

CH 2 – Differential Equations

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Introduction to Differential Equations

Info – Differential Equations

A differential equation (DE), is an equation involving an unknown function and its derivatives. The term ordinary differential equation (ODE) refers to a differential equation involving single-variable functions, whereas the term partial differential equation (PDE) refers to a differential equation involving multivariable functions (i.e., functions with multiple inputs).

An ODE is expressed

$$F(x, y, y', y'', \dots, y^n) = 0$$

for some $n \in \mathbb{N}$

The order of a DE is the order of the **highest derivative** that appears in the equation.

A function $y = \varphi(x)$ is a solution to the differential equation $F(x, y, y', y'', \dots, y^n) = 0$ if

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), \dots, \varphi^n(x)) = 0$$

The graph of a solution to a DE is called a **solution curve**

The complete collection of solutions to a DE, including any arbitrary constants, is called its general solution. A particular solution to a DE is one in which all arbitrary constants have been specified.

A differential equation together with one or more initial conditions is known as an initial value problem (IVP)

Examples:

1. First order differential equation $\frac{dy}{dx} = x + y$
2. Second order differential equation $\frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = \ln x + y$
3. Solve $\frac{dy}{dx} = \sin x \implies y = -\cos x + C, c \in \mathbb{R}$
4. $\frac{dy}{dx} = \sin x, y(0) = 2 \implies -\cos x + C \implies -\cos x + 3$
5. $y' = x + y$

$$y = -1 - x + 2e^x \implies y' = -1 + 2e^x \implies (x + y) = x + (-1 - x + 2e^x) = -1 + 2e^x = y'$$

6. $y = -1 - x - 5e^x$

$$y' = -1 - 5e^x \implies x + (-1 - x - 5e^x) = -1 - 5e^x = y'$$

7. $y = -5 - x + 2e^x$

$$y' = -1 + 2e^x \implies x + (-5 - x + 2e^x) = -5 + 2e^x \neq y'$$

Thus $y = -1 - x + Ce^x$, for $C \in \mathbb{R}$ is always a solution

8. Determine all real numbers k s.t. $x(t) = \sin(kt)$ is a solution to the second-order differential equation $\frac{d^2y}{dx^2} = -2x$

$$\begin{cases} x'(t) = k \cos(kt) \\ x''(t) = -k^2 \sin(kt) \end{cases} \implies -k^2 \sin kt = 0 \quad 2 \sin kt$$

$$(k^2 - 2) \sin(kt) = 0$$

$$k = \pm\sqrt{2}, k = 0$$

Direction Fields

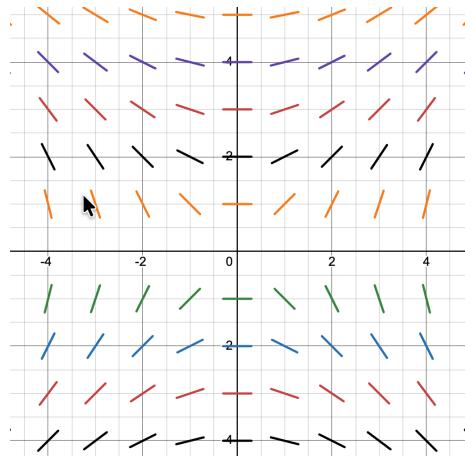
Info – Direction Field

A direction field for the differential equation $y' = F(x)$ displays short line segments of slope $F(x, y)$ at various points in the Cartesian plane

 Tip – Direction field plotter for DE: <https://www.desmos.com/calculator/p7vd3cdmei>

Examples:

1. $\frac{dy}{dx} = \frac{x}{y}$

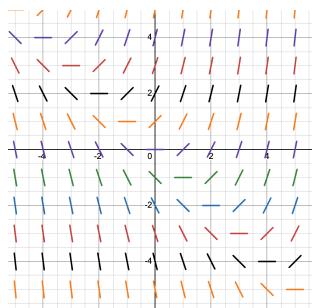


We can obtain this when plugging-in numbers.

