CH 2 — Sequence and Limits

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Triangle Inequality

$$|x - y| \leq |x - z| + |z - y|$$

for $x, y, z \in \mathbb{R}$

Idea: the straight-line distance is shortest.
Without loss of generality assume $x \leq y$; swapping x, y preserves the statement.

Number-line proof by cases:

- Case 1 $z \leq x \leq y$: $|x - y| \leq |z - y| \leq |x - z| + |z - y|$
- Case 2 $x \leq z \leq y$: $|x - y| = |x - z| + |z - y|$
- Case 3 $x \leq y \leq z$: $|x - y| \leq |x - z| + |z - y|$

Triangle Inequality 2
For all $a, b \in \mathbb{R}$

$$|a + b| \leq |a| + |b|$$

Proof:
apply the triangle inequality to $x = a$, $y = -b$, $z = 0$.

Quick check
Is $|a - b| \leq |a| - |b|$ for all a, b ?
No
Example:
$a = 10$, $b = -9$ gives $|10 - (-9)| = 19$, while $|10| - |-9| = 1$
Hence this statement is false.

Interval translations

1. $|x - a| < \delta \Rightarrow x \in (a - \delta, a + \delta)$
2. $|x - a| \leq \delta \Rightarrow x \in [a - \delta, a + \delta]$
3. $0 \leq |x - a| \leq \delta \Rightarrow x \in (a - \delta, a) \cup (a, a + \delta)$

Practice

1) Solve $|2x - 5| < 3$

$-3 < 2x - 5 < 3 \Rightarrow 1 < x < 4$

Answer: $x \in (1, 4)$

2) Solve $2 < |x + 7| \leq 3$

Split into $|x + 7| > 2$ and $|x + 7| \leq 3$

Answer: $x \in [-10, -9) \cup (-5, -4]$

3) Solve $\frac{|x+2|}{|x-2|} > 5$

Consider regions $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$ and track signs of $x + 2$ and $x - 2$

Answer: $x \in \left(\frac{4}{3}, 2\right) \cup (2, 3)$

Infinite Sequences

A sequence is an ordered list $a_1, a_2, a_3, \ldots$; write $\{a_n\}_{n=1}^{\infty}$

A subsequence chooses indices $n_1 < n_2 < \ldots$, yielding $a_{n_1}, a_{n_2}, \ldots$

The tail with cutoff k is $a_k, a_{k+1}, a_{k+2}, \ldots$

Convergence (definition)

IMPORTANT

We say $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there exists N such that $n > N \Rightarrow |a_n - L| < \varepsilon$

Examples

1) Show $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$

Choose $N = \frac{1}{\varepsilon^3}$

Then $n > N \Rightarrow \left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon$

2) Show $\lim_{n\to\infty} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$

Estimate $\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| \leq \frac{5}{16n+4}$

Pick $N > \frac{5}{16\varepsilon} - \frac{1}{4}$

Theorem (Equivalent definitions of the limit of a sequence)

IMPORTANT

For a sequence (a_n) and a number L, the following are equivalent

1) $\lim_{n\to\infty} a_n = L$

2) For every $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains a tail of $\{a_n\}$

3) For every $\varepsilon > 0$, only finitely many n satisfy $|a_n - L| \geq \varepsilon$

4) Every interval (a, b) containing L contains a tail of $\{a_n\}$

5) Given any interval (a, b) containing L, only finitely many terms of $\{a_n\}$ lie outside (a, b)

Example 1

Show $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$

Side work:

$\left| \frac{1}{\sqrt[3]{n}} \right| < \varepsilon \Rightarrow \frac{1}{\sqrt[3]{n}} < \varepsilon \Rightarrow \sqrt[3]{n} > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon^3}$

Proof

Let $\varepsilon > 0$ and choose $N = \frac{1}{\varepsilon^3}$

If $n > N$ then $\left| \frac{1}{\sqrt[3]{n}} \right| < \frac{1}{\sqrt[3]{N}} = \frac{1}{\sqrt[3]{\frac{1}{\varepsilon^3}}} = \varepsilon$

Example 2

Prove $\lim_{n\to\infty} \frac{3n^2+2n}{4n^2+n+1} = \frac{3}{4}$

Rough work

$$\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| = \frac{|5n-3|}{16n^2+4n+4} \leq \frac{5n}{16n^2+4n} = \frac{5}{16n+4}$$

Proof

Given $\varepsilon > 0$, pick $N = \frac{5}{16\varepsilon} - \frac{1}{4}$

Then for $n > N$

$$\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| < \frac{5}{16n+4} \leq \frac{5}{16N+4} < \varepsilon$$

Limits

Thinking quesiton:
Can a sequence converge to two different limits $L \neq M$?

