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1 Show that wk is the smallest topology on $\mathscr{X}$ such that each x^* in $\mathscr{X}^*$

Must show that an arbitrary open set in wk can be generated by some collection of sets of the form $x^{*-1}(V)$.

Start with an arbitrary open set U. Since X with wk is a locally convex set, $\bigcap_{i=1}^{n}\{x\epsilon\mathscr{X} : p_j(x - x_0) < \varepsilon_j\} \subseteq U$ for some finite list of ε and p, where $p_{x^*} = |\langle x, x^*\rangle|$.

$\bigcap_{i=1}^{n}\{x \in \mathscr{X} : p_j(x - x_0) < \varepsilon_j\}$ is open and the intersection of pre-images of open sets from $\mathbb{F}$ (why?).

Since U is generated by the subbase of preimages of the collection of x^* and U is arbitrary, all open sets of wk are generated in this way, thus is the smallest possible topology.

2 Show that wk* is the smallest topology on $\mathscr{X}^*$ such that for each x in $\mathscr{X}$, $x^* \mapsto \langle x, x^*\rangle$

This purported topology σ is generated by open sets that are preimages of open sets of functions of the form $x^* \mapsto \langle x, x^*\rangle$. By continuity of composition, they are also the preimages of open sets of functions of the form $x^* \mapsto |\langle x, x^*\rangle|$. Open sets of $\mathbb{R}$ are generated by open balls, so the pre-images are generated by sets of the form $|\langle x, x^*\rangle| < \varepsilon$.

A set U of $\mathscr{X}^*$ is weakly open if and only if for every x_0^* in U there is an ε and there are $x_1, ...x_n$ in $\mathscr{X}$ such that

$$\bigcap_{i=1}^{n}\{x^* \in \mathscr{X}^* : |\langle x_k, x^* - x_0^*\rangle| < \varepsilon_j\} \subseteq U$$

Every such set U can be generated as part of σ and therefore is the smallest topology

3 Prove Theorem 1.3

Theorem 1.3 is proven in the text!

4 Let $\mathscr{X}$ be a complex LCS and let $\mathscr{X}^*_{\mathbb{R}}$ denote the collection of all continous real linear functionals on $\mathscr{X}$. Use the elements of $\mathscr{X}^*_{\mathbb{R}}$ to define seminorms on $\mathscr{X}$ and let $\sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}})$ be the corresponding topology. Show that $\sigma(\mathscr{X}, \mathscr{X}^*) = \sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}})$

Goal is to show that an open set in $\sigma(\mathscr{X}, \mathscr{X}^*)$ is open in $\sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}})$ and vice versa.

Since $\mathscr{X}^*_{\mathbb{R}}$ is a strict subset of $\mathscr{X}^*$, $\sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}}) \subset \sigma(\mathscr{X}, \mathscr{X}^*)$ (since $\sigma(\mathscr{X}, \mathscr{X}^*)$ are those functions that make all linear functionals continuous, including the real linear functionals.)

Thus we need to show that $\sigma(\mathscr{X}, \mathscr{X}^*) \subset \sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}})$

Take the functions $f_r, f_i : \mathbb{C} \mapsto \mathbb{R}$ that map the complex numbers to their real, imaginary parts, respectively.

Open sets around x_0 in $\sigma(\mathscr{X}, \mathscr{X}^*)$ are generated by sets of the form $\bigcap_{i=1}^{n}\{x \in \mathscr{X} : |\langle x - x_0, x_k^*\rangle| < \varepsilon_k\}$ for some finite set of x_k^* and ε_k. To simplify consider open sets generated by a single complex semi-norm x_0^*. We will show that these open sets are also generated by two real semi-norms, x_0^* composed with f_r and f_i.

Firstly, observe that the open sets on the complex plane as a 1-dimensional complex vector space are the same as the open sets in $\mathscr{R}^2$ as a 2-dimensional real vector space since the homeomorphism between them simply ignores the complex structure (is this sufficient?). Since there's a unique combination of real part and imaginary part for each complex function, the images of the corresponding functions will be the same, hence the pre-images will be the same.

The same will be true given the finite intersection of sets generated by a finite number of seminorms. Thus any open set in $\sigma(\mathscr{X}, \mathscr{X}^*)$ is also open in $\sigma(\mathscr{X}, \mathscr{X}^*_{\mathbb{R}})$

5 If $A \subseteq \mathscr{X}$...

5.1 A° is convex and balanced

$A^\circ \equiv \{x^* \in \mathscr{X}^* : |\langle a, x^*\rangle| \leq 1 \text{ for all } a \text{ in } A\}$

To show that A° is balanced, given x^* in A° and $|\alpha| \leq 1$, we have to show that αx^* is also in A°.

If $|\langle a, x^*\rangle| \leq 1$, then $|\langle a, \alpha x^*\rangle| = |\alpha||\langle a, x^*\rangle| \leq 1$, hence αx^* is in A°

To show that A° is convex, we have to show that given arbitrary x_1^* and x_2^* in A°, then an arbitrarily chosen x_3 in $\{tx_1^* + (1-t)x_2^* : 0 \leq t \leq 1\}$ is also in A°, say $t_3x_1^* + (1-t_3)x_2^*$ (where t_3 was chosen arbitrarily in $0 \leq t \leq 1\}$).

Hence we need to prove that, for all a in A, $|\langle a, t_3x_1^* + (1-t_3)x_2^*\rangle| \leq 1$

If $|\langle a, x_1^* \rangle| \leq 1$ and $|\langle a, x_2^* \rangle| \leq 1$, then simply by multiplication of both sides $t_3|\langle a, x_1^* \rangle| \leq t_3$ and $(1-t_3)|\langle a, x_2^* \rangle| \leq 1 - t_3$ since $0 \leq t_3 \leq 1$ (to not flip the signs). Adding the inequalities we get $t_3|\langle a, x_1^* \rangle| + (1-t_3)|\langle a, x_2^* \rangle| \leq 1$.

SOMEHOW: prove $|\langle a, t_3 x_1^* + (1-t_3)x_2^* \rangle| \leq t_3|\langle a, x_1^* \rangle| + (1-t_3)|\langle a, x_2^* \rangle|$ (Cauchy inequality + convexity of the reals?)

5.2 If $A_1 \subseteq A$, then $A^\circ \subseteq A_1^\circ$

If $x^* \in A^\circ$, then $|\langle a, x^* \rangle| \leq 1$ for all a in A. Since $A_1 \subseteq A$, $|\langle a, x^* \rangle| \leq 1$ for all a in A_1 as well. Therefore $x^* \in A^\circ$. Since x^* was chosen arbitrarily, $A^\circ \subseteq A_1^\circ$.

6 If $A \subseteq \mathscr{X}$, show that A is weakly bounded if and only if A° is absorbing in $\mathscr{X}^*$

Assume A is weakly bounded.

For every open set U containing 0, there is an $\varepsilon > 0$ such that $\varepsilon A \subseteq U$ (see definition 2.5 in IV. §2, pg 106). Take one such open set U_0 and its associated ε_0. There exists $x_1^*, ..., x_n^*$ and $\hat{\varepsilon}_0$ such that

$$\bigcap_{i=1}^{n} \{x \in \mathscr{X} : |\langle x, x_k^* \rangle| < \hat{\varepsilon}\} \subseteq U_0$$

hmmm, still stuck here.

For an arbitrary x^*, we have to show that $|\langle a, tx^* \rangle| \leq 1$ for all a in A and $0 \leq t < \varepsilon$.