An Introduction to Quantum Computers and Hidden Subgroup Problems

Postquantum Cryptography Reading Group

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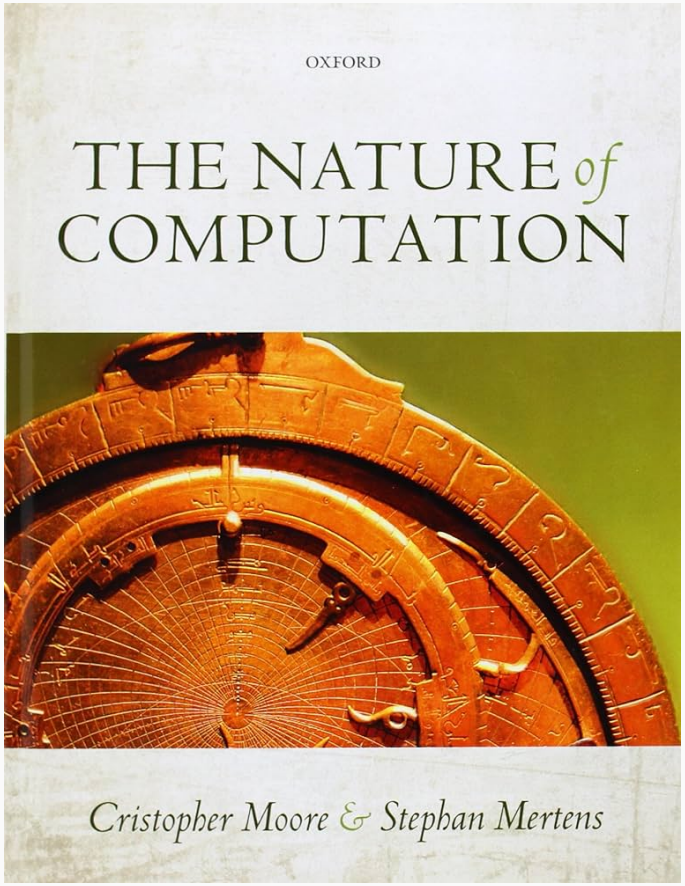
The plan

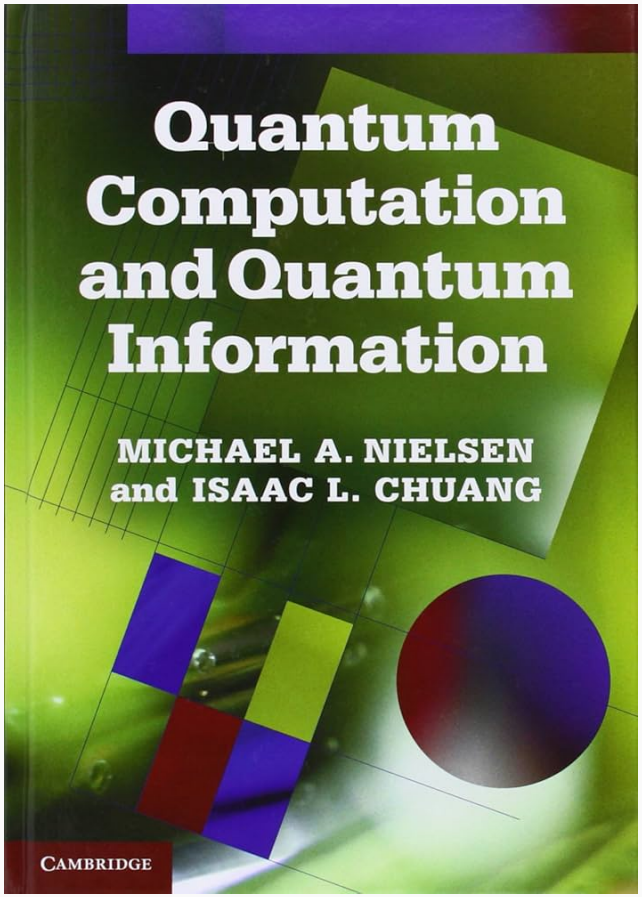
1. An overview of quantum for quantum computing
2. Some quantum algorithms:
    - The Deutsch problem and phase kickback
    - Quantum Fourier transforms and the Deutsch-Jozsa problem
    - Simon's problem
    - Shor's algorithm
3. Hidden subgroup problems on finite Abelian groups

Some good resources

This presentation is heavily taken from:

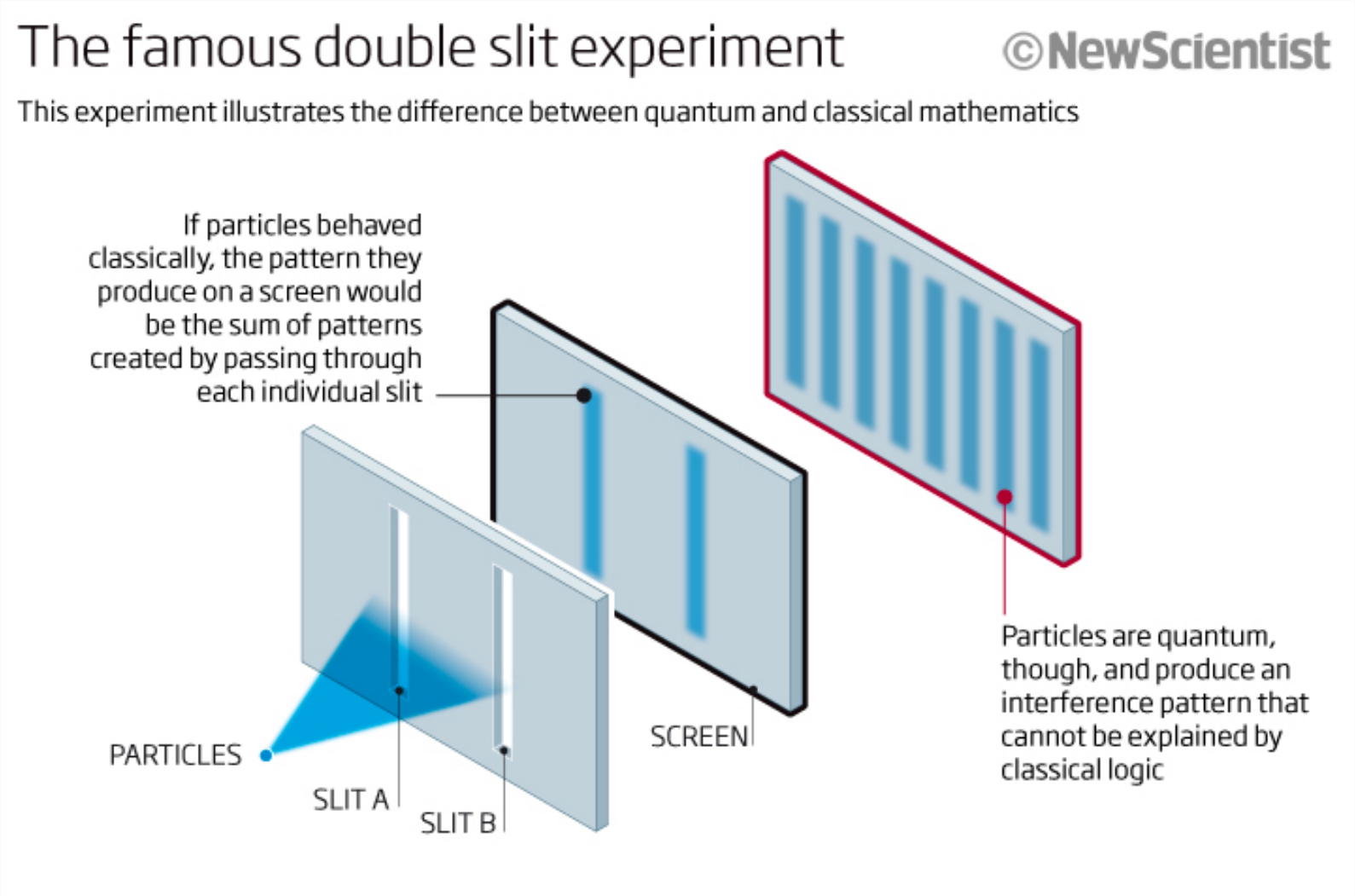


(a) Chapter 15



(b) Section I and II

The Double Slit Experiment



Bits and qubits

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A bit has two states: 0 or 1, true or false, on or off, etc. Physically, this is voltage on or off in a circuit.

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The quantum analogue is a *qubit*, and could physically be the spin of an electron (up or down) or the polarization of a photon (horizontal or vertical).

A qubit has a *continuum of states*.

Bits and qubits

A classical computer with m bits has 2^m states.

We can view each *state as a basis vector for a 2^m-dimensional vector space*, and *each computation as a $2^m \times 2^m$ matrix* acting on the state.

A program is the composition of all the matrices describing the computations.

Bits and qubits

Suppose we have two bits x_1 and x_2. Our *computational basis states* are

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The operation $x_2 \mapsto x_1$ (e.g. $|10\rangle \mapsto |11\rangle$) is described by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Bits and qubits

If now our operation is

$$\begin{cases} x_2 \mapsto x_1 & \text{with probability } \frac{1}{2} \\ x_2 \mapsto x_2 \text{ (do nothing)} & \text{with probability } \frac{1}{2} \end{cases}$$

we have a corresponding *stochastic matrix*

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

Vectors in the state space are now probability distributions, e.g.

$$U|10\rangle = \frac{1}{2}(|10\rangle + |11\rangle)$$

and the computer has an equal chance of being in either $|10\rangle$ or $|11\rangle$.