No, we are saying then $\varepsilon < \frac{|L-M|}{2}$
If $a_n \to L$ a tail of the sequence lies in $(L-\varepsilon, L+\varepsilon)$ so only finite many terms can lie in the interval $(M-\varepsilon, M+\varepsilon)$, that is $a_n \nrightarrow M$

IMPORTANT

Theorem (Uniqueness of Limits):
Let $\{a_n\}$ be a sequence. If $\{a_n\}$ has limit L, then the value L is unique.

We say that a sequence **diverges to ∞** if for every $m > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N, a_n > m$.
We say that a sequence **diverges to ∞** if any interval of the form (m, ε) for some $m > 0$ contains a tail of $\{a_n\}$. We write that $\lim_{n\to\infty} a_n = \infty$

We say that a sequence **diverges to −∞** if for every $m < 0$, there exists $N \in \mathbb{N}$ such that for all $n > N, a_n < m$

We say that a sequence **diverges to −∞** if any interval of the form (m, ε) for some $m < 0$ contains a tail of $\{a_n\}$. We write that $\lim_{n\to\infty} a_n = -\infty$

Thinking questions:
1. If a seuqnce consists of non-negative terms, is the limit non-negative?
   ANS: YES Suppose not, then $a_n \to L, a_n > 0, \forall n$. Consider $\varepsilon < \frac{|L|}{2}$. Then $(L-\varepsilon, L+\varepsilon)$ only contains negative numbers, so it can't include a tail of a_n, contradiction.
2. If a sequence consists of positive terms, is the limit positive?
   ANS: NO, consider the sequence $\left\{\frac{1}{n}\right\}$, $\lim_{n\to\infty} \frac{1}{n} = 0$

Examples: Prove that $\lim_{n\to\infty}$ Let $m > 0$ and consider the interval m, ∞. If $n > \sqrt[3]{m}$ then $n^3 > m$ and ao $n^3 \in (m, \infty)$. So choose $k = \lceil \sqrt[3]{m} \rceil + 1$, then the tails lies in (m, ∞)

Limit Laws

IMPORTANT

Let $\{a_n\}, \{b_n\}$ be sequences with $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ for some $a, b \in \mathbb{R}$ then:
1. For any $c \in \mathbb{R}$, if $a_n = c$ for all n then $c = a$
2. For any $c \in \mathbb{R}$, if $\lim_{n\to\infty} ca_n = ca$
3. $\lim_{n\to\infty}(a_n + b_n) = a + b$
4. $\lim_{n\to\infty} a_n b_n = ab$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$
6. If $a_n \geq 0$ for all n and $\alpha > 0$, then $\lim_{n\to\infty} a_n^\alpha = a^\alpha$
7. For any $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = a$

Prove the Sum of Sequences Rule

Proof

$a_n \to a, b_n \to b$
$\forall \varepsilon > 0, \exists M, N \in \mathbb{R}, \forall n > M, n > N, |a_n - a| < \varepsilon, |b_n - b| < \varepsilon$
$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n + b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

IMPORTANT

Tandem Convergence Theorem:
If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$

Examples:
Evaluate the following limits

1) $\lim_{n\to\infty} \frac{3n^2+2n}{4n^2+n+1}$
$= \lim_{n\to\infty} \frac{n^2\left(3+\frac{2}{n}\right)}{n^2\left(4+\frac{1}{n}+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{3+\frac{2}{n}}{4+\frac{1}{n}+\frac{1}{n^2}} = \frac{\lim_{n\to\infty} 3+\lim_{n\to\infty} \frac{2}{n}}{\lim_{n\to\infty} 4+\lim_{n\to\infty} \frac{1}{n}+\lim_{n\to\infty} \frac{1}{n^2}} = \frac{3+0}{4+0+0} = \frac{3}{4}$
2) $\lim_{n\to\infty} \sqrt{n^2+n} - n$, We have indeterminate form $[\infty - \infty]$
$= \lim_{n\to\infty} \sqrt{n^2+n} - n \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{n\left(\sqrt{1+\frac{1}{n}}+1\right)} =$
$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+\lim_{n\to\infty} \frac{1}{n}}+1} = \frac{1}{1+0+1} = \frac{1}{2}$
3) Let the sequence $\{a_n\}$ be defined recursively by $a_1 = 16$ and for all $n > 2$, $a_n = \frac{1}{2}\left(a_{n-1} + \frac{260}{a_{n-1}}\right)$. Given that $\lim_{n\to\infty} a_n$ exists, compute is value
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}\left(a_{n-1} + \frac{260}{a_{n-1}}\right) = \frac{1}{2}\left(\lim_{n\to\infty} a_{n-1} + \frac{260}{\lim_{n\to\infty}} a_{n-1}\right)$
$= \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{260}{\lim_{n\to\infty}} a_n\right)$

