

CH 5 - Applications of Derivatives

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Related Rates

Tip — Steps for Related Rates Questions

1. Draw diagram
2. Identify **changing** quantities
3. Find **constant** quantities (if possible)
4. Derive equations relating the quantities that are changing
5. **Implicitly differentiate** the key equations
6. Solve for the desired rate of change, substituting in known quantities.
7. **Concluding statement** (and also check units)

Example:

1. Laindon is taking a hot air balloon ride. A giant fan is blowing hot air into the balloon in a rate of $200 \frac{\text{m}^3}{\text{min}}$. Assuming that at any given point in time the balloon sphere, find the rate at which the radius of the balloon is changing when the diameter is 12 m.

ANS:

1. Picture: The problem is trivial so the graph is omitted
2. Changing variable: Volume(m^3), Radius(m), time(t)
3. Constant quantities: $\frac{dV}{dt} = 200 \frac{\text{m}^3}{\text{min}}$
4. Key Equation: $V = \frac{4}{3}\pi r^3(t)$
5. Implicit Differentiation: $\frac{dV}{dt} = 4\pi r^2(t) \cdot \frac{dr}{dt}$
6. $\frac{dr}{dt} \Big|_{r=6} = \frac{1}{4\pi(6)^2} \cdot 200 = \frac{200}{144\pi} = \frac{25}{18\pi} \frac{\text{m}}{\text{min}}$
7. Concluding statement: When the diameter of the balloon is 12m, the rate of change of the radius is expanding by $\frac{25}{18\pi} \frac{\text{m}}{\text{min}}$
2. The construction workers building M4 accidentally left a 20 foot ladder propped up against a concrete wall that is 80 feet in height. The base of the ladder begins to slide away from the wall at a rate of 2ft/sec, and the top begins to move down as a result. When the base of the ladder is 14 ft from the wall, how fast is the top of the ladder sliding down the wall?

ANS:

1. Picture is omitted and left as an exercise for the reader
2. Changing variable: Distance from wall of base of ladder (m), Height where ladder touches the wall (m)
3. Constant quantities : $\frac{dx}{dt} = 2$
4. Key Equation: $x^2 + y^2 = 20^2$

5. Implicit Differentiation: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$
6. $\frac{dy}{dt} = -\frac{14}{\sqrt{400-14^2}} \cdot 2 = -\frac{14}{\sqrt{51}} \frac{\text{ft}}{\text{sec}}$
7. Concluding statement: When the base of ladder is 14cm, the top of the ladder is falling at a speed of $\frac{14}{\sqrt{51}} \frac{\text{ft}}{\text{sec}}$

Extrema

Info – Extrema

Let $f(x)$ be a function defined on an interval I , and let $c \in I$. We say f has

1. A **global minimum** on I at $x = c$ if $f(c) \leq f(x) \forall x \in I$
2. A **global maximum** on I at $x = c$ if $f(c) \geq f(x) \forall x \in I$
3. A **global extremum** on I at $x = c$ if f has either a global minimum or global maximum.
 - Every point on a constant function is both a global minimum and global maximum
 - Every global extremum can be a local extremum in some interval

Examples:

1. Find all global extrema of $f(x) = x^2$ on $[0, 1]$
 - The global minimum be $x = 0$ because $f(0) \leq f(x) \forall x \in [0, 1]$
 - The global maximum DNE as the end point is missing. That is infinitely numbers lie on the interval $[0, 1]$
2. Find all global extrema of $f(x) = \frac{1}{x}$ on $[-1, 1]$
 - The global extrema DNE as $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$.

Info – Extreme Value Theorem (Existence Thm)

Assume that $f(x)$ is continuous on the closed interval $[a, b]$. Then **there exist** two numbers $c_1, c_2 \in [a, b]$ s.t. $f(c_1) \leq f(x) \leq f(c_2) \forall x \in [a, b]$.

