Fourier Methods - AM2071

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January 2022

Lecture 1 + 2

Definitions

- Vector Space
- Linearly independent set
- Inner product
- Orthogonal set

Definition - Vector Space

A vector space V over a field $\mathbb{K}$ is a non-empty set on which there are defined two operations

1. addition $+ : V \times V \to V$
2. multiplications of a scalar $\cdot : \mathbb{K} \times V \to V$

i Closure $\underline{x} + \underline{y} \in V \quad \forall \underline{x}, \underline{y} \in V$

ii Commutativity $\underline{x} + \underline{y} = \underline{y} + \underline{x} \quad \forall \underline{x}, \underline{y} \in V$

iii Associativity $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z} \quad \forall \underline{x}, \underline{y}, \underline{z} \in V$

iv Identity $\exists \underline{0} \in V$ such that $\underline{x} + \underline{0} = \underline{x} \quad \forall \underline{x} \in V$

v Inverse $\forall \underline{x} \in v \ \exists \ \underline{y} \in V$ such that $\underline{x} + \underline{y} = 0$ ie. $\underline{y} = -\underline{x}$

and under "$\cdot$"

i Closure $\alpha \cdot \underline{x} \in V \ \forall \alpha \in \mathbb{K}, \underline{x} \in V$

ii Distributive Laws $(\alpha + \beta) \cdot \underline{x} = \alpha \cdot \underline{x} + \beta \underline{y}$

iii Identity $1 \cdot \underline{x} = \underline{x} \quad \forall \underline{x} \in V$

Note: A vector space (just by itself) has no product between vectors

Definition - Linearly Independent

A subset W of vectors in V is said to be linearly independent set if every non-empty selection of finitely many elements of W is linearly independent:

If $\underline{w}, \ldots, \underline{w}_n \in W$ and $\sum_{i=1}^{n} \alpha_i \underline{w}_i = \underline{0}$, then we must have $\alpha_i = 0, i = 1, \ldots, n$

Definition - Basis

B is called a basis for V if B is linearly independent set and every non-zero vector $\underline{x} \in V$ can be written as a linear combination of finitely many elements of B with non-zero scalars as coefficients.

Definition - Inner Product

We define an inner product on a vector space V over a vector space V over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) to be a mapping $<,>: V \times V \to \mathbb{K}$ with following properties

i $< \underline{x} + \underline{y}, \underline{z} >=< \underline{x}, \underline{y} >=< \underline{x}, \underline{z} >$

ii $< \alpha\underline{x} + \underline{y} >= \alpha < \underline{x}, \underline{y} >$

iii $< \underline{x}, \underline{y} >=< \underline{y}, \underline{x} >$

iv $< \underline{x}, \underline{x} > \quad > 0$ and $< \underline{x}, \underline{x} >= 0 \iff \underline{x} = \underline{0}$

Note: An inner product can always induce a norm: a real-valued function $||$ such that $||: V \to \mathbb{R}$ such that

i $|| \underline{x} || \geq 0$ and $|| \underline{x} ||= 0 \iff \underline{x} = \underline{0}$

ii $|| \alpha\underline{x} ||=| \alpha ||| \underline{x} ||$

iii $|| \underline{x} + \underline{y} || \leq || \underline{x} || + || \underline{y} ||$ - Triangle Inequality

How? Choose $|| \underline{x} ||:= \sqrt{< \underline{x}, \underline{x} >}$ (positive square root)

Definition - Orthonormal

A set of vectors $\underline{x}_1, \ldots, \underline{x}_n$ from an inner product space is said to be an orthongal set if $\underline{x} \neq \underline{0} \forall_i$ nd $<: \underline{x}_i, \underline{x}_j >= 0 \ \forall i \neq j$
If in addition $< \underline{x}_i, \underline{x}_j >= 1 \ \forall_i$ the set is said to be orthonormal.

Short-hand, alternative notation $< \underline{x}_i, \underline{x}_j >= \delta_{ij}$ (Kronecker delta) $= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$

Example - Vectorspaces

(a) n tuples in $\mathbb{R}^n$: field $\mathbb{K} = \mathbb{R}$ and obvious definitions of "+" and "·"
(b) $C[a, b]$: space of continuous real-valued functions on $[a, b]$
(c) space of infinite sequences in $\mathbb{R} : x_{11}, x_{21}, \ldots, x_{n1}, x_{n+1}, \ldots$.

