

# CH 1 – Integration

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## Definite Integrals

### Info – Riemann Sums

Given  $f(x)$  that is defined over  $[a, b]$  with  $a < b$ , the area under function  $f(x)$  can be found by

#### 1. Left-Endpoint Riemann Sum

$$L_n = \sum_{i=0}^{n-1} f(x_i^*) \Delta x$$

- Underestimates Increasing Functions

#### 2. Right-Endpoint Riemann Sum

$$R_n = \sum_{i=1}^n f(x_i^*) \Delta x$$

- Overestimates Increasing Functions

where

- $\Delta x = \frac{b-a}{n}$  under regular partition
- $x_i^* = a + i\Delta x = a + i\frac{b-a}{n}$

$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n$  for  $f(x)$  on interval  $[a, b]$

Regular Partition means interval  $[a, b]$  is equally partitioned into  $n$  rectangles with identical width

Example:

Estimate area under the curve for  $f(x) = x^2$  on  $x \in [0, 1]$

$$R_n = \sum_{i=1}^n \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^n \frac{1}{n} \left(\frac{i}{n}\right)^2 = \frac{1}{n^3} \sum_{i=1}^n i^2 = \frac{(n+1)(2n+1)}{6n}$$

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n} = \frac{1}{3}$$

### Info – Definite Integral

$f(x)$  defined on  $x \in [a, b]$  with regular partition with  $n$  subintervals

The definite integral of  $f(x)$  on  $[a, b]$  is defined

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(a + i\Delta x) \Delta x$$

A function is integrable on  $x \in [a, b]$  provided that the limit of Riemann Sum exists and has the same value regardless of the choice of  $x_i^*$

### Info – Integrability Theorem for Continuous Functions

Integrability:  $\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n$

1. If  $f$  is continuous of  $[a, b]$  then  $f$  is integrable on  $[a, b]$
2.  $f$  is bounded on  $[a, b]$  and has a **finite** number of discontinuities, then  $f$  is integrable on  $[a, b]$

That is continuity implies integrability and the other way is false

Examples:

1.  $f(x) = x^2$
2.  $f(x) = \begin{cases} 2 & \text{if } x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}$ , note that  $f(x)$  is discontinuous
3.  $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$  on  $[0, 1]$ 
  - $x_i^*$  is rational

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x^* + i) \Delta x = 1$$

•  $x_i^*$  is irrational  $\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x^* + i) \Delta x = 0$

Thus not integrable

For geometric interpretation, Riemann Sums and Definite Integrals measures the “signed” area where there is no more than 1 inflection point

- A positive result of  $w$  implies the area under the curve above  $x$ -axis is  $w$
- A negative result of  $w$  implies the area under the curve under  $x$ -axis is  $w$

### Info – Parity of Functions and Definite Integrals

Let  $f(x)$  be bounded and integrable on  $[-a, a]$

1. If  $f(x)$  is odd function, then

$$\int_{-a}^a f(x) dx = 0$$

2. If  $f(x)$  is even function where  $\int_0^a f(x) dx = w$

$$\int_{-a}^a f(x) dx = 2w$$

Examples:

1.  $\int_1^3 x^2 - 3x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(1 + \frac{2i}{n}) \cdot \frac{2}{n} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ (1 + \frac{2i}{n})^3 - 3(1 + \frac{2i}{n}) \right] \cdot \frac{2}{n}$   
 $= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( -\frac{4i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n} \right) = -\frac{10}{3}$
2.  $\int_0^5 x^3 - 2 dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(i \frac{5}{n}) \frac{5}{n} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ (\frac{5}{n}) \left( (i \frac{5}{n})^3 - 2 \right) \right] = \frac{583}{4}$

### Info – Basic Property of Definite Integral

Let  $f(x), g(x)$  be integrable on  $[a, b]$

1. For any  $c \in \mathbb{R}$ , the function  $cf(x)$  is integrable and

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx$$

2. The function  $f + g$  is integrable and

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

3. If  $m, M \in \mathbb{R}$  and  $m \leq f(x) \leq M \forall x \in [a, b]$ , then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

4. If  $f(x) \geq 0 \forall x$ , then

$$\int_a^b f(x) dx \geq 0$$

5. If  $f(x) \leq g(x) \forall x \in [a, b]$ , then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

6. The function  $|f|$  is integrable on  $[a, b]$  and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

7. Bound fliping

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

8.  $\int_a^a f(x) dx = 0$

### Info – Separation of Domain of Definite Integral

If  $f(x)$  is also integrable on an interval containing  $a, b, c$ , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