In other words, there is a global minimum at $x = c_1$ and a global maximum at $x = c_2$

Info – Local Extrema

Let f be a function. We say that f has

1. a **local minimum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \leq f(x) \forall x \in (a, b)$
2. a **local maximum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(c) \geq f(x) \forall x \in (a, b)$
3. a **local extremum** at $x = c$ if there is either a local minimum or a local maximum

Warning – Local Extrema

If c is an endpoint of the domain of f , c can never be a local extremum, even if it is a global extremum

Info – Fermat's Theorem

If there is a local extremum for $f(x)$ at $x = c$ and $f'(c)$ exists, hence $f'(c) = 0$. That is we cannot put an open interval around the point.

Examples:

1. Does the converse of Fermat's Theorem hold? That is if $f'(0) = 0$, then is a local extremum at $x = c$.

This is false. Let $f(x) = x^3$, $f'(x) = 3x^2$, $f'(0) = 0$ but is not a local extremum on any interval containing $x = 0$

2. Why is it worth mentioning $f'(c)$ has to exist?

It is important because it is like saying $f(x)$ is differentiable at $x = c$. If not, let $f(x) = |x|$. $f(x)$ is continuous. It has a local minimum at $x = 0$ but $f'(0)$ DNE as it is not differentiable.

Info – Critical Points

We say that a function f has a **critical point** at $x = c$ if $f'(c) = 0$ or $f'(c)$ = DNE for $c \in$ the domain of f . These are our candidates for local extrema.

Tip – Closed Interval Method

Let $f(x)$ be continuous function on $[a, b]$.

1. Calculate $f(a)$ and $f(b)$
2. Find $f'(x)$
3. Find all the critical points of f on $[a, b]$
4. Calculate $f(c)$

Example:

$$f(x) = \frac{1}{3}x^3 - 3\sqrt[3]{x} \text{ on } [-8, 1]$$

$$f(-8) = -\frac{512}{3} - 3(-2) = -\frac{496}{3}$$

$$f(1) = \frac{1}{3} - 3 = -\frac{8}{3}$$

$$f'(x) = x^2 - x^{-\frac{2}{3}}$$

$$f'(c) = 0 \implies c^2 - c^{-\frac{2}{3}} = 0 \implies c^{\frac{8}{3}} = 1 \implies c = -1, 1. f'(c) = \text{DNE} \implies c = 0$$

$$f(0) = 0$$

$$f(-1) = -\frac{1}{3} + 3 = \frac{8}{3} \text{ Global maximum at } x = -1, \text{ global minimum at } f(-8)$$

Info – Rolle's Theorem (Existence Thm)

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b) = k \in \mathbb{R}$, then there exists $c \in (a, b)$ s.t. $f'(c) = 0$

Proof

If $f(x) = k \forall x \in [a, b]$, any value of c works.

Otherwise, $\exists x_0 \in [a, b]$ s.t. $f(x_0) \neq k$. Since f is continuous on $[a, b]$, it attains a maximum/minimum on $[a, b]$.

Since $f(x_0) \neq k \implies f(x_0) > k \leftrightarrow f(a), f(b)$ are not maximum, or $f(x_0) < k \leftrightarrow f(a), f(b)$ are not minimum. So one of maximum or minimum is in (a, b) , thus differentiable at some c .

By Fermat's Theorem, $f'(c) = 0$ or $f'(c) = \text{DNE}$. But f is differentiable on $(a, b) \implies f'(c)$ exists.

$$\therefore f'(c) = 0$$

Info – Mean Value Theorem (Existence Thm)

If f is continuous on $[a, b]$ and differentiable on (a, b) , then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

$$\text{Let } h(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$$

$$h(a) = f(a) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (a-a) \right] = 0$$

$$h(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{b-a} \cdot (b-a) \right] = 0$$

Since $h(b) = h(a) \stackrel{\text{Rolle's Theorem}}{\implies} \exists c \in (a, b)$ s.t. $h'(c) = 0$

That is $h'(x) = f'(x) \frac{f(b)-f(a)}{b-a} \implies f'(x) = \frac{f(b)-f(a)}{b-a}$

Finally, $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \leftrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

