1 Let A be a commutative ring. Let M be a module, and N a submodule. Let $N = Q_1 \cap ... \cap Q_r$ be a primary decomposition of N. Let $\bar{Q}_i = Q_i/N$. Show that $0 = \bar{Q}_1 \cap ... \bar{Q}_r$ is a primary decomposition of 0 in M/N. State and prove the converse.

1. Each $\bar{Q}_i$ is primary

Given $a \in A$, a_{M/Q_i} is either injective or nilpotent, we must show that, given $a \in A$, $a_{(M/N)/(Q_i/N)}$ is either injective or nilpotent.

The function a_{M/Q_i} is a particular function from M/Q_i to itself. Via the isomorphism $\sigma : M/Q_i \mapsto (M/N)/(Q_i/N)$, define $\hat{a}$ as a function from $(M/N)/(Q_i/N)$ to itself (Is just quoting the isomorphism theorem sufficient here?)

Since they are both multiplication by a, $a_{(M/N)/(Q_i/N)}$ and $\hat{a}$ are the same function on $(M/N)/(Q_i/N)$. Thus if a_{M/Q_i} is injective (resp. nilpotent) then $a_{(M/N)/(Q_i/N)}$ is injective (resp. nilpotent).

2. Their intersection is $0 = \bar{Q}_1 \cap ... \cap \bar{Q}_r$

Assume this is false.

Take element $a \neq (0) \in \bar{Q}_1 \cap ... \cap \bar{Q}_r$. (Better way to write this?) The pre-image of a under the canonical homomorphism $M \mapsto M/N$ is also not in N, since N is exactly the kernel of this homomorphism.

However it a *is* in each $\bar{Q}_i$ so its preimage has to be in Q_i, so has to be in N, a contradiction.

3. If 0 is primary decom, then N is primary decomp

Showing that Q_i is primary given that $\bar{Q}_i$ is primary is identical to part 1 by following the isomorphism $M/Q_i \cong (M/N)/(Q_i/N)$ the other direction.

The proof that the intersection is N is also directly analogous to part 2, shown here:

Assume it is false.

Take some x not in N but in $N = Q_1 \cap ... \cap Q_r$. Under the canonical homomorphism the image of x is in each of $\bar{Q}$, thus is 0. However this is a contradiction since the image of x is non-zero.

2 Let $\mathfrak{p}$ be a prime ideal and $\mathfrak{a}$, $\mathfrak{b}$ be ideals of A. If $\mathfrak{ab} \subset \mathfrak{p}$, show that $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

If $\mathfrak{a} \not\subset \mathfrak{p}$ and $\mathfrak{b} \not\subset \mathfrak{p}$, prove that $\mathfrak{ab} \not\subset \mathfrak{p}$

Pick a in $\mathfrak{a}$ and b in $\mathfrak{b}$ but not in $\mathfrak{p}$. Since $\mathfrak{p}$ is prime, ab cannot be in $\mathfrak{p}$.

3 Let $\mathfrak{q}$ be a primary ideal. Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals, and assume $\mathfrak{a}\mathfrak{b} \subset \mathfrak{q}$. Assume that $\mathfrak{b}$ is finitely generated. Show that $\mathfrak{a} \subset \mathfrak{q}$ or there exists some positive integer n such that $\mathfrak{b}^n \subset \mathfrak{q}$.

If $\mathfrak{a} \not\subset \mathfrak{q}$ and there is no integer n such that $\mathfrak{b}^n \not\subset \mathfrak{q}$, prove that $\mathfrak{a}\mathfrak{b} \not\subset \mathfrak{q}$

Pick a in $\mathfrak{a}$ but not in $\mathfrak{q}$. Pick b in $\mathfrak{b}$ such that there is no integer n that b^n is in $\mathfrak{q}$. Since $\mathfrak{q}$ is primary.... hmm can't seem to bring this one together.

4 Let A be Noetherian and let $\mathfrak{q}$ be a $\mathfrak{p}-$primary ideal. Show that there exists some $n \geq 1$ such that $\mathfrak{p}^n \subset \mathfrak{q}$

$\mathfrak{q}$ being $\mathfrak{p}$-primary means that the radical of $\mathfrak{q}$ is $\mathfrak{p}$. Hence $\sqrt{\mathfrak{q}} = \{r \in A | r^n \in \mathfrak{q}\} = \mathfrak{p}$

This means for every p in p, there's an n such that p^n is in q. Note that if p^n is in q, then p^N where $N > n$ is also in q (ideal absorbs multiplication from the ring).

Now just need to show there is a finite such n. This can been seen by observing that A Noetherian, so eventually raising to a power has to "stabilize", so there is some finite smallest n at which this occurs.

8. Let A be a local ring. Show that any idempotent $\neq 0$ in A is necessary the unit element.

We have to show that $e * e = e \Rightarrow e = 1$

If e is a unit, we're done (Left multiplication by e^{-1} shows $e = 1$).

So assume e is not a unit. Then it must be in the maximal ideal $\mathfrak{m}$ (otherwise it would generate a proper ideal not contained in m).