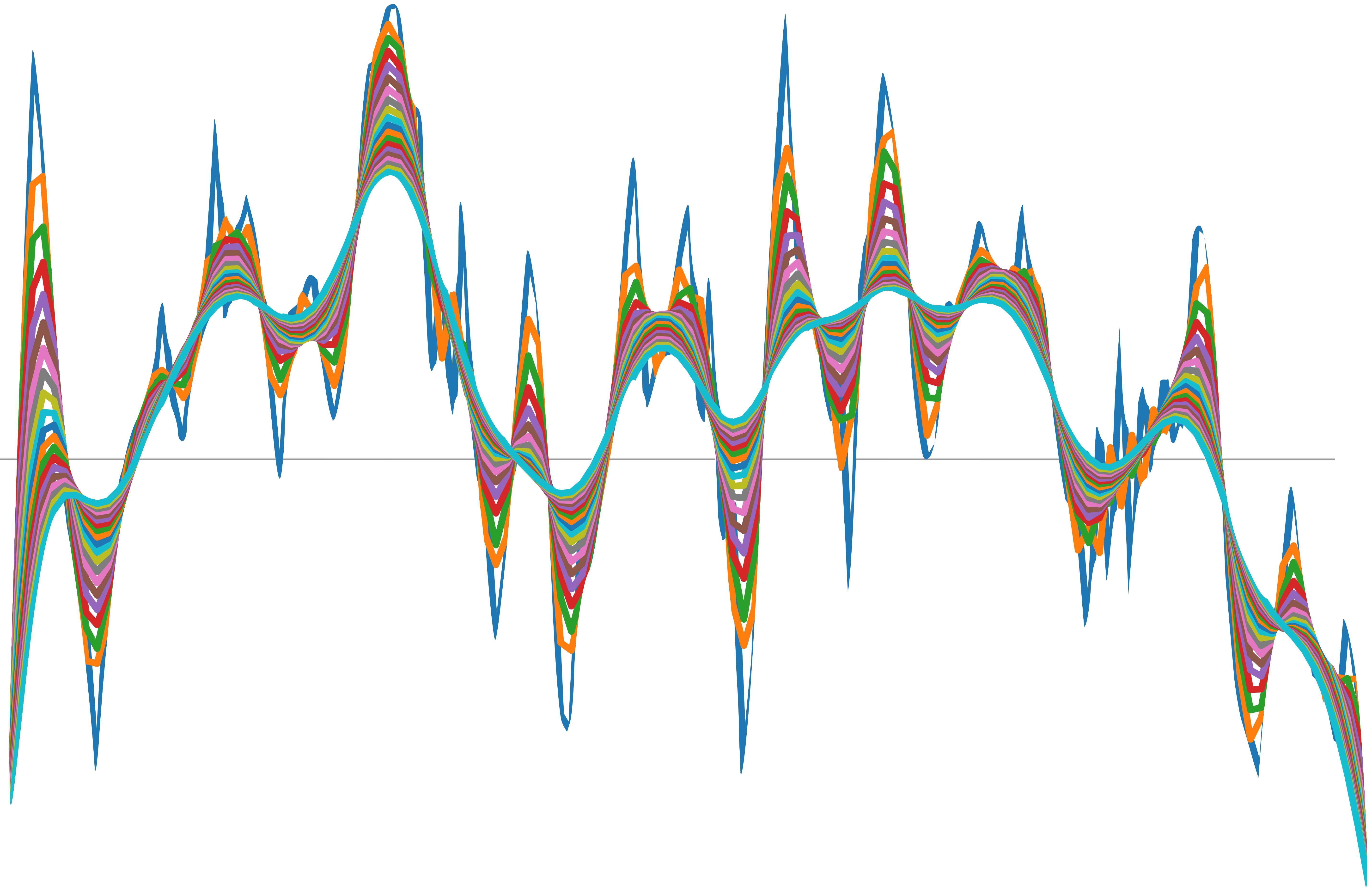
Multigrid Methods

The Basics



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What are we trying to do…

- Solve problems of the form

$$Ax = b$$

- Solve this problem **iteratively**:

$$x_1 \leftarrow x_0 + v$$

- Solve this problem **inexpensively:**
  - The update should be "good"
  - Finding the update should be "cheap"

Objectives — high level

- Construct a multigrid method for **your** problem
- Interpret the effectiveness of a multigrid method
- Identify *why* a method works — or — *why* a method does not
- Recognize different forms of multigrid, their pitfalls and their uses

Objectives — today

- **Create** a two level multigrid method
- **Illustrate** the main components of a multigrid method
- **Calculate** the effectiveness of a multigrid method
- **Highlight** some limitations of multigrid
- **Outline** some approaches to fixing multigrid (and why)
- **Identify** key pieces in moving to unstructured problems

A cautionary example

Guess and look for and update $\quad x_1 = x_0 + \text{update}$

Updating with the error would be **ideal** $\quad x_1 = x_0 + e_0$

Or in another form $\quad x_1 = x_0 + A^{-1} r_0$

Not practical so… $\quad x_1 = x_0 + D^{-1} r_0$

1 iteration

$\boxed{\text{Jacobi} \sim 1 \text{ SpMV}}$

x^* solution to $Ax = b$

$e_0 = x^* - x_0$ error

$r_0 = b - Ax_0$ residual

$= Ae_0$

$Ae_0 = r_0$ error equation

A reminder — projection methods

- Take a guess

$$x_0$$

- Look for an update that is the "best":

$$x_1 \leftarrow x_0 + u$$

- Minimize over a smaller space

$$\min_{u \in span\{V\}} \|x^* - x_1\|$$

- Then $u = Vy$

$$V^T V y = V^T e_0$$

- So the update looks like

$$x_1 = x_0 + V(V^T V)^{-1} V^T e_0$$

A reminder — projection methods

- Instead, look at the A-norm:

$$\min_{u \in span\{V\}} \|x^* - x_1\|_A$$

- Then $u = Vy$

$$V^T A V y = V^T A e_0$$

$$V^T A V y = V^T r_0$$

- So that

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

A reminder — projection methods

$$x_1 = x_0 + u$$

$$x_1 = x_0 + V(V^T AV)^{-1} V^T r_0$$

- What about the error

$$\begin{aligned} x^* - x_1 &= x^* - x_0 - V(V^T AV)^{-1} V^T r_0 \\ e_1 &= e_0 - V(V^T AV)^{-1} V^T A e_0 \\ &= \left(I - V(V^T AV)^{-1} V^T A\right) e_0 \end{aligned}$$

A-orthogonal projection onto the range of V

Model Problem

- A model problem

$$-u_{xx} = f$$
$$u(0) = u(1) = 0$$

- Finite differences

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i \quad i = 1, \dots, n \quad u_0 = u_{n+1} = 0$$

- A model matrix problem

$$A = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & \\ -1 & \ddots & \ddots \\ & \ddots & \end{bmatrix}$$

Model Problem

- A special matrix:

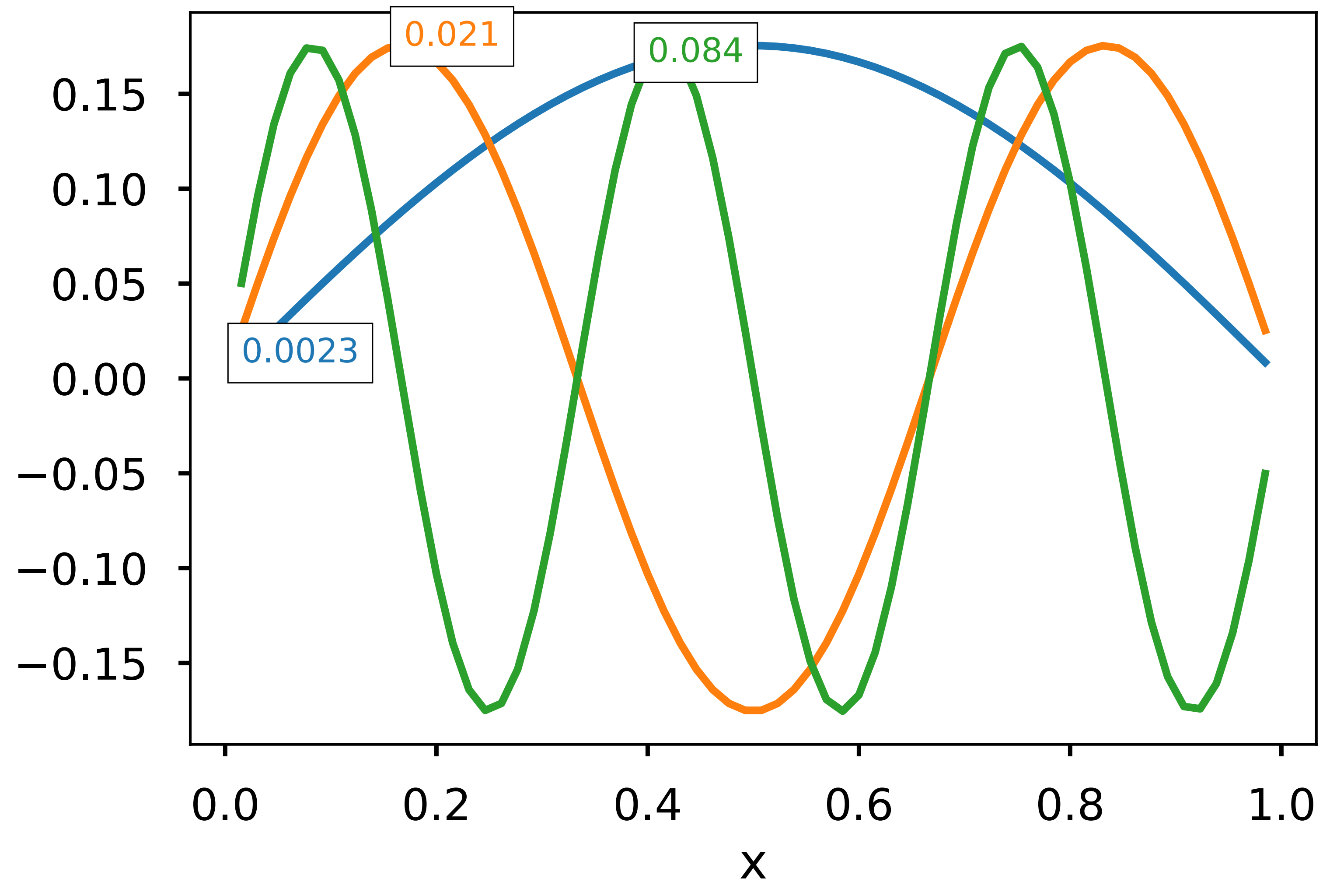
$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & & \\ & & & -1 & 2 \end{bmatrix}$$

The eigenvalues

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2(n+1)}\right)$$

range from (0,4]
(or from h^2 to 4)
The eigenvectors are Fourier modes:

$$(v_k)_j = \sin\left(\frac{(j+1) * k\pi}{n+1}\right)$$



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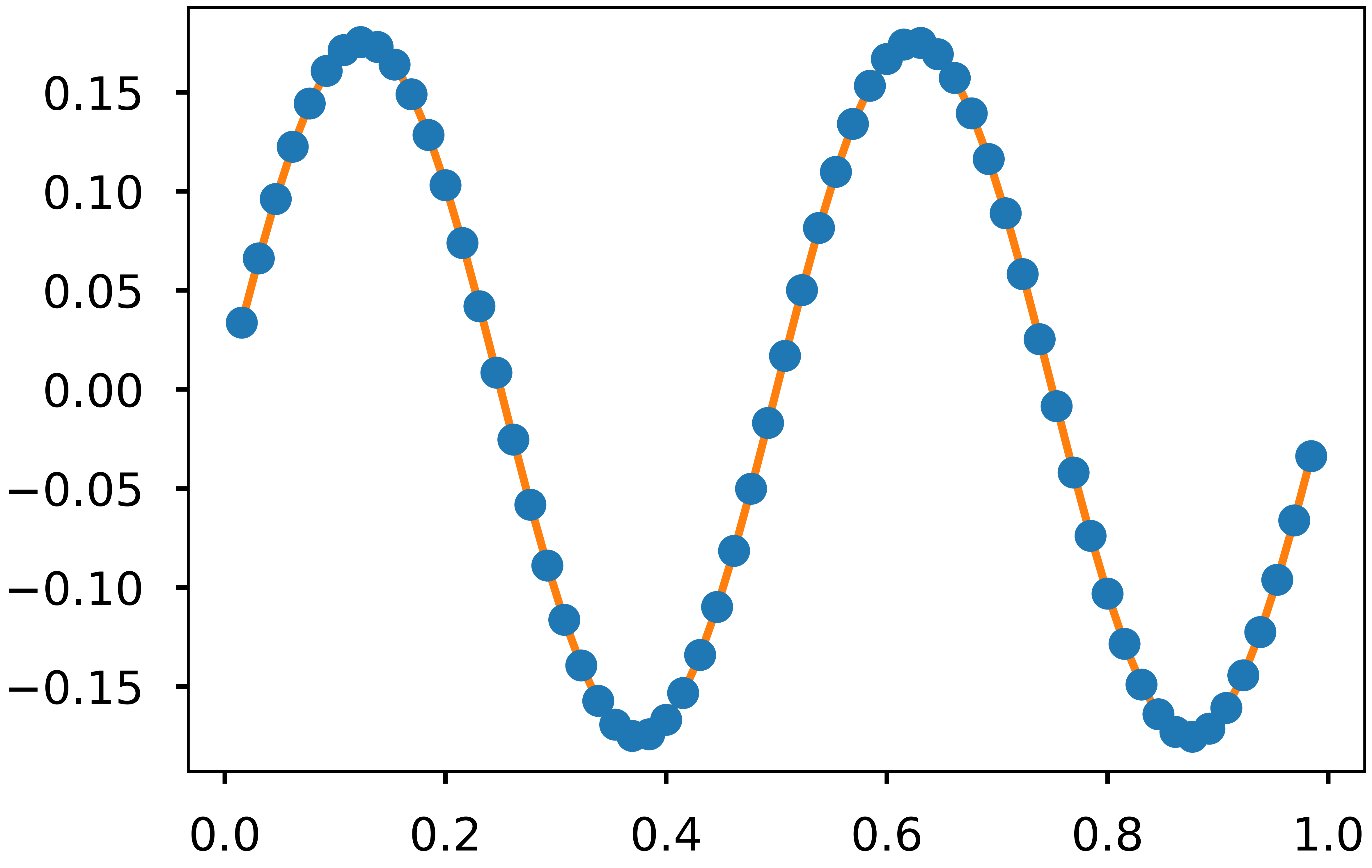
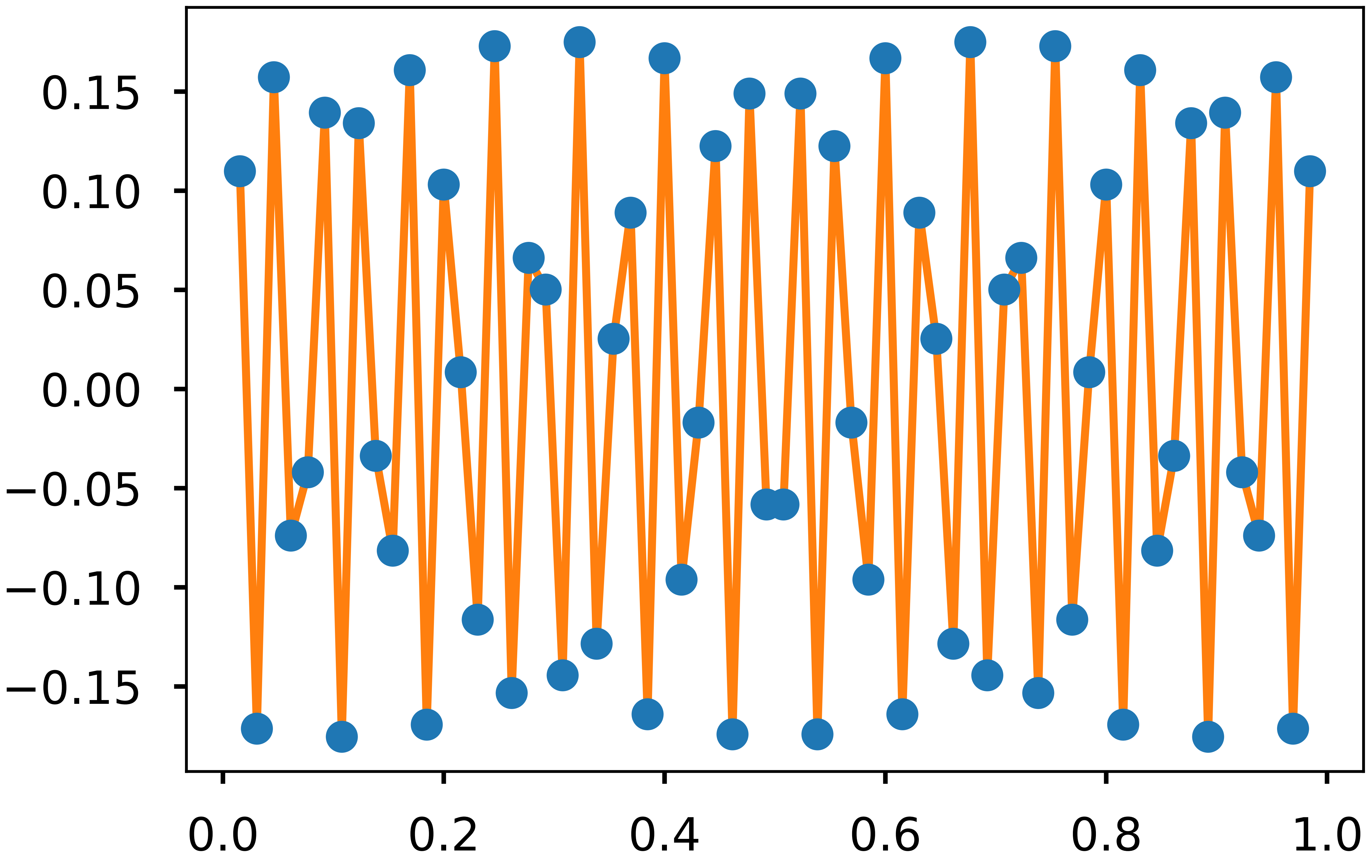
Take a look: `01-model-problem.ipynb`

Smooth Mode ~ low Fourier Modes

- We will talk a lot about "smoothness" — how much variation there is **per** grid point.





First relaxation scheme: Jacobi

- The discretization at point i:

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i$$

- Solving for the variable at this point (eliminating the residual):

$$u_i \leftarrow \frac{1}{2}\left(u_{i-1} + u_{i+1} + h^2 f_i\right)$$

- In matrix form:

$$u \leftarrow (I - D^{-1}A)u + h^2 D^{-1} f$$

$$u \leftarrow u + D^{-1} r$$

- And the error:

$$e \leftarrow Ge \qquad \boxed{G = I - D^{-1}A}$$

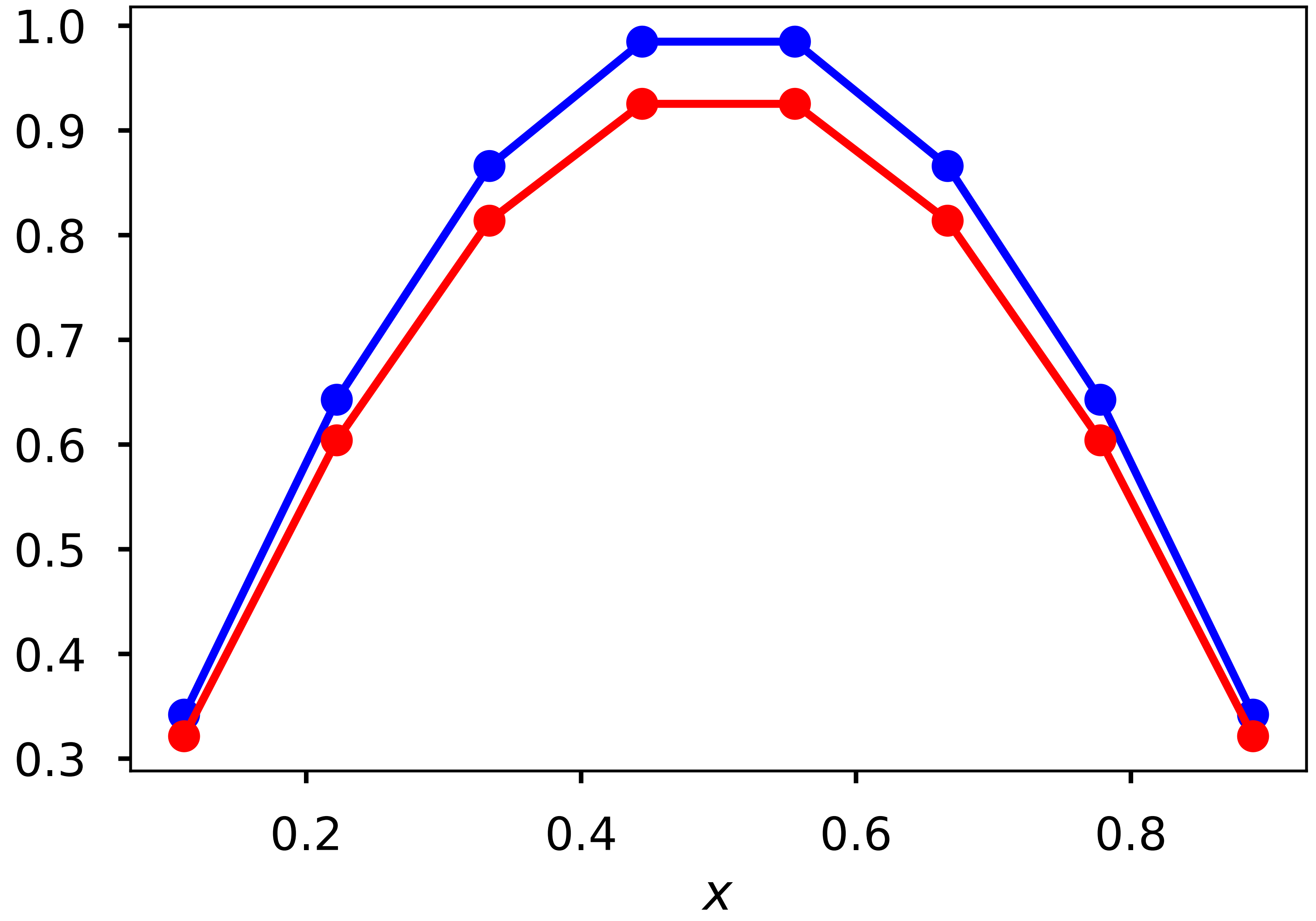
The “error propagation operator”

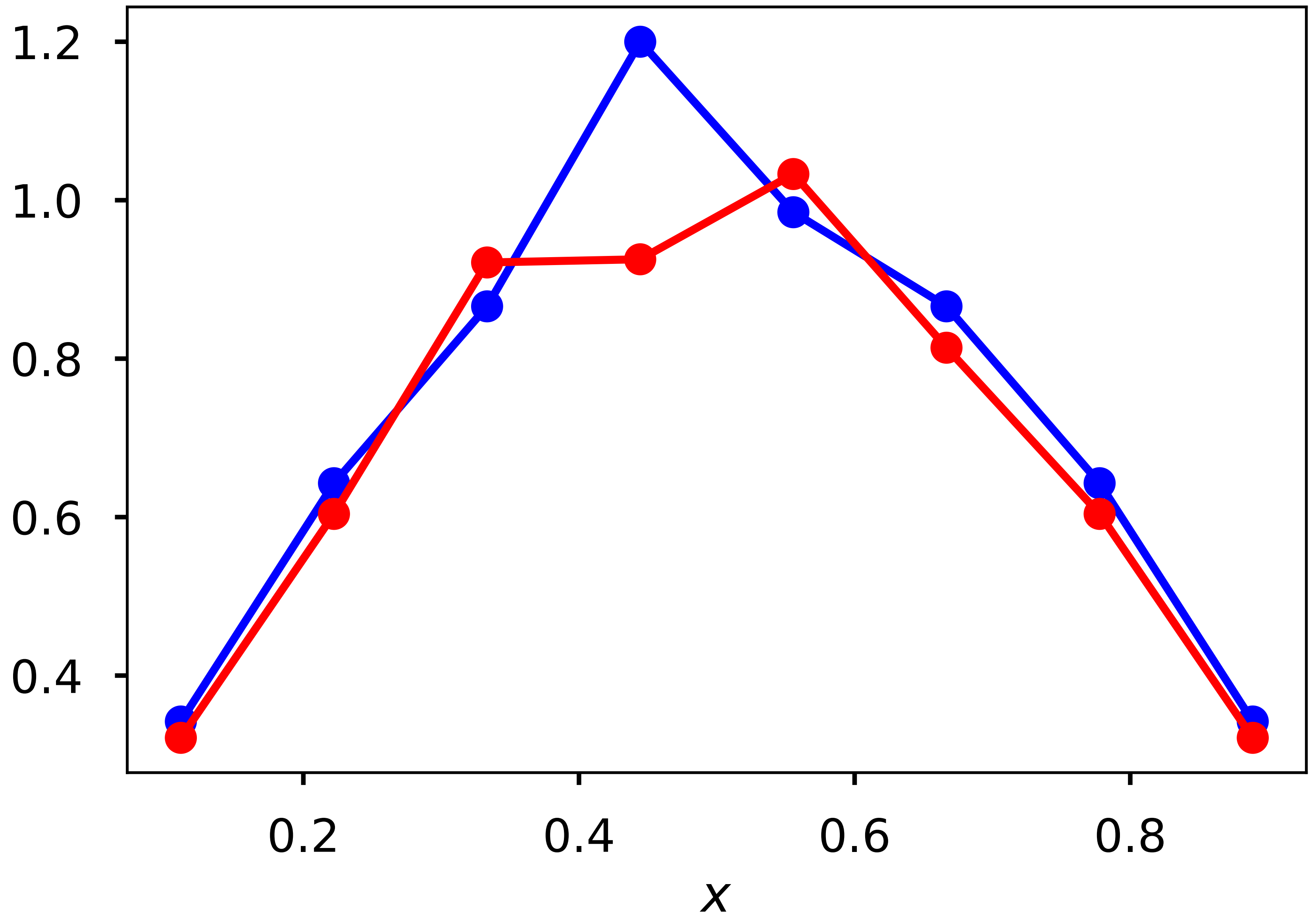
What does Jacobi do to error?

$$e^{new} \leftarrow (I - D^{-1}A)e^{old}$$

$$e_i^{new} \leftarrow \frac{1}{2}\left(e_{i-1}^{old} + e_{i+1}^{old}\right)$$

- It averages…
- Consider smooth and oscillatory error:





