) = f'(x) \pm g'(x)$
3.  $w(x) = f(x)g(x) \implies w'(x) = f'(x)g(x) + f(x)g'(x)$
4. If  $g(x) \neq 0$ ,  $w(x) = \frac{f(x)}{g(x)} \implies w'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
5. If  $f(x) = x^\alpha$  for some  $\alpha \in \mathbb{R} \setminus \{0\} \implies f'(x) = \alpha x^{\alpha-1}$
6.  $w(x) = (g \circ f)(x) = g(f(x)) \implies w'(x) = g'(f(x)) \cdot f'(x) \sim \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

### Warning – Power Rule

If  $x = 0$ ,  $x^{-1}$  does not make sense so that is why  $\alpha \in \mathbb{R} \setminus \{0\}$

## Proof

We suppose that  $f(x), g(x)$  are differentiable, so that the limits:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \text{ exists}$$

1. Product rule:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{w(x+h) - w(x)}{h} &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \\ \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} &= \lim_{h \rightarrow 0} \frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{(g(x+h) - g(x))f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \rightarrow 0} g(x+h) + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h \rightarrow 0} f(x) \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

2. Quotient rule

$$\lim_{h \rightarrow 0} \frac{w(x+h) - w(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} = \lim_{h \rightarrow 0} \frac{g(x)(f(x+h)-f(x))}{\frac{h}{g(x)g(x+h)}} - \frac{f(x)(g(x+h)-g(x))}{\frac{h}{g(x)g(x+h)}} \\
&= \frac{\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h \rightarrow 0} g(x) - \lim_{h \rightarrow 0} f(x) \cdot \lim_{h \rightarrow 0} \frac{g(x+h)-g(x)}{h}}{\lim_{h \rightarrow 0} g(x)g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}
\end{aligned}$$

□

## Basic Derivatives



### Info – Basic Trig Derivatives

$$\begin{aligned}
\frac{d}{dx} \tan x &= \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx} \csc x &= \frac{d}{dx} \frac{1}{\sin x} = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\csc x \cot x \\
\frac{d}{dx} \sec x &= \frac{d}{dx} \frac{1}{\cos x} = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x \\
\frac{d}{dx} \cot x &= \frac{d}{dx} \frac{\cos x}{\sin x} = \frac{\cos' x \sin x - \cos x \sin' x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\csc^2 x
\end{aligned}$$



### Info – Exponential/Logarithmic Derivatives

For  $a^x, x > 0$ :

$$\begin{aligned}
\frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln(a)} = e^{x \ln(a)} \cdot \ln a = a^x \ln a \\
\frac{d}{dx} \log_a x &= \frac{d}{dx} \frac{\ln(x)}{\ln(a)} = \frac{1}{x} \cdot \frac{1}{\ln a} = \frac{1}{x \ln a}
\end{aligned}$$

Example:

- $\frac{d}{dx} x^3 e^{2x} \cos x = 3x^2 e^{2x} \cos x + 2x^3 e^{2x} \cos x - x^3 e^{2x} \sin x$
- $\frac{d}{dx} 3^{\csc x} = 3^{\csc x} \ln 3 \cdot -\csc x \cot x = -3^{\csc x} \csc x \cot x \ln 3$
- $\frac{d^{67}}{dx^{67}} \sin x$ . Note that  $\sin' x = \cos x, \sin'' x = -\sin x, \sin''' x = \cos x, \frac{d^4}{dx^4} \sin x = \sin x /$   
 $67 \bmod 4 \equiv 3$ , that is  $\frac{d^{67}}{dx^{67}} \sin x = -\cos x$
- $\frac{d}{dx} \frac{x}{(1+e^{x^2})^3} = \frac{d}{dx} x \cdot (1+e^{x^2})^{-3} = \frac{1}{(1+e^{x^2})^3} - 3((1+e^{x^2})^{-4} \cdot x^2 e^{x^2} \cdot 2x) = \frac{1}{(1+e^{x^2})^3} - \frac{6x^3 e^{x^2}}{(1+e^{x^2})^4}$
- $\frac{d}{dx} x^x$   
 $\frac{d}{dx} x^x = x^x \ln x \cdot (\ln x + 1)$   
 $\frac{d}{dx} x^{f(x)} = x^{f(x)} \cdot (\ln x \cdot f'(x) + x) = x^{x^x} \cdot x^x (\ln^2(x) + \ln(x) + x^{x-1})$

## Linear Approximation

With the assumption of  $f(x)$  is continuous at  $x = a$ , we can derive

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \leftrightarrow f'(a)(x - a) = f(x) - f(a) \leftrightarrow f(x) = f'(a)(x - a) + f(a)$$

Which is the linear approximation of  $f(x)$  near  $x = a$

### Info – Linear Approximation

Let  $f(x)$  be differentiable at  $x = a$ . The **linear approximation** to  $f(x)$  at  $x = a$  is given by

$$L_a^f(x) = f'(a)(x - a) + f(a)$$

If it is clear what function  $f$  we are talking about, we sometimes denote  $L_a(x)$  instead.

