1 If $\mathscr{H}$ is a Hilbert space, $\mathscr{A} = \mathscr{B}(\mathscr{H})$ is a C*-algebra where for each A in $\mathscr{B}(\mathscr{H})$, A*=the adjoint of A

VIII.1.1 (pg 236)

$\langle Ah, k\rangle = \langle h, A^*k\rangle = \overline{\langle A^*k, h\rangle}$

$\langle A^*k, h\rangle = \langle k, A^{**}h\rangle = \overline{\langle A^{**}h, k\rangle}$

The above are justified by the definition of the adjoint and inner product, for all values h and k in $\mathscr{H}$.

Taking the complex conjugate of the bottom row shows that $\langle Ah, k\rangle = \langle A^{**}h, k\rangle$, hence A=A**.

$\langle (AB)^*h, k\rangle = \langle h, (AB)k\rangle = \langle h, A(B(k))\rangle = \langle A^*h, Bk\rangle = \langle B^*A^*h, k\rangle$

Hence, $(AB)^* = B^*A^*$

$\langle (\alpha A + B)^*h, k\rangle = \langle h, (\alpha A + B)k\rangle = \langle h, \alpha Ak + Bk\rangle$

$\langle \bar{\alpha}A^*h + B^*h, k\rangle$ (kind of ran out of steam by this point)

2 If X is a compact Hausdorff space, show that X is totally disconnected if and only if C(X) is the closed linear span of its projections ($\equiv$ hermitian idempotents)

VIII.2.3 (pg 239)

Assume X is totally disconnected. Consider a bounded function f. Ignoring net vs sequence issues (TODO: is this justified?), we must show that f can be written as the convergence of $a_1e_1 + a_2e_2 + ...$ where $a_n \in \mathbb{C}$ and e_n are hermitian idempotents.

In this context, a hermitian idempotent is one whose range is $\{0, 1\}$, since those are the only complex numbers equal to their own square. Consider a series of function e_{q_1}, e_{q_2}, where for each $q_i \in X$, define

$$e_{q_i}(x) = \begin{cases} 1 & x = q_i \\ 0 & x \neq q_i \end{cases}$$

f can therefore be written as a countably infinite linear sum of such a series, thus is in the closed linear span of the projections.

Hm, I didn't use the fact that X is totally disconnected so something is missing here.