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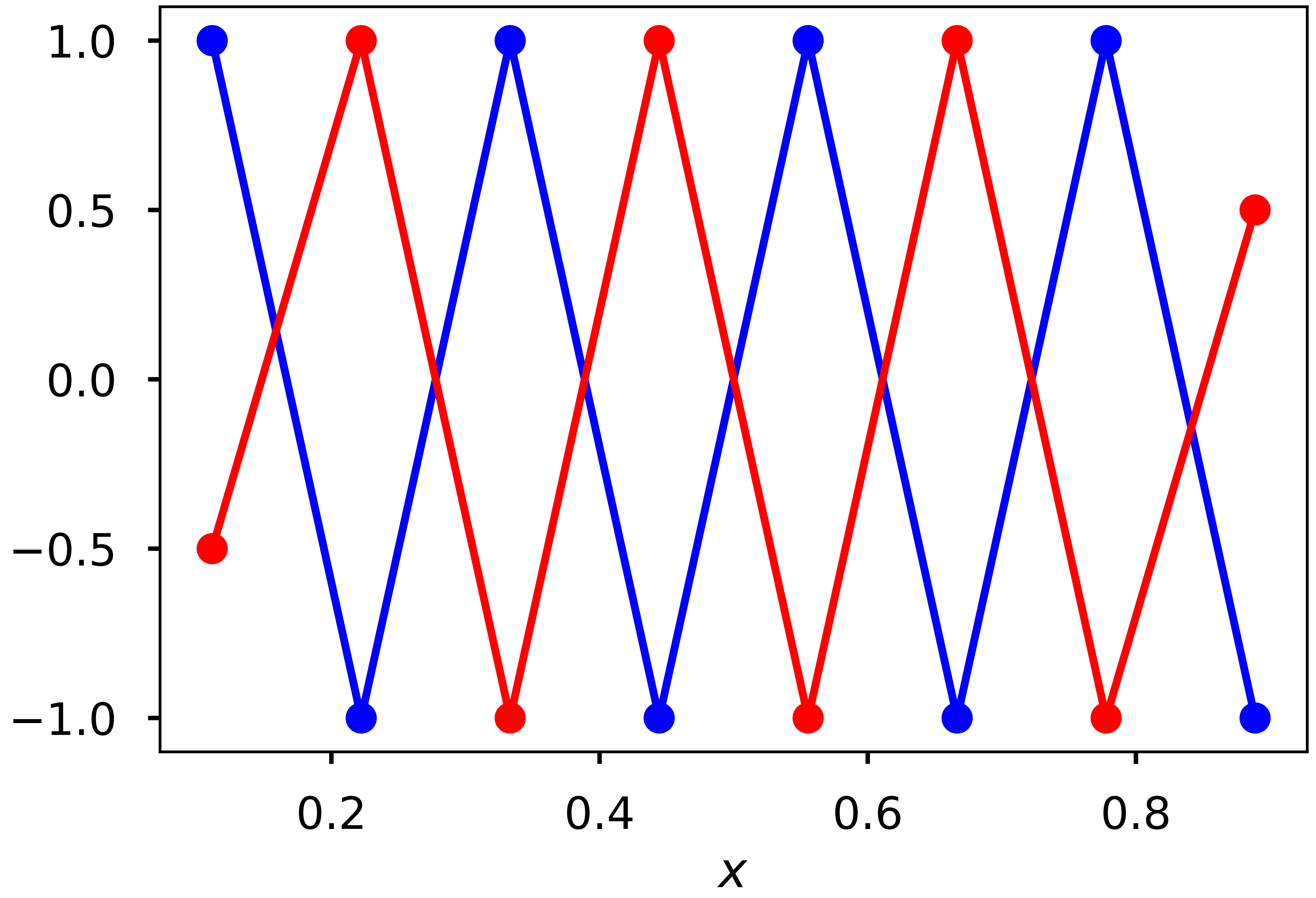
But…

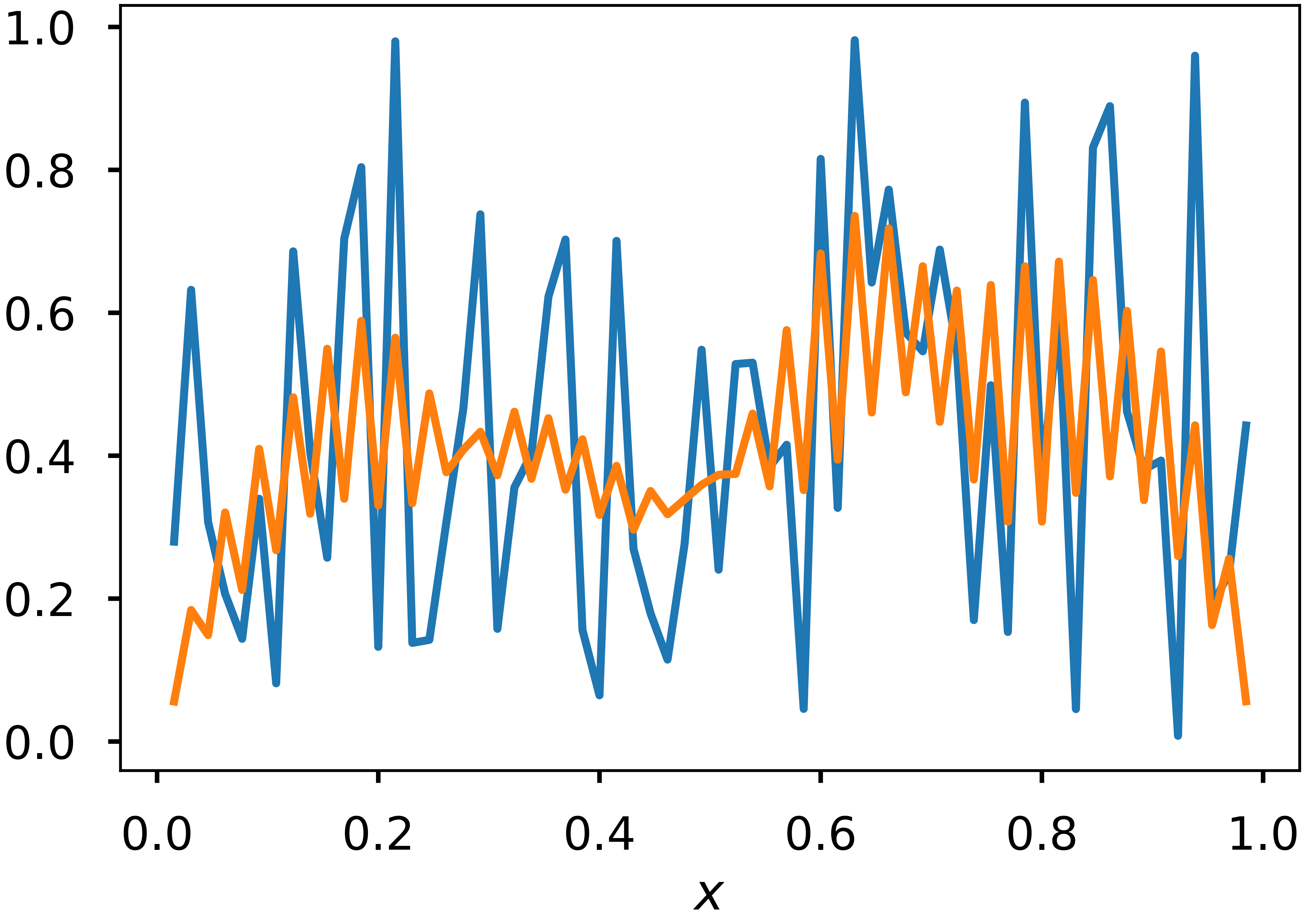


- Jacobi “averages out” certain error very quickly.
- But stagnates on very smooth error or very oscillatory error

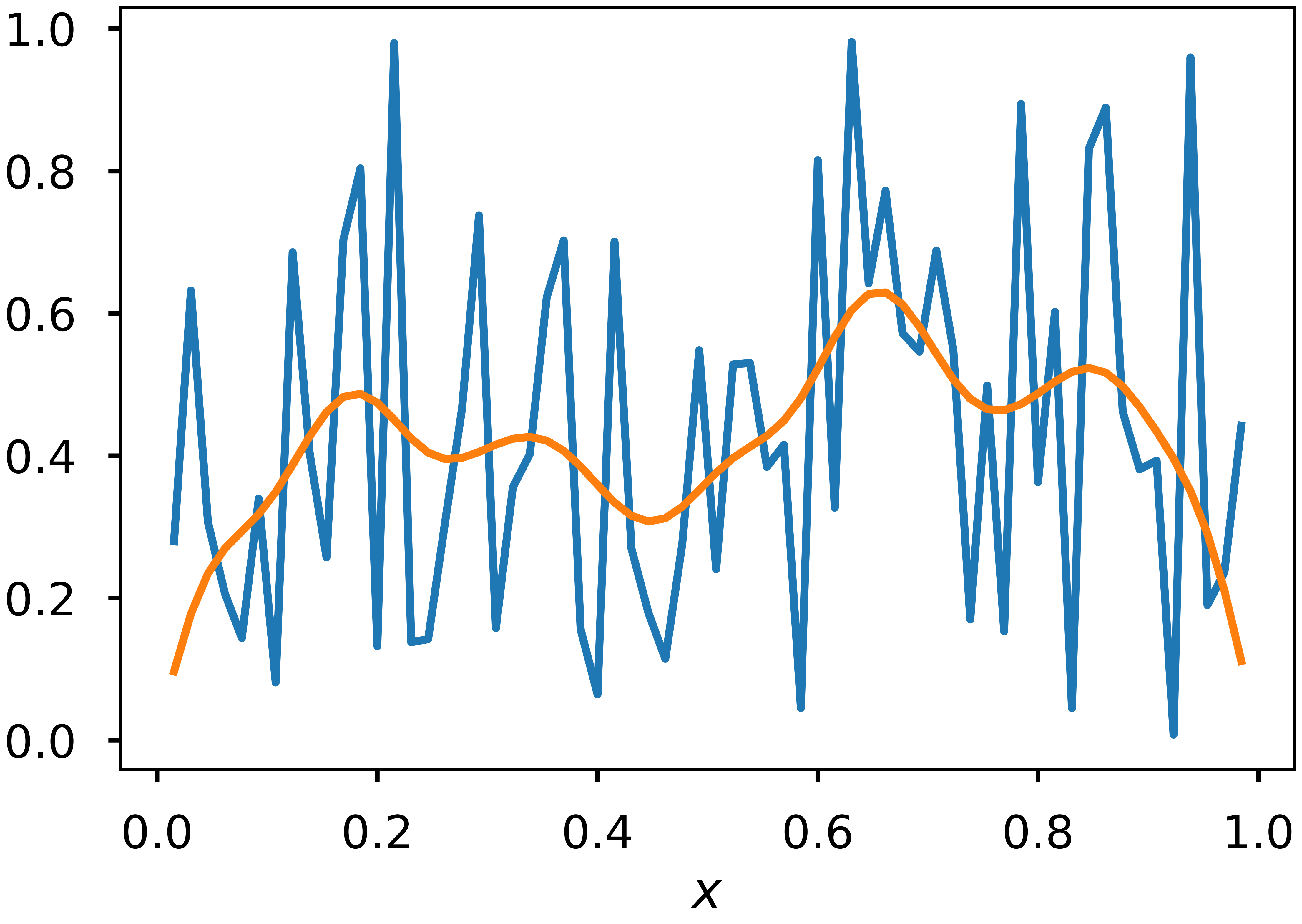
From Jacobi to weighted-Jacobi

- Random initial guess (random error):

$$u \leftarrow u + D^{-1} r$$



$$u \leftarrow u + (2/3) D^{-1} r$$



Weighted Jacobi

- Weighted Jacobi

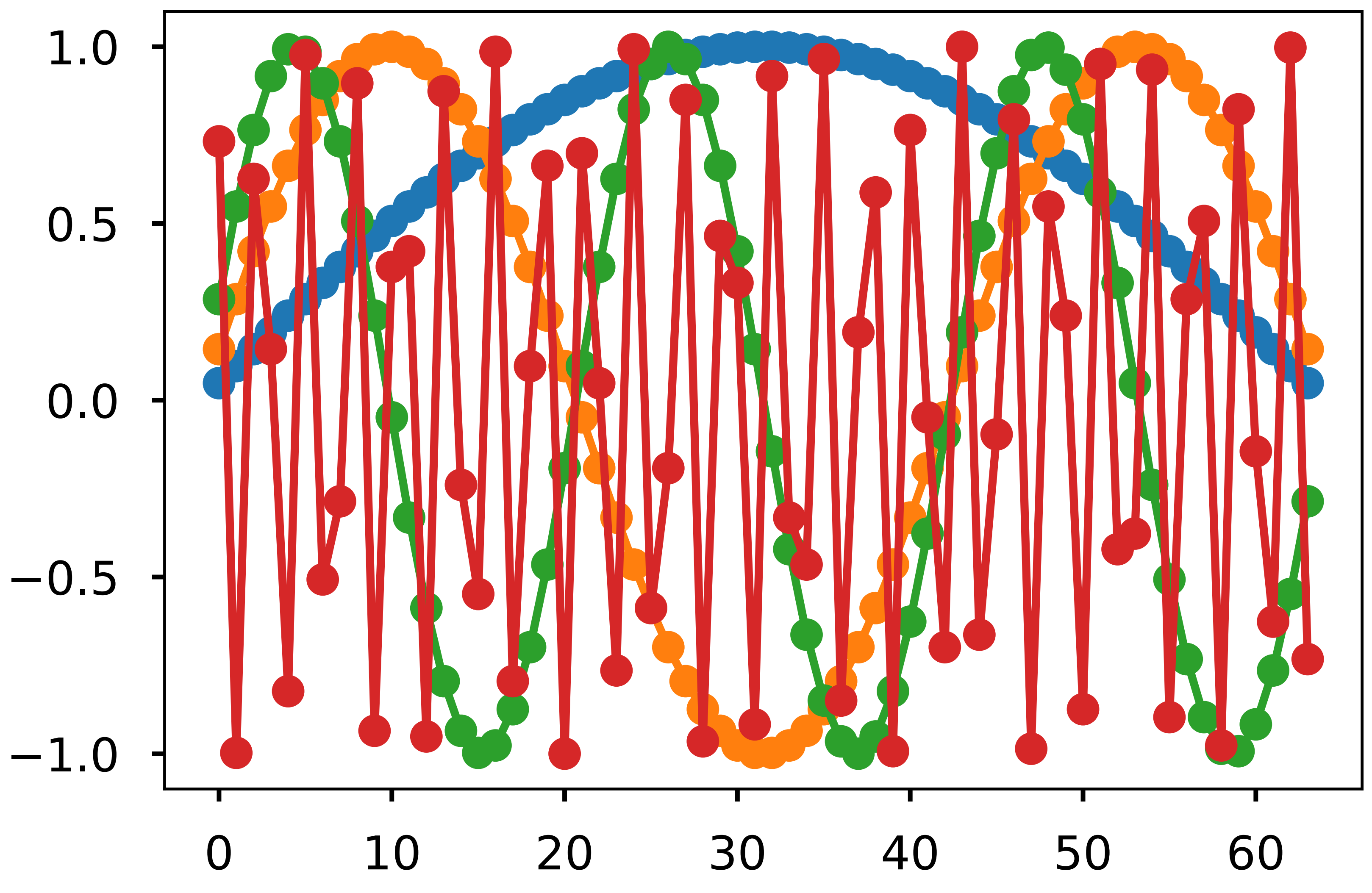
$$u \leftarrow u + (2/3)D^{-1}r$$

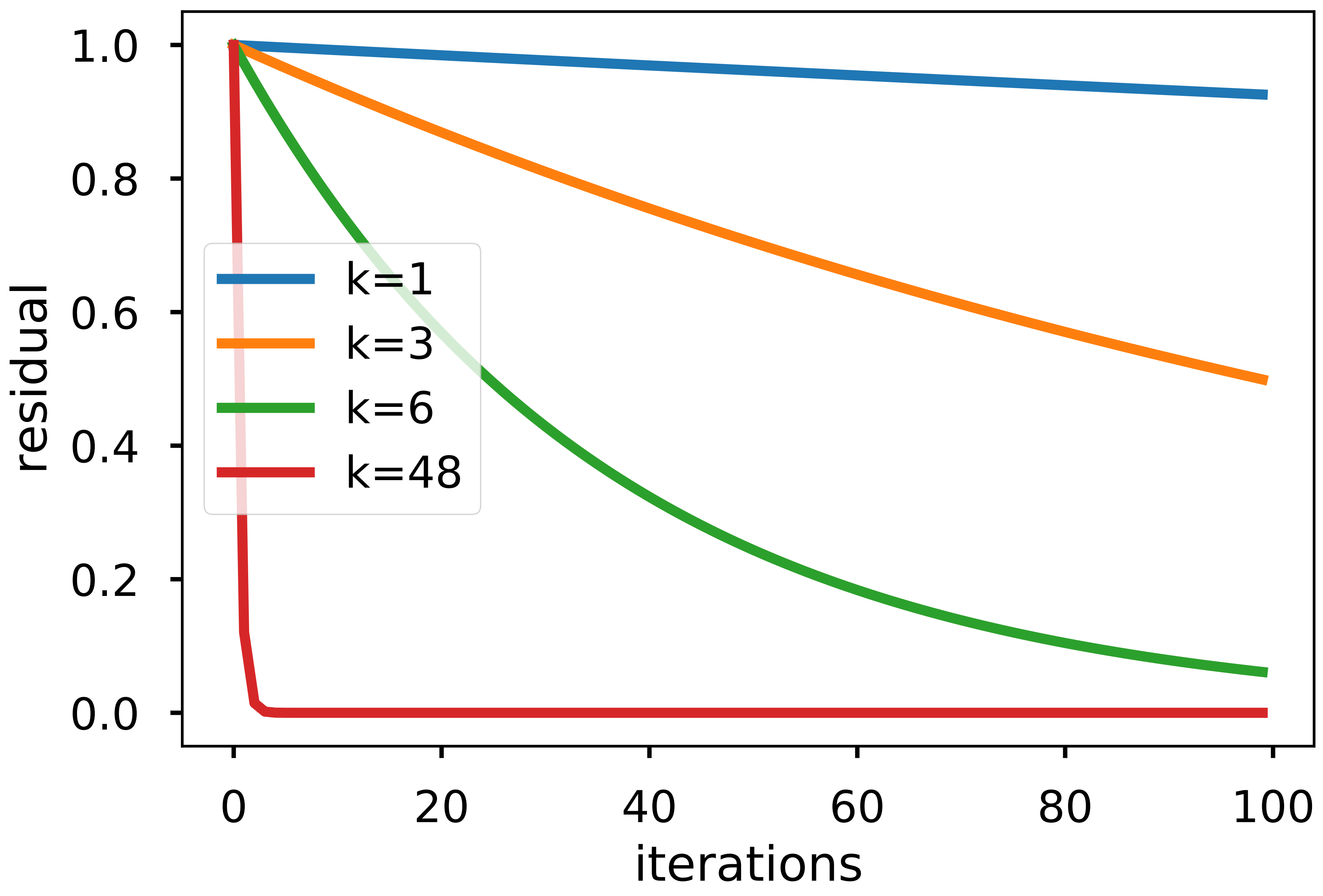
- What does this do to other modes?
  - Are smooth modes damed slowly?
  - What about oscillatory modes?

- Why did we pick 2/3 — that seemed like a lucky guess?!

Weighted Jacobi

- If we picked 4 modes, 1, 3, 6, and 48
- Then smooth modes still dampen less quickly than higher ones





Error Propagation

- Let's consider an initial error

$$e_0 = \sum_{k=1}^{n} c_k v_k$$

- And the weight Jacobi iteration matrix

$$e \leftarrow (I - \omega D^{-1} A)e = Ge$$

- From ν iterations we have

$$G^\nu e_0 = \sum_{k=1}^{n} c_k G^\nu v_k$$

$$= \sum_{k=1}^{n} c_k \lambda_k^\nu v_k$$

- As a result, mode **k** is reduced by the magnitude of λ_k in every pass

Fundamental Theorem of Iteration

$$G = I - M^{-1}A$$

- Convergent ($G^n \to 0$) if and only if $\rho(G) \leq 1$
- Suppose A is s.p.d. $$\frac{\|e_n\|}{\|e_0\|} \leq \|G^n\| \leq \|G\|^n \approx 10^{-d}$$
- How many iterations do we need to guarantee the reduction of the error by d digits?

$$n \approx -\frac{d}{\log_{10} \rho(G)}$$

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Relaxation

- Convergence **factor**

$$\|G\| \text{ or } \rho(G)$$

- Convergence **rate**

$$-\log_{10} \rho(G)$$

- For Jacobi:

$$G = I - D^{-1}A \longrightarrow \lambda_k = 1 - \frac{1}{2} \cdot 4 \cdot \sin^2\left(\frac{k\pi}{2(n+1)}\right)$$

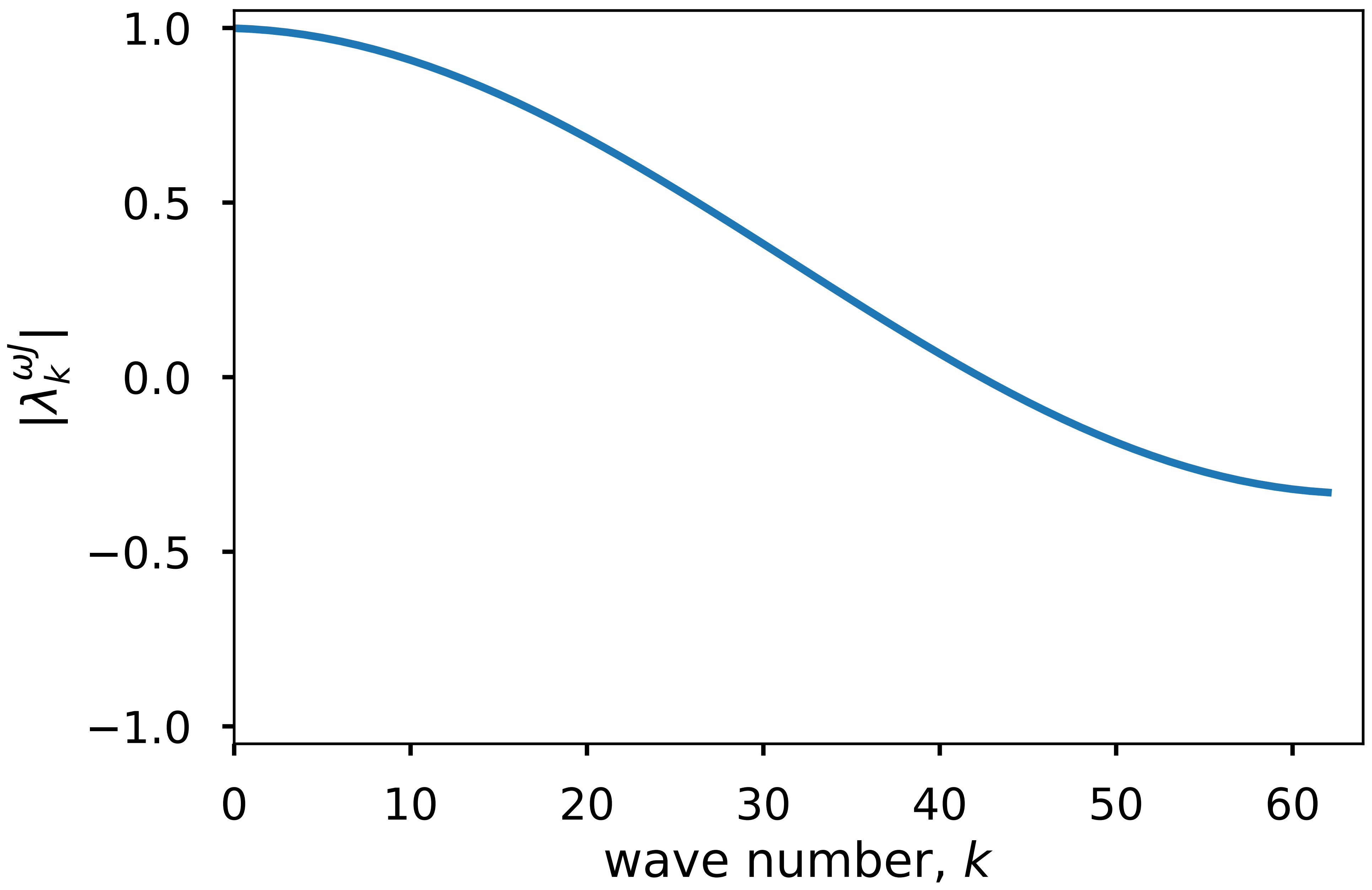
$$G = I - (2/3)D^{-1}A \longrightarrow \lambda_k = 1 - \frac{2}{3} \cdot \frac{1}{2} \cdot 4 \cdot \sin^2\left(\frac{k\pi}{2(n+1)}\right)$$

