1 If $\mathscr{A}$ is a Banach algebra with identity and $a \in \mathscr{A}$ and is nilpotent (that is, $a^n = 0$ for some n), then $\sigma(a) = \{0\}$

Firstly, assume a is invertible. Then

$$\begin{aligned} a^n &= 0 \\ (a^{-1})^n a^n &= (a^{-1})^n 0 \\ 1 &= 0 \end{aligned}$$

a contradiction, so $0 \in \sigma(a)$.

Next, we need to show that $a - \alpha$ for any $\alpha \in \mathbb{C}$ is invertible.

If $1 + a + a^2 ...$ converges, it will converge to $1/(1-a)$ by the geometric series. Since $a^n = 0$, all terms $\geq n$ vanish, hence the series converges, and $1/(1-a) \in \mathscr{A}$

Since $\mathscr{A}$ is closed under multiplication, it also contains $1/(a-1)$, hence $a-1$ is invertible, so $1 \notin \sigma(a)$.

TODO: Extend to the other $\alpha \in \mathbb{C} \setminus 0$ (maybe some simple algebra trick?)