CH 5 - Applications of Derivatives

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Related Rates

💡 **Tip** — Steps for Related Rates Questions

1. Draw diagram
2. Identitfy **changing** quantities
3. Find **constant** quantities (if possible)
4. Derive equations relating the quantities that are changing
5. **Implicitly differentiate** the key equations
6. Solve for the desired rate of change, substituting in known quantities.
7. **Concluding statement** (and also check units)

Example:

1. Laindon is taking a hot air ballon ride. A giant fan is blowing hot air into the ballon in a rate of $200 \frac{\text{m}^3}{\text{min}}$. Assuming that at any given point in time the balloon sphere, find the rate at which the radius of the balloon is changing when the diameter is 12 m.

   ANS:

   1. Picture: The problem is trivial so the graph is omitted
   2. Changing variable: Volume(m^3), Radius(m), time(t)
   3. Constant quantities: $\frac{\mathrm{d}V}{\mathrm{d}t} = 200 \frac{\text{m}^3}{\text{min}}$
   4. Key Equation: $V = \frac{4}{3}\pi r^3(t)$
   5. Implicit Differentiation: $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2(t) \cdot \frac{\mathrm{d}r}{\mathrm{d}t}$
   6. $\frac{\mathrm{d}r}{\mathrm{d}t}\Big|_{r=6} = \frac{1}{4\pi(6)^2} \cdot 200 = \frac{200}{144\pi} = \frac{25}{18\pi} \frac{\text{m}}{\text{min}}$
   7. Concluding statement: When the diameter of the balloon is 12m, the rate of change of the radius is expanding by $\frac{25}{18\pi} \frac{\text{m}}{\text{min}}$

2. The construction workers building M4 accidentally left a 20 foot ladder propped up against a concrete wall that is 80 feet in height. The base of the ladder begins to slide away from the wall at a rate of 2ft/sec, and the top begins to move down as a result. When the base of the ladder is 14 ft from the wall, how fast is the top of the ladder sliding down the wall?

   ANS:

   1. Picture is omitted and left as an exercise for the reader
   2. Changing variable: Distance from wall of base of ladder (m), Heing where ladder touches the wall (m)
   3. Constant quantities : $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$
   4. Key Equation: $x^2 + y^2 = 20^2$

5. Implicit Differentiation: $2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t}$
6. $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{14}{\sqrt{400-14^2}} \cdot 2 = -\frac{14}{\sqrt{51}}\frac{\mathrm{ft}}{\mathrm{sec}}$
7. Concluding statement: When the base of ladder is 14cm, the top of the ladder is falling at a speed of $\frac{14}{\sqrt{51}}\frac{\mathrm{ft}}{\mathrm{sec}}$

Extrema

> 🐳 **Info** — **Extrema**
>
> Let $f(x)$ be a funciton defined on an interval I, and let $c \in I$. We say f has
>
> 1. A **global minimum** on I at $x = c$ if $f(c) \leq f(x) \forall x \in I$
> 2. A **global maximum** on I at $x = c$ if $f(c) \geq f(x) \forall x \in I$
> 3. A **global extremum** on I at $x = c$ if f has either a global minimum or global maximum.
>
> - Every point on a constant function is both a global minimum and global maximum
> - Every global extremum can be a local extremum in some interval

Examples:

1. Find all global extrema of $f(x) = x^2$ on [0, 1)
   - The global minimum be $x = 0$ because $f(0) \leq f(x)\ \forall x \in [0, 1)$
   - The global maximum DNE as the end point is missing. That is infinitely numbers lie on the interval [0, 1]
2. Find all global extrema of $f(x) = \frac{1}{x}$ on $[-1, 1]$
   - The global extrema DNE as $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

> 🐳 **Info** — **Extreme Value Theorem (Existence Thm)**
>
> Assume that $f(x)$ is continuous on the closed interval $[a, b]$. Then **there exist** two numbers $c_1, c_2 \in [a, b]$ s.t. $f(c_1) \leq f(x) \leq f(c_2)\ \forall x \in [a, b]$.
>
> In other words, there is a global minimum at $x = c_1$ and a global maximum at x_{c_2}

> 🐳 **Info** — **Local Extrema**
>
> Let f be a function. We say that f has
>
> 1. a **local minimum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \leq f(x)\ \forall x \in (a, b)$
> 2. a **local maximum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \geq f(x)\ \forall x \in (a, b)$
> 3. a **local extremum** at $x = c$ if there is either a local minimum or a loacl maximum

> ⚠️ **Warning** — Local Extrema
>
> If c is an endpoint of the domain of f, c can never be a local extremum, even if it is a global extremum

🐳 **Info — Fermat's Theorem**

If there is a local extremum for $f(x)$ at $x = c$ and $f'(c)$ exists, hence $f'(c) = 0$. That is we cannot put an open interval around the point.

