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## 1 Show that wk is the smallest topology on $\mathcal{X}$ such that each $x^*$ in $\mathcal{X}^*$

Must show that an arbitrary open set in wk can be generated by some collection of sets of the form  $x^{*-1}(V)$ .

Start with an arbitrary open set  $U$ . Since  $X$  with wk is a locally convex set,  $\bigcap_{i=1}^n \{x \in \mathcal{X} : p_j(x - x_0) < \varepsilon_j\} \subseteq U$  for some finite list of  $\varepsilon$  and  $p$ , where  $p_{x^*} = |\langle x, x^* \rangle|$ .

$\bigcap_{i=1}^n \{x \in \mathcal{X} : p_j(x - x_0) < \varepsilon_j\}$  is open and the intersection of pre-images of open sets from  $\mathbb{F}$  (why?).

Since  $U$  is generated by the subbase of preimages of the collection of  $x^*$  and  $U$  is arbitrary, all open sets of wk are generated in this way, thus is the smallest possible topology.

## 2 Show that wk\* is the smallest topology on $\mathcal{X}^*$ such that for each $x$ in $\mathcal{X}$ , $x^* \mapsto \langle x, x^* \rangle$

This purported topology  $\sigma$  is generated by open sets that are preimages of open sets of functions of the form  $x^* \mapsto \langle x, x^* \rangle$ . By continuity of composition, they are also the preimages of open sets of functions of the form  $x^* \mapsto |\langle x, x^* \rangle|$ . Open sets of  $\mathbb{R}$  are generated by open balls, so the pre-images are generated by sets of the form  $|\langle x, x^* \rangle| < \varepsilon$ .

A set  $U$  of  $\mathcal{X}^*$  is weakly open if and only if for every  $x_0^*$  in  $U$  there is an  $\varepsilon$  and there are  $x_1, \dots, x_n$  in  $\mathcal{X}$  such that

$$\bigcap_{i=1}^n \{x^* \in \mathcal{X}^* : |\langle x_i, x^* - x_0^* \rangle| < \varepsilon_j\} \subseteq U$$

Every such set  $U$  can be generated as part of  $\sigma$  and therefore is the smallest topology

## 3 Prove Theorem 1.3

Theorem 1.3 is proven in the text!

4 Let  $\mathcal{X}$  be a complex LCS and let  $\mathcal{X}_{\mathbb{R}}^*$  denote the collection of all continuous real linear functionals on  $\mathcal{X}$ . Use the elements of  $\mathcal{X}_{\mathbb{R}}^*$  to define seminorms on  $\mathcal{X}$  and let  $\sigma(\mathcal{X}, \mathcal{X}_{\mathbb{R}}^*)$  be the corresponding topology. Show that  $\sigma(\mathcal{X}, \mathcal{X}^*) = \sigma(\mathcal{X}, \mathcal{X}_{\mathbb{R}}^*)$

Goal is to show that an open set in  $\sigma(\mathcal{X}, \mathcal{X}^*)$  is open in  $\sigma(\mathcal{X}, \mathcal{X}_{\mathbb{R}}^*)$  and vice versa.

Since  $\mathcal{X}_{\mathbb{R}}^*$  is a strict subset of  $\mathcal{X}^*$ ,  $\sigma(\mathcal{X}, \mathcal{X}_{\mathbb{R}}^*) \subset \sigma(\mathcal{X}, \mathcal{X}^*)$  (since  $\sigma(\mathcal{X}, \mathcal{X}^*)$  are those functions that make all linear functionals continuous, including the real linear functionals.)

Thus we need to show that  $\sigma(\mathcal{X}, \mathcal{X}^*) \subset \sigma(\mathcal{X}, \mathcal{X}_{\mathbb{R}}^*)$

Take the functions  $f_r, f_i : \mathbb{C} \mapsto \mathbb{R}$  that map the complex numbers to their real, imaginary parts, respectively.

Open sets around  $x_0$  in  $\sigma(\mathcal{X}, \mathcal{X}^*)$  are generated by sets of the form  $\bigcap_{i=1}^n \{x \in \mathcal{X} : |\langle x - x_0, x_k^* \rangle| < \varepsilon_k\}$  for some finite set of  $x_k^*$  and  $\varepsilon_k$ . To simplify consider open sets generated by a single complex semi-norm  $x_0^*$ . We will show that these open sets are also generated by two real semi-norms,  $x_0^*$  composed with  $f_r$  and  $f_i$ .