Bits and qubits

A qubit has states in $\mathbb{C}^2$, such as the *computational basis states*

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Quantum particles can also be in complex linear combinations (*superpositions*) of states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

α and β are known as the *amplitudes*. As they are complex, they can interfere *constructively and destructively*.

Bits and qubits

If we measure a qubit, we either measure 0 or 1; *nothing else!*

If the qubit has the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then

$$\mathbb{P}(\text{Measuring 0}) = |\alpha|^2$$

$$\mathbb{P}(\text{Measuring 1}) = |\beta|^2$$

$$\text{Total Probability} = |\alpha|^2 + |\beta|^2 = 1 = \|\psi\|^2$$

The state of a qubit is unobservable; when measuring, we only get information about the state.

Bits and qubits

We describe operations on our qubits through matrices acting on our state.

We need our matrices to preserve total probability. Computations are therefore described by *unitary matrices*, as these preserve the inner product.

Unitary matrices are invertible, which is interpreted as *reversible processes* and they *cannot create or destroy information*.

Some example of unitary operators

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

σ_x acts as a classical NOT-gate, e.g $\sigma_x|0\rangle = |1\rangle$, $\sigma_x|1\rangle = |0\rangle$.

σ_y and σ_z are more quantum as they introduce phase changes:

$$\sigma_y|0\rangle = i|1\rangle, \sigma_y|1\rangle = -i|0\rangle$$

$$\sigma_z|0\rangle = |0\rangle, \sigma_z|1\rangle = -|1\rangle$$

Some example of unitary operators

Hadamard matrix:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Maps $|0\rangle$ and $|1\rangle$ to superpositions of the two where each state is equally likely (*uniform superposition*);

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) =: |+\rangle$$
$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) =: |-\rangle$$

$|\pm\rangle$ are the eigenvectors of σ_x and are called the X-basis.

Multiple qubits

If we have N qubits, their states live in $(\mathbb{C}^2)^{\otimes N}$.

Our N-qubit basis vectors are the tensor products of single qubit each basis vector; e.g.

$$|10\rangle := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |1\rangle \otimes |0\rangle =: |1, 0\rangle$$

As

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \otimes \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} u_0 v_0 \\ u_0 v_1 \\ u_1 v_0 \\ u_1 v_1 \end{pmatrix}$$

the amplitudes of the joint state $|u, v\rangle$ are the products of the amplitudes

Quantum computers can be faster than classical computers due to

- *Parallelism* – e.g. qubits being in a superposition of states
- *Interference* – amplitudes are complex, so can combine constructively and destructively

Deutsch's algorithm

Let $f: \{0,1\} \to \{0,1\}$. Does $f(0) = f(1)$?

Classically, can determine through two *queries*: calculating $f(0)$ and $f(1)$ separately.

Deutsch's algorithm

Consider a two qubit computer in state $|x, y\rangle$. Define the unitary map

$$U_f|x, y\rangle = |x, y \oplus f(x)\rangle$$

where $\oplus$ is addition mod 2.

If x is in the uniform superposition $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and y in the state $|0\rangle$, then

$$U_f|+, |0\rangle\rangle = \frac{1}{\sqrt{2}}\left(|0, f(0)\rangle + |1, f(1)\rangle\right)$$

We have found information about $f(0)$ and $f(1)$ with only one computation; this is *parallelism*.

Deutsch's algorithm

We can improve this by using *interference* to be better than classical computation. Notice that

$$U_f|x,y\rangle = |x\rangle \otimes \sigma_x^{f(x)}|y\rangle$$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is an *eigenvector for* σ_x with eigenvalue -1, so

$$U_f|x,-\rangle = (-1)^{f(x)}|x,-\rangle.$$

Now preparing x in the state $|+\rangle$ gives

$$\begin{aligned} U_f|+,-\rangle &= \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \otimes |-\rangle \\ &\equiv \frac{|0\rangle + (-1)^{f(0)\oplus f(1)}|1\rangle}{\sqrt{2}} \otimes |-\rangle. \end{aligned}$$

If we now measure x as $|+\rangle$ then $f(0) = f(1)$.

Deutsch's algorithm

This is known as *phase kickback*: we prepare our 'output qubit' y to be an eigenvector whose eigenvalue affects the phase of the 'input qubit'.

By then measuring the 'input' qubit x and not caring about the 'output' qubit y, we learn about f.