Examples:

1. Does the converse of Fermat's Theorem hold? That is if $f'(0) = 0$, then is a local extremum at $x = c$.

   This is false. Let $f(x) = x^3$, $f'(x) = 3x^2$, $f'(0) = 0$ but is not a local extremum on any interval containing $x = 0$

2. Why is it worth mentioning $f'(c)$ has to exist?
   It is important because it is like saying $f(x)$ is differentiable at $x = c$. If not, let $f(x) = |x|$. $f(x)$ is continuous. It has a local minimum at $x = 0$ but $f'(0)$ DNE as it is not differentiable.

🐳 **Info — Critical Points**
We say that a function f has a **critical point** at $x = c$ if $f'(c) = 0$ or $f'(c) = \text{DNE}$ for $c \in$ the domain of f. These are our candidates for local extrema.

💡 **Tip — Closed Interval Method**
Let $f(x)$ be continuous function on $[a, b]$.
1. Calculate $f(a)$ and $f(b)$
2. Find $f'(x)$
3. Find all the critical points of f on $[a, b]$
4. Calculate $f(c)$

Example:

$f(x) = \frac{1}{3}x^3 - 3\sqrt[3]{x}$ on $[-8, 1]$

$f(-8) = -\frac{512}{3} - 3(-2) = -\frac{496}{3}$
$f(1) = \frac{1}{3} - 3 = -\frac{8}{3}$

$f'(x) = x^2 - x^{-\frac{2}{3}}$

$f'(c) = 0 \implies c^2 - c^{-\frac{2}{3}} = 0 \implies c^{\frac{8}{3}} = 1 \implies c = -1, 1$. $f'(c) = \text{DNE} \implies c = 0$

$f(0) = 0$

$f(-1) = -\frac{1}{3} + 3 = \frac{8}{3}$ Global maximum at $x = -1$, global minimum at $f(-8)$

🐳 **Info — Rolle's Theorem (Existence Thm)**

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = k \in \mathbb{R}$, then there exists $c \in (a, b)$ s.t. $f'(c) = 0$

Proof

If $f(x) = k\ \forall x \in [a, b]$, any value of c works.

Otherwise, $\exists x_0 \in [a, b]$ s.t. $f(x_0) \neq k$. Since f is continuous on $[a, b]$, it attains a maximum/minimum on $[a, b]$.

Since $f(x_0) \neq k \Longrightarrow f(x_0) > k \longleftrightarrow f(a), f(b)$ are not maximum, or $f(x_0) < k \longleftrightarrow f(a), f(b)$ are not minimum. So one of maximum or minimum is in (a, b), thus differentiable at some c.

By Fermat's Theorem, $f'(c) = 0$ or $f'(c) = \text{DNE}$. But f is differentiable on $(a, b) \Longrightarrow f'(c)$ exists.

$\therefore f'(c) = 0$

> 🐳 **Info — Mean Value Theorem (Existence Thm)**
>
> If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t.
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

Let $h(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x - a)\right]$

$h(a) = f(a) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (a - a)\right] = 0$

$h(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (b - a)\right] = 0$

Since $h(b) = h(a) \overset{\text{Rolle's Theorem}}{\Longrightarrow} \exists c \in (a, b)$ s.t. $h'(c) = 0$

That is $h'(x) = f'(x)\frac{f(b)-f(a)}{b-a} \Longrightarrow f'(x) = \frac{f(b)-f(a)}{b-a}$

Finally, $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \longleftrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

Tip: the construction of $\left[f(a) + \frac{f(b)-f(a)}{b-a}(x - a)\right]$ is the linear approximation of $f(x)$ near a