Tip: the construction of $\left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$ is the linear approximation of $f(x)$ near a

Antiderivative

Info – Antiderivative

Given a function $f(x)$, an **antiderivative** is a function $F(x)$ s.t. $F'(x) = f(x)$. If $F'(x) = f(x)$ for all $x \in I$ for some interval I , then $F(x)$ is an antiderivative of $f(x)$ on I

$$\text{e.g. } \frac{d}{dx}(\ln(\cos x)) = -\frac{1}{\cos x} \sin x = -\frac{-\sin x}{\cos x} = \tan x$$

Note: one function can have infinitely many antiderivatives, that is why we insist **an antiderivative** of $f(x)$

Info – Constant Function Theorem

Suppose that $f'(x) = 0 \forall x \in I$ for some interval I . Then $\exists \alpha \in \mathbb{R}$ s.t. $f(x) = \alpha \forall x \in I$

Proof

Let $x_1 < x_2 \in I$.

Since f is differentiable on I , it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) .

By MVT, $\exists c \in (x_1, x_2)$ s.t. $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. But $f'(c) = 0$ since $f'(x) = 0$ on I .

Thus, $0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \wedge x_2 - x_1 \neq 0 \implies f(x_2) - f(x_1) = 0 \iff f(x_2) = f(x_1)$.

Since x_1, x_2 are arbitrary, therefore f is constant on I

Info – Antiderivative Theorem

Suppose that $F'(x) = G'(x) \forall x \in I$ for some interval I . Then there exists $\alpha \in \mathbb{R}$ s.t. $F(x) = G(x) + \alpha \forall x \in I$

Proof

Let $h(x) = F(x) - G(x)$. Then $h'(x) = F'(x) - G'(x) = 0$ on I .

By the CFT, $h(x) = \alpha$ for some $\alpha \in \mathbb{R}$, so $F(x) - G(x) = \alpha \implies F(x) = G(x) + \alpha \quad (\forall x \in I)$

First Derivatives

Info – Definition of Increasing/Decreasing

Let I be in interval and $x_1, x_2 \in I$, then $f(x)$

- **increasing** on I if $f(x_1) \leq f(x_2) \forall x_1 < x_2$
- **decreasing** on I if $f(x_1) \geq f(x_2) \forall x_1 < x_2$
- **strictly increasing** on I if $f(x_1) < f(x_2) \forall x_1 < x_2$
- **strictly decreasing** on I if $f(x_1) > f(x_2) \forall x_1 < x_2$

Note: a constant function is both increasing and decreasing but not strictly

Info – Increasing/Decreasing Function Theorem

Let I be an interval

1. If $f'(x) \geq 0 \forall x \in I$, then $f(x)$ is increasing on I
2. If $f'(x) > 0 \forall x \in I$, then $f(x)$ is strictly increasing on I
3. If $f'(x) \leq 0 \forall x \in I$, then $f(x)$ is decreasing on I
4. If $f'(x) < 0 \forall x \in I$, then $f(x)$ is strictly decreasing on I

Proof

Let $x_1 < x_2 \in I$. $f'(x) > 0$ on I , so it exists on $I \Rightarrow f$ is differentiable on (x_1, x_2) , continuous on $[x_1, x_2]$.

By MVT, $\exists c \in (x_1, x_2)$ s.t. $f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Given $f'(x) > 0 \forall x \in I$ and $x_1 < x_2$ and $f(x_1) < f(x_2)$

Since x_1, x_2 are arbitrary, thus f is strictly increasing on I

Proof for increasing, strictly decreasing and decreasing is similar thus be omitted.

Question: If f is strictly increasing on $I \Rightarrow f'(x) > 0 \forall x \in I$?

ANS: No, counterexample $f(x) = x^3$

Question: If f is strictly decreasing on $I \Rightarrow f'(x) < 0 \forall x \in I$?

ANS: No, counterexample $f(x) = -\sqrt[3]{x}$



Info – Bounded Derivative Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) .