Let $L = \lim_{n\to\infty} a_n$, then $L = \frac{1}{2}\left(L + \frac{260}{L}\right) \Leftrightarrow L^2 = \frac{1}{2}L^2 + 260 \Leftrightarrow L \pm \sqrt{260}$

Since a_n consists of non-negative terms, thus its limit converges to a value that is non-negative.
Thus, $\lim_{n\to\infty} a_n = \sqrt{260}$

IMPORTANT

Squeeze Theorem:
If $a_n \geq b_n \geq c_n$ for all $n \in \mathbb{N}$ with $n \geq M$ for some $M \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ for some $L \in \mathbb{R}$, then $\lim_{n\to\infty} b_n = L$

Proof

Since $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ for any $\varepsilon > 0, \exists N_a, N_c \in \mathbb{R} : n > N_a, n > N_c$. $|a_n - L| < \varepsilon, |c_n - L| < \varepsilon$. Let $N = \max(N_a, N_c)$ but $a_n \geq b_n \geq c_n$, so $a_n \in (L-\varepsilon, L+\varepsilon), b_n \in (L-\varepsilon, L+\varepsilon), c_n \in (L-\varepsilon, L+\varepsilon)$
$\therefore \lim_{n\to\infty} b_n = L$

4) $\lim_{n\to\infty} \frac{\sin(n)}{n}$

$-1 \leq \sin(n) \leq 1$ for any $n \in \mathbb{N}$, so $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}, \forall n \in \mathbb{N}$
$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

By Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

5) $\lim_{n\to\infty} \frac{4+(-1)^n}{n^3+n^2-1}$

$\frac{3}{n^3+n^2-1} \leq \frac{4+(-1)^n}{n^3+n^2-1} \leq \frac{5}{n^3+n^2-1}$
$\lim_{n\to\infty} \frac{3}{n^3+n^2-1} = \lim_{n\to\infty} \frac{5}{n^3+n^2-1} = 0$
By Squeeze Theorem, $\lim_{n\to\infty} \frac{4+(-1)^n}{n^3+n^2-1} = 0$
$\lim_{n\to\infty} \frac{4+(-1)^n+(-1)^{n^2+n+2}}{n^3+n^2+100}$ can be solved similarly

Definitions

A sequence $\{a_n\}$ is
1. increasing if $a_n \leq a_{n+1} \forall n \in \mathbb{N}$
2. decreasing if $a_n \geq a_{n+1} \forall n \in \mathbb{N}$
3. monotoinc if it is increasing or decreasing

A set $S \subset \mathbb{R}$ is
1. bounded above it there exists some $\alpha \in \mathbb{R}$ with $a \leq \alpha \forall x \in S$, and we call such an α an upper bound for S. The least upper bound is the smallest such α
2. bounded below it there exists some $\beta \in \mathbb{R}$ with $a \geq \beta \forall x \in S$, and we call such an β an upper bound for S. The greatest lower bound is the largest such β
3. bounded if it is both bounded above and bounded below

If a set $S \subset \mathbb{R}$ is bounded above, it has a least upper bound. If it is bounded below, it has a greatest lower bound.

Greatest lower bound and least upper bound do not have be in part of the set

IMPORTANT

Theorem(Monotone Convergence Theorem): Let $\{a_n\}$ be an increasing sequence. If $\{a_n\}$ is bounded above, it converges to its least upper bound, otherwise to ∞

Proof

Let $\{a_n\}$ be increasing, bounded above. Then it has a lowest upper bound say L. Suppose $\lim_{n\to\infty} a_n \neq L$. So there is some bad ε s.t. no tail of $\{a_n\}$ lies in $(L-\varepsilon, L+\varepsilon)$. But then no term from a_n. lies in $(L-\varepsilon, L+\varepsilon)$ since a_n is increasing. Hence $L-\varepsilon$ is an upper bound for $\{a_n\}$, but $L-\varepsilon < L$ and L is the least upper bound of $\{a_n\}$is a contradiction. The assumption of $\lim_{n\to\infty} a_n \neq L$ is false. $\therefore \lim_{n\to\infty} a_n = L$

Let $\{a_n\}$ be a decreasing sequence. If $\{a_n\}$is bounded below, it converges to its greatest lower bound, otherwise it diverges to $-\infty$

Proof

Let $L =$ greatest lower bound of $\{a_n\}$ since $\{a_n\}$ is decreasing, $\{-a_n\}$ is increasing with lowest upper bound is $-L$. By the Monotone Convergence Theorem, it is true.