TODO: Finish

The same will be true given the finite intersection of sets generated by a finite number of seminorms. Thus any open set in  $\sigma(\mathcal{X}, \mathcal{X}^*)$  is also open in  $\sigma(\mathcal{X}, \mathcal{X}_{\mathbb{R}}^*)$

## 5 If $A \subseteq \mathcal{X} \dots$

### 5.1 $A^\circ$ is convex and balanced

$A^\circ \equiv \{x^* \in \mathcal{X}^* : |\langle a, x^* \rangle| \leq 1 \text{ for all } a \text{ in } A\}$

To show that  $A^\circ$  is balanced, given  $x^*$  in  $A^\circ$  and  $|\alpha| \leq 1$ , we have to show that  $\alpha x^*$  is also in  $A^\circ$ .

If  $|\langle a, x^* \rangle| \leq 1$ , then  $|\langle a, \alpha x^* \rangle| = |\alpha| |\langle a, x^* \rangle| \leq 1$ , hence  $\alpha x^*$  is in  $A^\circ$

To show that  $A^\circ$  is convex, we have to show that given arbitrary  $x_1^*$  and  $x_2^*$  in  $A^\circ$ , then an arbitrarily chosen  $x_3$  in  $\{tx_1^* + (1-t)x_2^* : 0 \leq t \leq 1\}$  is also in  $A^\circ$ , say  $t_3x_1^* + (1-t_3)x_2^*$  (where  $t_3$  was chosen arbitrarily in  $0 \leq t \leq 1$ ).

Hence we need to prove that, for all  $a$  in  $A$ ,  $|\langle a, t_3x_1^* + (1-t_3)x_2^* \rangle| \leq 1$

If  $|\langle a, x_1^* \rangle| \leq 1$  and  $|\langle a, x_2^* \rangle| \leq 1$ , then simply by multiplication of both sides  $t_3|\langle a, x_1^* \rangle| \leq t_3$  and  $(1-t_3)|\langle a, x_2^* \rangle| \leq 1-t_3$  since  $0 \leq t_3 \leq 1$  (to not flip the signs). Adding the inequalities we get  $t_3|\langle a, x_1^* \rangle| + (1-t_3)|\langle a, x_2^* \rangle| \leq 1$ .

SOMEHOW: prove  $|\langle a, t_3x_1^* + (1-t_3)x_2^* \rangle| \leq t_3|\langle a, x_1^* \rangle| + (1-t_3)|\langle a, x_2^* \rangle|$  (Cauchy inequality + convexity of the reals?)

**5.2 If  $A_1 \subseteq A$ , then  $A^\circ \subseteq A_1^\circ$**

If  $x^* \in A^\circ$ , then  $|\langle a, x^* \rangle| \leq 1$  for all  $a$  in  $A$ . Since  $A_1 \subseteq A$ ,  $|\langle a, x^* \rangle| \leq 1$  for all  $a$  in  $A_1$  as well. Therefore  $x^* \in A_1^\circ$ . Since  $x^*$  was chosen arbitrarily,  $A^\circ \subseteq A_1^\circ$ .

**6 If  $A \subseteq \mathcal{X}$ , show that  $A$  is weakly bounded if and only if  $A^\circ$  is absorbing in  $\mathcal{X}^*$**

Assume  $A$  is weakly bounded.

For every open set  $U$  containing 0, there is an  $\varepsilon > 0$  such that  $\varepsilon A \subseteq U$  (see definition 2.5 in IV. §2, pg 106). Take one such open set  $U_0$  and its associated  $\varepsilon_0$ . There exists  $x_1^*, \dots, x_n^*$  and  $\hat{\varepsilon}_0$  such that

$$\bigcap_{i=1}^n \{x \in \mathcal{X} : |\langle x, x_k^* \rangle| < \hat{\varepsilon}\} \subseteq U_0$$

For an arbitrary  $x^*$ , we have to show that  $|\langle a, tx^* \rangle| \leq 1$  for all  $a$  in  $A$  and  $0 \leq t < \varepsilon$ .