1 Let A be a commutative ring. Let M be a module, and N a submodule. Let $N = Q_1 \cap ... \cap Q_r$ be a primary decomposition of N. Let $\bar{Q}_i = Q_1/N$. Show that $0 = \bar{Q}_1 \cap ...\bar{Q}_r$ is a primary decomposition of 0 in M/N. State and prove the converse.

Firstly, let's take a concrete example, $A = \mathbb{Z}$, $M = \mathbb{Z}$, $N = 12\mathbb{Z}$.

One primary decomposition (the one that is in some sense minimal) of $12\mathbb{Z}$ is $4\mathbb{Z}$, $3\mathbb{Z}$.

$N = 12\mathbb{Z}$ itself is not primary since $3_{\mathbb{Z}_{12}}$ is neither injective (it sends both 0 and 4 to 0) nor nilpotent (no power of the homomorphism brings 1 to 0, that is to say $3^n \bmod 12 \neq 0$).

So now $\bar{Q}_1 = 4\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_3$ and $\bar{Q}_2 = 3\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_4$. Since $M/N = \mathbb{Z}_12$, $\mathbb{Z}_4$ and $\mathbb{Z}_3$ are in fact primary decompositions with an intersection at 0.

2 Let $\mathfrak{p}$ be a prime ideal and $\mathfrak{a}$, $\mathfrak{b}$ be ideals of A. If $\mathfrak{ab} \subset \mathfrak{p}$, show that $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

Take a_i to be an arbitrary member of $\mathfrak{a}$ and b_i to be an arbitrary member of $\mathfrak{b}$. If $a_i b_i$ is in $\mathfrak{p}$, then, since $\mathfrak{p}$ is prime, either a_i is in $\mathfrak{p}$ or b_i is.

Say a_i is. Now we have to show that all of $\mathfrak{a}$ is thus in $\mathfrak{p}$.

3 Let $\mathfrak{q}$ be a primary ideal. Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals, and assume $\mathfrak{ab} \subset \mathfrak{q}$. Assume that $\mathfrak{b}$ is finitely generated. Show that $\mathfrak{a} \subset \mathfrak{q}$ or there exists some positive integer n such that $\mathfrak{b}^n \subset \mathfrak{q}$.

Same argument as above, just replacing prime for prime powers.

4 Let A be Noetherian and let $\mathfrak{q}$ be a $\mathfrak{p}-$primary ideal. Show that there exists some $n \geq 1$ such that $\mathfrak{p}^n \subset \mathfrak{q}$

I'm having trouble understanding what it means to be $\mathfrak{p}-$primary.