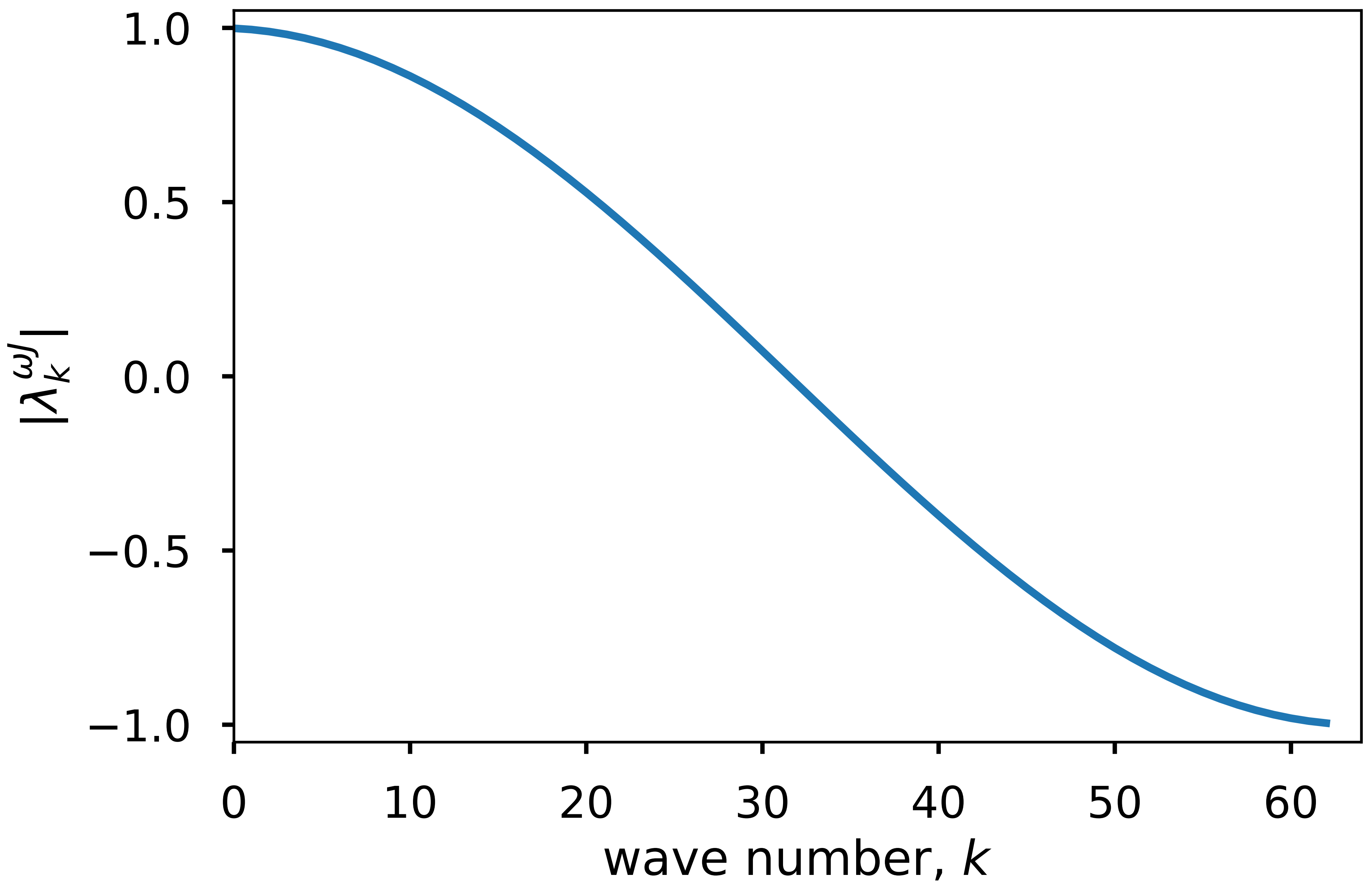
The spectral radius is the same, but…

Weighted Jacobi

- If we look at the spectrum:
  - Weighted Jacobi dampens modes that are highly oscillatory

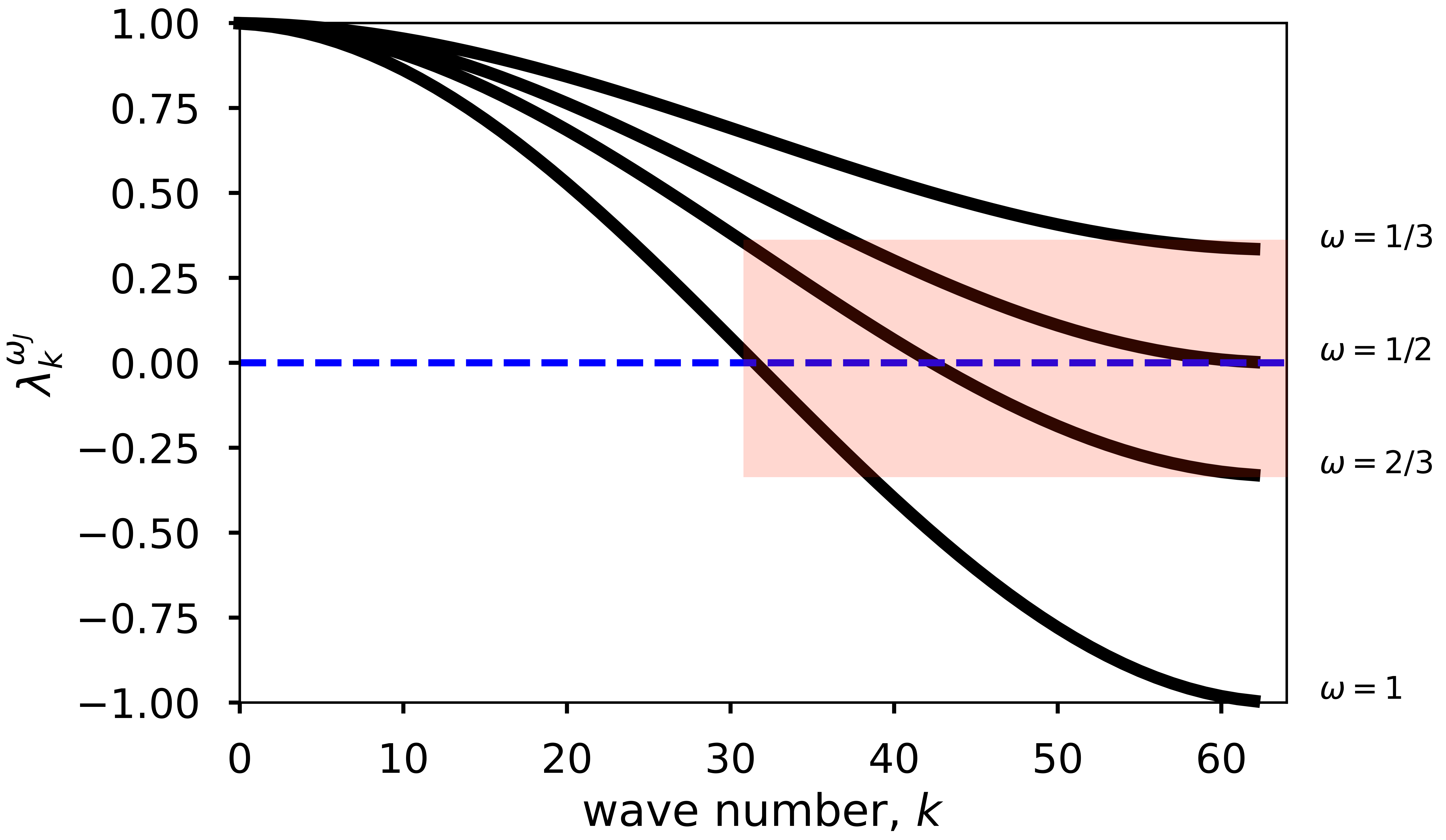
$$\omega = 2/3 \qquad\qquad \omega = 1$$





Weighted Jacobi

- Selecting 2/3 balances the reduction in error in the **high** modes



The multigrid smoothing factor

- The **smoothing factor** of relaxation method G is the maximum magnitude of the upper half of the spectrum:

$$\max_{k \in [n/2, n]} |\lambda_k^G|$$

- A common feature:

Oscillatory modes are quick to converge
Smooth modes are slow to converge

Multigrid Step #1: pick a smoother

- For $\omega = 2/3$

$$|\lambda_{n/2}| = |\lambda_n| = 1/3$$

- For $\omega = 1$

$$|\lambda_{n/2}| = |\lambda_n| = 1$$

- Jacobi is not a smoother (weighted *is*)

An important observation on “smoothness”

- So far, we’ve mainly looked at

$$Au = 0$$

- In general we need to consider

$$Au = f$$

- If we **smooth** with

$$u \leftarrow u + \omega D^{-1} r$$

then the **error** is smooth, not (necessarily the solution).

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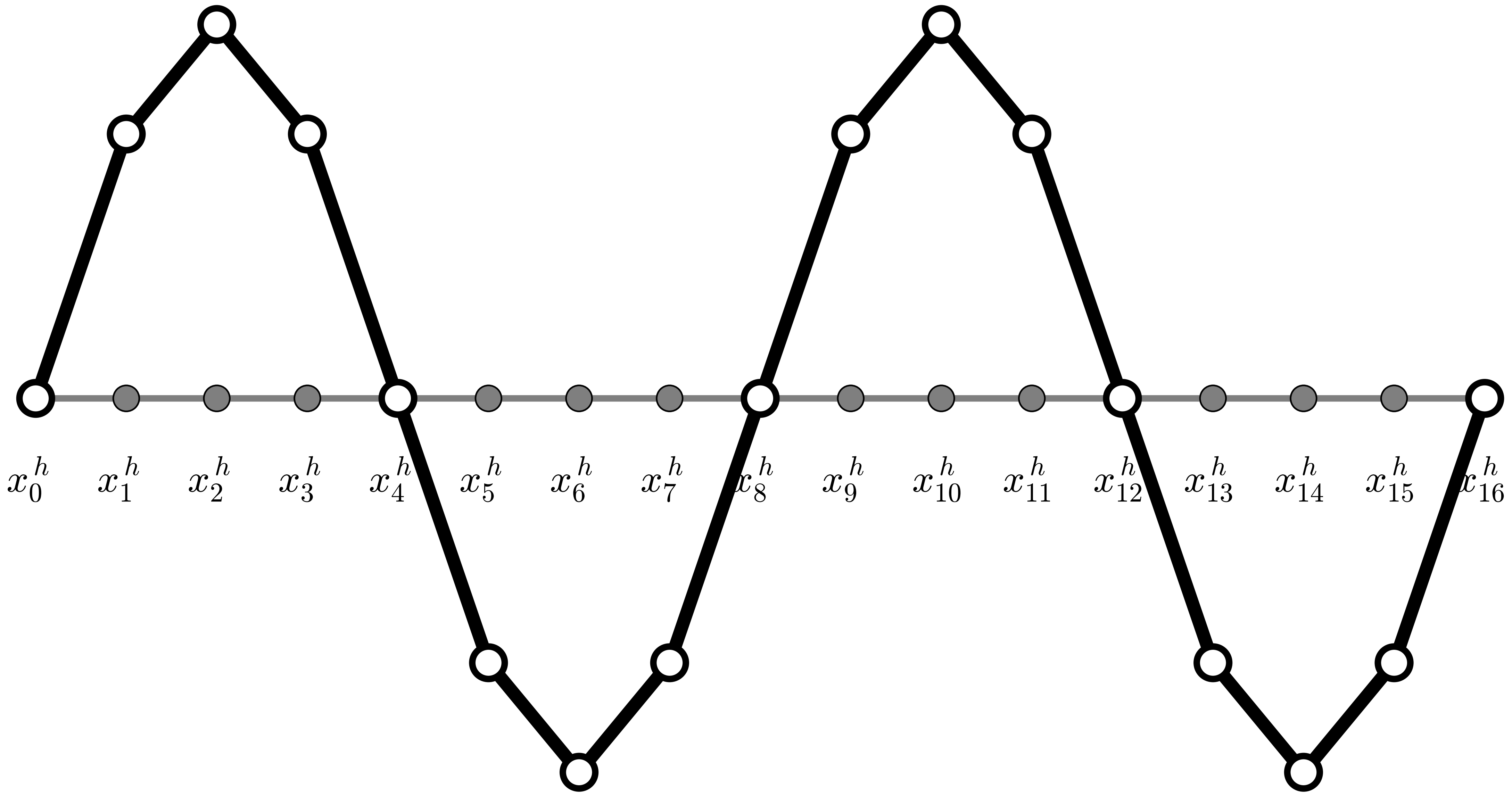
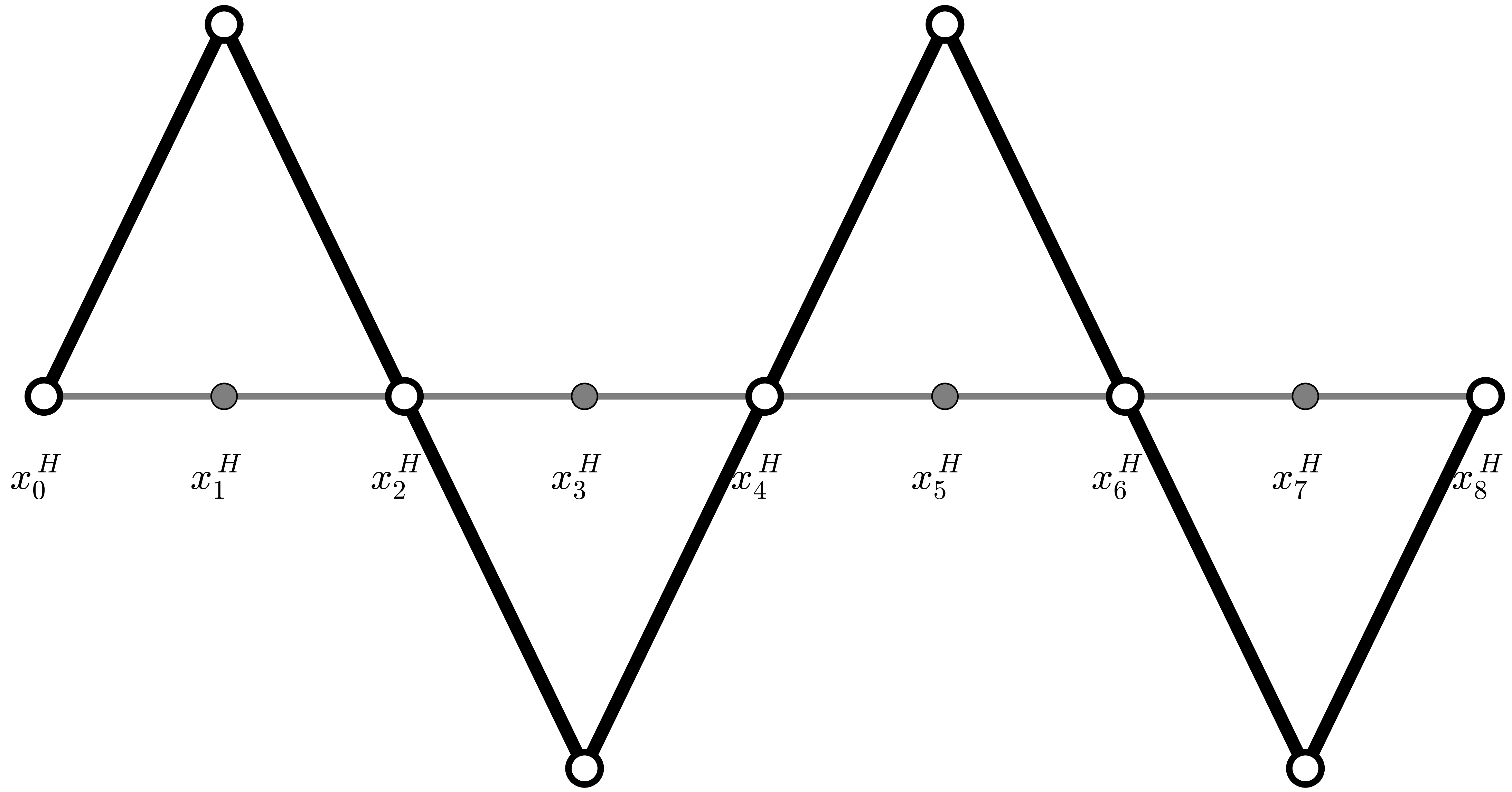
Coarse Grids

- Smooth modes look like oscillatory modes when sampled on a coarse grid
- 4-mode of 15 **versus** 4-mode of 7

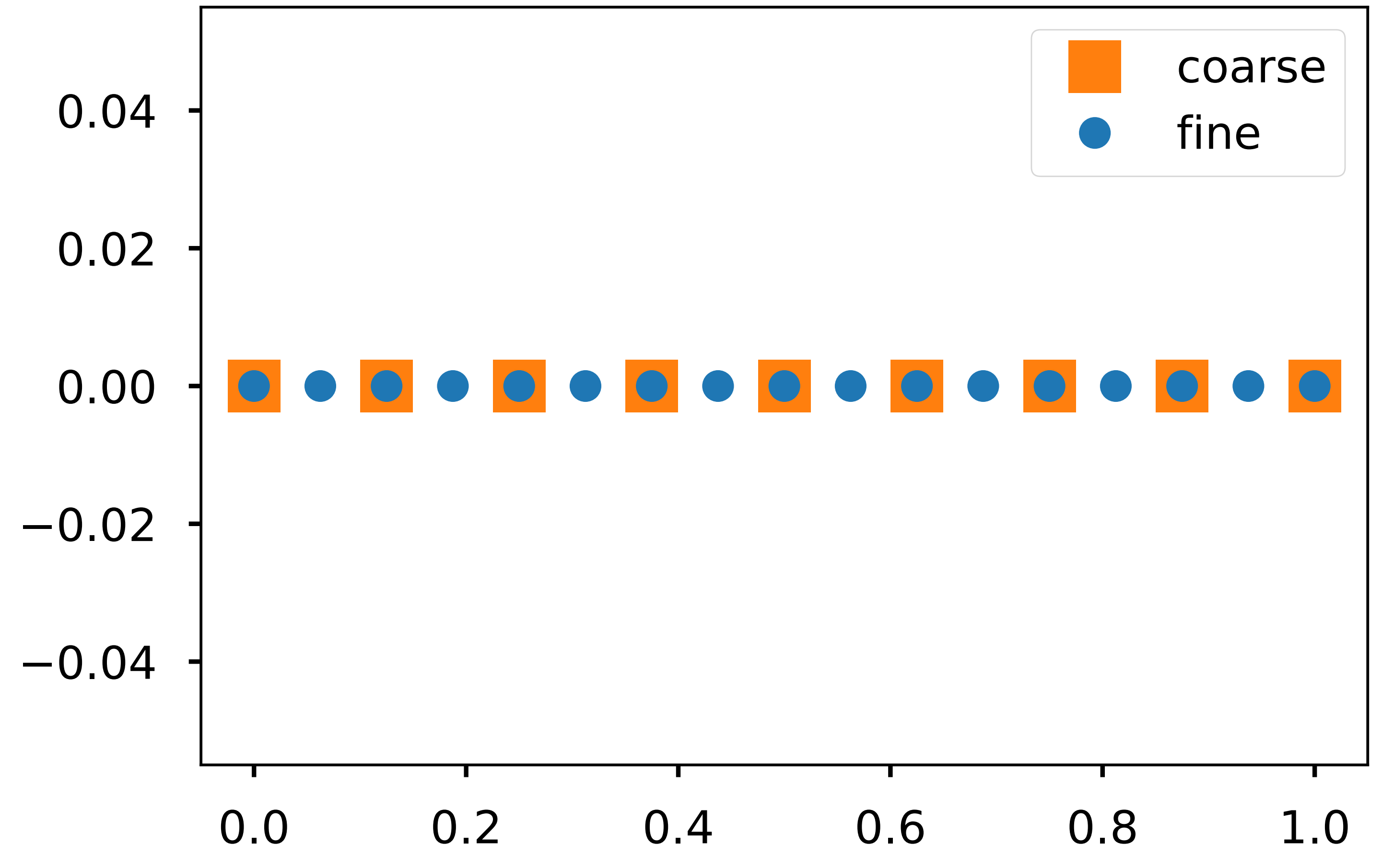


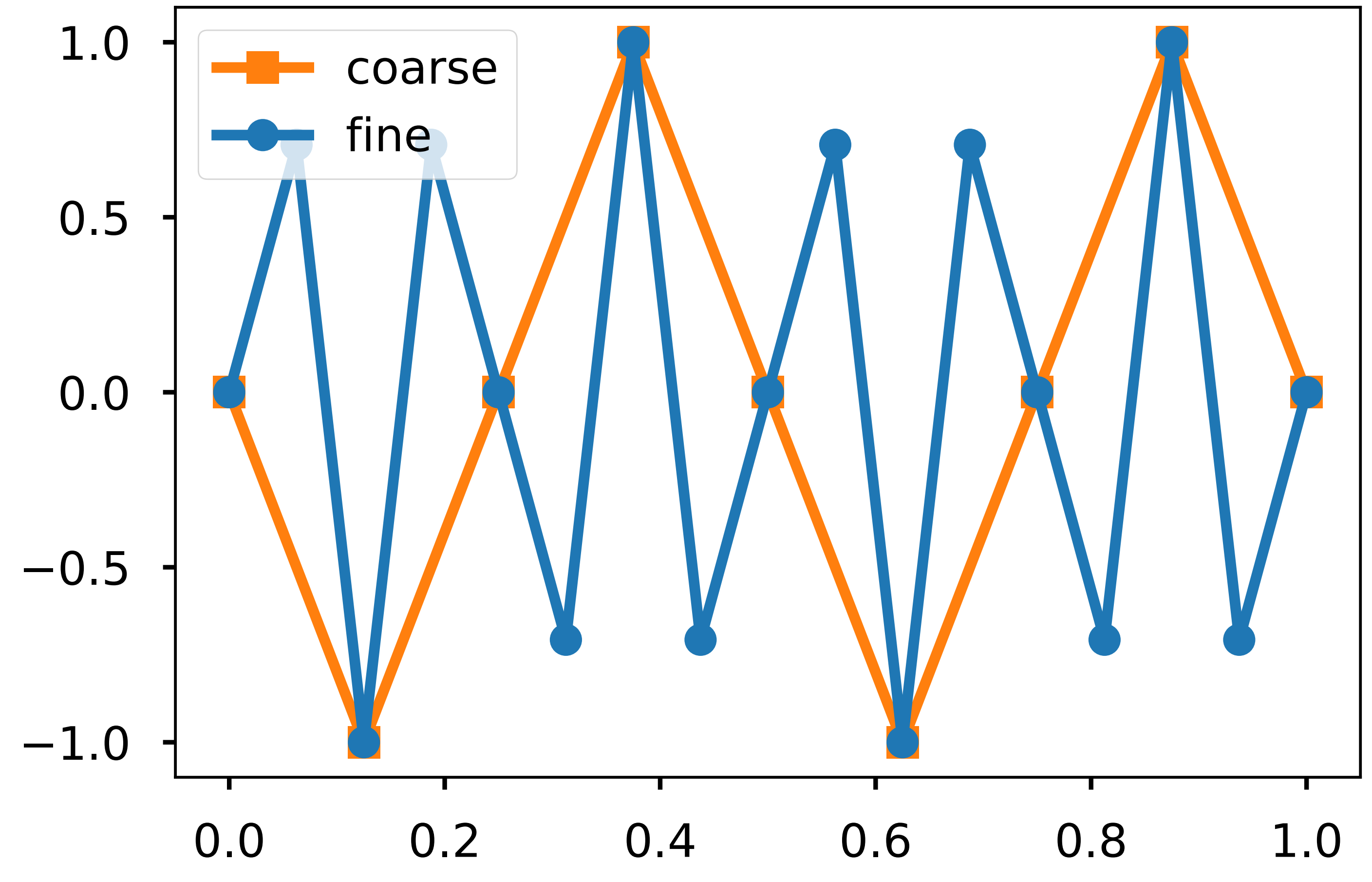


This looks like a "smooth" mode

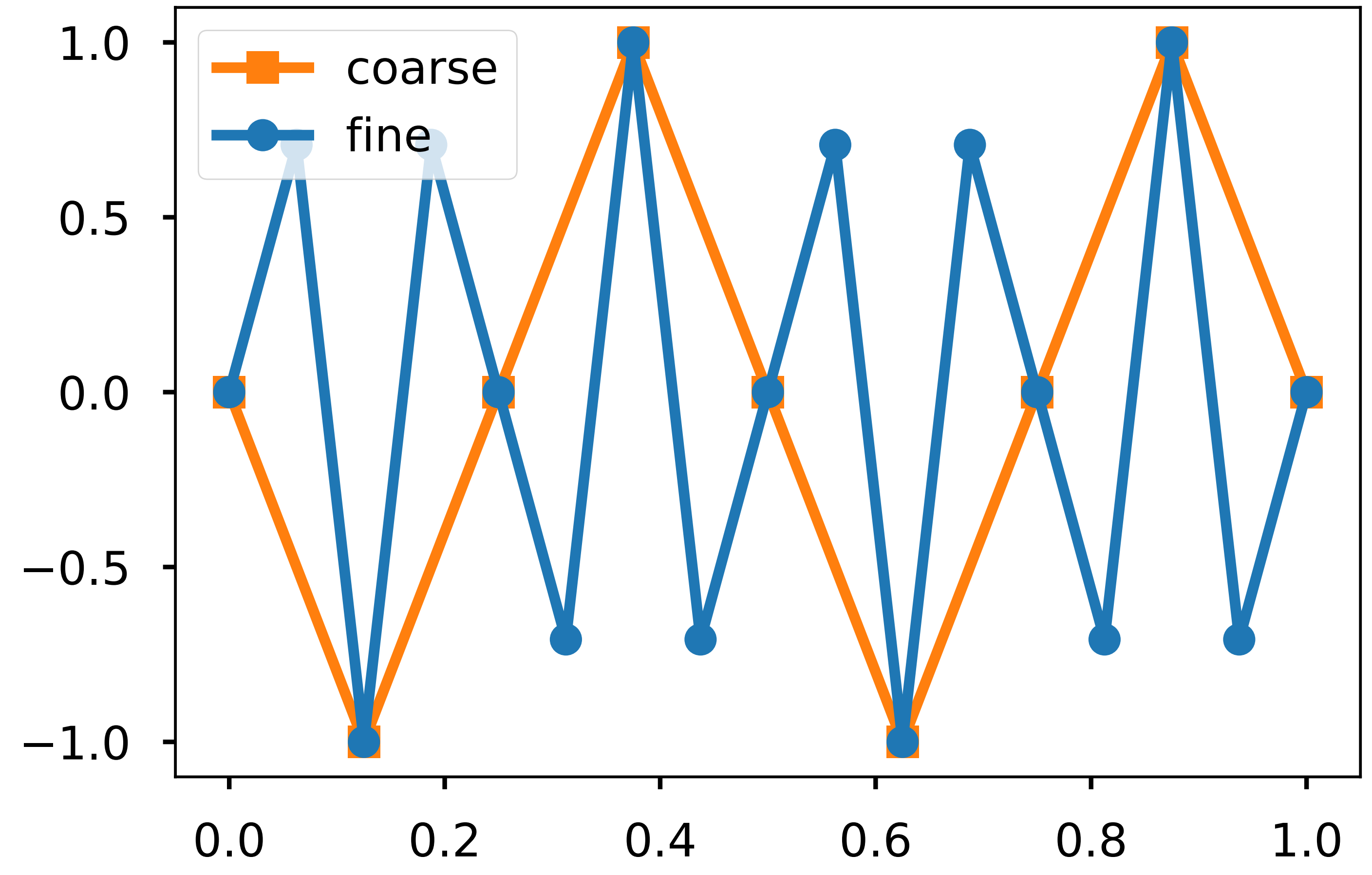
This looks like an "oscillatory" mode

Coarse modes





mode 4 of 15



mode 12 of 15

$$(v_k)_j = \sin\frac{jk\pi}{n+1}$$ Fine mode

$$(v_k)_{2j} = \sin\frac{2jk\pi}{n+1}$$ Fine mode (every other)