Proof by Induction

IMPORTANT

Let $P(n)$ be a statement over the natural numbers $\mathbb{N}$

1) Prove the basic case $P(1)$ is true
2) Prove that if $P(n)$ is true, then $P(n+1)$ is true $\forall n \in \mathbb{N}$
3) Apply 2) repeatedly starting at $P(1)$

Prove a recursive sequence $\{a_n\}$ converges:
1) Show that $\{a_n\}$ is monotone
2) Show that $\{a_n\}$ is bounded above if increasing or bounded below if decreasing.
3) By the Monotone Convergence Theorem, $\lim_{n\to\infty} a_n$ exists. Use limit laws to solve for it, keeping in mind that the initial term and whether {an} is increasing or decreasing will tell you which solution is admissible if there are multiple.

Example:

1) Find the limit of the sequence (a_n) given by

$$a_1 = 1, \quad a_n = \sqrt{3 + 2a_{n-1}} \text{ for } n \geq 2.$$

Proof

Let $P(n)$ be the statement that $a_n \leq a_{n+1}$.

Base Case: $P(1)$, $a_1 = 1, a_2 = \sqrt{5}$, so $a_1 < a_2$.

Inductive Hypothesis: $P(a) \to P(a+1)$ suppose $P(n)$ is true for some n.

Then $a_n < a_{n+1}$; we want to show that $a_{n+1} \leq a_{n+2}$. From $a_n \leq a_{n+1}$:

$$2a_n \leq 2a_{n+1}$$

$$3 + 2a_n \leq 3 + 2a_{n+1}$$

$$\sqrt{3+2a_n} \leq \sqrt{3+2a_{n+1}}$$

$$a_{n+1} \leq a_{n+2}$$

By inducion, $P(n)$ is true for all $n \in \mathbb{N}$, so (a_n) is increasing.

Step2: Choose upper bound to be big to make proof easier.

Let $P(n)$ be the statement that $a_n \leq 100$. $P(1)$ is true since $a_1 = 1 < 100$.

Suppose $P(n)$ is true for some n. Then $a_n \leq 100$.

We want to show $a_{n+1} \leq 100$:

$$a_{n+1} = \sqrt{3+2a_n} \leq \sqrt{3+2*100} < \sqrt{10000} = 100.$$

By induction $P(n)$ ...

Since (a_n) is increasing and bounded above. By MCT, a converges to least upper bound. So, let

$$a_n \to L, a_{n+1} = \sqrt{3+2a_n}$$

.

$$\lim_{n\to\infty} a_{n+1} = \sqrt{3+2\lim_{n\to\infty} a_n} \to L = \sqrt{3+2L}.$$

So $L = -1, 3 = \lim_{n\to\infty} a_n$. Since the sequence is increasing, we choose $L = 3$

$\square$

2) Find the limit of the sequence $\{b_n\}$ given by

$$b_1 = 4, \quad b_n = \frac{7+b_{n-1}}{22} \text{ for } n \geq 2.$$

Notice that the sequence is a decreasing sequence.
Let $P(n)$ be the statement that $b_n > b_{n+1} \forall n \in \mathbb{N}$ and $b_n \geq \frac{1}{3} \forall n$.

Base case: $b_1 = 4 \geq \frac{1}{3}$ and $b_2 = \frac{1}{2} < 4 = b_1$.

Inductive hypothesis: suppose $P(n)$ is true for some $n = k$. Then

$$b_{n+1} = \frac{7+b_k}{22} \geq \frac{7+\frac{1}{3}}{22} > \frac{1}{3}.$$

We want to show $b_k \geq b_{k+1} \leftrightarrow 22b_k \geq 7 + b_k \leftrightarrow 21b_k \geq 7 \leftrightarrow b_k > \frac{1}{3}$, so $P(k+1)$ holds.
By Induction, $P(n)$ holds for all n.

By MCT, (b_n) converges to L, since $b_n = \frac{7+b_{n-1}}{22}$, so

$$\lim_{n\to\infty} b_n = \frac{7+\lim_{n\to\infty} b_{n-1}}{22} \to L = \frac{7+L}{22} \to 22L = 7 + L \to L = \frac{1}{3}.$$