Antiderivative

> 🐳 **Info — Antiderivative**
>
> Given a function $f(x)$, an **antiderivative** is a function $F(x)$ s.t. $F'(x) = f(x)$. If $F'(x) = f(x)$ for all $x \in I$ for some interval I, then $F(x)$ is an antiderivative of $f(x)$ on I
>
> e.g. $\frac{\mathrm{d}}{\mathrm{d}x} - \ln(\cos x)) = -\frac{1}{\cos}x \sin x = -\frac{-\sin x}{\cos x} = \tan x$
>
> Note: one function can have infinitely many antiderivatives, that is why we insist **an antiderivative** of $f(x)$

> 🐳 **Info — Constant Function Theorem**
>
> Suppose that $f'(x) = 0\ \forall x \in I$ for some interval I. Then $\exists \alpha \in \mathbb{R}$ s.t. $f(x) = \alpha\ \forall x \in I$

Proof

Let $x_1 < x_2 \in I$.

Since f is differentiable on I, it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

By MVT, $\exists c \in (x_1, x_2)$ s.t. $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ But $f'(c) = 0$ since $f'(x) = 0$ on I.

Thus, $0 = \frac{f(x_2)-f(x_1)}{x_2-x_1} \land x_2 - x_1 \neq 0 \Longrightarrow f(x_2) - f(x_1) = 0 \Longleftrightarrow f(x_2) = f(x_1)$.

Since x_1, x_2 are arbitrary, therefore f is constant on I

> 🐳 **Info — Antiderivative Theorem**
>
> Suppose that $F'(x) = G'(x) \forall x \in I$ for some interval I. Then there exists $\alpha \in \mathbb{R}$ s.t. $F(x) = G(x) + \alpha \forall x \in I$

Proof

Let $h(x) = F(x) - G(x)$. Then $h'(x) = F'(x) - G'(x) = 0$ on I.

By the CFT, $h(x) = \alpha$ for some $\alpha \in \mathbb{R}$, so $F(x) - G(x) = \alpha \Longrightarrow F(x) = G(x) + \alpha \quad (\forall x \in I)$

First Derivatives

> 🐳 **Info — Definition of Increasing/Decreasing**
>
> Let I be in interval and $x_1, x_2 \in I$, then f(x)
> - **increasing** on I if $f(x_1) \leq f(x_2)\ \forall x_1 < x_2$
> - **decreasing** on I if $f(x_1) \geq f(x_2)\ \forall x_1 < x_2$
> - **strictly increasing** on I if $f(x_1) < f(x_2)\ \forall x_1 < x_2$
> - **strictly decreasing** on I if $f(x_1) > f(x_2)\ \forall x_1 < x_2$
>
> Note: a constant function is both increasing and decreasing but not strictly

> 🐳 **Info — Increasing/Decreasing Function Theorem**
> Let I be an interval
>
> 1. If $f'(x) \geq 0 \forall x \in I$, then $f(x)$ is increasing on I
> 2. If $f'(x) > 0 \forall x \in I$, then $f(x)$ is strictly increasing on I
> 3. If $f'(x) \leq 0 \forall x \in I$, then $f(x)$ is decreasing on I
> 4. If $f'(x) < 0 \forall x \in I$, then $f(x)$ is strictly decreasing on I

Proof

Let $x_1 < x_2 \in I$. $f'(x) > 0$ on I, so it exists on $I \implies f$ is differentiable on (x_1, x_2), continuous on $[x_1, x_2]$.

By MVT, $\exists c \in (x_1, x_2)$ s.t. $f'(x) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Given $f'(x) > 0\ \forall x \in I$ and $x_1 < x_2$ and $f(x_1) < f(x_2)$

Since x_1, x_2 are arbitrary, thus f is strictly increasing on I

Proof for increasing, strictly decreasing and decreasing is similar thus be omitted.

Question: If f is strictly increasing on $I \implies f'(x) > 0\ \forall x \in I$?

ANS: No, counterexample $f(x) = x^3$

Question: If f is strictly decreasing on $I \implies f'(x) < 0\ \forall x \in I$?

ANS: No, counterexample $f(x) = -\sqrt[3]{x}$

🐳 Info — Bounded Derivative Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).