Example - Inner product

Concept familiar from vectors as n tuples living in $\mathbb{R}^n$: dot product $<\underline{x}, \underline{y}>= \sum_{i=1}^{n} x_i y_i = \underline{x} \cdot \underline{y}$.
Same idea transferred to function space:
The inner product of two functions f_1 and f_2 on an interval $[a, b]$ is defined as

$$< f_1, f_2 >= \int_a^b f_1(x) f_2(x) dx$$

$\Rightarrow$ f_1 and f_2 are orthogonal if $< f_1, f_2 >= 0$

Lecture 3

Examples

Example (a)

$f_1(x) = x^2, f_2(x) = x^3$ on $[-1, 1]$

$$\begin{aligned} < f_1, f_2 > &= \int_{-1}^{1} x^2 x^3 dx \\ &= \int_{-1}^{1} x^5 dx \\ &= \frac{x^6}{6}\bigg|_{-1}^{1} \\ &= \frac{1}{6} - \frac{(-1)^6}{6} = 0 \end{aligned}$$

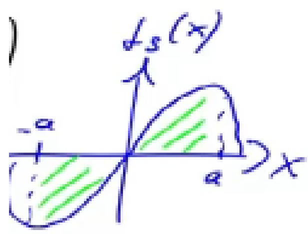
Example (b)

f_1 even function, f_2 odd function on $[-a, a]$

$$\begin{aligned} f_1(x) &= f_1(-x) \\ f_2(x) &= -f_2(-x) \end{aligned}$$

$$
\begin{aligned}
< f_1, f_2 > &= \int_{-a}^{a} \underbrace{f_1(x)f_2(x)}_{=f_3(x)} dx = \int_{-a}^{a} \underbrace{f_3(x)}_{\text{odd function}} \quad dx \\
&= \int_{-0}^{0} f_3(x)dx + \int_{0}^{a} f_3(x) \\
&= \underbrace{-\int_{0}^{a} f_3(-y)(-dy)}_{\text{inverted + variable change}} + \int_{0}^{a} f_3(x) \\
&= \underbrace{\int_{0}^{a} f_3(y)dy}_{f_3(-y) = -f_3(y)} + \int_{0}^{a} f_3 dx \\
&= 0
\end{aligned}
$$

Visually



Example (c)

Show that the set $\{1, \cos x, \cos 2x, \cos 3x, \dots\} = \{\cos nx\}$ where $n = 01, 2, \dots$ is orthogonal on $[-\pi, \pi]$

$$\{\phi_0, \phi_1, \dots\} = \cos(n\pi)$$

Consider $\phi_0(x) = 1, \phi_n(x) = \cos(nx)$
Show that $\int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)dx = 0$ for $n \neq 0$ and $\int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)dx = 0$ for $m \neq n$

$$
\begin{aligned}
< \phi_0(x), \phi_n(x) > = \int_{-\pi}^{\pi} \cos(nx)dx &= \frac{1}{n} \sin(nx)\Big|_{=\pi}^{\pi} \\
&= \frac{1}{n} \left(\sin(n\pi) - \sin(-n\pi)\right) \approx 0
\end{aligned}
$$

$$
\begin{aligned}
< \phi_m(x), \phi_n(x) > &= \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} [\cos((m-n)x) + \cos((m+n)x)]dx \\
&= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} + \frac{\sin((m+n)x)}{m+n}\right] \\
&= 0
\end{aligned}
$$

Find the norm of elements of the set $\{1, \cos x, \cos 2x, \dots\}$

$$
\| \phi_0(x) \|^2 = \int_{-\pi}^{\pi} | \cdot | \, dx = \int_{-\pi}^{\pi} dx = 2\pi \Rightarrow \| \phi_0(x) \| = \sqrt{2\pi}
$$

$$
\| \phi_n(x) \|^2 = \int_{-\pi}^{\pi} [\cos(nx)]^2 dx = \frac{1}{2}\int_{-\pi}^{\pi} [1+\cos(2nx)]dx = \pi \Rightarrow \| \phi_n \| = \sqrt{\pi}, n > 0
$$

Therefore $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots\right\}$ is an orthogonal set.
Is this a basis of the space of continous functions on $[-\pi, \pi]$? No, becasuse it is not a complete set.
"Complete": There is no other function (expect $f(x) = 0$) that is orthogonal to every numbers of the set.