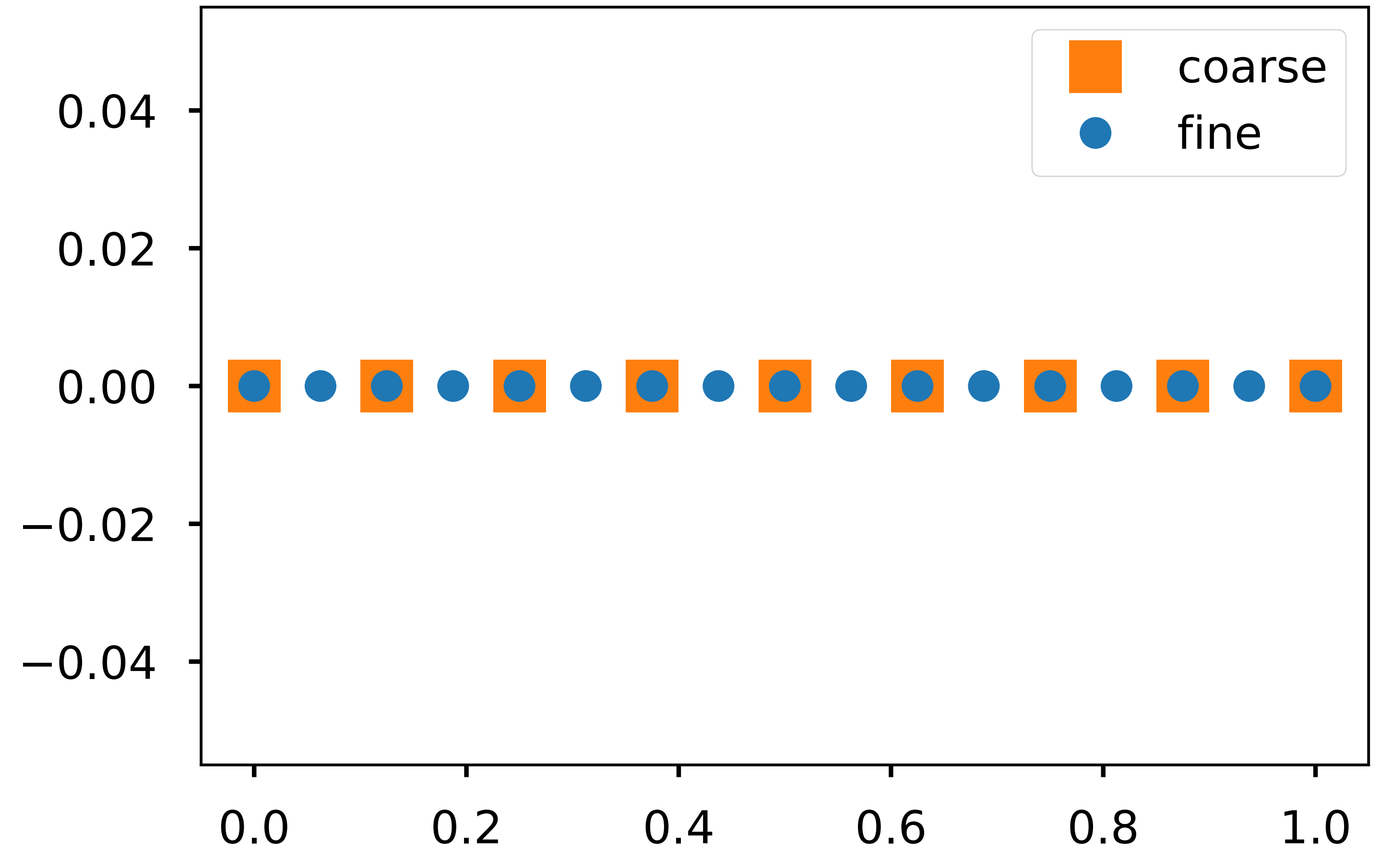
$$= \sin\frac{jk\pi}{(n+1)/2}$$ Coarse mode

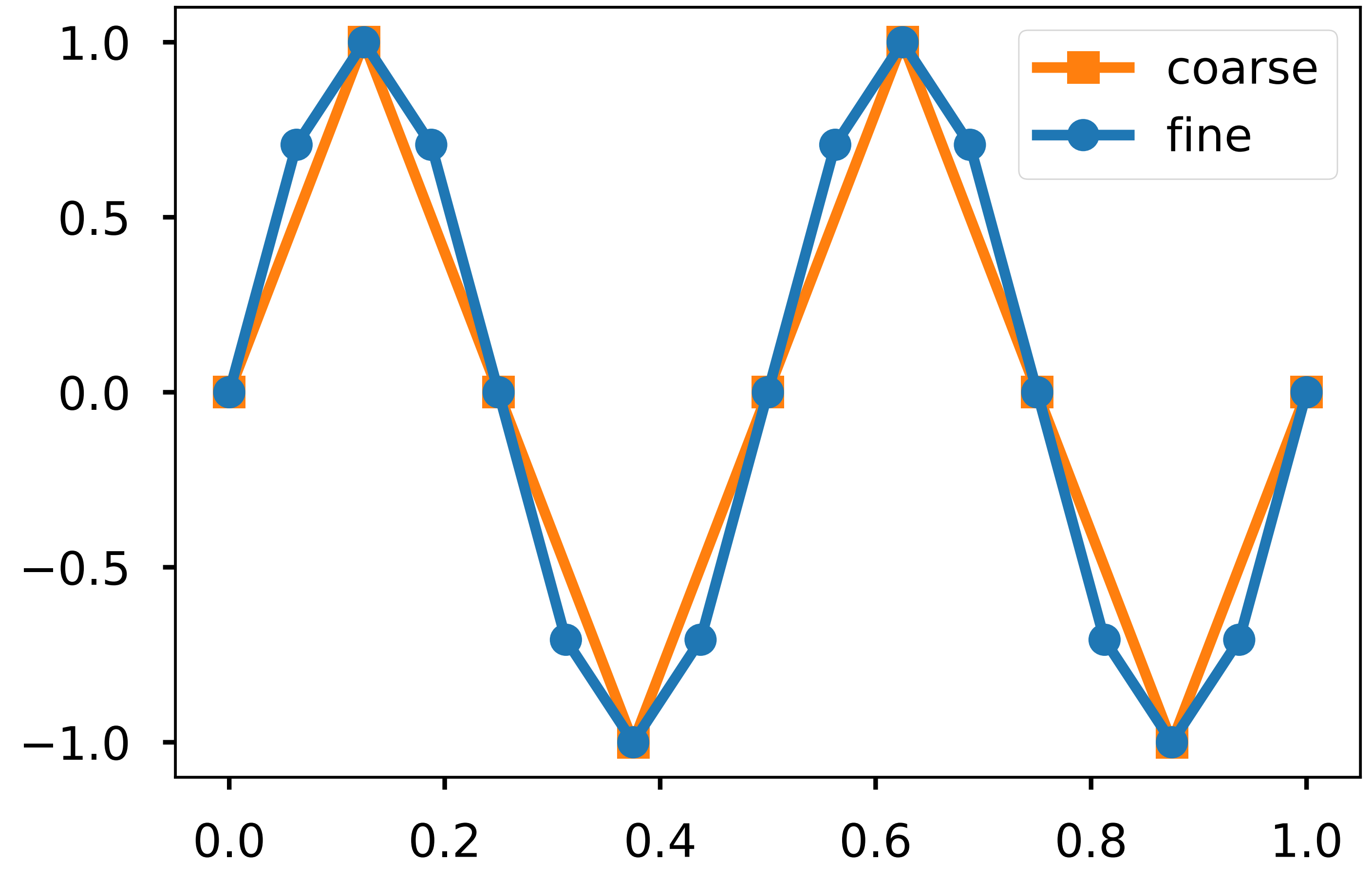
$$= (\hat{v}_k)_j$$

For low modes, k-modes are preserved

Take a look: `04-coarse-modes.ipynb`

Coarse modes





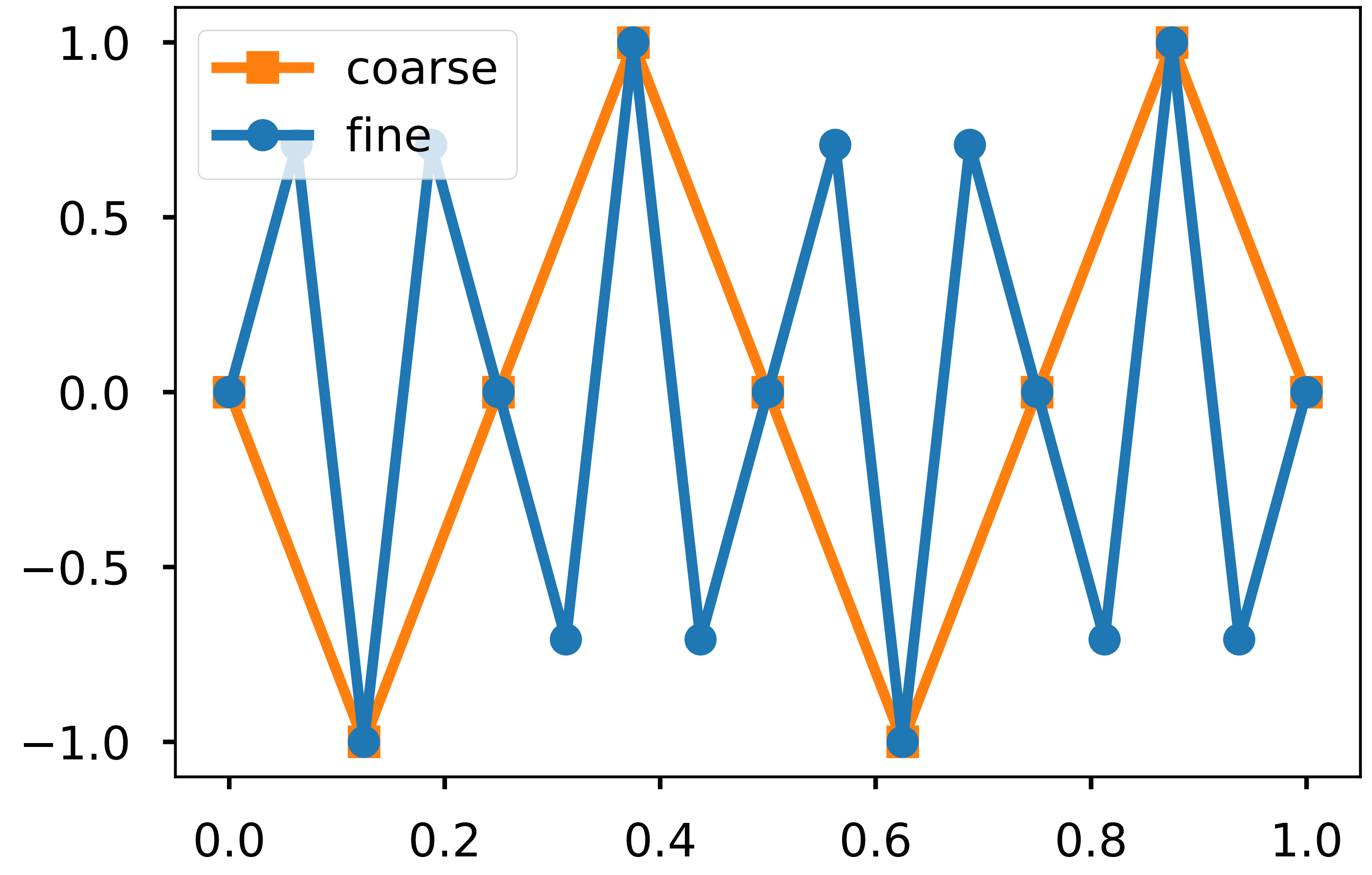
mode 4 of 15

$$(v_k)_{2j} = \sin \frac{2jk\pi}{n+1}$$ Fine mode

$$= -\sin \frac{2j(n-k)\pi}{n+1}$$ Fine mode (every other)

$$= -\sin \frac{j(n-k)\pi}{(n+1)/2}$$ Coarse mode

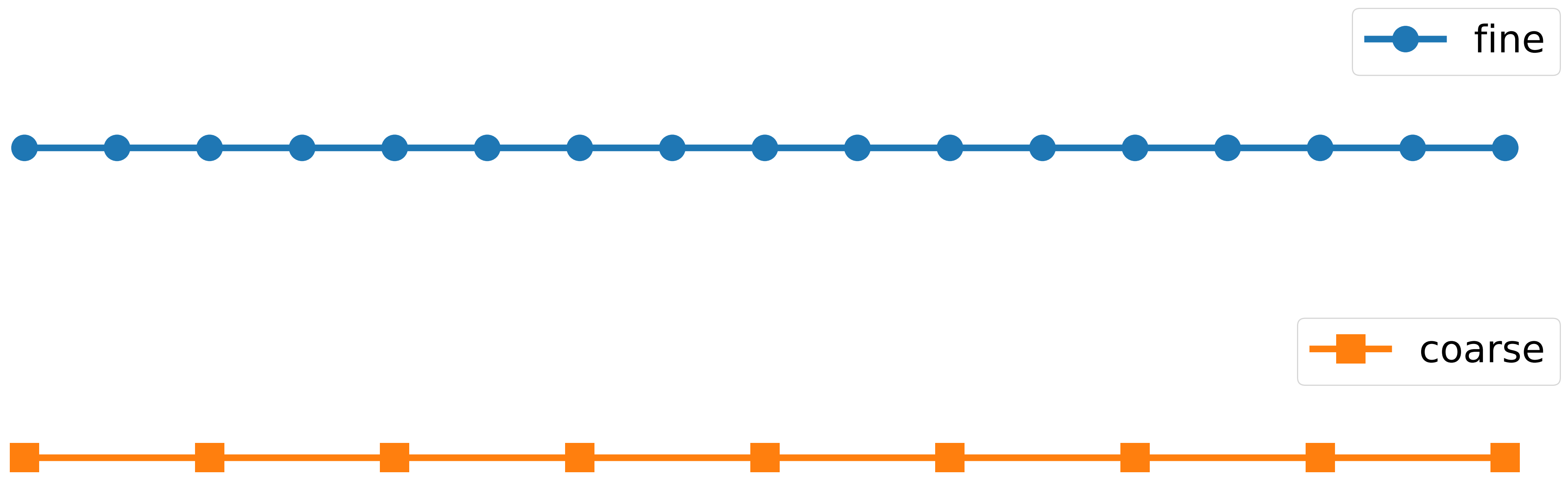
$$= -(\hat{v}_{n-k})_j$$



mode 12 of 15

For high modes, k-modes are *aliased*

Questions to resolve…



- How to transfer between **fine** and **coarse**?
- What do we do “solve” on a coarse grid?

What to use to transfer…

- Return to our projection problem:

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

- If we have a smoothing property, then smoothing a few times will leave smooth error.
- This is A-orthogonal projection onto V
- Let’s construct V from say piecewise cont. linears

Interpolation

- Consider coarse grid

$$\Omega^{2h}$$

- and fine grid

$$\Omega^h$$

- Construct an operator

$$P : \Omega^{2h} \to \Omega^h$$

- Such that

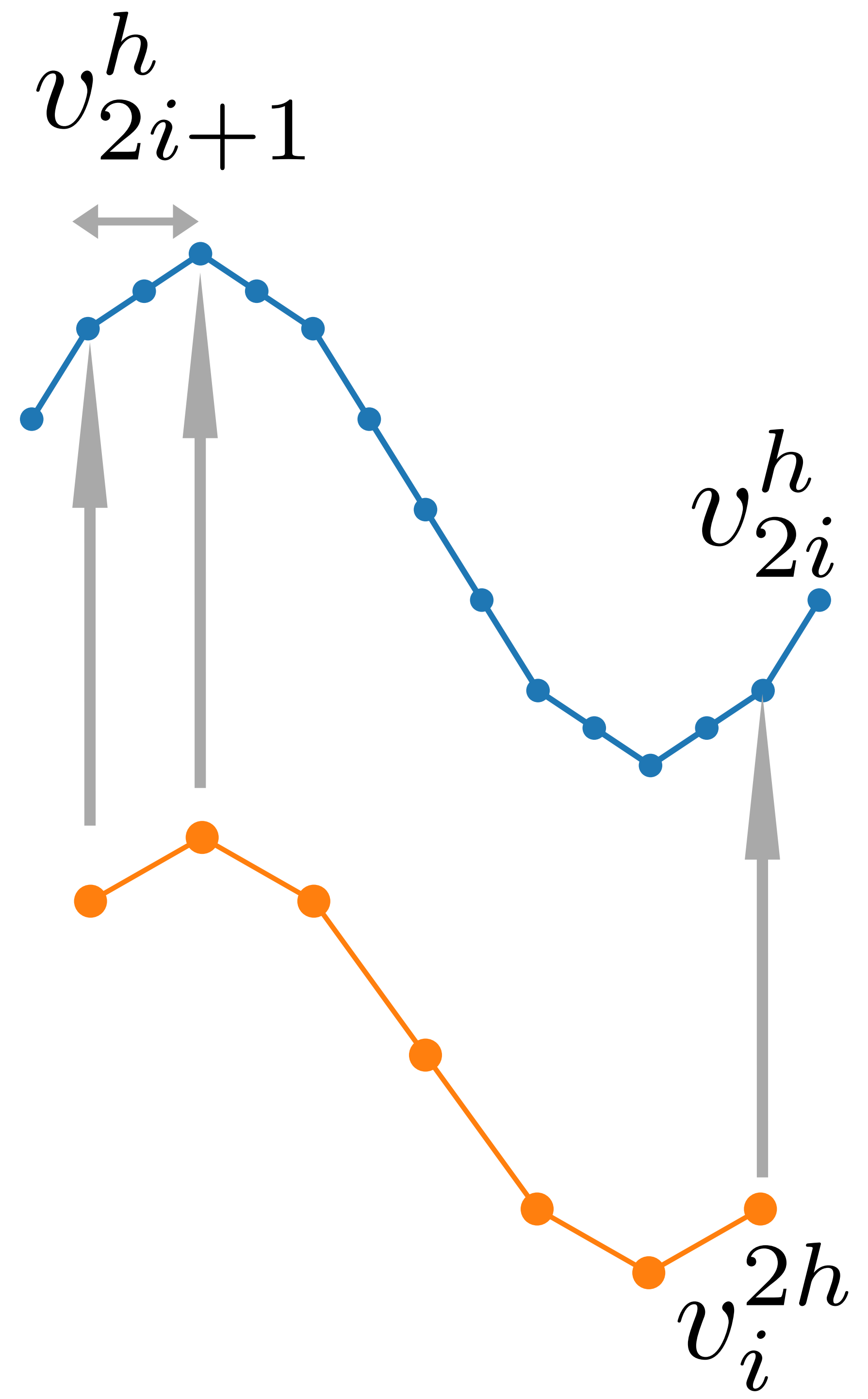
$$Pv^{2h}$$

  is continuous and piecewise linear

Interpolation

$$v^h_{2i} = v^{2h}_i$$ Injection

$$v^h_{2i+1} = \frac{1}{2}(v^{2h}_i + v^{2h}_{i+1})$$ Average (linear interp)

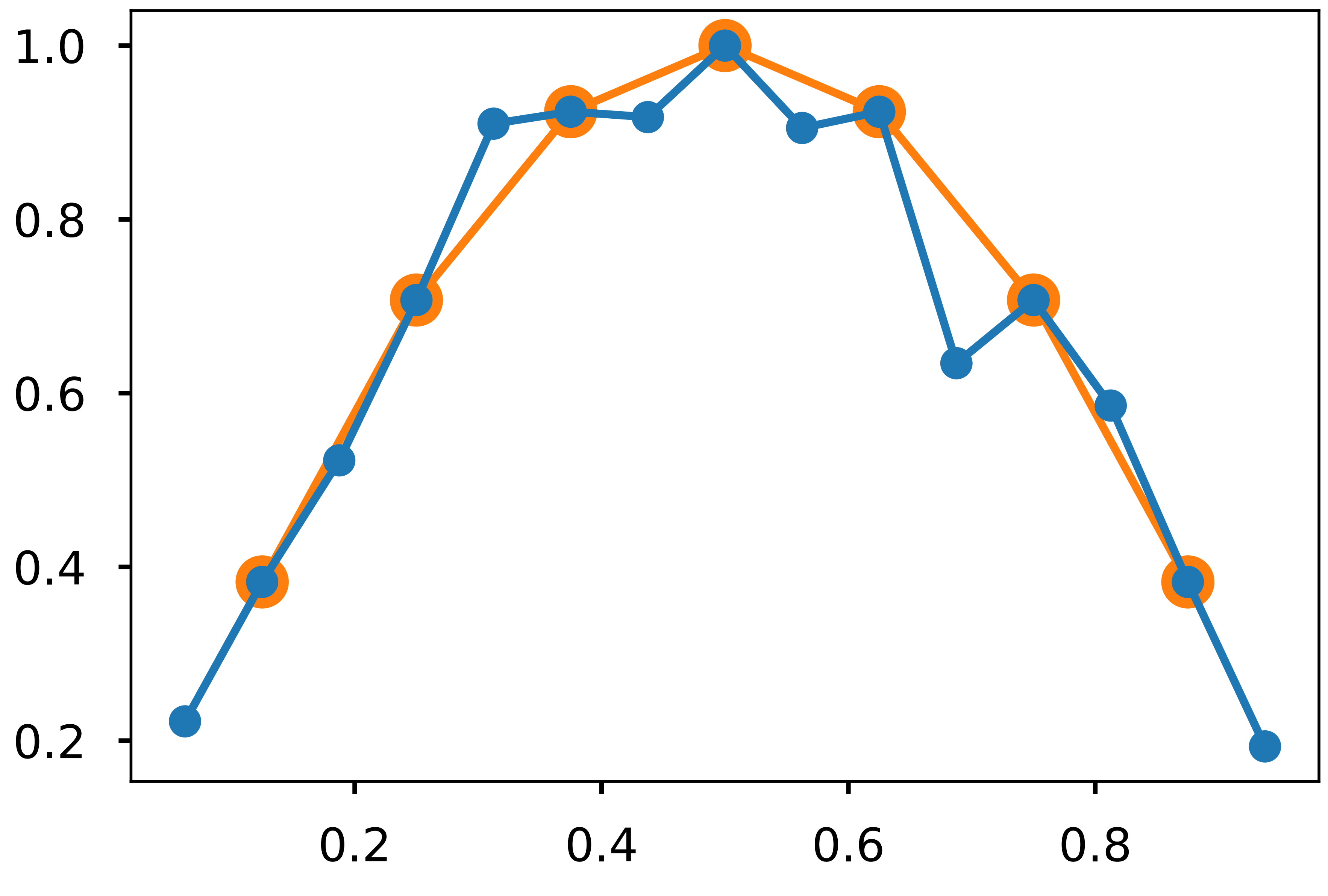


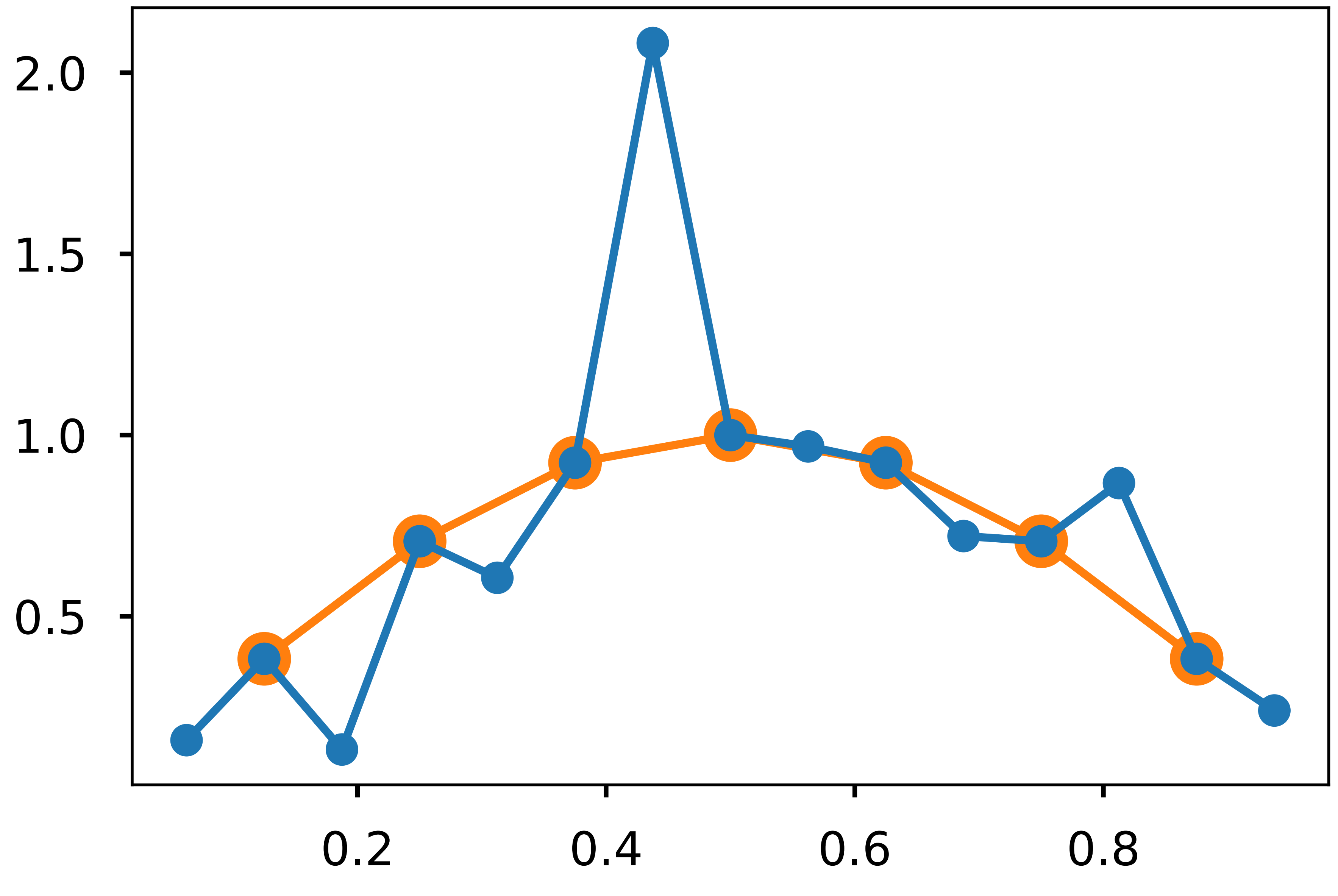
- Values at points common to both grids are reused (injected

Interpolation

- Things that are smooth are interpolated well (smooth error!)

