1 Let A be a commutative ring. Let M be a module, and N a submodule. Let $N = Q_1 \cap ... \cap Q_r$ be a primary decomposition of N. Let $\bar{Q}_i = Q_i/N$. Show that $0 = \bar{Q}_1 \cap ... \bar{Q}_r$ is a primary decomposition of 0 in M/N. State and prove the converse.

1. Each $\bar{Q}_i$ is primary

Given $a \in A$, a_{M/Q_i} is either injective or nilpotent, we must show that, given $a \in A$, $a_{(M/N)/(Q_i/N)}$ is either injective or nilpotent.

The function a_{M/Q_i} is a particular function from M/Q_i to itself. Via the isomorphism $\sigma : M/Q_i \mapsto (M/N)/(Q_i/N)$, define $\hat{a}$ as a function from $(M/N)/(Q_i/N)$ to itself (TODO: justify).

We must show that $a_{(M/N)/(Q_i/N)}$ and $\hat{a}$ are the same function. (TODO: do that) Thus if a_{M/Q_i} is injective (resp. nilpotent) then $a_{(M/N)/(Q_i/N)}$ is injective (resp. nilpotent).

2. Their intersection is $0 = \bar{Q}_1 \cap ... \cap \bar{Q}_r$

Assume false.

Take element $a \neq (0) \in \bar{Q}_1 \cap ... \cap \bar{Q}_r$. The pre-image of a under the canonical homomorhism $M \mapsto M/N$ is also not in N (WHY).

However it a *is* in each $\bar{Q}_i$ so its preimage has to be in Q_i, so has to be in N, a contradiction.

4. If 0 is primary decom, then N is primary decomp

Showing that Q_i is primary given that $\bar{Q}_i$ is primary is identical to part 1 by following the isomorphism $M/Q_i \cong (M/N)/(Q_i/N)$ the other direction.

The proof that the intersection is N is also directly analogous to part 2, shown here:

2 Let $\mathfrak{p}$ be a prime ideal and $\mathfrak{a}$, $\mathfrak{b}$ be ideals of A. If $\mathfrak{ab} \subset \mathfrak{p}$, show that $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

Take a_i to be an arbitrary member of $\mathfrak{a}$ and b_i to be an arbitrary member of $\mathfrak{b}$. If $a_i b_i$ is in $\mathfrak{p}$, then, since $\mathfrak{p}$ is prime, either a_i is in $\mathfrak{p}$ or b_i is.

Say a_i is. Now we have to show that all of $\mathfrak{a}$ is thus in $\mathfrak{p}$.

3 Let $\mathfrak{q}$ be a primary ideal. Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals, and assume $\mathfrak{a}\mathfrak{b} \subset \mathfrak{q}$. Assume that $\mathfrak{b}$ is finitely generated. Show that $\mathfrak{a} \subset \mathfrak{q}$ or there exists some positive integer n such that $\mathfrak{b}^n \subset \mathfrak{q}$.

Same argument as above, just replacing prime for prime powers.

4 Let A be Noetherian and let $\mathfrak{q}$ be a $\mathfrak{p}-$primary ideal. Show that there exists some $n \geq 1$ such that $\mathfrak{p}^n \subset \mathfrak{q}$

I'm having trouble understanding what it means to be $\mathfrak{p}-$primary.