1 If $\mathscr{A}$ is a Banach algebra with identity and $a \in \mathscr{A}$ and is nilpotent (that is, $a^n = 0$ for some n), then $\sigma(a) = \{0\}$

Firstly, assume a is invertible. Then

$$\begin{aligned} a^n &= 0 \\ (a^{-1})^n a^n &= (a^{-1})^n 0 \\ 1 &= 0 \end{aligned}$$

a contradiction, so $0 \in \sigma(a)$.

Next, $||(a+1) - 1|| = ||a|| \leq ||a^n||^{1/n} = 0 < 1$, so by Lemma 2.1, $a+1$ is invertible, so $a+1 \notin \sigma(a)$. TODO: extend argument to all values a + somehow?