### Info – Average Value of Function

Let  $f$  be a function that is continuous on an interval  $[a, b]$  with  $a < b$ . The **average value of  $f$  on  $[a, b]$**  is defined as

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

Examples:

- Determine the average value of  $f(x) = 1 - x^2$  on  $[-1, 1]$

$$f_{\text{avg}} = \frac{1}{1-(-1)} \int_{-1}^1 f(x) dx = \int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1-(\frac{i}{n})^2}{n}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 1 - \frac{i^2}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \left( n - \frac{1}{n^2} \left( \frac{n(n+1)(2n+1)}{6} \right) \right) = 1 - \frac{1}{3} = \frac{2}{3}$$

- Suppose that  $f, g$  are integrable on  $[-1, 1]$ ,  $\int_1^{-1} f(t) dt = 5$ , and  $g$  is an even function with  $\int_0^1 g(t) dt = 2$ .

$$\int_{-1}^1 3f(x) - g(x) dx = 3 \int_{-1}^1 f(x) dx - \int_{-1}^1 g(x) dx = -3 \int_1^{-1} f(x) dx - 2 \int_0^1 g(x) dx = -19$$

### Info – Fundamental Theorem of Calculus (FTC - 1)

Let  $a \in \mathbb{R}$ . If  $f$  is continuous on an open interval  $I$  containing  $a$ , then the function

$$G(x) = \int_a^x f(t) dt$$

is differentiable  $\forall x \in I$  and  $G'(x) = f(x)$ . That is,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

### General Extended FTC 1

Let  $f$  be continuous,  $g, h$  be differentiable

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f'(h(x))h'(x) - f'(g(x))g'(x)$$

### Proof

Given  $x \in I$ , from the definition of the derivative, we have

$$G'(x) = \lim_{h \rightarrow 0} \frac{G(x+h) - G(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(t) dt - \int_a^x f(t) dt}{h} = \lim_{h \rightarrow 0} \frac{\int_a^x f(t) dt + \int_x^{x+h} f(t) dt - \int_a^x f(t) dt}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt.$$

For all  $h \neq 0$ , sufficiently close to 0, and  $h > 0$   $f$  is continuous on  $[x, x+h]$ .

$\forall h, \exists c = c(h)$  in  $[x, x+h]$  s.t.

$$f(c(h)) = \frac{1}{h} \int_x^{x+h} f(t) dt$$

Since  $x \leq c(h) \leq x + h$ , by Squeeze Theorem,  $\lim_{h \rightarrow 0} c_h = x$ , thus

$$G'(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt = \lim_{h \rightarrow 0} (c_h) = f(x)$$

□

Examples

$$1. G(x) = \int_0^x \frac{1}{1+t^2} dt$$

Since  $f(t) = \frac{1}{1+t^2}$  is continuous on  $\mathbb{R}$ , by FTC 1

$$G'(x) = \frac{d}{dx} \int_0^x \frac{1}{1+t^2} dt = \frac{1}{1+x^2}$$

$$2. H(x) = \int_2^{e^x} \cos(t^2) dt$$

Since  $f(t) = \cos(t^2)$  is continuous on  $\mathbb{R}$ , by FTC 1

$$H'(u) = \frac{d}{du} \int_2^u \cos(t^2) dt \cdot \frac{du}{dx} = \frac{du}{dx} \cos(u^2) \stackrel{u=e^x}{=} e^x \cos(e^{2x})$$

3. Assume  $f$  is continuous and  $g, h$  differentiable

$$G(x) = \int_{g(x)}^{h(x)} f(t) dt = \int_{g(x)}^0 f(t) dt + \int_0^{h(x)} f(t) dt = - \int_0^{g(x)} f(t) dt + \int_0^{h(x)} f(t) dt$$

$$G'(x) = - \frac{d}{dx} \int_0^{g(x)} f(t) dt + \frac{d}{dx} \int_0^{h(x)} f(t) dt \stackrel{\text{by FTC 1}}{=} -f'(g(x))g'(x) + f'(h(x))h'(x)$$

### Info – Fundamental Theorem of Calculus (FTC - 2)

If  $f, F$  are continuous on  $[a, b]$  and  $F'(x) = f(x) \forall x \in (a, b)$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

### Proof

Let  $F$  be any antiderivative of  $f$ . Then  $F(x)$  and the antiderivative  $G(x) = \int_a^x f(t) dt$  have the relation that  $G(x) = F(x) + C$

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b f(t) dt - \int_a^a f(t) dt \\ &= G(b) - G(a) \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) \end{aligned}$$