Suppose that $m \leq f'(x) \leq M \forall x \in (a, b)$. Then

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

$$\forall x \in [a, b]$$

Proof

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) .

It also applies to $[a, x_1]$.

Case1:

For $x_1 \in (a, b]$

By MVT, $\exists c \in (a, x_1)$ s.t. $f'(c) = \frac{f(x_1) - f(a)}{x_1 - a}$

Since $m \leq f'(c) \leq M$ so $m(x_1 - a) \leq f(x_1) - f(a) \leq M(x_1 - a)$.

Then $m(x_1 - a) + f(a) \leq f(x_1) \leq M(x_1 - a) + f(a)$

Case 2:

When $x = a$, $m(x - a) + f(a) = f(a)$ and similar applies to $M(x - a)$.

Resulting to $f(a) \leq f(a) \leq f(a)$

Example:

Prove that $\sqrt{51} \in [7 + \frac{1}{8}, 7 + \frac{1}{7}]$.

Let $f(x) = \sqrt{x}$ and let $[a, b]$ be $[49, 64]$

Since $f(x)$ is continuous on $[49, 64]$ and differentiable on $(49, 64)$

By Bounded Derivative Theorem, $f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$

where $m \leq f'(x) \leq M$ on $[49, 64] \forall x$.

$$\begin{aligned}f'(x) &= \frac{1}{2\sqrt{x}}, \frac{1}{\sqrt{64}} \leq \frac{1}{2\sqrt{x}} \leq \frac{1}{2\sqrt{49}} \Rightarrow 7 + \frac{1}{2} \cdot \frac{1}{8}(51 - 49) \leq \sqrt{51} \leq \sqrt{49} + \frac{1}{2} \cdot \frac{1}{7}(51 - 49) \\&\Rightarrow 7 + \frac{1}{8} \leq \sqrt{51} \leq 7 + \frac{1}{7}\end{aligned}$$

Info – Comparison via First Derivative Theorem

Assume $f(x)$ and $g(x)$ are continuous at $x = a$ with $f(a) = g(a)$. Then

- f and g are differentiable for $x > a$ and $f'(x) \leq g'(x) \forall x > a \Rightarrow f(x) \leq g(x) \forall x > a$
- f and g are differentiable for $x < a$ and $f'(x) \leq g'(x) \forall x < a \Rightarrow f(x) \geq g(x) \forall x < a$

Example:

Show that $x - \frac{1}{2}x^2 < \ln(1 + x) \forall x > 0$

Let $f(x) = x - \frac{1}{2}x^2, g(x) = \ln(1 + x)$

At $x = 0, f(0) = 0, g(0) = 0$, that is $f(0) = g(0)$

All functions are differentiable on $(0, \infty)$

$$f'(x) = 1 - x, g'(x) = \frac{1}{1+x}.$$

Since $x > 0, (1-x)(1+x) < 1 \Rightarrow 1 - x^2 < 1 \Rightarrow -x^2 < 0$

Thus by CFDT, $x - \frac{1}{2}x^2 < \ln(1 + x) \forall x > 0$

Second Derivatives

Info – Concavity and Second Derivative Theorem

1. We say f is **concave up** on an interval I if for all $a, b \in I$, the secant line between $(a, f(a))$ and $(b, f(b))$ lies **above** the graph of $f(x)$.
2. We say f is **concave down** on an interval I if for all $a, b \in I$, the secant line between $(a, f(a))$ and $(b, f(b))$ lies **below** the graph of $f(x)$.
3. If $f''(x) > 0 \forall x$ in an interval I then $f(x)$ is **concave up** on I
4. If $f''(x) < 0 \forall x$ in an interval I then $f(x)$ is **concave down** on I

Example:

$f(x) = |x|$ is neither concave up nor concave down as the secant line lies on the graph if a and b are same side of the absolute function.

