1 If $B = \text{ball} l^\infty$, show that $d(\phi, \psi) = \sum_{j=1}^{\infty} 2^{-j} |\phi(j) - \psi(j)|$ defines a metric on B and that this metrics defines the weak-star topology on B

d is a metric:

$d(\phi, \phi) = \sum_{j=1}^{\infty} 2^{-j} |\phi(j) - \phi(j)| = 0$

$d(\phi, \psi) = \sum_{j=1}^{\infty} 2^{-j} |\phi(j) - \psi(j)| = \sum_{j=1}^{\infty} 2^{-j} |\psi(j) - \phi(j)| = d(\psi, \phi)$

Given an arbitrary non-negative number j, by the triangle inequality:

$$|(\phi(j) - \psi(j)) + (\psi(j) - \rho(j))| \leq |\phi(j) - \psi(j)| + |\psi(j) - \rho(j)| \tag{1}$$

$$|\phi(j) - \rho(j)| \leq |\phi(j) - \psi(j)| + |\psi(j) - \rho(j)| \tag{2}$$

$$\tag{3}$$

Therefore, since j was chosen arbitrarily:

$$\sum_{j=1}^{\infty} 2^{-j} |\phi(j) - \rho(j)| \leq \sum_{j=1}^{\infty} 2^{-j} |\phi(j) - \psi(j)| + \sum_{j=1}^{\infty} 2^{-j} |\psi(j) - \rho(j)| \tag{4}$$

$$d(\phi, \rho) \leq d(\phi, \psi) + (\psi, \rho) \tag{5}$$

We now have to show that given ϕ in B and $\varepsilon > 0$, $\{\psi \in$B: $d(\phi, \psi) < \varepsilon\}$ is open in the weak* topology.

A set U of B is weakly open if and only if for every x_0^* in U there is an ε and there are $x_1, ..., x_n$ in l^1 such that

$$\bigcap_{i=1}^{n} \{x^* \in B : |\langle x_k, x^* - x_0^* \rangle| < \varepsilon_j\} \subseteq U$$