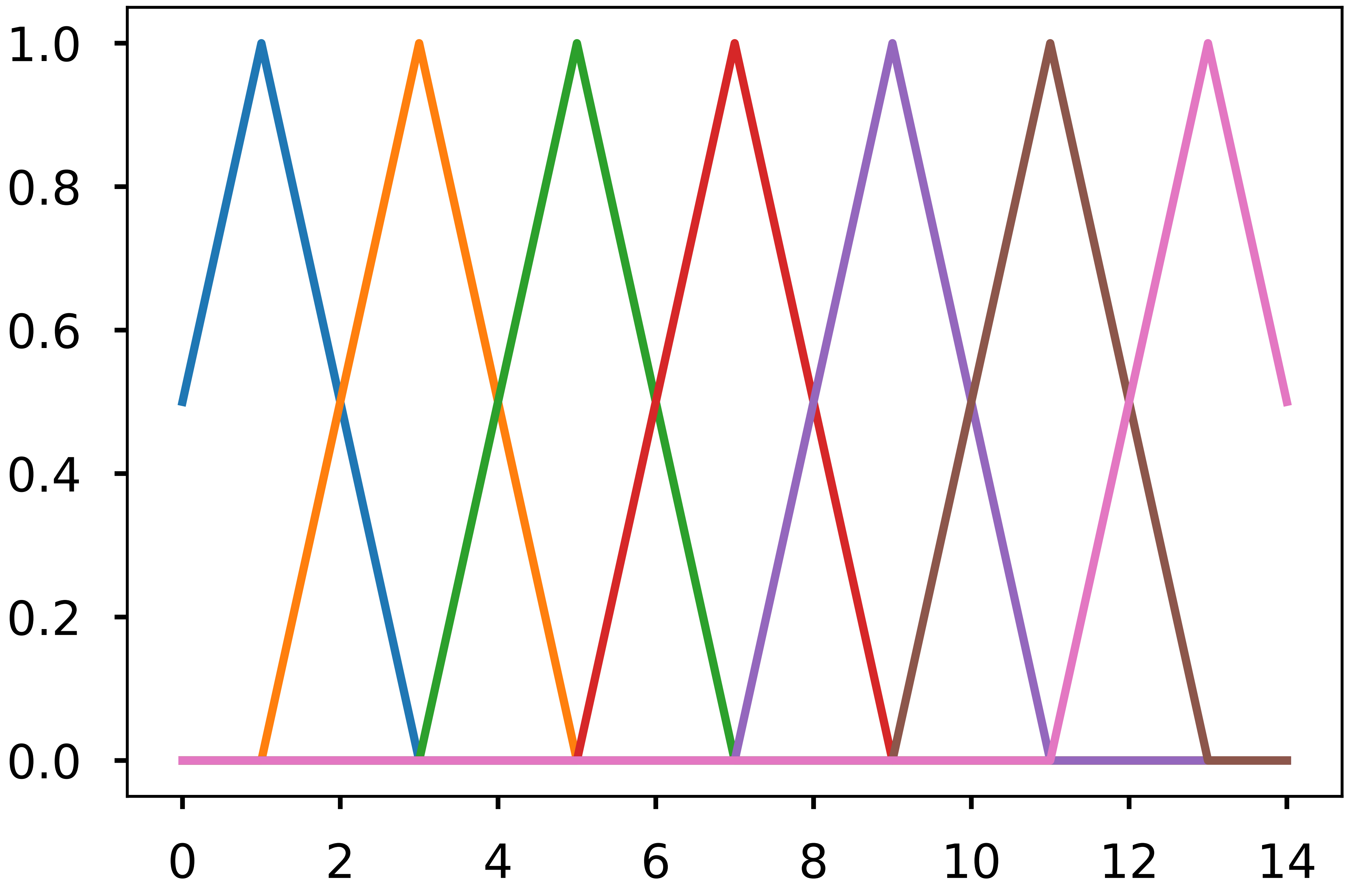
- Things that are not smooth are not interpolated well (un-smoothed error!)





In matrix form

$$P = \frac{1}{2}\begin{bmatrix} 1. & & & & & & \\ 2. & & & & & & \\ 1. & 1. & & & & & \\ & 2. & & & & & \\ & 1. & 1. & & & & \\ & & 2. & & & & \\ & & 1. & 1. & & & \\ & & & 2. & & & \\ & & & 1. & 1. & & \\ & & & & 2. & & \\ & & & & 1. & 1. & \\ & & & & & 2. & \\ & & & & & 1. & 1. \\ & & & & & & 2. \\ & & & & & & 1. \end{bmatrix}$$



- The columns of P are basis functions (right)
- The fine grid vectors, Pv, are linear combinations of these functions
- Notice: P is full rank!

The coarse grid operator

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

(sub)space defined by P

- If V is defined by $span\{P\}$ —or just P,
- Then V^T defines **restriction** as P^T
- And the **coarse level operator** is defined by $P^T A P$

Take a look: `06-multigrid-two-level.ipynb`

Two level method

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

$$x_1 = x_0 + P(P^T A P)^{-1} P^T r_0$$

- Given $\quad x_0$ — What should this be?
- Smooth a few times $\quad x_0 \leftarrow x_0 + \omega D^{-1} A r_0$ — How many times
- Form residual $\quad r_0 = b - A x_0$
- Restrict the residual $\quad P^T r_0$
- Solve the coarse problem $\quad P^T A P \hat{e}_c = P^T r_0 \qquad \boxed{A_c \hat{e}_c = r_c}$ — What does this mean?
- Interpolate the approx error $\quad P \hat{e}_c$ — What are we interpolating?
- Correct the initial guess $\quad x_0 + P \hat{e}_0$ — Are we done?

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Notes on variants

- An alternative to restriction, is injection

$$\left[\begin{array}{ccccccccccccccc}
0 & 1 & 0 & & & & & & & & & & & & \\
 & & 0 & 1 & 0 & & & & & & & & & & \\
 & & & & 0 & 1 & 0 & & & & & & & & \\
 & & & & & & 0 & 1 & 0 & & & & & & \\
 & & & & & & & & 0 & 1 & 0 & & & & \\
 & & & & & & & & & & 0 & 1 & 0 & & \\
 & & & & & & & & & & & & 0 & 1 &
\end{array}\right]$$

- Or a weighted transpose of linear interpolation (to restrict constants exactly)

$$\frac{1}{4}\left[\begin{array}{ccccccccccccccc}
1 & 2 & 1 & & & & & & & & & & & & \\
 & & 1 & 2 & 1 & & & & & & & & & & \\
 & & & & 1 & 2 & 1 & & & & & & & & \\
 & & & & & & 1 & 2 & 1 & & & & & & \\
 & & & & & & & & 1 & 2 & 1 & & & & \\
 & & & & & & & & & & 1 & 2 & 1 & & \\
 & & & & & & & & & & & & 1 & 2 & 1
\end{array}\right]$$

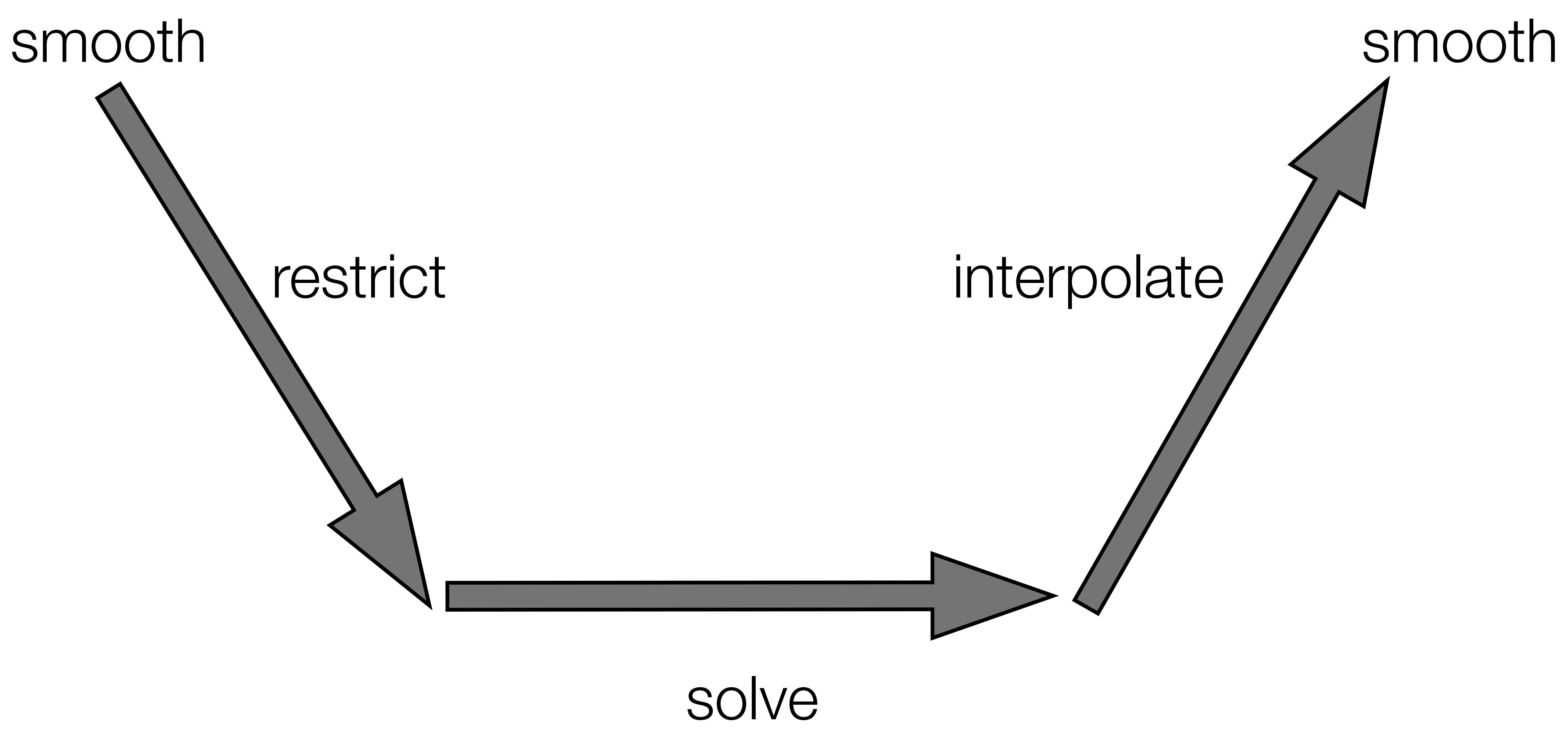
- An alternative to $A_c = P^T A_h P$ is to rediscretize $A_c = A_{2h}$

$$x_1 \leftarrow x_0 + P A_{2h}^{-1} R r_0$$

Algorithm: two-level multigrid

Input: initial guess

1. Smooth ν_{pre} times on $Au = f$
2. Compute $r = f - Au$
3. Compute $r_c = Rr$
4. Solve $A_c e_c = r_c$
5. Interpolate $\hat{e} = Pe_c$
6. Correct $u \leftarrow u + \hat{e}$
7. Smooth ν_{post} times on $Au = f$



A two-level “V” cycle

How Accurate is Multigrid?

- Consider the exact solution to the PDE u^*

$$-u'' = f$$

- The exact solution to the **discrete** problem u_h^*

$$Au = b$$

- The approximate **discrete** solution $u_h \approx u_h^*$

- Define

$u^* - u_h^*$ Discretization error

$u_h^* - u_h$ Algebraic error

How Accurate is Multigrid?

- Would like the error bounded

$$\|u^* - u_h\| \leq \|u^* - u_h^*\| + \|u_h^* - u^h\|$$
$$\leq \varepsilon$$

- To achieve this,

  — force the discretization (grid space) so that

$$\|u^* - u_h^*\| \leq ch^2 \leq \frac{\varepsilon}{2}$$

  — and the algebraic error (convergence) up to the discretization error

$$\|u_h^* - u^h\| \leq \frac{\varepsilon}{2}$$

Multigrid convergence

- Convergence factor of a cycle — the factor by which the error (residual) is reduced (in some norm) in each iteration

$$\gamma$$

  …assume this is independent of n

- Wish to have m cycles such that

$$\gamma^m \sim \mathcal{O}(n^{-2}) \leq \frac{\varepsilon}{2}$$

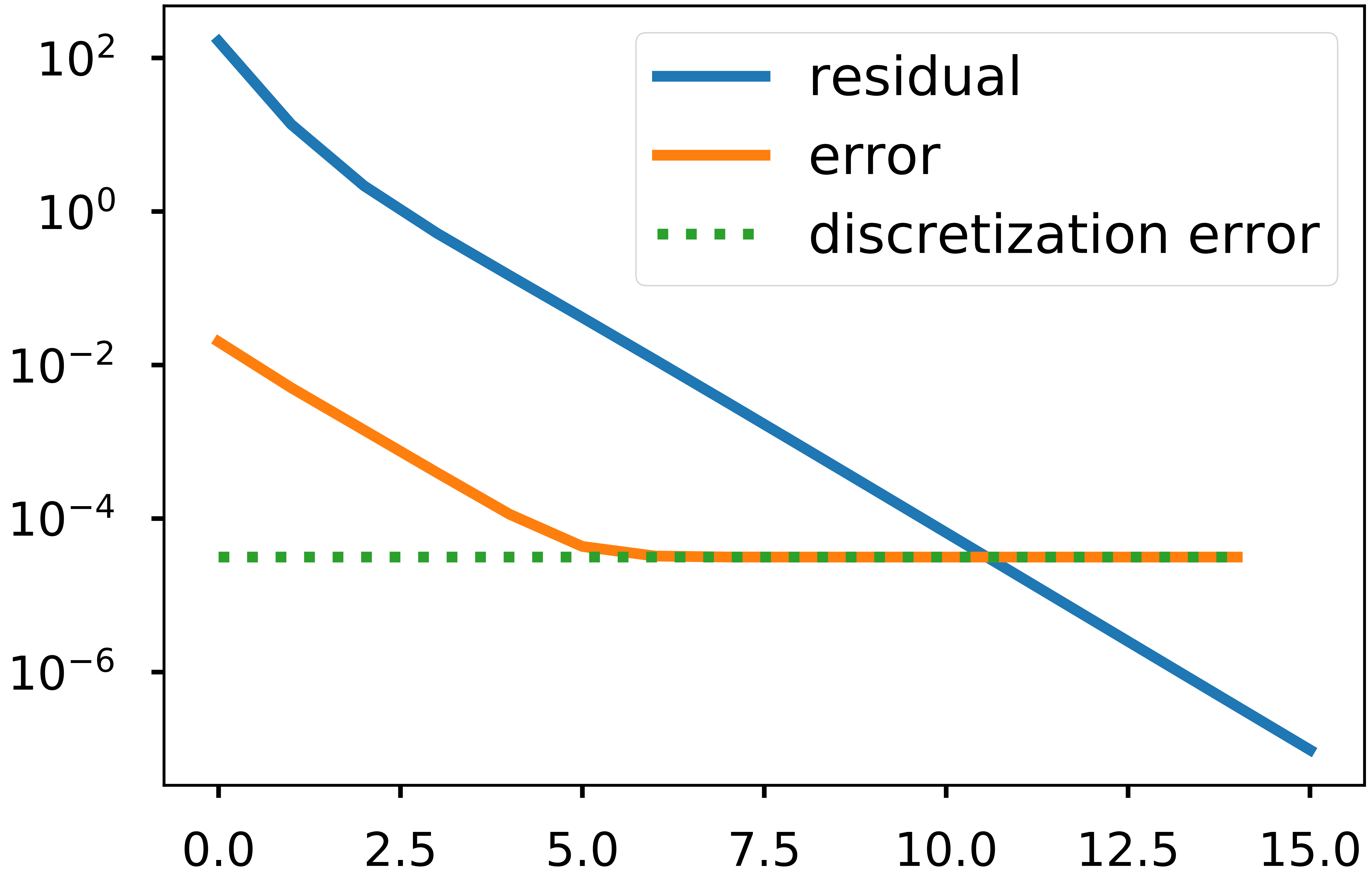
- Then we need

$$m \sim \mathcal{O}(\log n)$$

  cycles

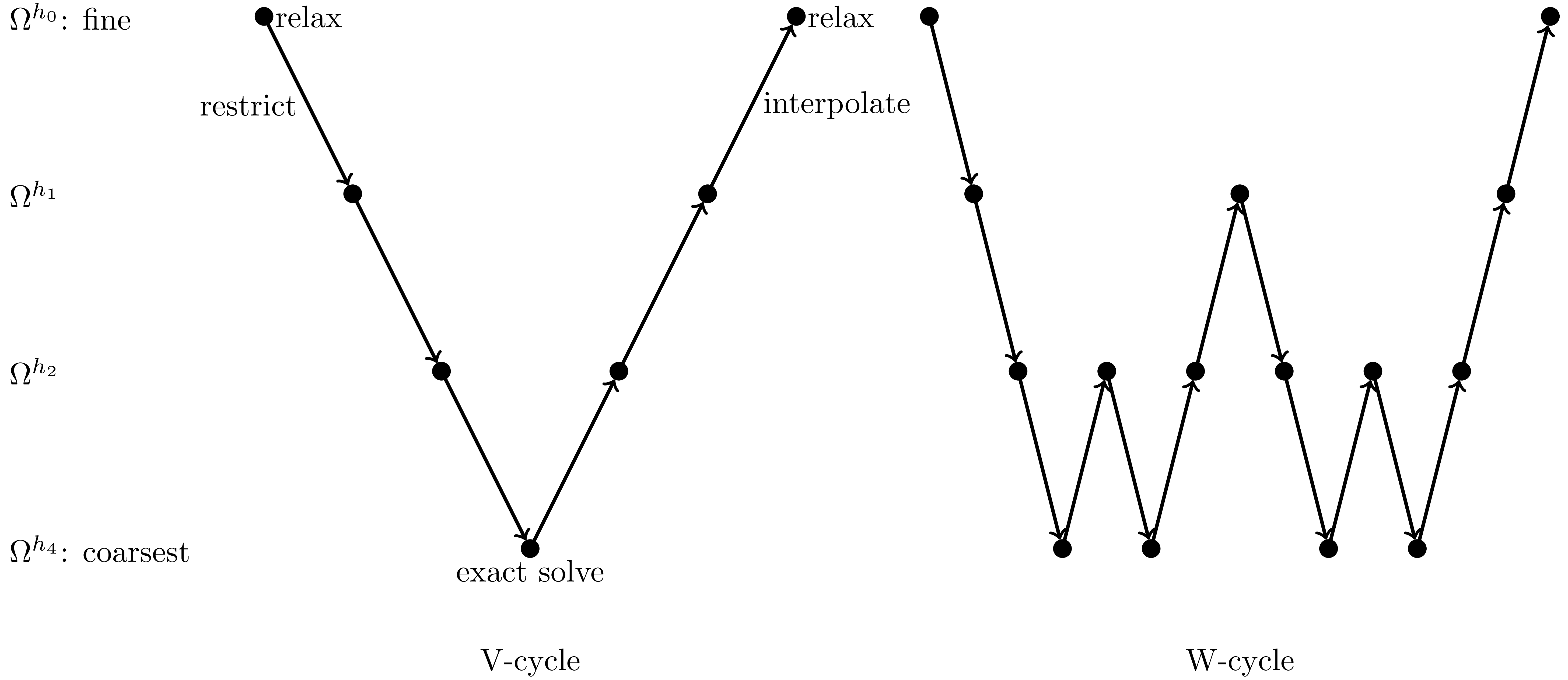
Take a look: `08-multigrid-v-cycle.ipynb`

Algebraic error



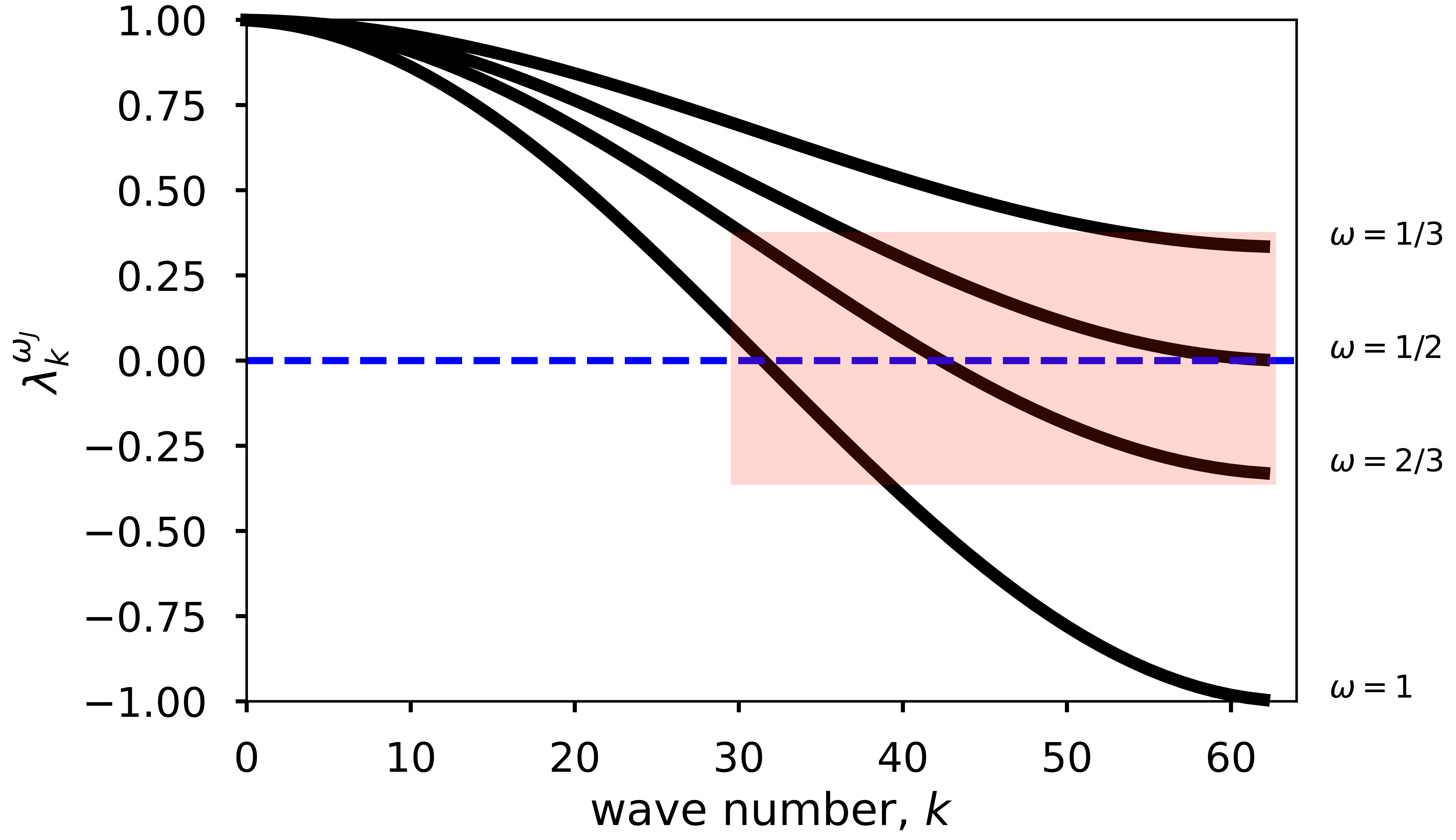
- The total error is limited by the discretization error

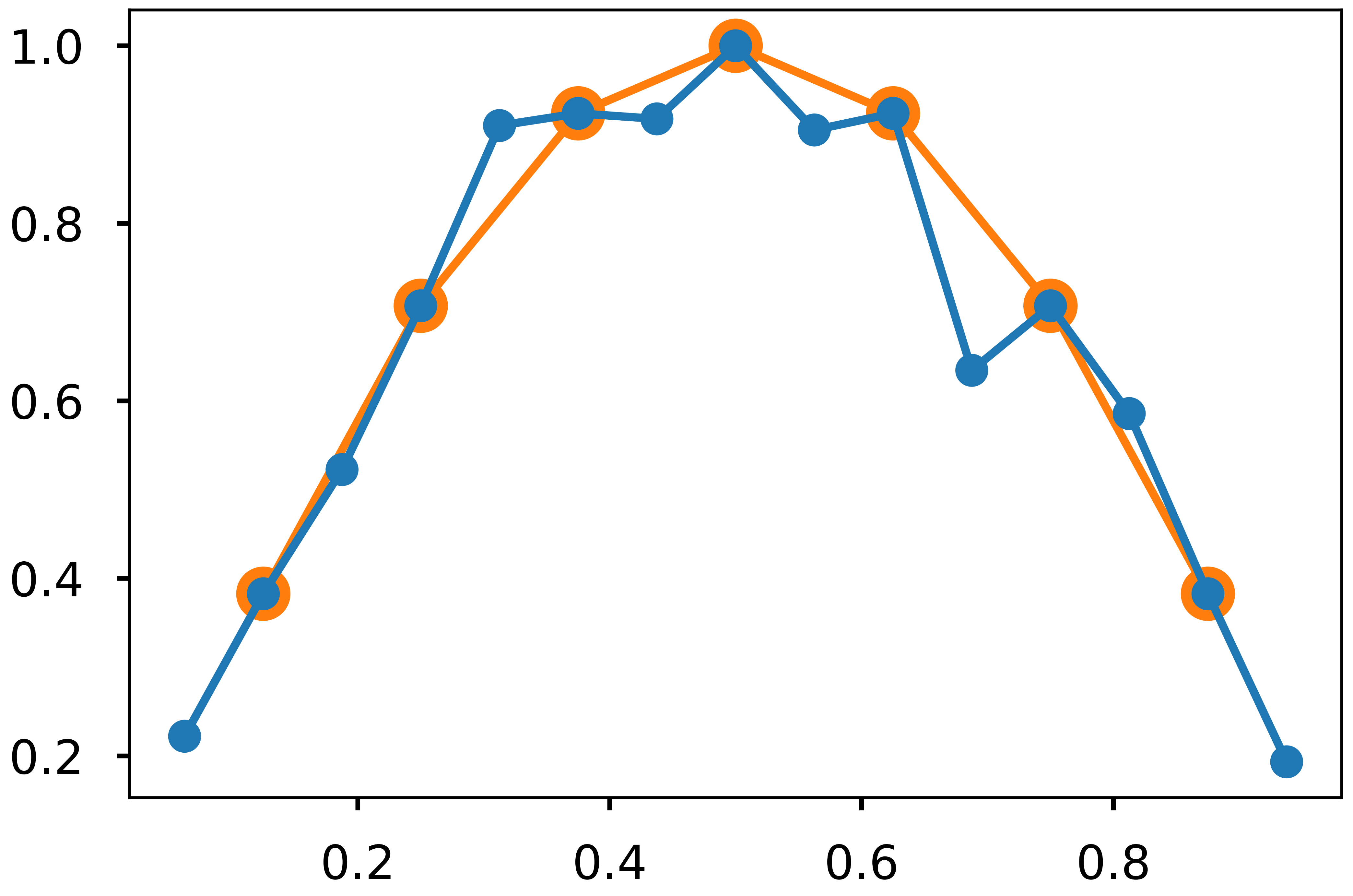
The Multigrid V-Cycle and W-Cycle



- Two-grid cycle can expose issues with coarser interpolation
- W-Cycle can account for inadequate coarser level solves
- **Exact solve?** Usually a pseudo-inverse