Info – Point of Inflection

A point $(c, f(x))$ is called a **point of inflection** of $f(x)$ if $f(x)$ is continuous at $x = c$ and the concavity of $f(x)$ changes at $x = c$

If $f''(x)$ is continuous at c and $(c, f(c))$ is a point of inflection, then $f''(c) = 0$ or DNE

However, the converse is false: $f(x) = x^4$, $f''(0) = 0$, but the concavity does not change

Examples:

1. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$

$f''(x) < 0 \forall x \in (-\infty, 0)$, concave down

$f''(x) > - \forall x \in (0, \infty)$, concave up

$f''(x) = 0 \implies 6x = 0 \implies x = 0$.

2. $g(x) = \frac{1}{x}$, $g'(x) = -\frac{1}{x^2}$, $g''(x) = \frac{2}{x^3}$.

$g''(x) < 0 \forall x \in (-\infty, 0)$, concave down

$g''(x) > - \forall x \in (0, \infty)$, concave up

$g''(x)$ is discontinuous at $x = 0$, therefore, $x = 0$ is not an inflection point.

⚠ Warning – If a function is concave down before $x = c$, and concave up after $x = c$, it is not necessary that there exists an inflection point. Notably: $f(x) = \frac{1}{x}$

Derivative Tests

Info – First Derivative Test

Let $f(x)$ has a critical point at $x = c$ and suppose that $f(x)$ is continuous at c . If there is an interval (a, b) containing c s.t.

1. $f'(x) \geq 0$ on (a, c) and $f'(x) \leq 0$ on (c, b) , then f has a local maximum at c
2. $f'(x) \leq 0$ on (a, c) and $f'(x) \geq 0$ on (c, b) , then f has a local minimum at c

Otherwise c is neither a local maximum nor a local minimum

Info – Second Derivative Test

Suppose that $f'(c) = 0$ and $f''(x)$ is continuous at c , then

1. if $f''(c) < 0$, then there is a local maximum for f at c
2. if $f''(c) > 0$, then there is a local minimum for f at c
3. if $f''(c) = 0$, then it is inconclusive, that is, there might be a local maximum, a local minimum, or neither.

Comparison:

FDT:

- Requires an interval, that is points around c
- Requires lesser steps of differentiation
- Conclusive as long as the constraints are satisfied

SDT:

- Requires an interval, that is the point at c
- Requires more steps of differentiation
- Inconclusive in some certain cases $f''(x) = 0$

Examples:

1. Classify all critical points of $f(x) = x^3 - 13x + 12$

$$f'(x) = 3x^2 - 13, f''(x) = 6x$$

$$f'(x) = 3x^2 - 13 \implies f'(x) \geq 0 \quad \forall x \in \left(-\infty, -\sqrt{\frac{13}{3}}\right] \cup \left[\sqrt{\frac{13}{3}}, \infty\right)$$

$$\text{and } f'(x) \leq 0 \quad \forall x \in [-\sqrt{\frac{13}{3}}, \sqrt{\frac{13}{3}}]$$

$$f'(0) < 0, f'(3) > 0, f'(-3) > 0$$

That is, by FDT, we have a local maximum at $x = -\sqrt{\frac{13}{3}}$ and a local minimum at $x = \sqrt{\frac{13}{3}}$

$$f''(x) = 6x, f''(x) < 0 \quad \forall x \in (-\infty, 0), f''(x) > 0 \quad \forall x \in (0, \infty).$$

By SDT, we have a local maximum at $x = -\sqrt{\frac{13}{3}}$ and a local minimum at $x = \sqrt{\frac{13}{3}}$

2. Find all extrema of $f(x) = x\sqrt[3]{x-4}$ on interval $[0, 5]$

$$f'(x) = \sqrt[3]{x-4} + \frac{1}{3}x(x-4)^{-\frac{2}{3}}$$

$$f''(x) = \frac{1}{3}(x-4)^{-\frac{2}{3}} + \frac{1}{3}(x-4)^{-\frac{2}{3}} - \frac{2}{9}(x-4)^{-\frac{5}{3}} = \frac{2}{3}(x-4)^{-\frac{2}{3}} \left(1 - \frac{x}{3(x-4)}\right)$$

$$f'(4) = \text{DNE}, f'(x) = 0 \iff 1 + \frac{x}{3(x-4)} \iff x = 3$$

$$f'(0) < 0, f'(5) > 0, f'(3.5) > 0$$

By FDT, $x = 3$ is a local minimum

SDT is inconclusive as $x = 4$ as $f''(4) = \text{DNE}$

$$f(0) = 0, f(3) = -3, f(4) = 0, f(5) = 5$$

The global maximum is at $x = 5$ and global minimum at $x = 3$ on the interval $[0, 5]$