Deutsch's algorithm

Now consider $f: \{0,1\}^n \to \{0,1\}$. We use a $(n+1)$-qubit computer in the state $|\vec{x}, y\rangle$ where $\vec{x} \in \{0,1\}^n$.

Consider again

$$U_f|\vec{x}, y\rangle = |\vec{x}, y \oplus f(\vec{x})\rangle = |\vec{x}\rangle \otimes \sigma_x^{f(x)}|y\rangle.$$

Let's try phase kickback again. Prepare the input qubits $\vec{x}$ in a uniform superposition

$$\frac{1}{\sqrt{2^n}} \sum_{\vec{x}} |\vec{x}\rangle$$

and the output qubit y in $|-\rangle$ to get

$$U_f \frac{1}{\sqrt{2^n}} \sum_{\vec{x}} |\vec{x}, -\rangle = \left(\frac{1}{\sqrt{2^n}} \sum_{\vec{x}} (-1)^{f(\vec{x})} |\vec{x}\rangle \right) \otimes |-\rangle.$$

Deutsch's algorithm

The state

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{\vec{x}} (-1)^{f(\vec{x})} |\vec{x}\rangle$$

again contains the values of $f(\vec{x})$ in the phases of $\vec{x}$'s amplitudes.

Questions:

1. What basis should we measure $|\psi\rangle$ in?
2. What can we learn when measuring $|\psi\rangle$ about f?

The Quantum Fourier Transform

Discrete Fourier transform: For $x_0, \cdots, x_{N-1} \in \mathbb{C}$,

$$x_j \mapsto y_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi ijk}{N}} x_k$$

Quantum Fourier transform: Defined by mapping the computational basis states

$$|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi ijk}{N}} |k\rangle$$

where $0 \leq j \leq N-1$.

QFT on $\mathbb{Z}_2^n$

Going back to $\mathbb{Z}_2^n$, let $|\phi\rangle = \sum_{\vec{x}} a_x |x\rangle$. We want to describe a_x as a linear combination of basis vectors oscillating at specific frequencies.

Each frequency $\vec{k} \in \mathbb{Z}_2^n$ has the basis function $(-1)^{\vec{k}\cdot\vec{x}}$ and basis state

$$|\vec{k}\rangle = \frac{1}{\sqrt{2^n}} \sum_{\vec{x}} (-1)^{\vec{k}\cdot\vec{x}} |x\rangle$$

giving us that

$$|\phi\rangle = \sum_{\vec{k}} \tilde{a}_{\vec{k}} |k\rangle, \qquad \tilde{a}_{\vec{k}} = \frac{1}{\sqrt{2^n}} \sum_{|x\rangle} (-1)^{\vec{k}\cdot\vec{x}} a_{\vec{x}}$$

This is *equivalent to applying* $H^{\otimes n}$.

For our state

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{\vec{x}} (-1)^{f(\vec{x})} |\vec{x}\rangle$$

We apply the QFT $H^{\otimes n}$ to $|\psi, 1\rangle$ to get the frequency bits $k_1 k_2 \cdots k_n$ written in the computational basis.

Measuring a given $\vec{k}$ has the probability

$$\mathbb{P}(\vec{k}) = \left|\tilde{a}_{\vec{k}}\right|^2 = \left| \frac{1}{2^n} \sum_{\vec{x}} (-1)^{\vec{k}\cdot\vec{x} + f(\vec{x})} \right|^2$$

If $n = 1$, we have the Deutsch problem. We have two frequencies $k = 0$ and $k = 1$.

If $f(0) = f(1)$ then $\mathbb{P}(0) = 1$, $\mathbb{P}(1) = 0$.

If $f(0) \neq f(1)$ then $\mathbb{P}(0) = 0$, $\mathbb{P}(1) = 1$.

The Deutsch-Jozsa Problem

For $f: \{0,1\}^n \to \{0,1\}$ suppose either is true:

1. f is constant
2. f is balanced: $f(\vec{x}) = 0$ for half of all $\vec{x}$, $f(\vec{x}) = 1$ for the other half.

Which is true?

The probability for observing the frequency of $(0, 0, \cdots, 0)$ is

$$\tilde{a}_{\vec{0}} = \frac{1}{2^n} \sum_{\vec{x}} (-1)^{f(\vec{x})}$$

If f is constant, this is ± 1. If f is balanced, this is 0, so

$$\mathbb{P}(\vec{0}) = \left|\tilde{a}_{\vec{0}}\right|^2 = \begin{cases} 0 & \text{if } f \text{ balanced,} \\ \\ 1 & \text{if } f \text{ constant} \end{cases}$$

so we answer the Deutsch-Jozsa problem with just one observation.

The plan for next time

- Simon's problem
- Shor's algorithm
- Generalising our problems: the hidden subspace problem