1 Prove that the given set of functions converges weakly but not strongly

Consider $\mathscr{X} = C[0,1]$, the Banach Space of continuous functions on $[0,1]$ with the supremum norm.

Consider sequence of functions $(f_n)_{n\geq 2}$, where

$$f_n(x) = \begin{cases} xn & 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \frac{2}{n} < x \leq 1 \end{cases}$$

$D_n = \{\alpha \in \mathbb{F} : |\alpha| \leq 1\}$

(Starts at 0, goes up to 1 at 1/n, down to 0 at 2/n, and stays flat from there).

Denote $0 \in \mathscr{X}$ as function $z(x) = 0$ for $0 \leq x \leq 1$.

Claim is that $f_n \rightharpoonup z$, but that $f_n \not\to z$.

First, $||f_n - 0|| = ||f_n|| = 1$ for all n, so f_n does not converge to 0 (TODO: Prove it doesn't converge at all?)

Suppose L is a continuous linear functional such that $\lim_{n\to\infty} L(f_n) \neq 0$. Therefore there is some subsequence b_n and ε such that $L(b_n) > \varepsilon$. Choose a further subsequence of b_n, called c_n where the c_n have disjoint support

$d_n = \sum_{i=1}^{n} c_n$