Recap…



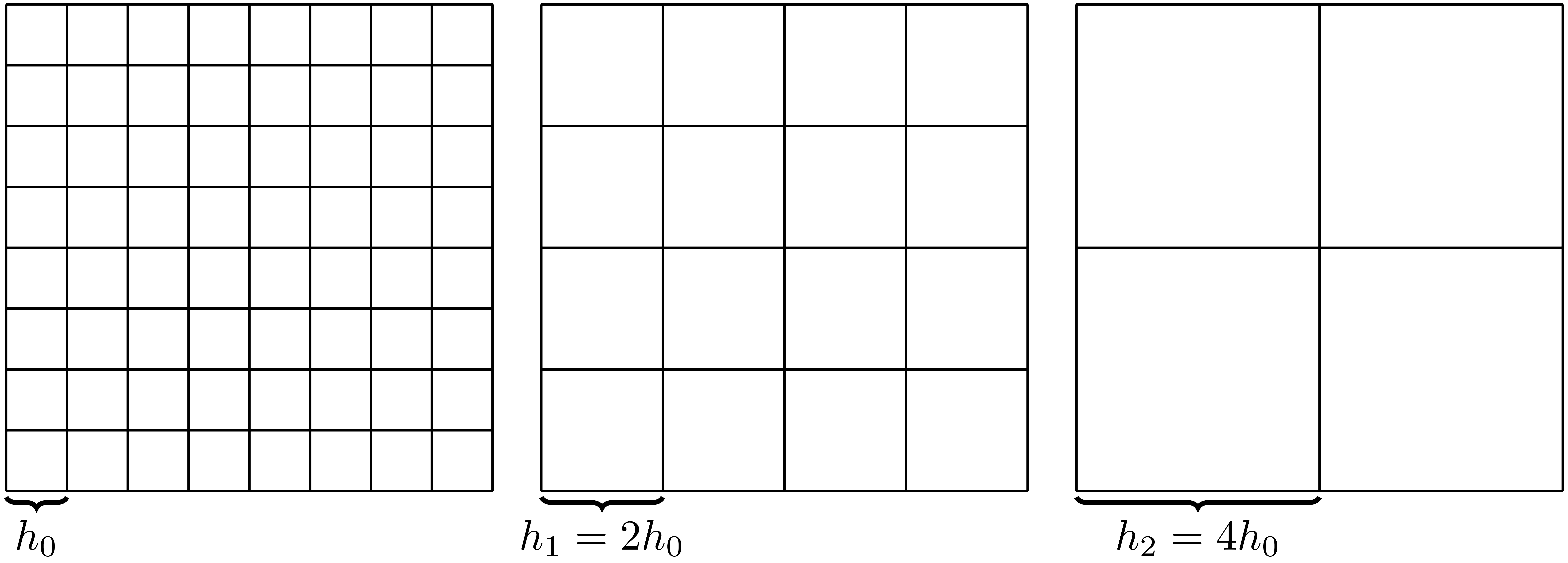


- Smoothing:
  Reduce high frequency error

- Coarse-grid correction:
  Reduce smooth-ish things in the range of interpolation

$$e_1 = e_0 - P(P^T A P)^{-1} P^T A G e$$

Multigrid in 2D



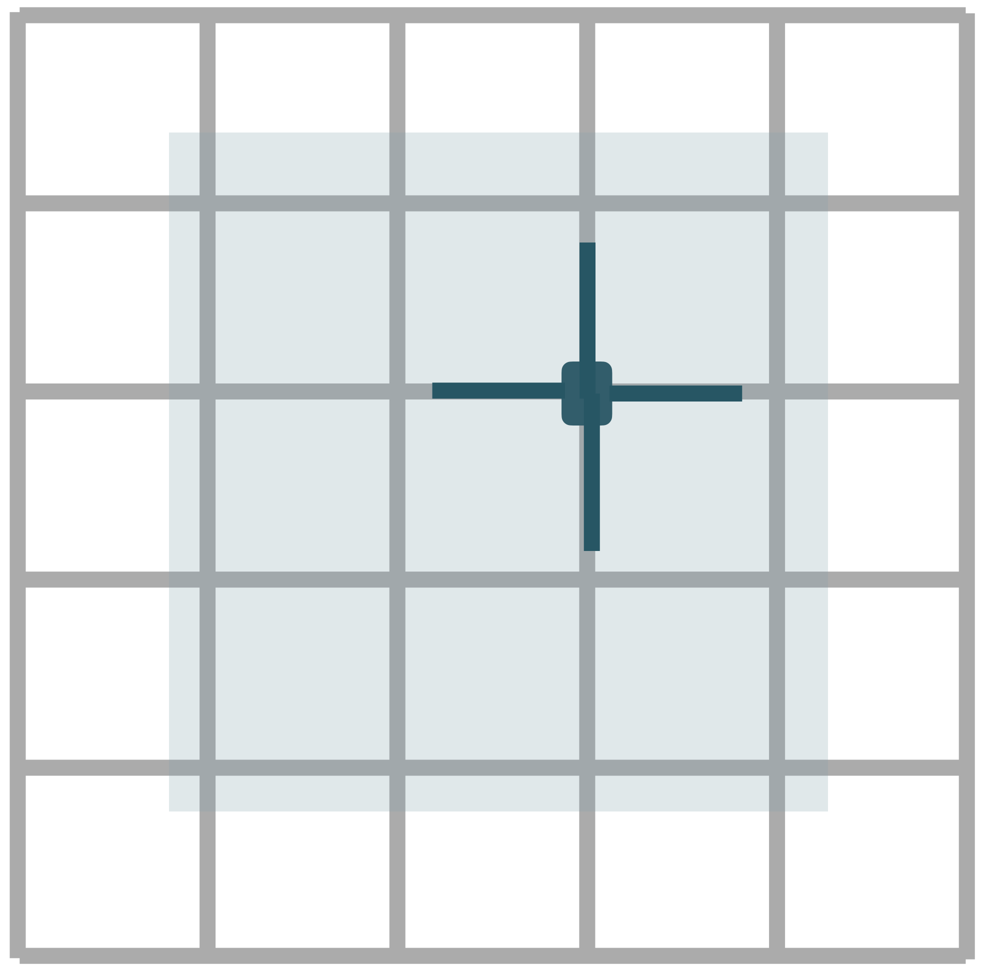
- Again, assume we have a series of uniform grids
- Relaxation remains the same (what is ω?)

Multigrid in 2D

- Model problem

$$
\begin{aligned}
-u_{xx} - u_{yy} &= f \\
u &= 0 \qquad \text{on boundary}
\end{aligned}
$$

- Results in the stencil / matrix

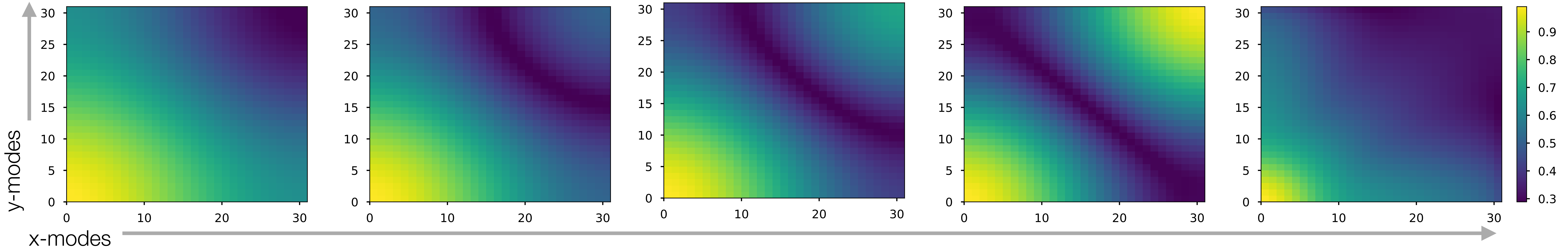


$$
\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}
\longrightarrow
\frac{1}{h^2}
\begin{bmatrix}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & & & & & \\
& -1 & 4 & -1 & & & -1 & & & & & & & & & \\
& & -1 & 4 & & & & -1 & & & & & & & & \\
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
& -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
& & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
& & & -1 & & & -1 & 4 & & & & -1 & & & & \\
& & & & -1 & & & & 4 & -1 & & & -1 & & & \\
& & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
& & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
& & & & & & & -1 & & & -1 & 4 & & & & -1 \\
& & & & & & & & -1 & & & & 4 & -1 & & \\
& & & & & & & & & -1 & & & -1 & 4 & -1 & \\
& & & & & & & & & & -1 & & & -1 & 4 & -1 \\
& & & & & & & & & & & -1 & & & -1 & 4
\end{bmatrix}
$$

Relaxation in 2D

$$\sin\left(\frac{k_x i\pi}{n+1}\right)\sin\left(\frac{k_y j\pi}{n+1}\right)$$

Convergence factor over 10 sweeps



$\omega = 0.5$ $\omega = 0.667$ $\omega = 0.8$ $\omega = 1.0$ GS

$$I - \omega D^{-1}A \qquad I - (D-E)^{-1}A$$

- weighted Jacobi: Same issue — need to select a parameter
- Gauss-Seidel improved
- Red-Black Gauss-Seidel and other schemes even more effective

Interpolation in 2D

- Bilinear interpolation, tensor of 1D interpolation

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

- Example 3x3 -> 7x7

$$\begin{bmatrix} 0.5 & & \\ 1 & & \\ 0.5 & 0.5 & \\ & 1 & \\ & 0.5 & 0.5 \\ & & 1 \\ & & 0.5 \end{bmatrix} \otimes \begin{bmatrix} 0.5 & & \\ 1 & & \\ 0.5 & 0.5 & \\ & 1 & \\ & 0.5 & 0.5 \\ & & 1 \\ & & 0.5 \end{bmatrix} \longrightarrow \frac{1}{4}\begin{bmatrix}
1 & & & & & & & & \\
2 & & & & & & & & \\
1 & 1 & & & & & & & \\
 & 2 & & & & & & & \\
 & 1 & 1 & & & & & & \\
 & & 2 & & & & & & \\
 & & 1 & & & & & & \\
2 & & & & & & & & \\
4 & & & & & & & & \\
2 & 2 & & & & & & & \\
 & 4 & & & & & & & \\
 & 2 & 2 & & & & & & \\
 & & 4 & & & & & & \\
 & & 2 & & & & & & \\
1 & & & 1 & & & & & \\
2 & & & 2 & & & & & \\
1 & 1 & & 1 & 1 & & & & \\
 & 2 & & & 2 & & & & \\
 & 1 & 1 & & 1 & 1 & & & \\
 & & 2 & & & 2 & & & \\
 & & 1 & & & 1 & & & \\
 & & & 2 & & & & & \\
 & & & 4 & & & & & \\
 & & & 2 & 2 & & & & \\
 & & & & 4 & & & & \\
 & & & & 2 & 2 & & & \\
 & & & & & 4 & & & \\
 & & & & & 2 & & & \\
 & \vdots & & \vdots & & & & &
\end{bmatrix}$$

A few observations so far: **One**

- Let's consider a V(1,1) cycle — weighted Jacobi, etc.

- The **error** propagation for this looks like

$$e_1 = \underbrace{G(I - P(P^T AP)^{-1}P^T A)G}_{M}e_0$$

$$G = I - \omega D^{-1} A$$

$$Me_k \leftarrow 0?$$

- One thing we can do, is consider bounds on each operation.

A few observations so far: **One**

- Take the operator

$$M = G(I - P(P^T AP)^{-1} P^T A)G$$

- And makes some bounds

$$\underbrace{\|I - P(P^T AP)^{-1} P^T A\|}_{\text{approximation property}} \underbrace{\|G\|^2}_{\text{smoothing property}}$$

- General $\|G\| \leq 1$

1D over [n/2,n]: $\|G\| \leq \frac{1}{3}$

2D over [n/2,n]: $\|G\| \leq \frac{3}{5}$

A few observations so far: **One**

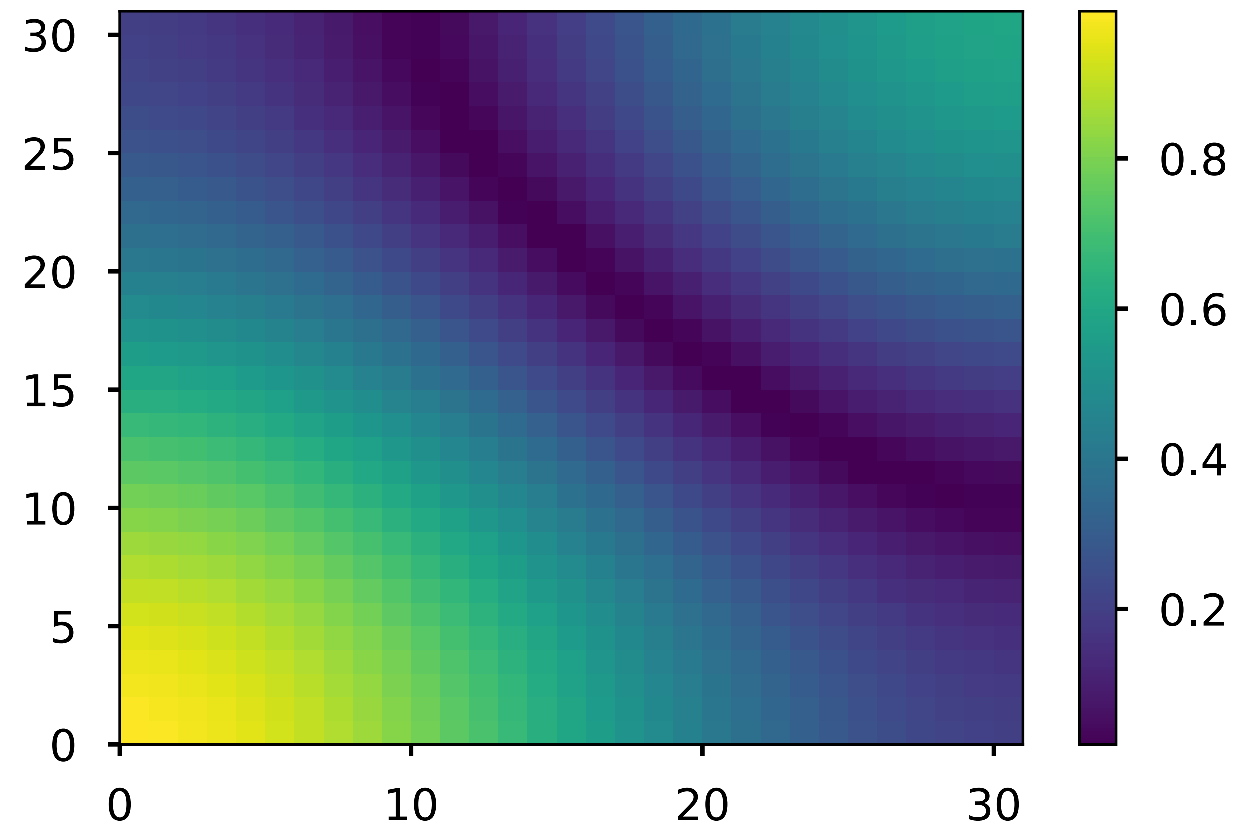
$$\|I - P(P^TAP)^{-1}P^TA\|\|G\|^2$$

approximation property   smoothing property

- General $\|G\| \leq 1$

1D over [n/2,n]: $\|G\| \leq \frac{1}{3}$

2D over [n/2,n]: $\|G\| \leq \frac{3}{5}$



- Also, if $w \quad \text{s.t.} \quad Aw \in \mathcal{N}(P^T)$

then $(I - P(P^TAP)^{-1}P^TA)w = w$

$\|\cdot\| \geq 1$

Luke Olson
http://lukeo.cs.illinois.edu/

A few observations so far: **Two**

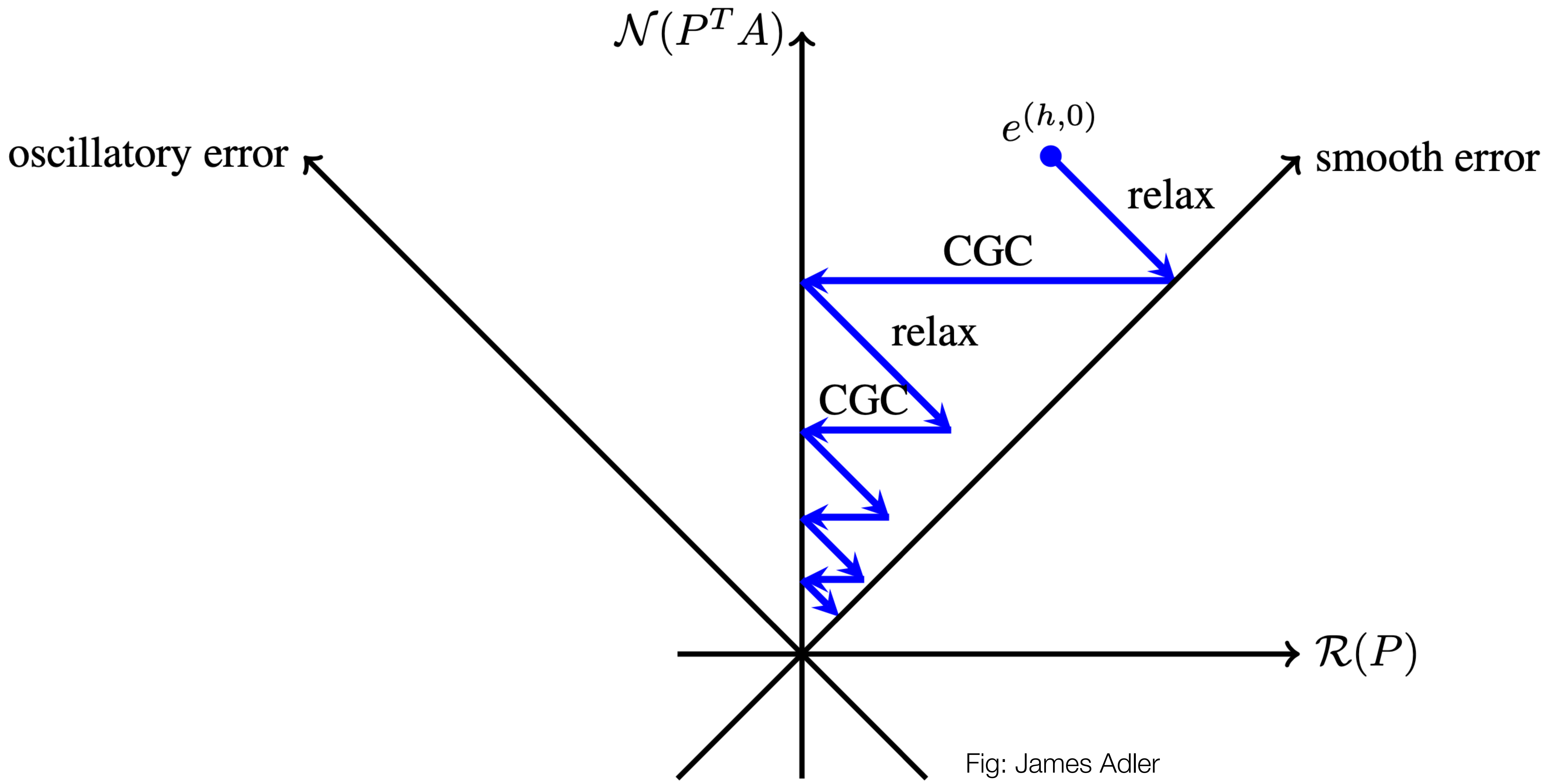


Fig: James Adler

- Complementary processes:
  - **relaxation:** targets for (Fourier) smoothing
  - **coarse grid correction:** targets things in the range of interpolation

A few observations so far: **Three**

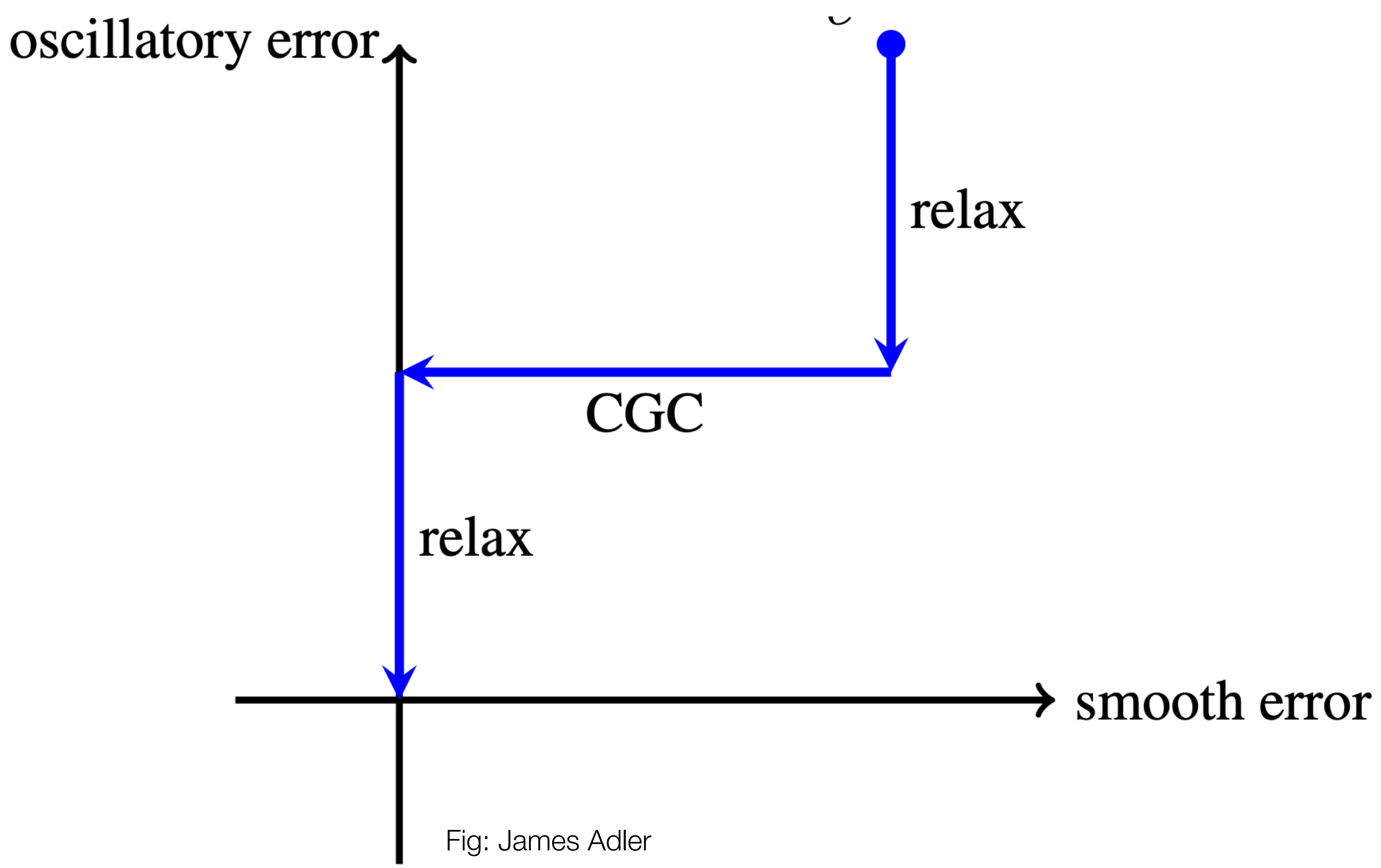


Fig: James Adler

$$e_1 \leftarrow (I - P(P^TAP)^{-1}P^TA)Ge_0$$

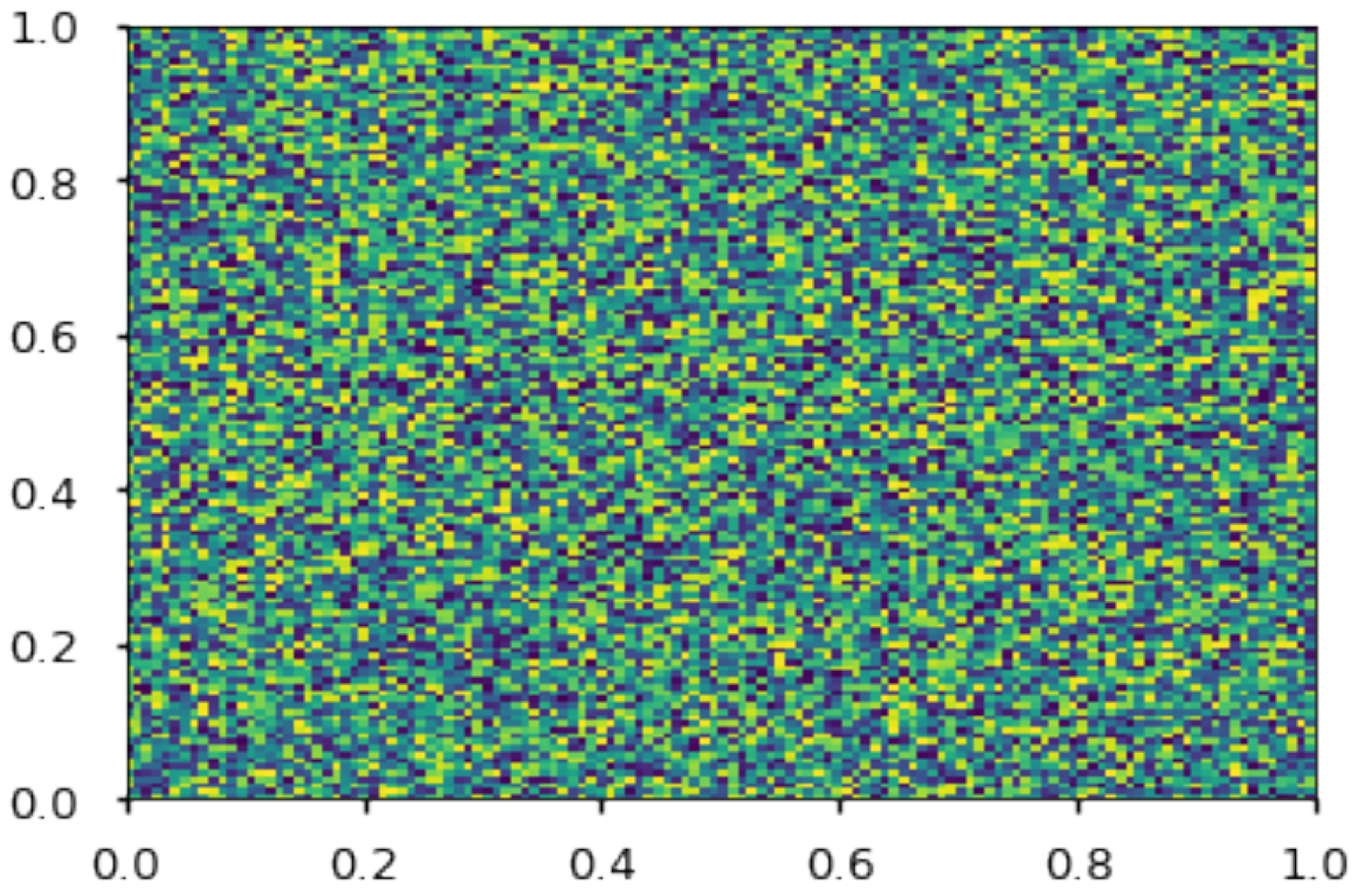
$$Ge_0 \in \mathcal{R}(P) \quad \Rightarrow \quad e_1 = 0$$

