1 Prove that the given set of functions converges weakly but not strongly

Consider $\mathscr{X} = C[0,1]$, the Banach Space of continuous functions on $[0,1]$ with the supremum norm.

Consider sequence of functions $(f_n)_{n\geq 2}$, where

$$f_n(x) = \begin{cases} xn & 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \frac{2}{n} < x \leq 1 \end{cases}$$

asdflkjasfd

(Starts at 0, goes up to 1 at 1/n, down to 0 at 2/n, and stays flat from there).

Denote $z \in \mathscr{X}$ as function $z(x) = 0$ for $0 \leq x \leq 1$.

Claim is that $f_n \rightharpoonup z$, but that $f_n \not\to z$.

First, $||f_n - z|| = ||f_n|| = 1$ for all n, so f_n does not converge to z (TODO: Prove it doesn't converge at all?)

Now, to show that $f_n \rightharpoonup 0$, we have to show that for every bounded linear functional L on $\mathscr{X}$, $L(f_n) \to 0$

A linear functional L_g indexed by an $g \in \mathscr{X}$ is given by $L_g(f) = \int |fg| dx$.

Define $\hat{f}_n$ as the function on $0 \leq x \leq 1$ that is 1 from $0 \leq 2/n$ and 0 onward. This function is not continuous so not in $\mathscr{X}$, but is strictly larger than f_n.

Take an arbitrary L_{g_0}. We will prove that $\int |\hat{f} g_0| dx \to 0$, which will imply $\int |f g_0| dx \to 0$, and thus all bounded linear functionals converge to z.

g_0 achieves its maximum M on $0 \leq x \leq 1$, thus the value of $\int |f_n g_0| dx$ is broken into two parts: $0 \leq 2/n$ where it is bounded by $(2/n)M$, and $2/nle1$, where it is 0. As $n \to \infty$, $(2/n)M \to 0$.