Suppose that $m \leq f'(x) \leq M\ \forall x \in (a, b)$. Then

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

$\forall x \in [a, b]$

Proof

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).

It also applies to $[a, x_1]$.

Case1:

For $x_1 \in (a, b]$

By MVT, $\exists\ c \in (a, x_1)$ s.t. $f'(c) = \frac{f(x_1)-f(a)}{x_1-a}$

Since $m \leq f'(c) \leq M$ so $m(x_1 - a) \leq f(x_1) - f(a) \leq M(x_1 - a)$.

Then $m(x_1 - a) + f(a) \leq f(x_1) \leq M(x_1 - a) + f(a)$

Case 2:

When $x = a$, $m(x - a) + f(a) = f(a)$ and similar applies to $M(x - a)$.

Resulting to $f(a) \leq f(a) \leq f(a)$

Example:

Prove that $\sqrt{51} \in \left[7 + \frac{1}{8}, 7 + \frac{1}{7}\right]$.

Let $f(x) = \sqrt{x}$ and let $[a, b]$ be [49,64]

Since $f(x)$ is continuous on $[49, 64]$ and differentiable on $[49, 64]$

By Bounded Derivative Theorem, $f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$

where $m \leq f'(x) \leq M$ on $[49, 64]$ $\forall x$.

$f'(x) = \frac{1}{2\sqrt{x}}, \frac{1}{\sqrt{64}} \leq \frac{1}{2\sqrt{x}} \leq \frac{1}{2\sqrt{49}} \Longrightarrow 7 + \frac{1}{2} \cdot \frac{1}{8}(51 - 49) \leq \sqrt{51} \leq \sqrt{49} + \frac{1}{2} \cdot \frac{1}{7}(51 - 49)$

$\Longrightarrow 7 + \frac{1}{8} \leq \sqrt{51} \leq 7 + \frac{1}{7}$

> 🐳 **Info — Comparison via First Derivative Theorem**
>
> Assume $f(x)$ and $g(x)$ are continuous at $x = a$ with $f(a) = g(a)$. Then
>
> - f and g are differentiable for $x > a$ and $f'(x) \leq g'(x)$ $\forall x > a \Longrightarrow f(x) \leq g(x)$ $\forall x > a$
> - f and g are differentiable for $x < a$ and $f'(x) \leq g'(x)$ $\forall x < a \Longrightarrow f(x) \geq g(x)$ $\forall x < a$

Example:

Show that $x - \frac{1}{2}x^2 < \ln(1 + x)$ $\forall x > 0$

Let $f(x) = x - \frac{1}{2}x^2, g(x) = \ln(1 + x)$

At $x = 0, f(0) = 0, g(0) = 0$, that is $f(0) = g(0)$

All functions are differentiable on $(0, \infty)$

$f'(x) = 1 - x, g'(x) = \frac{1}{1+x}$.

Since $x > 0, (1 - x)(1 + x) < 1 \Longrightarrow 1 - x^2 < 1 \Longrightarrow -x^2 < 0$

Thus by CFDT, $x - \frac{1}{2}x^2 < \ln(1 + x)$ $\forall x > 0$

Second Derivatives

> 🐳 **Info — Concavity and Second Derivative Theorem**
>
> 1. We say f is **concave up** on an interval I if for all $a, b \in I$, the secant line between $(a, f(a))$ and $(b, f(b))$ lies **above** the graph of $f(x)$.
> 2. We say f is **concave down** on an interval I if for all $a, b \in I$, the secant line between $(a, f(a))$ and $(b, f(b))$ lies **below** the graph of $f(x)$.
> 3. If $f''(x) > 0$ $\forall x$ in an inverval I then $f(x)$ is **concave up** on I
> 4. If $f''(x) < 0$ $\forall x$ in an inverval I then $f(x)$ is **concave down** on I

Example:

$f(x) = |x|$ is neither concave up nor concave down as the secant line lies on the graph if a and b are same side of the absolute function.