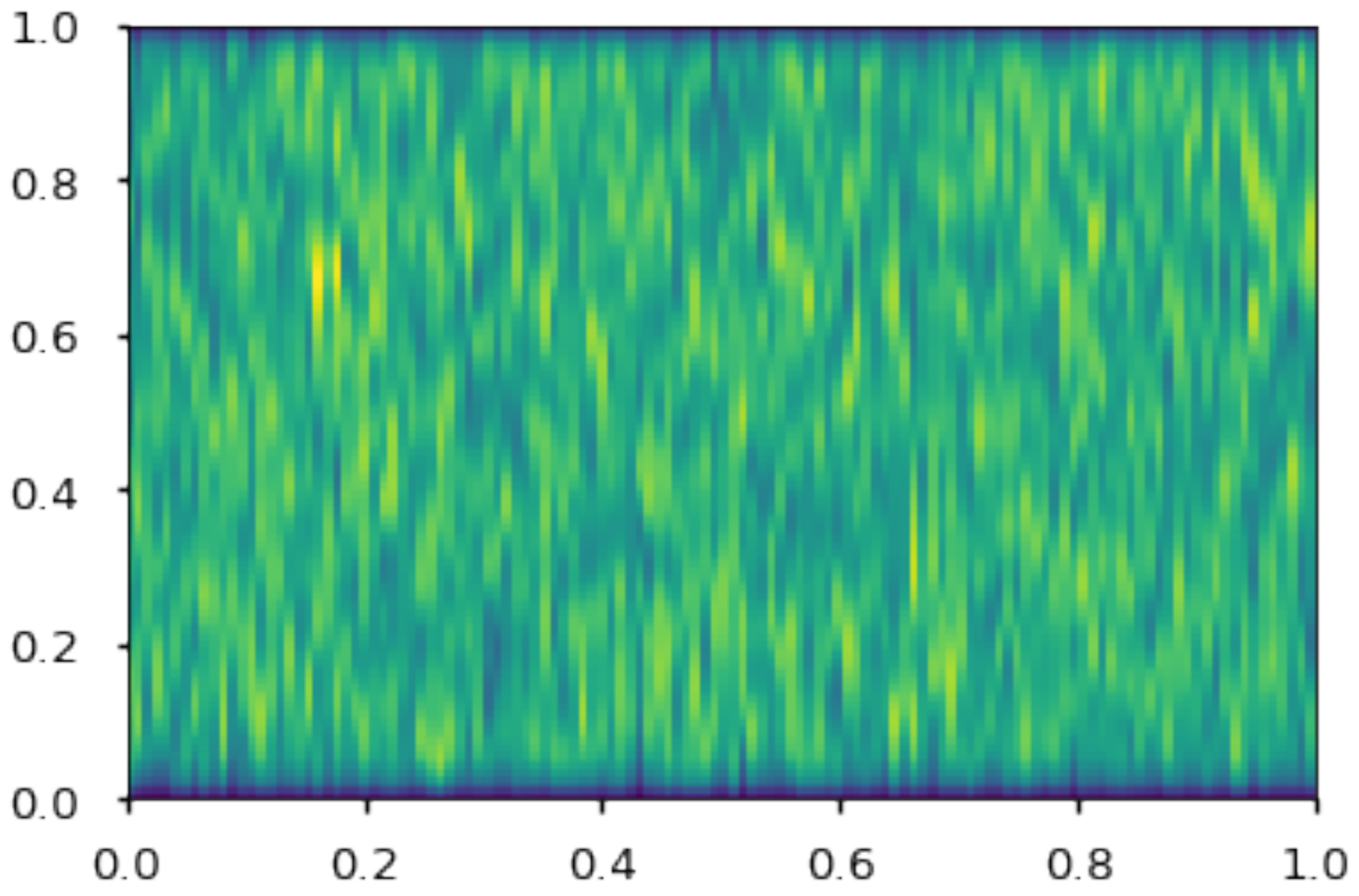
interpolation should capture what relaxation misses

What can go wrong?!

- Consider an *anisotropic problem*

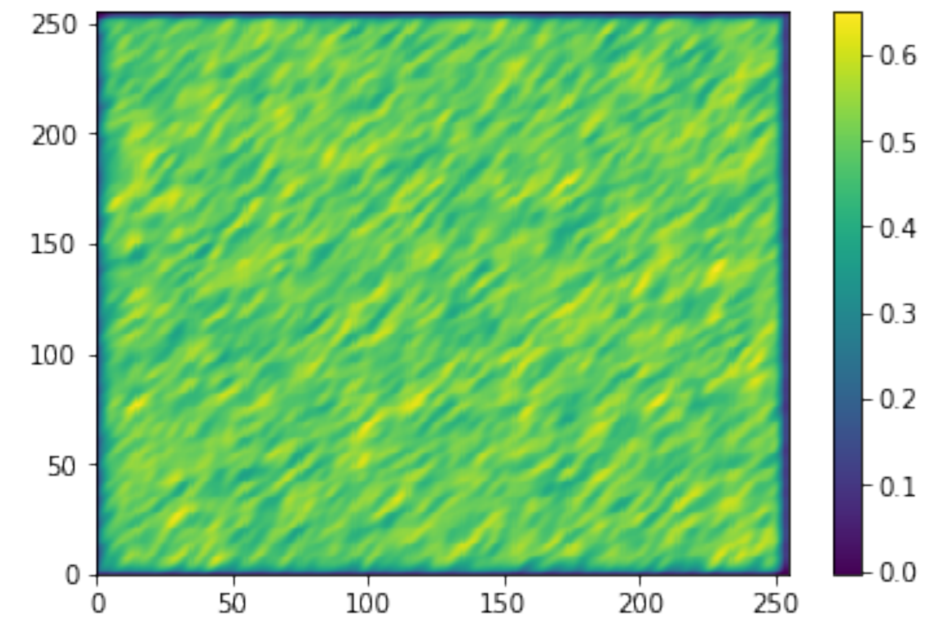
$$
\begin{aligned}
-\varepsilon u_{xx} - u_{yy} &= f \\
u &= 0 \qquad \text{on boundary}
\end{aligned}
\qquad
\begin{bmatrix} & -1 & \\ -\varepsilon & 2+2\varepsilon & -\varepsilon \\ & -1 & \end{bmatrix}
$$

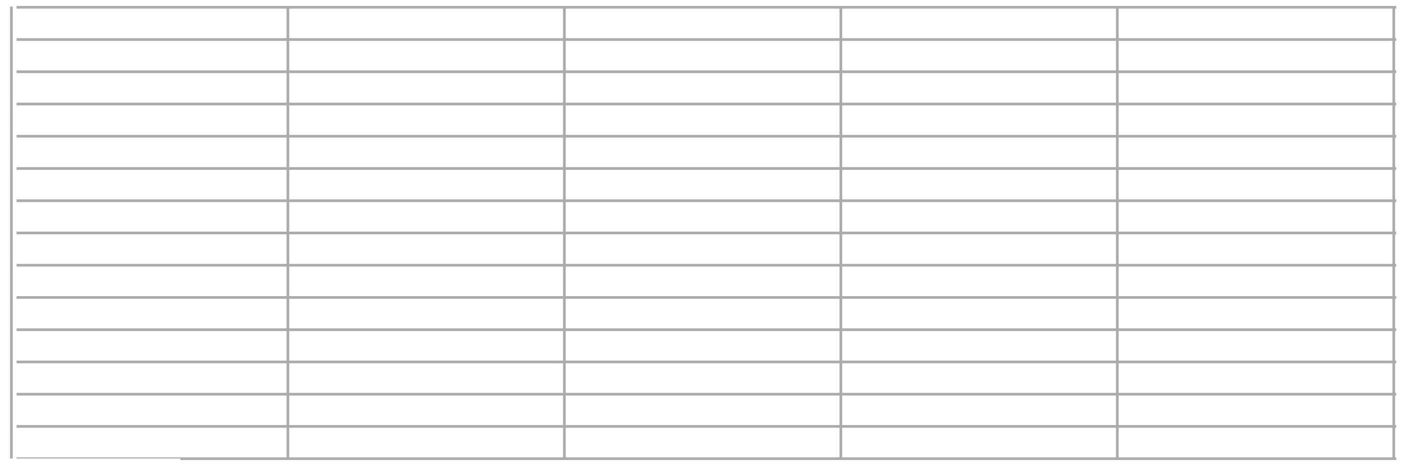


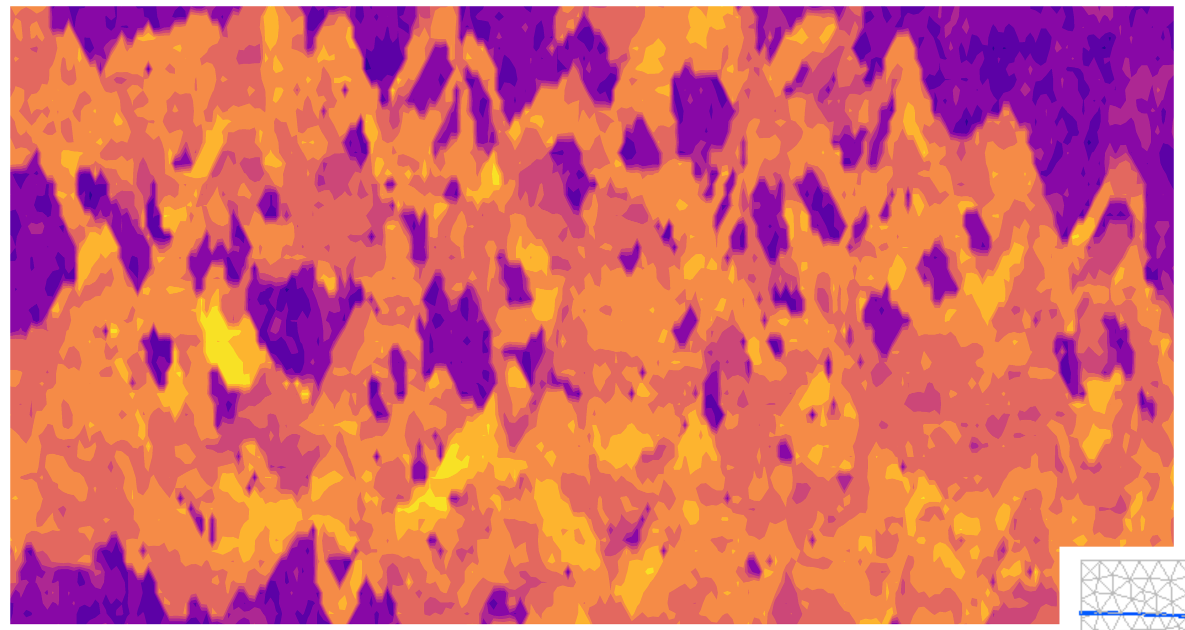


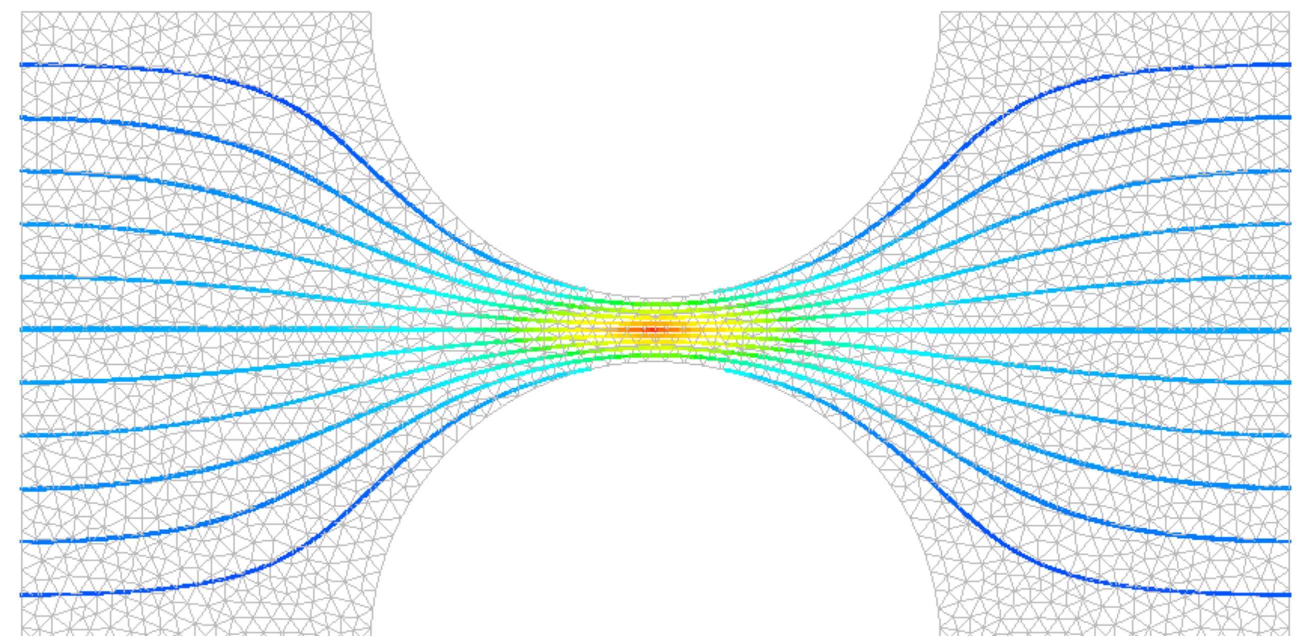
What can go wrong?!

- Anisotropy
- Mesh stretching
- Jumping coefficients
- Non-elliptic



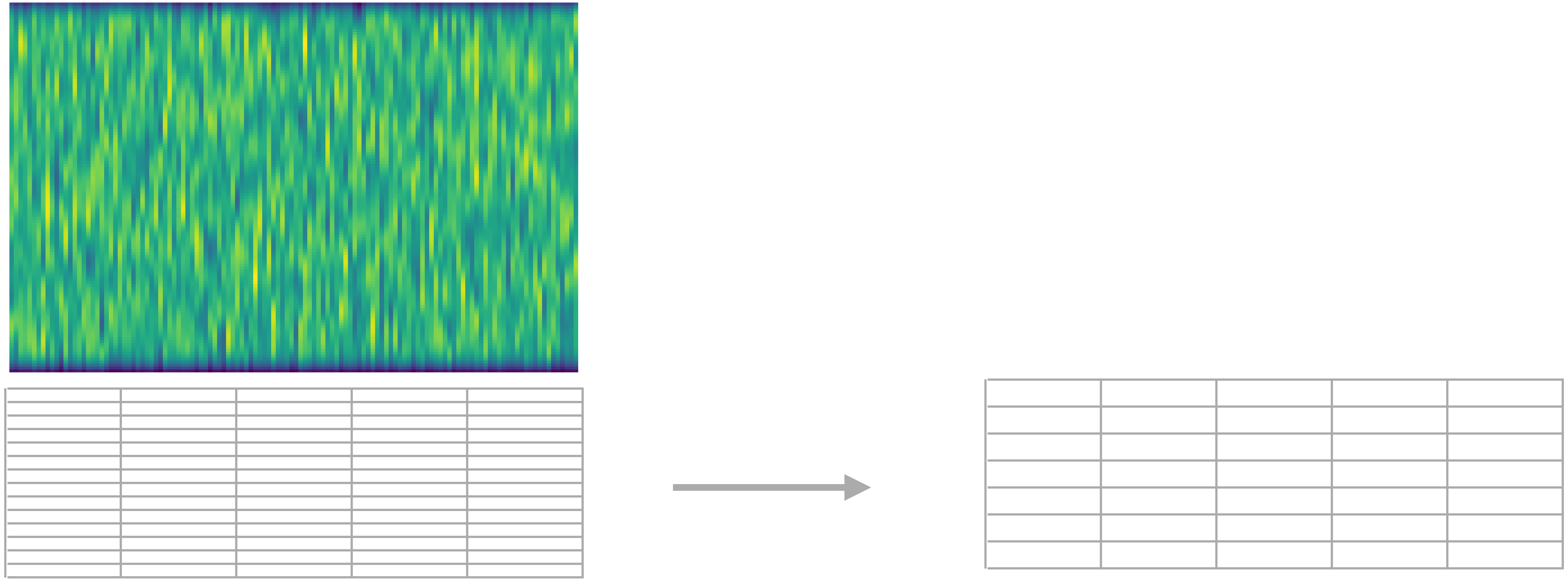






Options for more robust Multigrid

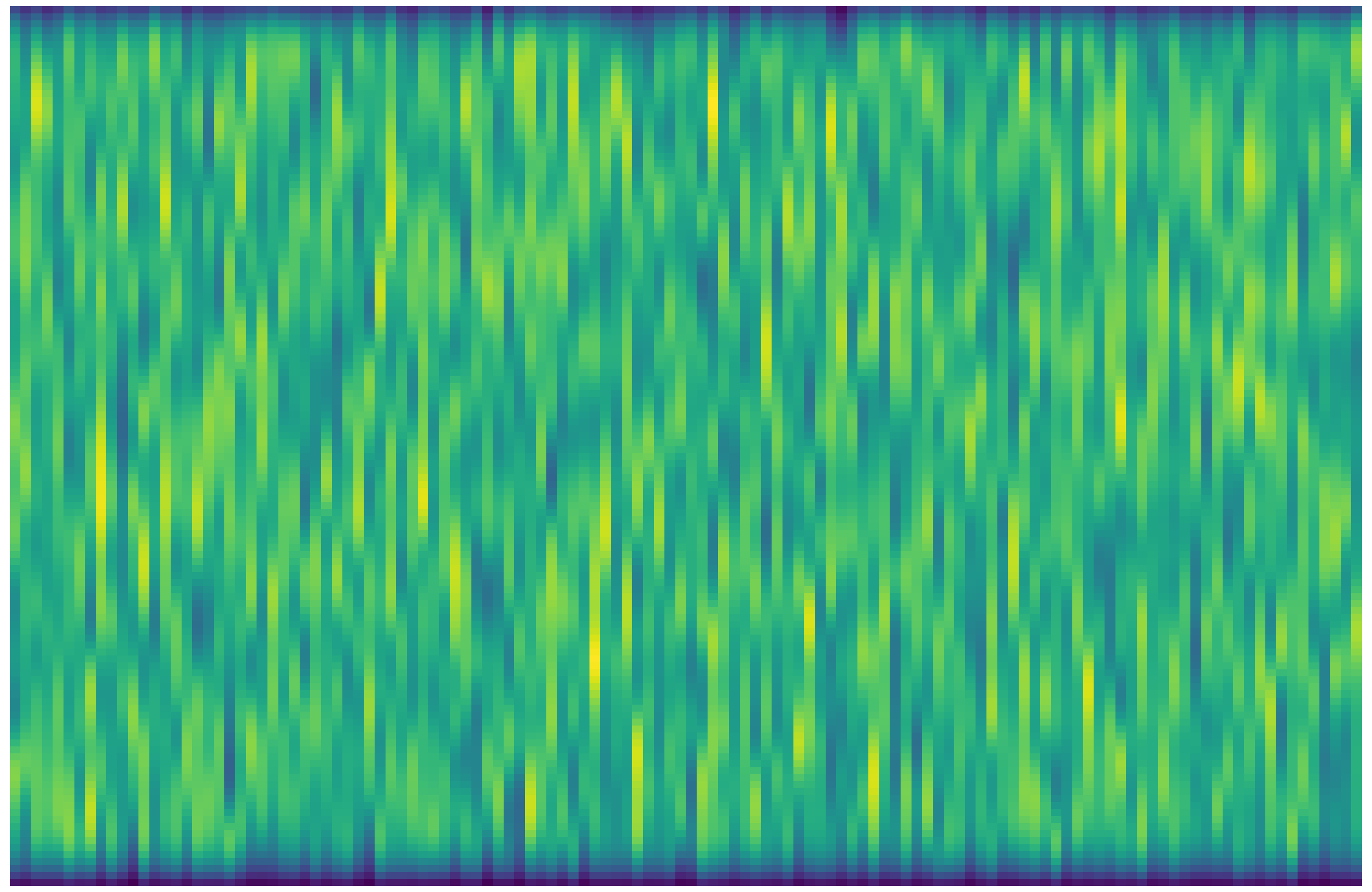
- **Semicoarsening:** Coarsen in the direction of smoothness

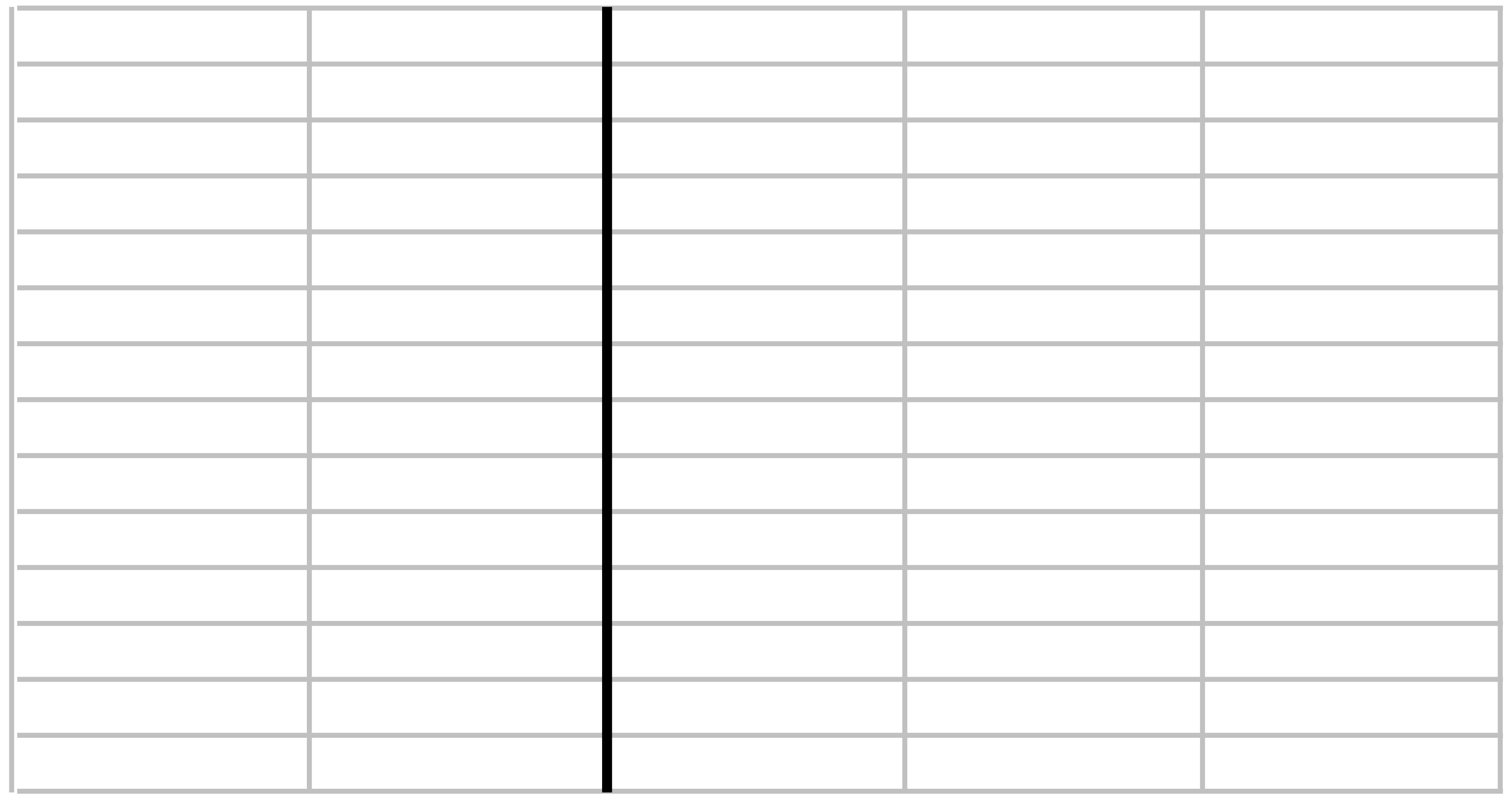


- Downside: if anisotropy varies in another direction, we need a different grid

Options for more robust Multigrid

- **Line/plane relaxation**
  Perform relaxation in groups (in a line)

