1 Show that wk is the smallest topology on $\mathscr{X}$ such that each x^* in $\mathscr{X}^*$

Must show that an arbitrary open set in wk can be generated by some collection of sets of the form $x^{*-1}(V)$.

Start with an arbitrary open set U. Since X with wk is a locally convex set, $\bigcap_{i=1}^{n}\{x\epsilon\mathscr{X} : p_j(x-x_0) < \varepsilon_j\} \subseteq U$ for some finite list of ε and p, where $p_{x^*} = |\langle x, x^*\rangle|$.

$\bigcap_{i=1}^{n}\{x \in \mathscr{X} : p_j(x-x_0) < \varepsilon_j\}$ is open and the intersection of pre-images of open sets from $\mathbb{F}$ (why?).

Since U is generated by the subbase of preimages of the collection of x^* and U is arbitrary, all open sets of wk are generated in this way, thus is the smallest possible topology.

2 Show that wk* is the smallest topology on $\mathscr{X}^*$ such that for each x in $\mathscr{X}$, $x^* \mapsto \langle x, x^*\rangle$

This purported topology σ is generated by open sets that are preimages of open sets of functions of the form $x^* \mapsto \langle x, x^*\rangle$. By continuity of composition, they are also the preimages of open sets of functions of the form $x^* \mapsto |\langle x, x^*\rangle|$. Open sets of $\mathbb{R}$ are generated by open balls, so the pre-images are generated by sets of the form $|\langle x, x^*\rangle| < \varepsilon$.

A set U of $\mathscr{X}^*$ is weakly open if and only if for every x_0^* in U there is an ε and there are $x_1, ...x_n$ in $\mathscr{X}$ such that

$$\bigcap_{i=1}^{n}\{x^* \in \mathscr{X}^* : |\langle x_k, x^* - x_0^*\rangle| < \varepsilon_j\} \subseteq U$$

Every such set U can be generated as part of σ and therefore is the smallest topology

3 Prove Theorem 1.3

Theorem 1.3 is proven in the text!

4 Let $\mathscr{X}$ be a complex LCS and let $\mathscr{X}^*_{\mathbb{R}}$ denote the collection of all continous real linear functionals on $\mathscr{X}$. Use the elements of $\mathscr{X}^*_{\mathbb{R}}$ to define seminorms on $\mathscr{X}$ and let $\sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}})$ be the corresponding topology. Show that $\sigma(\mathscr{X}, \mathscr{X}^*) = \sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}})$

Take the function f

$$f : \mathbb{C} \mapsto \mathbb{R}$$
$$f(x) = |x|$$

f defines a function $\hat{f}$ from $\mathscr{X}^*$ to $\mathscr{X}^*_{\mathbb{R}}$ via composition. We will show that $\hat{f}$ is a homeomorphism.

$\hat{f}$ is clearly continuous because it is the composition of two continuous functions.

Open sets around x_0^* in $\mathscr{X}^*$ are generated by sets of the form $\bigcap_{i=1}^n \{x^* \in \mathscr{X}^* : |\langle x_k, x^* - x_0^* \rangle| < \varepsilon_j\}$ for some finite set of x_k and ε_j. The preimage of this set under $\hat{f}^{-1}$ is ???, which is open, therefore $\hat{f}^{-1}$ is continuous.

5 If $A \subseteq \mathscr{X}$...

5.1 A° is convex and balanced

$A^\circ \equiv \{x^* \in \mathscr{X}^* : |\langle a, x^* \rangle| \leq 1 \text{ for all } a \text{ in } A\}$

To show that A° is balanced, given x^* in A° and $|\alpha| \leq 1$, we have to show that αx^* is also in A°.

If $|\langle a, x^* \rangle| \leq 1$, then $|\langle a, \alpha x^* \rangle| = |\alpha||\langle a, x^* \rangle| \leq 1$, hence αx^* is in A°

To show that A° is convex, we have to show that given arbitrary x_1^* and x_2^*, then all of $\{t x_1^* + (1-t) x_2^* : 0 \leq t \leq 1\}$ is in A°.

If $|\langle a, x_1^* \rangle| \leq 1$ and $|\langle a, x_2^* \rangle| \leq 1$, then $t|\langle a, x_1^* \rangle| \leq t$ and $(1-t)|\langle a, x_2^* \rangle| \leq 1-t$ as long as $0 \leq t \leq 1$ (to not flip the signs). Adding the inequalities we get $t|\langle a, x_1^* \rangle| + (1-t)|\langle a, x_2^* \rangle| \leq 1$.

6 If $A \subseteq \mathscr{X}$, show that A is weakly bounded if and only if A° is absorbing in $\mathscr{X}^*$

Assume A is weakly bounded.

Assume A° is absorbing in $\mathscr{X}^*$