Example

For example: $f(x) = \sin(x)$ on $[-\pi, \pi]$ is orthogonal to all elements of $\{1, \cos x, \dots\}$

Think of vector spaces $\underline{u} = c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3}$ with $\{v_1, v_2, v_3\}$ basis of $\mathbb{R}^3$ and $c_i = \frac{<\underline{u}, \underline{v_i}>}{\|\underline{v}_i\|^2}$ $i - 1, 2, 3$

If you try this for $f(x)$ you'll get

$$
\sin(x) = f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \dots
$$

$c_i = \frac{<f_1 \phi_i>}{\|\underline{\phi}_i\|^2} = 0$ with

$$
= 0
$$

Solve $< \sin x, \cos(nx) >$ explicitly or remember example (b).
$\sin(x)$: odd function $\{1, \cos x, \cos 2x, \dots\}$ even function

$$
\Rightarrow \int_{-\pi}^{\pi} \sin x \phi_n(x) dx = 0
$$

$\{1, \sin x, \cos x, \sin(2x), \cos(2x), \dots\}$ is a basis of the considered vector space.

Lecture 4

Fourier Series

Series expansion

Orthogonal series expansion:

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

with

$$c_n = \frac{\int_a^b f(x)\phi_n(x)}{\| \phi_n(x) \|^2}$$

and orthogonal set $[a, b]\{\phi_n(x)\}$ with $n = 0, 1, 2, \ldots$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{< f_1 \phi_n >}{\| \phi_n \|^2} \phi_n(x)$$

Example

Consider the orthogonal set $\{1, \cos \frac{\pi}{p} x, \cos \frac{2\pi}{p} x, \ldots, \sin \frac{\pi}{p} x, \sin \frac{2\pi}{p} x, \ldots\}$ on the interval $[-p, p]$

Then, any function of f on $[-p, p]$ can be expanded by this set

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{p} x\right) + b_n \sin\left(\frac{n\pi}{p} x\right) \right]$$

Project on all elements on the set to compute the coefficients $a_0, a_1, \ldots, b_1, \ldots$

(i)

For a_0: project $f(x)$ on $\phi_0(x) = 1$

$$\begin{aligned}
\int_{-p}^{p} f(x)dx &= \int_{-p}^{p} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right)\right]\right] dx \\
&= \frac{a_0}{2}\int_{-p}^{p} 1dx + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right)\right) \\
&= \frac{a_0}{2}x\Big|_{-p}^{p} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\frac{p}{n\pi}\sin\left(\frac{k\pi}{p}x\right)\Big|_{-p}^{p}}_{\sin(n\pi)-\sin(-n\pi)=0} + b_n(-\frac{p}{n\pi}\underbrace{\cos\left(\frac{n\pi}{p}x\right))\Big|_{-p}^{p}}_{\cos(n\pi)-\cos(-n\pi)=0}\right] \\
&= \frac{a}{2}(p-(-p)) \\
&= pa_0
\end{aligned}$$

$$\Rightarrow a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

(ii)

For a_n project

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right)\right]$$

on $\cos\left(\frac{m\pi}{p}x\right)$

$$\int_{-p}^{p} f(x)\cos\left(\frac{m\pi}{p}x\right)dx$$

$$= \underbrace{\frac{a_0}{2}\int_{-p}^{p}\cos\left(\frac{m\pi}{p}x\right)dx}_{=0} + \sum_{n=1}^{\infty}\left[a_n\int_{-p}^{p}\cos\left(\frac{m\pi}{p}x\right)\cos\left(\frac{n\pi}{p}x\right)dx + \underbrace{b_n\int_{-p}^{p}\cos\left(\frac{m\pi}{p}x\right)\sin\left(\frac{n\pi}{p}x\right)dx}_{=0}\right]$$

$$= \sum_{n=1}^{\infty} a_n \int_{-p}^{p}\cos\left(\frac{m\pi}{p}x\right)\cos\left(\frac{n\pi}{p}x\right)dx$$