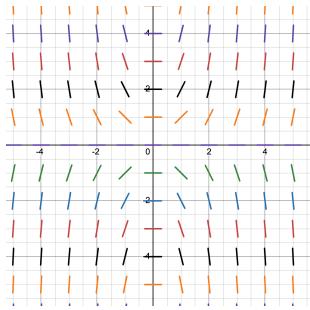
$$\text{At } (-2, 1), \frac{dy}{dx} \Big|_{(x,y)=(-2,1)} = \frac{x}{y} \Big|_{(x,y)=(-2,1)} \implies -\frac{2}{1} = -2$$

Hence a slope of -2 at $(-2, 1)$ and other points have the same method

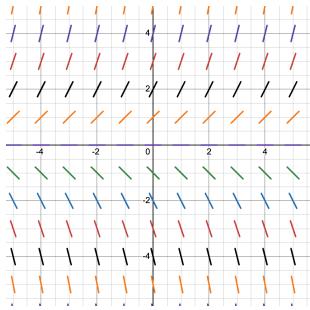
2. $y'x = x + y$



3. $y' = xy$



4. $y' = y$



Separable Differential Equations

Info – Separable Differential Equation

A first order differential equation is said to be separable if it is written in format

$$\frac{dy}{dx} = g(x)h(y)$$

Examples:

1. $y' = \sin x$

2. $y' = \frac{x}{y}$

3. $y' = 5y$

4. $y^2y' = 2y + xy$

5. $y' = x + y$ is not separable but linear, see in Linear First Order DEs

 **Tip – Solving Separable DE**

We have two cases

1. Determine any solution y with $h(y) = 0$
2. Find the solutions y where $h(y) \neq 0$ by evaluating

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

If possible, isolate y as a function of x in the resulting equation.

The general solution is the collection of all solutions obtained in Case 1 together with all solutions obtained in Case 2.

Examples:

1. $y' = y$

Case 1: $y = 0$

Case 2:

$$\begin{aligned} \int \frac{1}{y} dy &= \int 1 dx \\ \ln|y| &= x + C_1 \\ |y| &= e^{x+C_1} y = \pm e^{C_1} e^x = \pm C e^x, \quad C \in \mathbb{R} \end{aligned}$$

The general solution is $y = A e^x$, $A \in \mathbb{R}$

2. $y' = \frac{x}{y}$

Case 1: $h(y) = \frac{1}{y} \neq 0$

Case 2:

$$\begin{aligned} \int y dy &= \int x dx \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ y^2 &= x^2 + 2C \\ y &= \pm \sqrt{x^2 + 2C} \end{aligned}$$

The general solution $y = \pm \sqrt{x^2 + D}$, $D \in \mathbb{R}$

3. $y' = x y^2$, $y(0) = 3$

Case 1: $h(y) = 0 \implies y \equiv 0$

Case 2:

$$\int \frac{1}{y^2} dy = \int x dx \implies -\frac{1}{y} = \frac{x^2}{2} + C \implies y = -\frac{2}{x^2 + 2C}$$

The general solution is $y \equiv 0$ or $y = \frac{-2}{x^2 + D}$, $D \in \mathbb{R}$

For $y(0) = 3 \implies -\frac{2}{D} = 3 \implies D = -\frac{2}{3} \implies$ the unique solution is $y = \frac{-2}{x^2 - \frac{2}{3}}$

$$4. \ y' - x^2 - y^2 - 2xy + 1 = 0 \implies y' = (x+y)^2 - 1 \stackrel{v=x+y}{\implies} v' = v^2$$

Case 1: $h(v) = v^2 \equiv 0 \implies v \equiv 0$

$$\text{Case 2: } \int \frac{1}{v^2} dv = x \implies -\frac{1}{v} = x + C \implies v = -\frac{1}{x+C}, C \in \mathbb{R}$$

The general solution $v \equiv 0$ or $v = -\frac{1}{x+C}, C \in \mathbb{R}$

$$y = -x \text{ or } y = -x - \frac{1}{x+C}, C \in \mathbb{R}$$

Linear First-Order Differential Equations

Info – First-Order Linear Differential Equation

First order Linear DE has the form

$$A_0(x)y + A_1(x)y' = B(x) \quad A_1(x) \neq 0$$

Such an equation can be written in the form $y' + P(x)y = Q(x)$ called the standard form.

Info – Integrating Factor

Given a DE of the form $y' + P(x)y = Q(x)$,

$$\mu = e^{\int P(x) dx}$$

is called the integrating factor for the DE.

Tip – Solving First-Order Linear DE

Given $A_1(x)y' + A_0(x)y = B(x)$

1. Divide by $A_1(x)$ to rewrite the DE in standard form
2. Multiply the equation by the integrating factor.
3. Rewrite LHS as $(\mu(x)y)'$
4. Integrate in respect to x
5. Isolate y