### Info – Upper Bound Error of Linear Approximation

The error of linear approximation is defined as:

$$\text{error} = |f(x) - L_a^f(x)|$$

Assume that  $f(x)$  is such that  $|f''(x)| \leq M$  for each  $x$  in an interval  $I$  containing  $x = a$ . Then,

$$\text{error} = |f(x) - L_a^f(x)| \leq \frac{M}{2}(x - a)^2$$

for each  $x \in I$

Examples:

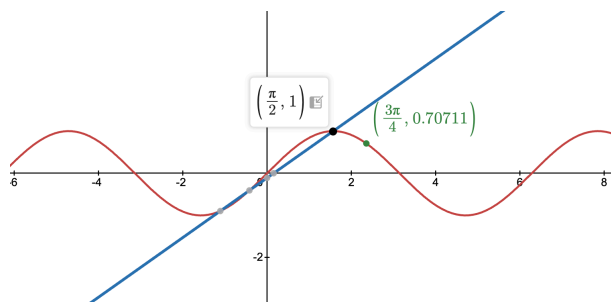
1. Find the linearization of  $\sqrt{x}$  at  $x = 4$  and use it to estimate  $\sqrt{4.01}$

$$L_a^f(x) = f'(a)(x - a) + f(a) \text{ where } f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

$$L_a^f(x) = \frac{1}{2\sqrt{x}}(x - 4) + 2 \Rightarrow L_a^f(0.01) = \frac{1}{2\sqrt{0.01}}(0.1) + 2 \approx 2.0024984$$

2. What factors could affect the error in linear approximation?

- The distance from  $x$  to  $a$ . (e.g.  $f(x) = \sin(x)$ ,  $x = 3\frac{\pi}{2}$ ,  $a = \frac{\pi}{4}$ .)



- The curvature (e.g.  $f(x) = e^x$ ,  $g(x) = e^{\frac{x}{10}}$ )

3. Find an upper bound on the error in using  $L_9$  to approximate  $f(x) = \sqrt{x}$  on  $[5, 13]$

If  $|f''(x)| \leq M$  on  $I$  then:  $\text{error} \leq \frac{M}{2}(x - a)^2 \forall x \in I$

$$f(x) = \sqrt{x}; f'(x) = \frac{1}{2\sqrt{x}}; f''(x) = -\frac{1}{4x^{\frac{3}{2}}}$$

$$|f''(x)| = \frac{1}{4x^{\frac{3}{2}}} \leq \frac{1}{4(5)^{\frac{3}{2}}} = \frac{1}{20\sqrt{5}}$$

$$\text{So the error} \leq \frac{1}{40\sqrt{5}}(x - 9)^2 = \frac{1}{40\sqrt{5}}(13 - 9)^2 = \frac{2}{5\sqrt{5}}$$

Example:

A poor group of students is writing a midterm. A nefarious professor decides to make things harder by cranking up the thermostat to uncomfortable levels. To make things as uncomfortable as possible, the professor increases the temperature according to an equation of the form  $T(t) = 2t + c$  for some constant  $c \in \mathbb{R}$ , where  $t$  is the time elapsed in minutes. Initially, at  $t = 0$ , the room is a cool  $21^\circ \text{C}$ . Estimate the change in the room's temperature 90 seconds after the professor alters the thermostat.

ANS:

$$T'(t) = 2^t \ln 2$$

$$\Delta T = T'(0) \cdot \Delta t = \ln 2 \cdot 1.5 = \left(3^{\frac{\ln 2}{2}}\right)^\circ \text{C}$$

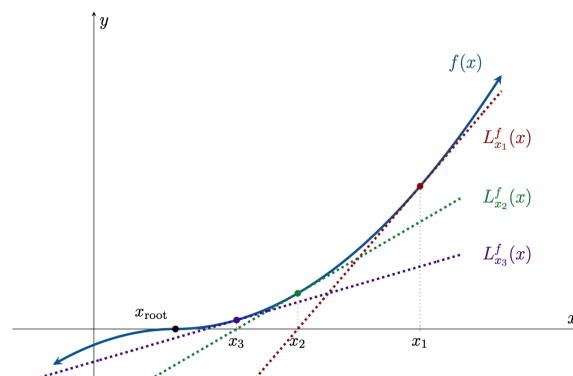
## Newton's Method

### Info – Newton's Method

1. Make an initial guess,  $x_1$ , of where the root of  $f(x)$ . IVT can be helpful.
2. Take the linear approximation,  $L_{x_1}^f(x)$  and find its root and call it  $x_2$
3. Repeat Step 3 at  $x_2$  to find  $x_3$

Note that NM converges faster than Bisection Method

The Visualization of the procedure



### Tip – Root Finding Formula for NM

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**⚠ Warning – Flaws of Newton's Method**

1. NM requires differentiability of at  $x_n$
2. Not always converges
  1. Consider where  $f'(x_n) = 0$ . The formula is not computable, and this is the case of a horizontal tangent line - which would not have a root for the next iteration.
  2. Consider  $f(x) = \sqrt[3]{5}$  for which Newton's Method will not work for any guess of root  $x_1 \neq 0$ , as the formula gives  $x_{n+1} = -6x_n$
  3. Consider  $f(x) = x^3 + x^2 - 3x + 3$  for which guess of  $x_1 = 0, x_2 = 1, x_3 = 0, \dots$  falls into an alternating cycle and unable to find a root.
  4. Some choices of  $x_1$  might lead to convergence to a different root than desired.

**🔍 Info – Comparison BM vs NM**

1. Bisection requires continuity, Newton requires differentiability
2. Bisection is guaranteed to converge, Newton does not guarantee
3. Newton's is faster if it works

Example:

1.  $x^2 - 4x - 7$  at  $x = 5$  to nearest thousandth

ANS:

Let  $x_1 = 5$ , given  $f'(x) = 2x - 4$

$$x_2 = x_1 - \frac{f(5)}{f'(5)} = 5 - \left(-\frac{1}{3}\right) = \frac{16}{3}$$

$$x_3 = x_2 - \frac{f(\frac{16}{3})}{f'(\frac{16}{3})} = \frac{16}{3} - \frac{1}{60}$$