# A whirlwind introduction to bounded linear operators on Hilbert spaces

MATH491: Mathematics Dissertation

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## Why are we here?

1. From vector spaces to Hilbert spaces.

2. Some types of linear operators.

3. The spectral theorem for compact self-adjoint linear operators.

# It all starts with vector spaces

Our fundamental building block at the foundation of everything is a *complex vector space* $V$.

We allow our vector spaces to be *infinite-dimensional*: there exists no finite basis for $V$.

MATH220 introduced the *inner product*,

$$\langle \cdot, \cdot \rangle : V \to \mathbb{C}.$$

and the *norm induced by the inner product:*

$$\|\cdot\| \coloneqq \sqrt{\langle \cdot, \cdot \rangle}$$

# The norm represents distance...

For all vectors $x$ and $y$, the value of

$$d(x, y) := \|x - y\|$$

can be thought of representing the distance between them!

We say that the *norm induces a metric* on $V$.

# …so we can define the convergence of vectors

**Convergence of vectors**

For some vector space $V$ with a norm $\|\cdot\|$, we say that a sequence of vectors $(x_n)$ in $V$ *converges* to $x \in V$ if

$$\|x_n - x\| \to 0 \quad \text{as } n \to \infty.$$

# Cauchy sequences in vector spaces

In $\mathbb{R}$ and $\mathbb{C}$, all Cauchy sequences converge.

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**No!**

# Hilbert Spaces

This motivates the definition of a Hilbert space:

**Hilbert Space**

A *Hilbert space* is a vector space $\mathcal{H}$ equipped such that we have

- Some inner product, $\langle \cdot, \cdot \rangle$,
- The norm induced by the inner product, $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$,
- Every Cauchy sequence converges (*completeness*).

# Examples of Hilbert Spaces

- *Any finite-dimensional vector space with an inner product* is a Hilbert space when equipped with the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.
- Let $\ell^2$ be the vector space of *complex square-summable sequences*:

$$\ell^2 = \left\{ (x_j) \, : \, \text{complex sequences } (x_j) \text{ with } \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}.$$

The following inner product

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}$$

makes $\ell^2$ into a Hilbert space when equipped with the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

# Operators

We'll be interested in studying a specific type of linear map on $\mathcal{H}$.

## Bounded operator

An *operator* is a linear map $T : \mathcal{H} \to \mathcal{H}$: for all $x, y \in \mathcal{H}$ and $\mu, \lambda \in \mathbb{C}$,

$$T(\mu x + \lambda y) = \mu T(x) + \lambda T(y).$$

We say that $T$ is *bounded* if there exists some $M > 0$ such that for every $x \in \mathcal{H}$,

$$\|T(x)\| \leq M\|x\|.$$

# Boundedness $\iff$ continuity

**Boundedness is equivalent to continuity**

An operator $T : \mathcal{H} \to \mathcal{H}$ is bounded if and only if $T$ is continuous.

**Finite Hilbert Spaces mean bounded operators**

If $\mathcal{H}$ is finite-dimensional then every operator $T : \mathcal{H} \to \mathcal{H}$ is bounded.

# Adjoints and their existence

**Adjoints**

For any operator $T : \mathcal{H} \to \mathcal{H}$, we say that $T^* : \mathcal{H} \to \mathcal{H}$ is its *adjoint* if

for all $x, y \in \mathcal{H}$,

$$\langle x, T(y) \rangle = \langle T^*(x), y \rangle .$$

$T$ is *self-adjoint* if $T = T^*$.

**Adjoints always exist uniquely**

If $T : \mathcal{H} \to \mathcal{H}$ is an operator, then there exists a unique adjoint $T^*$.

# Examples of an adjoint

- The *identity map* and the *zero map* are self-adjoint.
- For any *bounded* operator $T : \mathcal{H} \to \mathcal{H}$, the following are self-adjoint:
  - $T^*T$,
  - $T^*T$.
- The *right-shift operator* $R : \ell^2 \to \ell^2$,

$$R(x) = (0, x_1, x_2, \cdots)$$

has the *left-shift operator* as its adjoint: $L : \ell^2 \to \ell^2$,

$$L(y) = (y_2, y_3, y_4, \cdots)$$

.

# Compact Operators

## Compact Operator

An operator $T : \mathcal{H} \to \mathcal{H}$ is *compact* if

$$\text{For any bounded sequence } (x_j) \text{ in } \mathcal{H},$$

$$(T(x_j)) \text{ has a convergent subsequence.}$$

## Compact operators are bounded

If $T : \mathcal{H} \to \mathcal{H}$ is compact, then it is bounded.

The set of compact operators is a subspace of the set of bounded operators.

## Compactness and adjoints

$T : \mathcal{H} \to \mathcal{H}$ being compact is equivalent to $T^*$ being compact.

## Example of a compact operator

For any Hilbert space $\mathcal{H}$, then for all $x \in \mathcal{H}$, there exist *orthogonal* vectors $y, z \in \mathcal{H}$ with

$$x = y + z.$$

Let $P : \mathcal{H} \to \mathcal{H}$ be the *orthogonal projection*

$$P(x) = y.$$

Then,

- $P$ is *self-adjoint*,
- *$P$ is compact if and only if $\mathcal{H}$ is finite-dimensional.*

# Matrix Diagonalisation

From MATH220, we know that *real symmetric matrices* ($X^T = X$) can be *diagonalised*:

**Spectral Theorem for Real Symmetric Matrices**

Let $X \in \mathrm{M}_n(\mathbb{R})$ be a matrix. Then:

- $X$ has an orthonormal basis $\iff$ $X$ is symmetric.
- If $X$ has an orthonormal basis, let $P$ be the matrix whose columns form an orthonormal basis of $X$ and $D$ be the diagonal matrix of eigenvalues in the same order as $P$. Then,

$$X = PDP^T.$$

Can this be generalised to our operators?

# The Spectral Theorem

**The Spectral Theorem for Compact Self-Adjoint Operators**

Suppose $T : \mathcal{H} \to \mathcal{H}$ is a *compact* and *self-adjoint* operator. Then,

- $\mathcal{H}$ has an orthonormal basis $(v_n)$ of *eigenvectors* of $T$.
- For $n \in \mathbb{N}$ or $n = \infty$,

$$T(x) = \sum_{j=1}^{n} \lambda_j \langle x, v_j \rangle v_j,$$

  where $\lambda_j$ is the eigenvalue of the eigenvector $v_j$.

  If $n = \infty$, then $\lambda_j \to 0$ as $j \to \infty$.

Thanks for listening!

Any questions?

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