□

Example

$$\text{If } H(x) = \int_5^x x^2 dx, \int_1^2 = H(2) - H(1) = \int_5^2 x^2 dx - \int_5^1 x^2 dx = \int_1^2 x^2 dx$$

 **Info – Basic Integration Rules**

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad \forall r \neq -1$$

$$\int x^{-1} \, dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int -\csc^2 x \, dx = \cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

Examples:

$$1. \int e^{5x} \, dx = \frac{e^{5x}}{5} + C$$

$$2. \int \frac{t}{t+1} \, dt = \int 1 - \frac{1}{t+1} \, dt = t - \ln|t+1| + C$$

Examples:

$$1. \int_0^4 2x^2 - x \, dx = \frac{2}{3}x^3 - \frac{x^2}{2} \Big|_0^4 = \frac{2}{3}(4)^2 - \frac{4^3}{2} - 0 = \frac{128}{3} - 8 = \frac{104}{3}$$

$$\begin{aligned} 2. \int_1^3 \frac{x + |x - 2|}{x} \, dx &= \int_1^2 \frac{2}{x} \, dx + \int_2^3 \frac{2x - 2}{x} \, dx \\ &= 2 \ln|x| \Big|_1^2 + \int_2^3 2 - \frac{2}{x} \, dx \\ &= 2 \ln 2 + 2x - 2 \ln|x| \Big|_2^3 \\ &= 2 \ln 2 + 6 - 2 \ln 3 - 4 + 2 \ln 2 \\ &= 4 \ln 2 + 2 - 2 \ln 3 \end{aligned}$$

## Substitution Rule / U-Substitution

### Info – U-Substitution

Let  $f, g$  be functions s.t.  $g'(x)$  is continuous on  $a, b]$  and  $f$  is continuous on range of  $g$

$$\int f(g(x))g(x) \, dx = \int f(u) \, du \Big|_{u=g(x)}$$

Example:

$$1. \int 2x\sqrt{1+x^2} \, dx$$

$$\begin{aligned} \text{Let } u &= 1 + x^2 \Rightarrow du = 2x \, dx \Rightarrow dx = \frac{du}{2x} \stackrel{u=1+x^2}{\Rightarrow} \int u^{\frac{1}{2}} \, du = \frac{2}{3}u^{\frac{3}{2}} \stackrel{u=1+x^2}{\Rightarrow} +C \\ &= \frac{2}{3}(1 + x^2)^{\frac{3}{2}} + C \end{aligned}$$

$$2. \int x^2 e^{x^3} \, dx$$

$$\text{Let } u = x^3 \Rightarrow du = 3x^2 \, dx \Rightarrow dx = \frac{du}{3x^2} \stackrel{u=x^3}{\Rightarrow} \int \frac{1}{3}e^u \, du = \frac{1}{3}e^u \stackrel{u=x^3}{\Rightarrow} +C = \frac{1}{3}e^{x^3} + C$$

$$3. \int \frac{\cos(\ln x)}{x} \, dx$$

$$\text{Let } u = \ln x \Rightarrow du = \frac{1}{x} \, dx \Rightarrow dx = x \, du \stackrel{u=\ln x}{\Rightarrow} \int \cos u \, du = \sin u \stackrel{u=\ln x}{\Rightarrow} +C = \sin(\ln x) + C$$

$$4. \int \frac{x}{3\sqrt[3]{x+2}} \, dx$$

$$\text{Let } u = x + 2 \Rightarrow du = dx$$

$$\stackrel{u=x+2}{\Rightarrow} \int \frac{u-2}{3\sqrt[3]{u}} \, du = \int u^{\frac{2}{3}} \, du - 2 \int u^{-\frac{1}{3}} \, du = \frac{3}{5}u^{\frac{5}{3}} + 3u^{\frac{2}{3}} \stackrel{u=x+2}{\Rightarrow} +C = (x+2)^{\frac{5}{3}} + 3(x+2)^{\frac{2}{3}} + C$$

$$5. \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} \, dx$$

$$\text{Let } u = 1 + \sin x \Rightarrow du = \cos x \, dx \Rightarrow dx = \frac{du}{\cos x} \Rightarrow \int_1^2 \frac{1}{u} \, du = \ln u \Big|_1^2 = \ln 2$$

$$6. \int_0^{\frac{\pi}{3}} \tan x \, dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx$$

$$\text{Let } u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow \int_1^{\frac{1}{3}} -\frac{1}{u} \, du = \ln u \Big|_{\frac{1}{3}}^1 = -\ln \frac{1}{3}$$