$$2\cos\theta\cos\varphi = \cos(\theta-\varphi)+\cos(\theta+\varphi)$$

$$= \sum_{n=1}^{\infty}\frac{a_n}{2}\left[\underbrace{\int_{-p}^{p}\cos\left(\cos\frac{(m-n)\pi}{p}x\right)dx}_{\begin{cases}0 \text{ if } m\neq n\\ 2P \text{ if } m=n\end{cases}} + \underbrace{\int_{-p}^{p}\cos\left(\frac{(m+n)\pi}{p}x\right)dx}_{=0 \text{ see (i)}}\right]$$

$$= \sum_{n=1}^{\infty}\frac{a_0}{2}2P\delta_{nm} - \text{ Kronecker delta}$$

$$= a_m p$$

(iii)

For b_n project $f(x) = \frac{a_0}{2} + \sum_{n=1} a_n\cos\left(\frac{n\pi}{p}x\right) + b_n\sin\left(\frac{n\pi}{p}x\right)$ on $\sin\left(\frac{m\pi}{p}x\right)$:

$$\int_{-p}^{p} f(x)\sin\left(\frac{m\pi}{p}x\right)dx = \cdots = b_m P$$

Fill in these dots

Definition

The Fourier series of a function f defined on the interval $(-p,p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{p}x\right) + b_n\sin\left(\frac{n\pi}{p}x\right)\right]$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$
$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)dx$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)dx$$

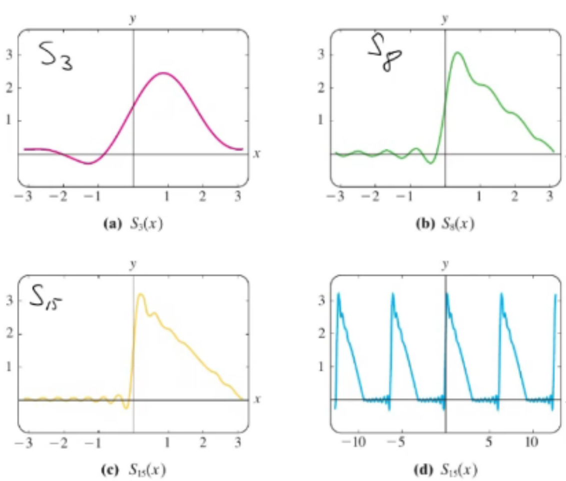
Example

$$f(x) = \begin{cases} 0 \text{ for } -\pi < x < 0 \\ \pi - x \text{ for } 0 \leq x < \pi \end{cases}$$

Details on calculations of a_0, a_n, b_n in examples and tutorial 1.

Partial sums of the Fourier Series

$$S_1(x) = \frac{\pi}{4} = \frac{a_0}{2}$$
$$S_2(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x$$
$$S_3(x) = \frac{a_0}{2} + \sum_{n=1}^{2}(a_n\cos(nx) + b_n\sin(nx)) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x$$
$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N-1}(a_n\cos(nx) + b_n\sin(nx))$$



(a) $S_3(x)$ (b) $S_8(x)$ (c) $S_{15}(x)$ (d) $S_{15}(x)$

periodic repitition of $f(x)$

The partial sums converge to $f(x)$ and the point(s) of discontinuity are approximately by the mid point $\frac{f(x^+)+f(x^-)}{2}$
$f(x^+) = \lim_{h\to 0} f(x+h)$ limit from the right
$f(x^-) = \lim_{h\to 0} f(x-h)$ limit from the left

Theorem: Convergence of Fourier Series

Let f and f' be piece wise continuous on the interval $(-p, p)$; that is, let f and f' be continuous expect at a fininte number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier series converges to the average.

$$\frac{f(x^+) + f(x^-)}{2}$$

where $f(x^+)$ and $f(x^-)$ denote the limit of f at x from the right and from the left respectively.