🐳 **Info** — **Point of Inflection**

A point $(c, f(x))$ is called a **point of inflection** of $f(x)$ if $f(x)$ is continuous at $x = c$ and the concavity of $f(x)$ changes at $x = c$

If $f''(x)$ is continuous at c and $(c, f(c))$ is a point of inflection, then $f''(x) = 0$ or DNE

However, the converse is false: $f(x) = x^4$, $f''(0) = 0$, but the concavity does not change

Examples:

1. $f(x) = x^3, f'(x) = 3x^2, f''(x) = 6x$

$f''(x) < 0 \ \forall x \in (-\infty, 0)$, concave down

$f''(x) > - \ \forall x \in (0, \infty)$, concave up

$f''(x) = 0 \Longrightarrow 6x = 0 \Longrightarrow x = 0$.

2. $g(x) = \frac{1}{x}, g'(x) = -\frac{1}{x^2}, g''(x) = \frac{2}{x^3}$.

$g''(x) < 0 \ \forall x \in (-\infty, 0)$, concave down

$g''(x) > - \ \forall x \in (0, \infty)$, concave up

$g''(x)$ is discontinuous at $x = 0$, therefore, $x = 0$ is not an infleciton point.

⚠️ **Warning** — If a function is concave down before $x = c$, and concave up after $x = c$, it is not necessary that there exists an infleciton point. Notably: $f(x) = \frac{1}{x}$

Derivative Tests

🐳 **Info** — **First Derivative Test**

Let $f(x)$ has a critical point at $x = c$ and suppose that $f(x)$ is continuous at c. If there is an interval (a, b) containing c s.t.

1. $f'(x) \geq 0$ on (a, c) and $f'(x) \leq 0$ on (c, b), then f has a local maximum at c
2. $f'(x) \leq 0$ on (a, c) and $f'(x) \geq 0$ on (c, b), then f has a local minimum at c

Otherwise c is neither a local maximum nor a local minimum

🐳 **Info** — **Second Derivative Test**

Suppose that $f'(c) = 0$ and $f''(x)$ is continuous at c, then

1. if $f''(c) < 0$, then there is a local maximum for f a c
2. if $f''(c) > 0$, then there is a local minimum for f at c
3. if $f''(x) = 0$, then it is inconclusive, that is, there might be a local maximum, a local minimum, or neither.

Comparison:

FDT:

- Requires an interval, that is points around c
- Requires lesser steps of differentiation
- Conclusive as long as the constraints are satisfied

SDT:

- Requires an interval, that is the point at c
- Requires more steps of differentiation
- Inconclusive in some certain cases $f''(x) = 0$

Examples:

1. Classify all critical points of $f(x) = x^3 - 13x + 12$

   $f'(x) = 3x^2 - 13, f''(x) = 6x$

   $f'(x) = 3x^2 - 13 \Longrightarrow f'(x) \geq 0 \ \forall x \in \left(-\infty, -\sqrt{\frac{13}{3}}\right] \cup \left[\sqrt{\frac{13}{3}}, \infty\right)$

   and $f'(x) \leq 0 \ \forall x \in [-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}}$

   $f'(0) < 0, f'(3) > 0, f'(-3) > 0$

   That is, by FDT, we have a local maximum at $x = -\sqrt{\frac{13}{3}}$ and a local minimum at $x = \sqrt{\frac{13}{3}}$

   $f''(x) = 6x, f''(x) < 0 \ \forall x \in (-\infty, 0), f''(x) > - \ \forall x \in (0, \infty)$.

   By SDT, we have a local maximum at $x = -\sqrt{\frac{13}{3}}$ and a local minimum at $x = \sqrt{\frac{13}{3}}$

2. Find all extrema of $f(x) = x\sqrt[3]{x-4}$ on interval $[0, 5]$

   $f'(x) = \sqrt[3]{x-4} + \frac{1}{3}x(x-4)^{-\frac{2}{3}}$

   $f''(x) = \frac{1}{3}(x-4)^{-\frac{2}{3}} + \frac{1}{3}(x-4)^{-\frac{2}{3}} - \frac{2}{9}(x-4)^{-\frac{5}{3}} = \frac{2}{3}(x-4)^{-\frac{2}{3}}\left(1 - \frac{x}{3(x-4)}\right)$

   $f'(4) = \text{DNE}, f'(x) = 0 \Longleftrightarrow 1 + \frac{x}{3(x-4)} \Longleftrightarrow x = 3$

   $f'(2), f'(5), f'(3.5)$

Optimization

L'Hôpital Rule