### Info – $f(ax)$

Let  $a \in \mathbb{R}, a \neq 0$  If  $\int f(x) dx = F(x) + C$ , then

$$\int f(ax) dx = \frac{1}{a} F(ax) + C$$

## Trigonometry Substitution

### Info – Trig-Sub

Integral contains	Substitution	Domain for $\theta$	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \tan \theta$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	$\sec^2 \theta - 1 = \tan^2 \theta$

### Tip – Half-Angle

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 + \sin(2x)}{2}$$

Example:

$$1. \int_0^1 \sqrt{1 - x^2} dx$$

Let  $u = \arcsin x \Rightarrow x = \sin u, \forall u \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow dx = \cos u du \Rightarrow$

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u} \cos u du = \int_0^{\frac{\pi}{2}} \cos^2 u du = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos(2u)) du \left. \frac{1}{2}(u + \sin(2u)) \right|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$2. \int \sqrt{1 - x^2} dx$$

Let  $u = \arcsin x \Rightarrow x = \sin u \quad \forall u \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow dx = \cos u du \Rightarrow$

$$\int \sqrt{1 - \sin^2 u} \cos u du = \int \cos^2 u du = \int \frac{1 + \cos 2u}{2} du \stackrel{u = \arcsin x}{=} \frac{1}{2} (\arcsin x + \sqrt{1 - x^2}) + C$$

$$3. \int \frac{1}{\sqrt{x^2 + 9}} dx$$

Let  $x = 3 \tan \theta \quad \forall \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow dx = 3 \sec^2 \theta d\theta \Rightarrow \int \frac{1}{\sqrt{9 \tan^2 \theta + 9}} 3 \sec^2 \theta d\theta = \int \sec \theta d\theta$

$$\int \sec \theta \cdot \frac{\sec \theta + \tan \theta}{\sec \theta - \tan \theta} d\theta = \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta - \tan \theta} d\theta \stackrel{u = \sec \theta + \tan \theta}{=} \int \frac{1}{u} du = \ln |\sec \theta + \tan \theta| + C$$

$$\stackrel{\theta = \arctan \frac{x}{3}}{\Rightarrow} \ln |\sec \arctan \frac{x}{3} + \tan \arctan \frac{x}{3}| + C = \ln \left( \frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right) + C$$

$$4. \int_1^2 \frac{\sqrt{x^2-1}}{x} dx$$

Let  $x = \sec \theta \quad \forall \theta \in [0, \frac{\pi}{2}) \cup [\pi, 3\frac{\pi}{2})$

$$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta} \sec \theta \tan \theta d\theta = \int_0^{\frac{\pi}{3}} \tan^2 \theta d\theta = \int_0^{\frac{\pi}{3}} \sec^2 \theta - 1 d\theta = \tan \theta - \theta \Big|_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}$$

$$5. \int \frac{1}{(5-4x-x^2)^{\frac{5}{2}}} dx = \int \frac{1}{(1-(2-x)^2)^{\frac{5}{2}}} dx$$

Let  $2-x = \sin \theta, \quad \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow dx = -\cos \theta d\theta$

$$= \int \frac{-\cos \theta}{(1-(\sin \theta)^2)^{\frac{5}{2}}} d\theta = \int \frac{-\cos \theta}{\cos^5 \theta} d\theta = \int -\sec^4 \theta d\theta = -\int (\sec^2 \theta)^2 d\theta = -\frac{2}{7} \sec^{\frac{7}{2}} \theta + C = -\frac{2}{7} \left( \frac{1}{\sqrt{5-4x-x^2}} \right)^{\frac{7}{2}} + C$$

## Integration By Parts

### Info – Integration by Parts

If  $f$  and  $g$  are differentiable functions of  $x$ , then

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

The rule of choosing parts being  $f(x)$  is

1. Logarithmic
2. Inverse trigonometric
3. Algebraic
4. Trigonometric
5. Exponential

## Examples

$$1. \int xe^x dx$$

$$\text{Let } \begin{cases} u=x \\ dv=e^x dx \end{cases} \Rightarrow \begin{cases} du=dx \\ v=e^x \end{cases} \Rightarrow xe^x - \int e^x = xe^x - e^x + C$$

$$2. \int x^2 \sin x dx$$

$$\text{Let } \begin{cases} u=x^2 \\ dv=\sin x dx \end{cases} \Rightarrow \begin{cases} du=2x dx \\ v=-\cos x \end{cases} \Rightarrow -x^2 \cos x + 2 \int x \cos x dx = -x^2 \cos x + 2(x \sin x + \cos x) + C = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

3.