Dirichlet conditions

1. Piece wise continuous on the interval $(-p, p)$
2. Continuous expect at a finite number of points
3. Finite discontinuities at these points

Notation

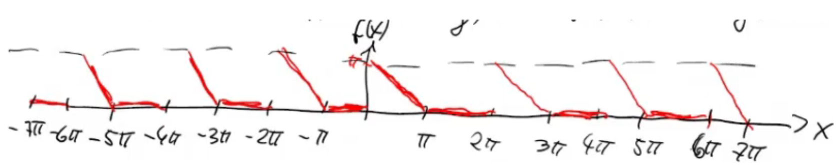
Derivative

$$f'(x) = \frac{d}{dx} f(x)$$

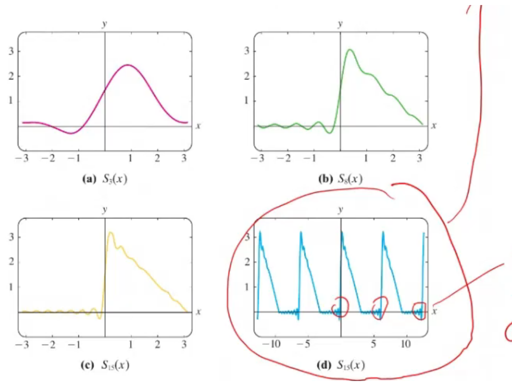
Looking at the example:

$$f(x) = \begin{cases} 0 \text{ for } -\pi < x < 0 \\ \pi - x \text{ for } 0 \leq x < \pi \end{cases}$$

Above example $f(0) \approx \frac{\pi}{2}$
What happens if we (periodically) add $f(x)$ to the right and left?



Does the theorem still apply?
Yes, but for a larger interval $(-mp, mp)$ Use the Fourier series (and its periodic properties) and repeat for new, additional subintervals.



overshooting close to points of discontinuity: Gibbs phenomenon

Lecture 5

Even and Odd Functions

Question? What if $f(x)$ is even or odd?

(1)

If f is even: $f(x) = f(-x)$
Then:

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\left(\frac{n\pi}{p}x\right)}_{\text{even}} dx = \frac{2}{p}\int_{0}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right) dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\left(\frac{n\pi}{p}x\right) dx}_{\text{even}} = 0$$

(2)

f is odd: Similarly: $a_n = 0 \quad n = 0, 1, 2, \ldots$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\left(\frac{n\pi}{p}x\right) dx}_{\text{even}} = \frac{2}{p}\int_{0}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right) dx$$

Cosine Series

The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}\right) x$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x) dx$$
$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi}{p}\right) x \ dx$$

Sine Series

The Fourier series of an odd function in the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}\right) x$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi}{p}\right) x \ dx$$

Half-Range Expansions

If we are only interested in $f(x)$ on $[0, p]$ we can extend f to $[-p, p]$ in different ways

1. mirroring with respect to y-axis $\Rightarrow$ even function $(b_n = 0)$
2. mirroring with respect to the origin $\Rightarrow$ odd function $(a_n = 0)$
3. transnational (shift to $[-p, 0]$) $\Rightarrow$ full Fourier series (compute a_n, b_n)

Helpful Tricks

Integration by parts

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)dx$$

Memorize this formula or derive it from the product rule

$$\frac{d}{dx}(u(x)v(x)) = \frac{du(x)}{dx}v(x) + u(x)\frac{dv(x)}{dx}$$
$$= u'v + uv'$$

Integrate on both sides

$$\int_a^b \frac{d}{dx}(u(x)v(x))dx = \int_a^b \frac{du(x)}{dx}v(x)dx + \int_a^b u(x)\frac{du(x)}{dx}dx$$

(ii)

Consider the following integral

$$2\int_0^1 \pi t \sin(n\pi t)dt = 2\pi \int_0^1 t\sin(n\pi t)dt$$

$$\begin{aligned} u &= t \\ u' &= 1 \\ v' &= \sin(n\pi t) \\ v &= -\frac{1}{n\pi}\cos(n\pi t) \end{aligned}$$

$$\begin{aligned} &= 2\pi\left[-\frac{1}{n\pi}t\cos(n\pi t)\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi t)dt\right] \\ &= 2\pi\left[-\frac{\cos(n\pi) - 0\cos(0)}{n\pi} + \frac{1}{n\pi}\frac{1}{n\pi}\sin(n\pi t)\bigg|_0^1\right] \\ &= 2\frac{(-1)(-1)^n}{n} \\ &= 2\frac{(-1)^{n+1}}{n} \end{aligned}$$

Lecture 6