Example:

$$1. \ y' + \frac{3}{x}y = 1$$

$$\begin{aligned} \mu(x) &= e^{3 \ln(x)} \\ &\implies x^3 y' + 3x^2 y = x^3 \end{aligned}$$

$$\frac{d}{dx} x^3 y = x^3$$

$$\int \frac{d}{dx} x^3 y \, dx = \int x^3 \, dx$$

$$y = \frac{1}{4}x + \frac{C}{x^3}$$

$$2. \ y' = x + y \implies y' - y = x$$

$$\begin{aligned} \mu(x) &= e^{-x} \\ &\implies e^{-x} y' - e^{-x} y = e^{-x} x \end{aligned}$$

$$\frac{d}{dx} e^{-x} y = x e^{-x}$$

$$y = -x - 1 + C e^x, C \in \mathbb{R}$$

3. $y' + 2xy = 1$

$$\begin{aligned} \mu(x) &= e^{x^2} \\ \implies e^{x^2} y' + 2xye^{x^2} &= e^{x^2} \\ \frac{d}{dx} e^{x^2} y &= e^{x^2} \\ y &= e^{-x^2} \int_0^1 e^{t^2} dt + Ce^{-x^2}, C \in \mathbb{R} \end{aligned}$$

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Applications

Mixing Problem

 **Tip – Formula - Mixing Problem**

$$\frac{dA}{dt} = (\text{rate of substance in}) - (\text{rate of substance out})$$

Q1:

Suppose that a tank has 1000L of salt water at an initial concentration of 0.1kg/L. Salt water at a concentration of 0.3kg/L flows into the tank at a rate of 10 L/min. The solution is kept mixed and drain out at a rate of 10 L/min.

Determine the amount of salt, in kg, in the tank at time t

Let $A(t)$ represent the amount of salt(kg) in the tank at time t (min)

$$\frac{dA}{dt} = \frac{ds_i}{dt} - \frac{ds_o}{dt} \text{ in kg/min}$$

$$\frac{ds_i}{dt} = 0.3 \cdot 10 = 3\text{kg/min}$$

$$\frac{ds_o}{dt} = \frac{A(t)}{1000} \cdot 10 = \frac{A}{100}\text{kg/min}$$

$$A(0) = 100\text{kg}$$

$$\frac{dA}{dt} = 3 - \frac{A}{100} = -\frac{A-300}{100} \implies \int \frac{1}{A-300} dA = \int -\frac{1}{100} dt \implies \ln|A-300| = -\frac{t}{100} + C$$

$$A = 300 \pm e^{-\frac{t}{100}+C}$$

$$\text{General solution } A = 300 + Be^{-\frac{t}{100}}, B \in \mathbb{R}$$

$$B = -200 \text{ since } A(0) = 100$$

$$\text{Particular solution } A = 300 - 200e^{-\frac{t}{100}}$$

Newton's Law of Heating/Cooling

💡 Tip – Formula Newton's Law

$T(t)$ represents the temperature of an object at time t , and T_s is the constant temperature of its surroundings, then there exists a constant $k > 0$

$$\frac{dT}{dt} = -k(T - T_s)$$

The general solution to this DE is given

$$T(t) = T_s + Ae^{-kt}, \quad A \in \mathbb{R}$$

If $T(0) > T_s$, the object will cool.

An equality suggests no change.

$$\frac{dT}{dt} = -k(T - T_s)$$

1. Constant solution: $T \equiv T_s$

$$\begin{aligned} 2. \int \frac{1}{T-T_s} dT &= \int -k dt \implies \ln|T - T_s| = -kt + C \\ &= T = \pm e^{-kt+C} + T_s = \pm e^{-kt} e^C + T_s = Ae^{-kt} + T_s, \quad A \in \mathbb{R} \end{aligned}$$

General Solution:

$$T_s + Ae^{-kt}, \quad A \in \mathbb{R}$$

Q2:

A cup of coffee has a temperature of 98°C in a room of 20°C . After 1 minute, the temperature of the coffee is 96°C . How long will it reach 80°C ?

$$T_s = 20, \exists k > 0 \text{ s.t. } \frac{dT}{dt} = -k(T - 20) \text{ holds.}$$

Initial conditions, $T(0) = 98, T(1) = 96$

The general solution will be $T = 20 + Ae^{-kt}$

With initial condition, we obtain $T = 20 + 78e^{\ln \frac{38}{39}t}$

$$80 = 20 + 78e^{\ln \frac{38}{39}t} \implies t = \frac{\ln(\frac{10}{13})}{\ln(\frac{38}{39})} \approx 10.1 \text{ min}$$

Population Growth

💡 Tip – Exponential Growth/Decay

The general solution to the exponential growth DE $\frac{dP}{dt} = kP$ is given

$$P(t) = Ae^{kt}, \quad A = P(0)$$

- $k > 0$ for growth
- $k < 0$ for decay

 **Tip – Logistic Growth**

- If P is significantly smaller than M , $\frac{dP}{dt} \approx kP$
- If $P \approx M$, $\frac{dP}{dt} \approx 0$
- If $P > M$, $\frac{dP}{dt} < 0$

Formula

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right)$$

General Solution:

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