# 1 Let $A$ be a commutative ring. Let $M$ be a module, and $N$ a submodule. Let $N = Q_1 \cap \ldots \cap Q_r$ be a primary decomposition of $N$. Let $\bar{Q}_i = Q_i/N$. Show that $0 = \bar{Q}_1 \cap \ldots \bar{Q}_r$ is a primary decomposition of $0$ in $M/N$. State and prove the converse.

1. Each $\bar{Q}_i$ is primary

Given $a \in A$, $a_{M/Q_i}$ is either injective or nilpotent, we must show that, given $a \in A$, $a_{(M/N)/(Q_i/N)}$ is either injective or nilpotent.

The function $a_{M/Q_i}$ is a particular function from $M/Q_i$ to itself. Via the isomorphism $\sigma : M/Q_i \mapsto (M/N)/(Q_i/N)$, define $\hat{a}$ as a function from $(M/N)/(Q_i/N)$ to itself (Is just quoting the isomorphism theorem sufficient here?)

Since they are both multiplication by $a$, $a_{(M/N)/(Q_i/N)}$ and $\hat{a}$ are the same function on $(M/N)/(Q_i/N)$. Thus if $a_{M/Q_i}$ is injective (resp. nilpotent) then $a_{(M/N)/(Q_i/N)}$ is injective (resp. nilpotent).

2. Their intersection is $0 = \bar{Q}_1 \cap \ldots \cap \bar{Q}_r$

Assume this is false.

Take element $a \neq (0) \in \bar{Q}_1 \cap \ldots \cap \bar{Q}_r$. (Better way to write this?) The pre-image of $a$ under the canonical homomorphism $M \mapsto M/N$ is also not in $N$, since $N$ is exactly the kernel of this homomorphism.

However it $a$ *is* in each $\bar{Q}_i$ so its preimage has to be in $Q_i$, so has to be in $N$, a contradiction.

3. If 0 is primary decom, then N is primary decomp

Showing that $Q_i$ is primary given that $\bar{Q}_i$ is primary is identical to part 1 by following the isomorphism $M/Q_i \cong (M/N)/(Q_i/N)$ the other direction.

The proof that the intersection is N is also directly analogous to part 2, shown here:

Assume it is false.

Take some $x$ not in $N$ but in $N = Q_1 \cap \ldots \cap Q_r$. Under the canonical homomorphism the image of $x$ is in each of $\bar{Q}$, thus is 0. However this is a contradiction since the image of $x$ is non-zero.

# 2 Let $\mathfrak{p}$ be a prime ideal and $\mathfrak{a}$, $\mathfrak{b}$ be ideals of $A$. If $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, show that $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

If $\mathfrak{a} \not\subset \mathfrak{p}$ and $\mathfrak{b} \not\subset \mathfrak{p}$, prove that $\mathfrak{a}\mathfrak{b} \not\subset \mathfrak{p}$

Pick $a$ in $\mathfrak{a}$ and $b$ in $\mathfrak{b}$ but not in $\mathfrak{p}$. Since $\mathfrak{p}$ is prime, $ab$ cannot be in $\mathfrak{p}$.

## 3 Let $\mathfrak{q}$ be a primary ideal. Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals, and assume $\mathfrak{ab} \subset \mathfrak{q}$. Assume that $\mathfrak{b}$ is finitely generated. Show that $\mathfrak{a} \subset \mathfrak{q}$ or there exists some positive integer $n$ such that $\mathfrak{b}^n \subset \mathfrak{q}$.

Assume that $\mathfrak{a} \not\subset \mathfrak{q}$. We will show that this implies there there exists some positive integer $n$ such that $\mathfrak{b}^n \subset \mathfrak{q}$. Take $a_0$ to be in $\mathfrak{a}$ but not in $\mathfrak{q}$.

An arbitrary element of $\mathfrak{b}$, being finitely generated, looks like $k_1b_1 + k_2b_2 + ... + k_rb_r$ (where $k_i$ are positive integers and $r$ and $n$ are unrelated). For each $k_ib_i$, since $a_0(k_ib_i)$ in $\mathfrak{q}$ but $a_0$ is not and $\mathfrak{q}$ is primary, $(k_ib_i)^{n_i} \in \mathfrak{q}$ for some finite $n_i$. Our goal is to find some $n$ large enough such that every term of $(k_1b_1 + k_2b_2 + ... + k_rb_r)^n$ is in $\mathfrak{q}$, since, being additively closed, that would imply $b^n$ itself were in $\mathfrak{q}$.

Take $n = \prod_{i=1}^{r} n_i$. Each term of $(k_1b_1 + k_2b_2 + ... + k_rb_r)^n$ is a homogenous monomial of degree $n$, thus in general looks like $\prod_{i=1}^{r}(k_ib_i)^{m_i}$ where the $m_i$'s sum to $n$.

If any $m_i \geq n_i$, then the term itself will be in $\mathfrak{q}$, since $(k_ib_i)^{n_i}$ is in $\mathfrak{q}$ as are its subsequent powers and anything multiplied by it, since ideals absorb multiplication.

It is impossible for all $m_i < n_i$, since then they would not fully sum to $n$, so therefore at least one is and the term is in $\mathfrak{q}$. Since $r$ is finite, so too is $n$, thus proving the statement.

## 4 Let $A$ be Noetherian and let $\mathfrak{q}$ be a $\mathfrak{p}-$primary ideal. Show that there exists some $n \geq 1$ such that $\mathfrak{p}^n \subset \mathfrak{q}$

$\mathfrak{q}$ being $\mathfrak{p}$-primary means that the radical of $\mathfrak{q}$ is $\mathfrak{p}$. Hence $\sqrt{\mathfrak{q}} = \{r \in A | r^n \in \mathfrak{q}\} = \mathfrak{p}$

Since $A$ is Noetherian, $\mathfrak{p}$ is finitely generated, so an arbitrarily element of $p$ is of the form $k_1p1 + k_2p_2 + ... + k_rp_r$, where $k_i$ and $r$ are positive integers.

For every $k_ip_i$, there's an $n_i$ such that $(k_ip_i)^{n_i}$ is in $\mathfrak{q}$. For the same reasoning as 3) above, $n = \prod_{i=1}^{r} n_i$ is such that this arbitrary $p$ is in $\mathfrak{q}$.

8. Let $A$ be a local ring. Show that any idempotent $\neq 0$ in $A$ is necessary the unit element.

We have to show that $e * e = e \Rightarrow e = 1$

If $e$ is a unit, we're done (Left multiplication by $e^{-1}$ shows $e = 1$).

So assume $e$ is not a unit. Then it must be in the maximal ideal $\mathfrak{m}$ (otherwise it would generate a proper ideal not contained in $\mathfrak{m}$). Not quite sure how to finish...