L'Hôpital Rule

Info – Indeterminate Forms

1. $\frac{0}{0}$
2. $\frac{\pm\infty}{\pm\infty}$
3. $0 \cdot \pm\infty$
4. $\infty - \infty$
5. 1^∞
6. ∞^0
7. 0^0

These forms signal to apply L'Hôpital.

1 and 2 are classic form, 3 to 7 need to be manipulated into classical form to apply L'Hôpital

Info – L'Hôpital Rule

Suppose that $f'(x), g'(x)$ exists near a , except possibly at $x = a$, and that $g'(x) \neq 0$ near a , except possibly at $x = a$.

Suppose that $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is of indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

whenever the right side exists or equals $\pm\infty$

You need to write $\stackrel{\text{LHR}}{=}$ just for the sake of this course

⚠ Warning – Case do not use LHR

$\lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{\text{LHR}}{=} \lim_{x \rightarrow 0} \cos x = 0$ **WRONG, CIRCULAR LOGIC, REMEMBER FUNDAMENTAL TRIG LIMIT**

Examples:

$$1. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} \stackrel{\text{LHR}}{=} \lim_{x \rightarrow \pi} = 2 \frac{x}{\cos x} = 2 \frac{\pi}{-1} = -2\pi$$

$$2. \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{\text{LHR}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \frac{1}{x} = \frac{1}{\infty} = 0$$

$$3. \lim_{x \rightarrow 0^+} \frac{\frac{1}{x^2}}{\ln x} \stackrel{\text{LHR}}{=} \lim_{x \rightarrow 0^+} \frac{-\frac{2}{x^3}}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} = -\frac{2}{x^2} = -\frac{2}{0} = -\infty$$

Info - $0 \cdot \pm\infty$

$$\lim_{x \rightarrow a} f(x)g(x) \stackrel{0 \cdot \pm\infty}{=} \lim_{x \rightarrow a} \frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} \stackrel{\text{LHR}}{\underset{\pm\infty}{\equiv}} \dots$$

Examples:

$$1. \lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} = \frac{\ln x}{\frac{1}{x}} \stackrel{\text{LHR}}{\equiv} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\frac{x^2}{x} = 0$$

$$2. \lim_{x \rightarrow -\infty} x^{\frac{5}{3}} \cdot e^x = \lim_{x \rightarrow -\infty} \frac{x^{\frac{5}{3}}}{e^{-x}} \stackrel{\text{LHR}}{\equiv} \lim_{x \rightarrow -\infty} \frac{\frac{5}{3}x^{\frac{2}{3}}}{-e^{-x}} \stackrel{\text{LHR}}{\equiv}$$

$$\lim_{x \rightarrow -\infty} \frac{\frac{10}{9}x^{-\frac{1}{3}}}{e^{-x}} = \frac{10}{9} \lim_{x \rightarrow -\infty} \frac{e^x}{x^{\frac{1}{3}}} = \frac{0}{-\infty} = 0$$

Info - $\infty - \infty$

$\lim_{x \rightarrow a} f(x) - g(x)$ needed to be algebraically manipulated using one of two

1. Using ln
2. Using conjugate/fractions

Example:

$$\lim_{x \rightarrow \infty} \ln x + \ln\left(\frac{67}{x+1}\right) = \lim_{x \rightarrow \infty} \ln(67 \frac{x}{x+1}) = \ln\left(\lim_{x \rightarrow \infty} 67 \frac{x}{x+1}\right) = \ln 67$$

Info - 1^∞

Curve Sketching

Optimization