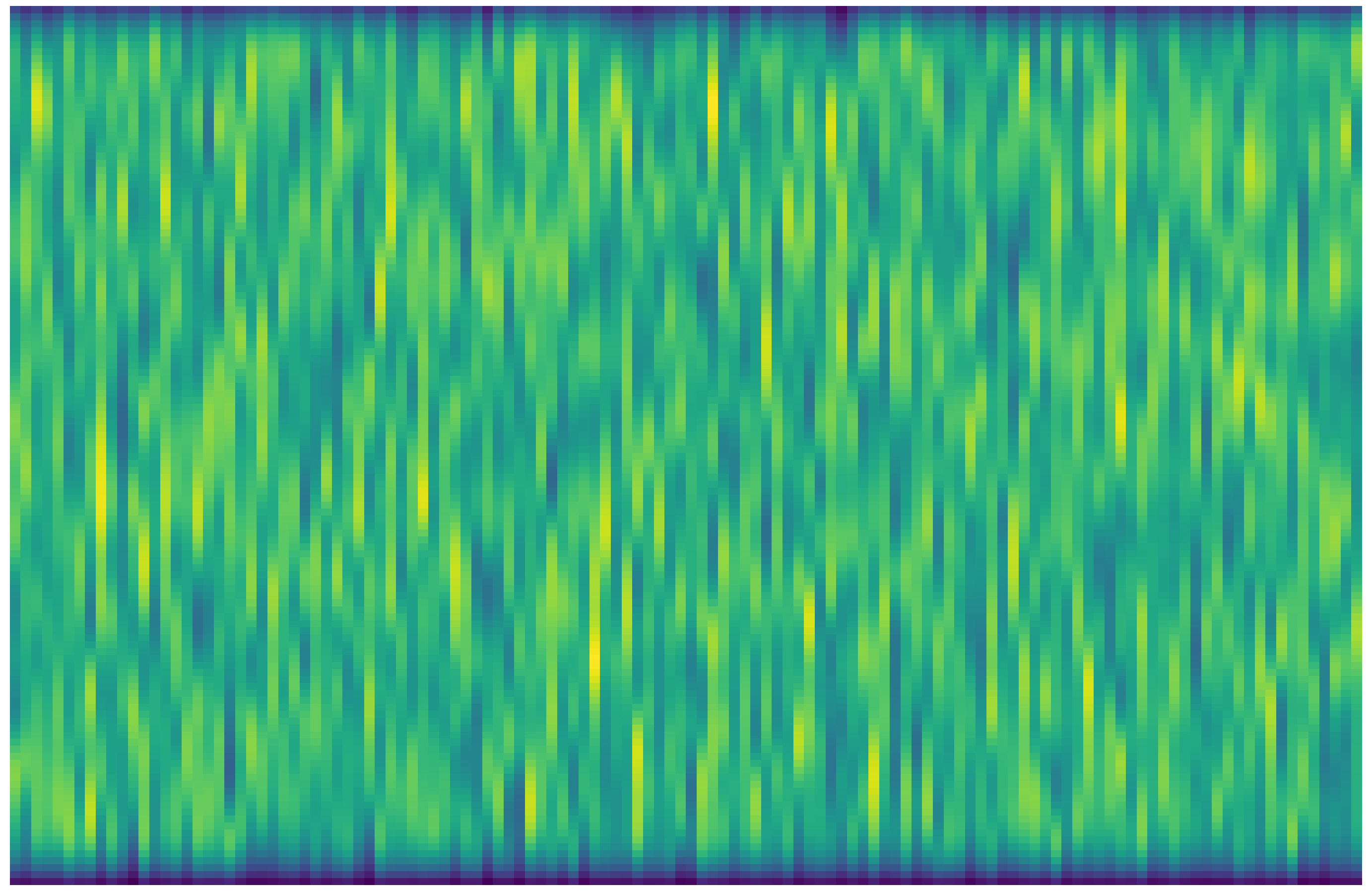


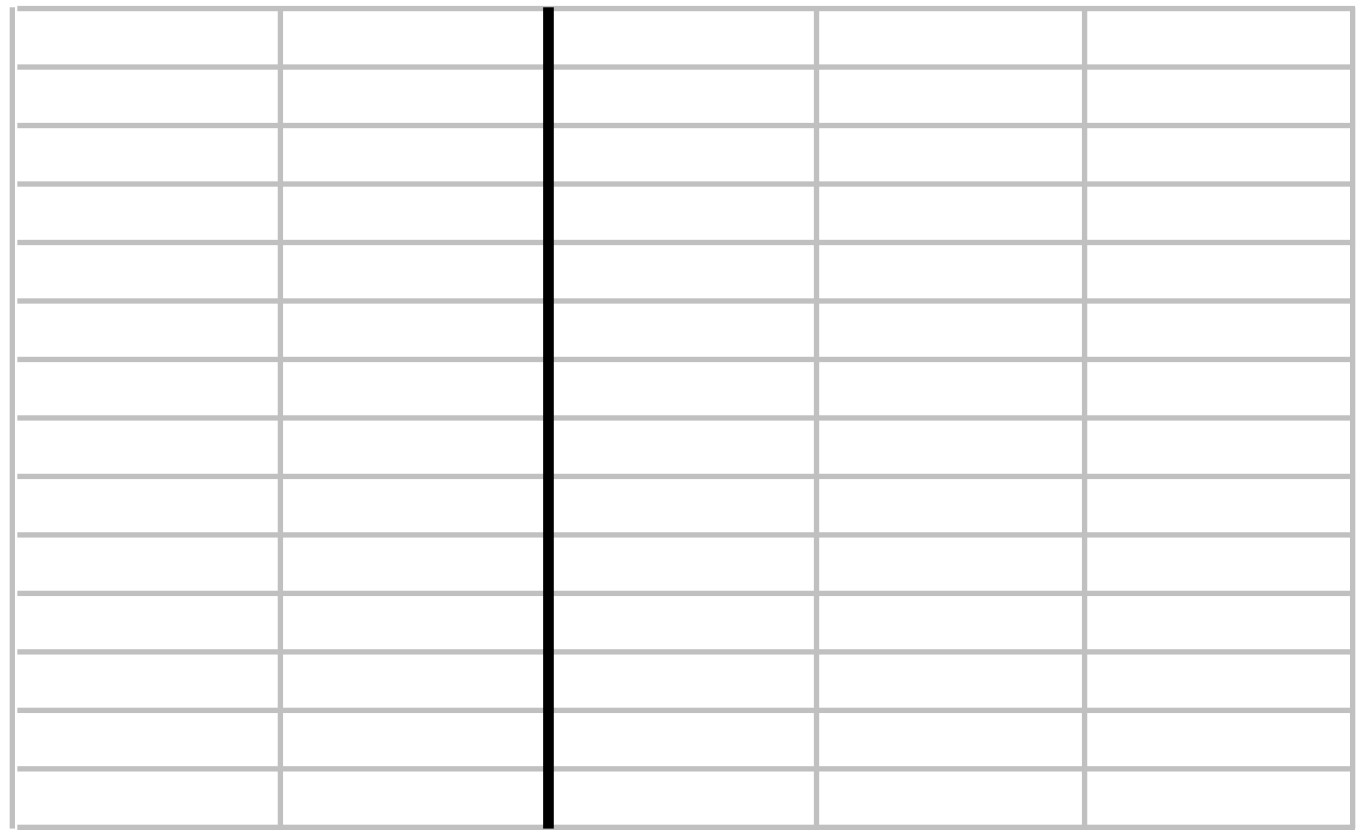


- Example:
  - smooth error in the y-direction (x-lines)
  - no smoothing in the x-direction (y-lines)

Options for more robust Multigrid

- **Line/plane relaxation**
- For each x-line (lines of strong anisotropy):
  - Eliminate the residual on the entire line
  - (Gauss-Siedel, by lines)

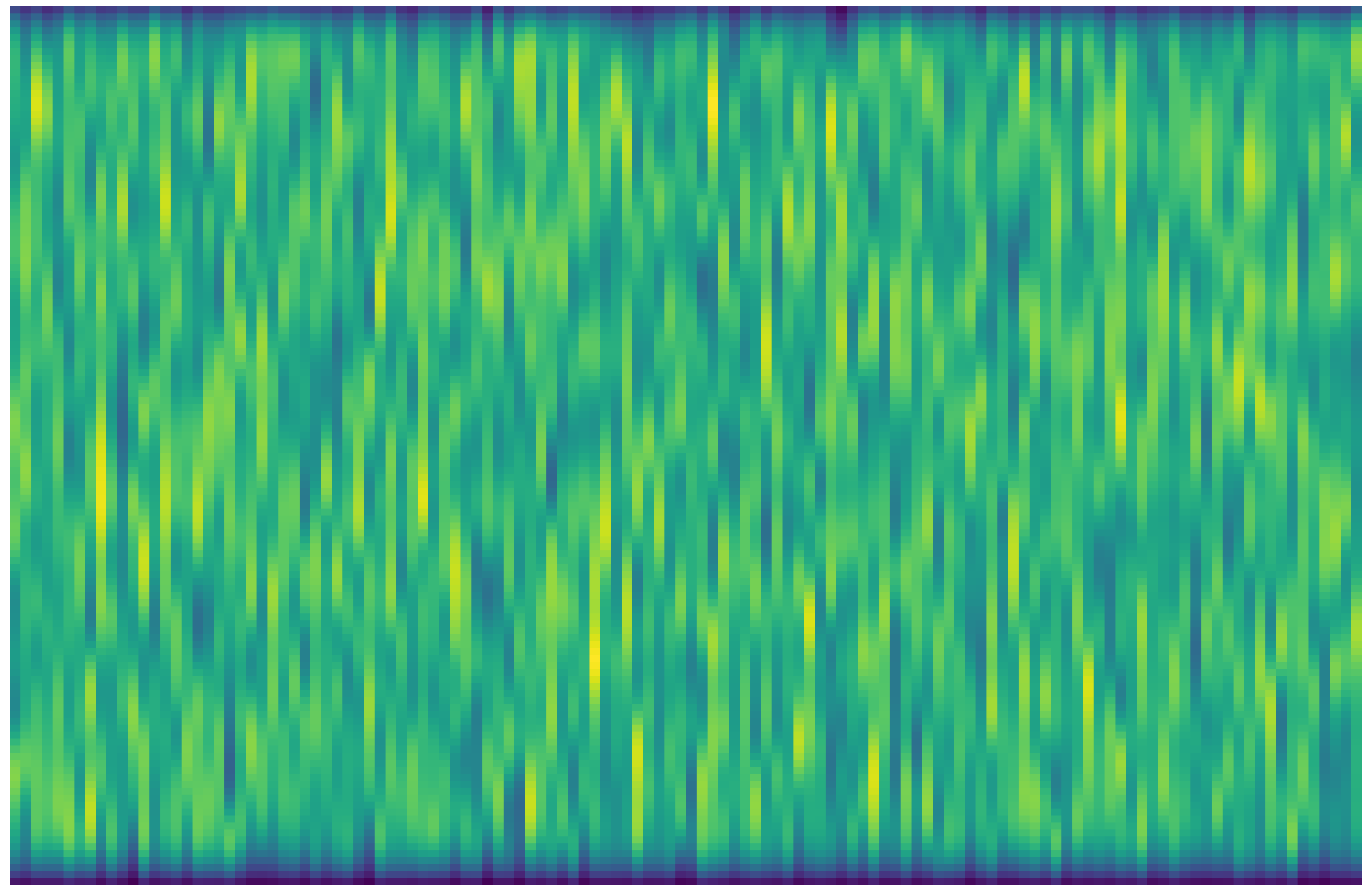


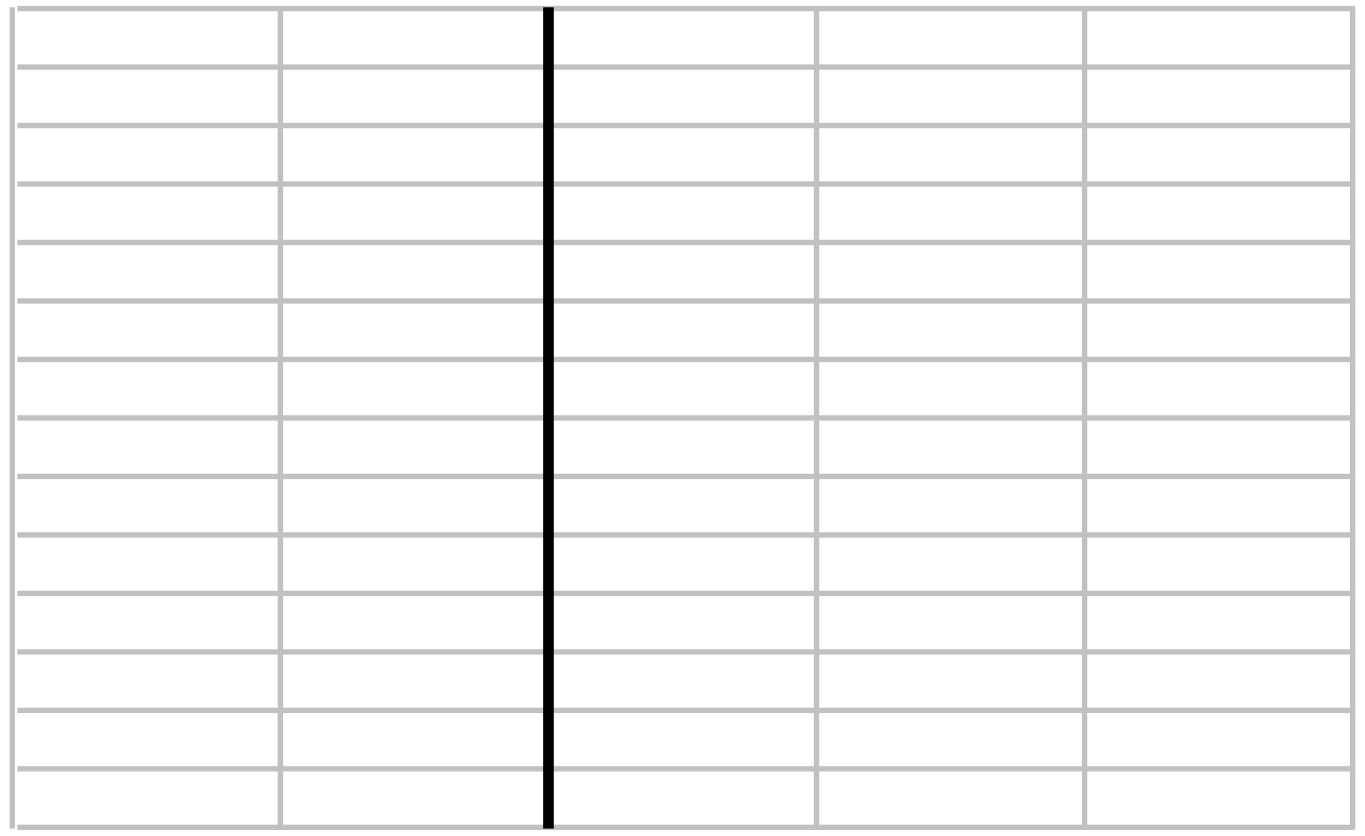


Options for more robust Multigrid

- **Line/plane relaxation**
- $$A = (n+1)^2 \begin{bmatrix} A_{1D} & -\varepsilon I & & & \\ -\varepsilon I & A_{1D} & -\varepsilon I & & \\ & -\varepsilon I & A_{1D} & -\varepsilon I & \\ & & \ddots & & \\ & & & -\varepsilon I & A_{1D} \end{bmatrix}$$

$$A_{1D} = (n+1)^2 \begin{bmatrix} 2+2\varepsilon & -1 & & \\ -1 & 2+2\varepsilon & -1 & \\ & & \ddots & \\ & & -1 & 2+2\varepsilon \end{bmatrix}$$





Take a look: `11-multigrid-2d-line-relaxation.ipynb`

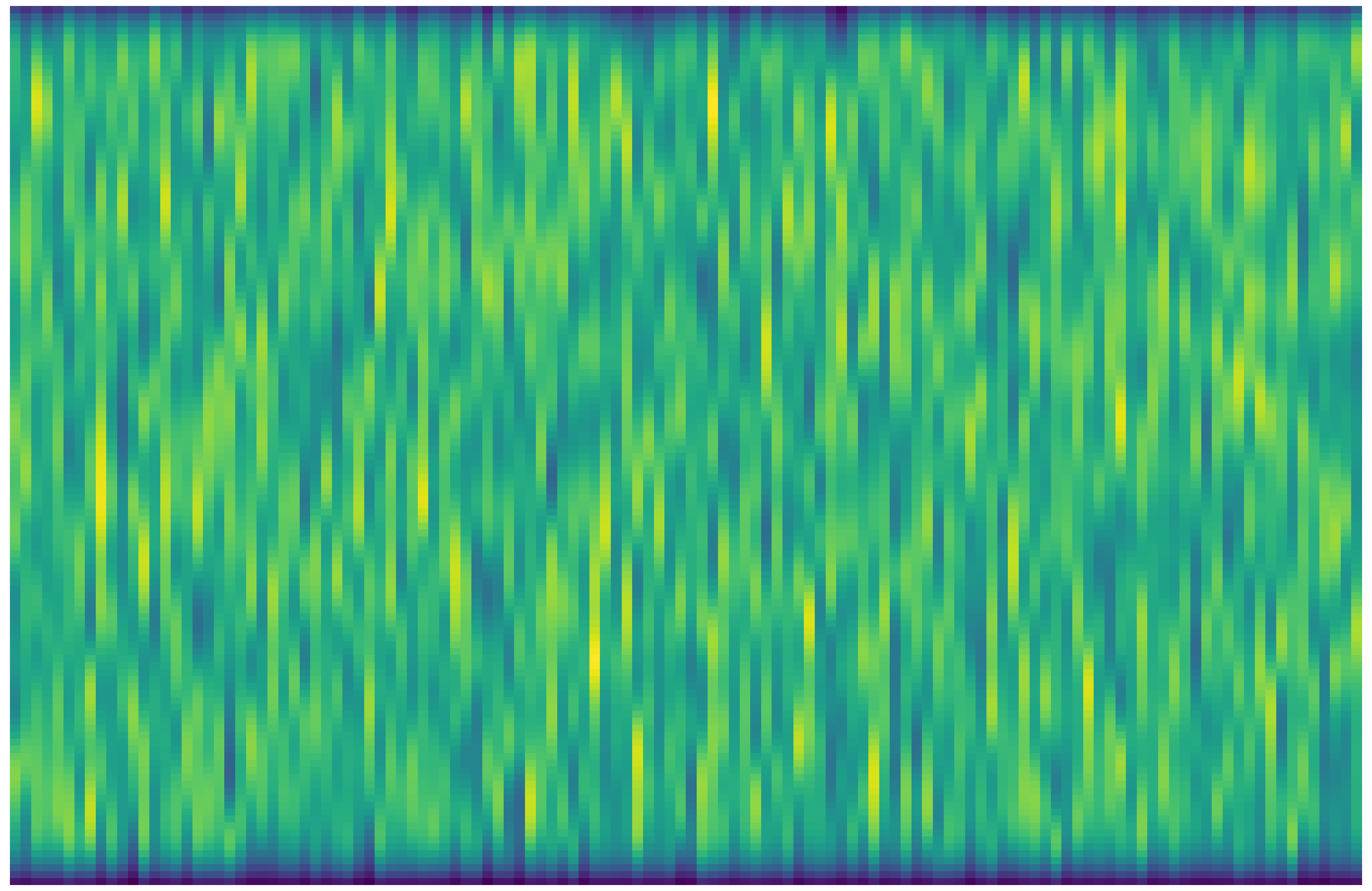
Options for more robust Multigrid

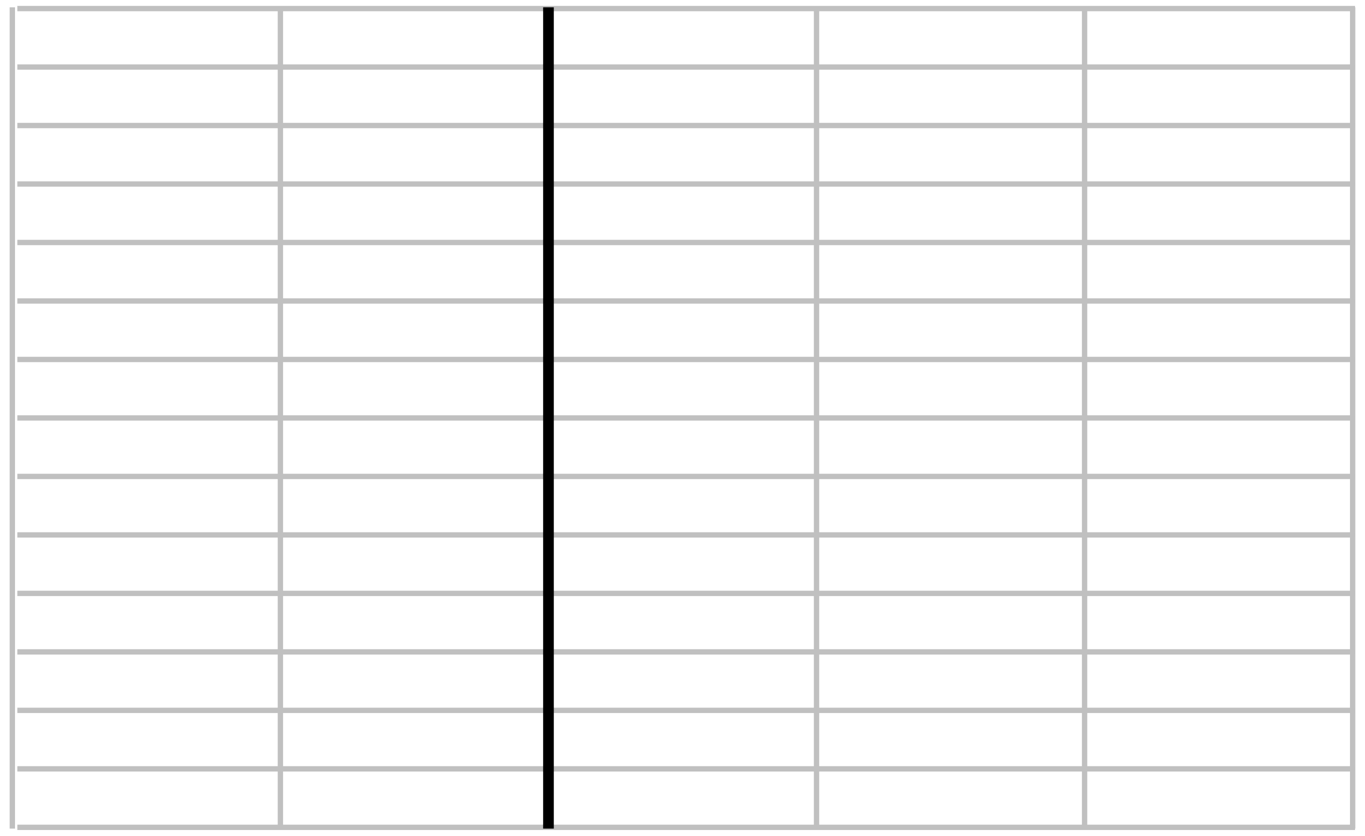
- **Line/plane relaxation**
- For each line, solve

$$A_{1D} v_k = g_k$$

$$v_k = \begin{bmatrix} \vdots \\ v_{k,j-1} \\ v_{k,j} \\ v_{k,j+1} \\ \vdots \end{bmatrix} \quad g_k = \begin{bmatrix} \vdots \\ f_{k,j-1} + \varepsilon\left(v_{k-1,j-1} + v_{k+1,j-1}\right) \\ f_{k,j} + \varepsilon\left(v_{k-1,j} + v_{k+1,j}\right) \\ f_{k,j+1} + \varepsilon\left(v_{k-1,j+1} + v_{k+1,j+1}\right) \\ \vdots \end{bmatrix}$$

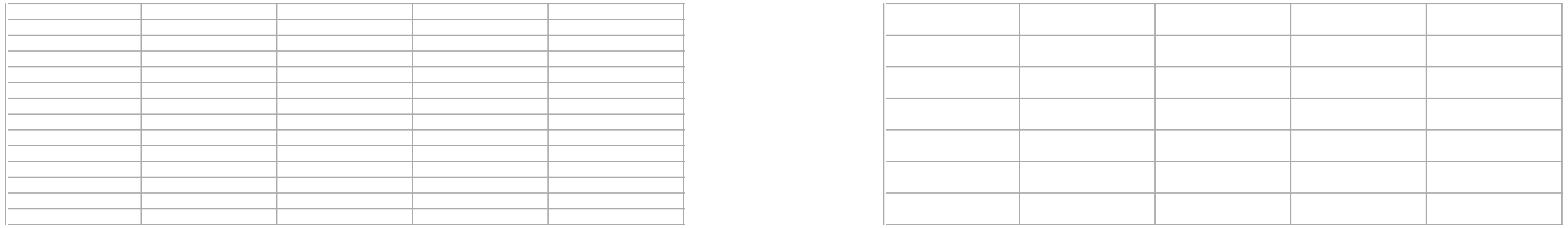
- In 3D, lines become planes…



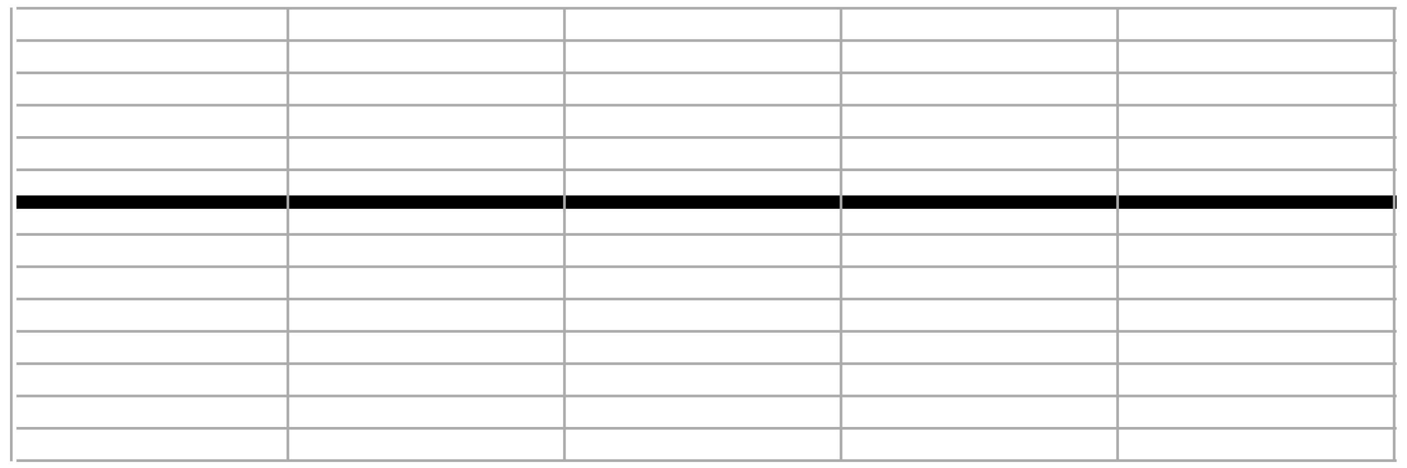


Options for more robust Multigrid

- **Semicoarsening**
  Coarsen in the direction of smoothness



- **Line/plane relaxation**
  Perform relaxation in groups (in a line)



- **Operator based Interpolation** (e.g. BoxMG)

$$Ae = 0$$

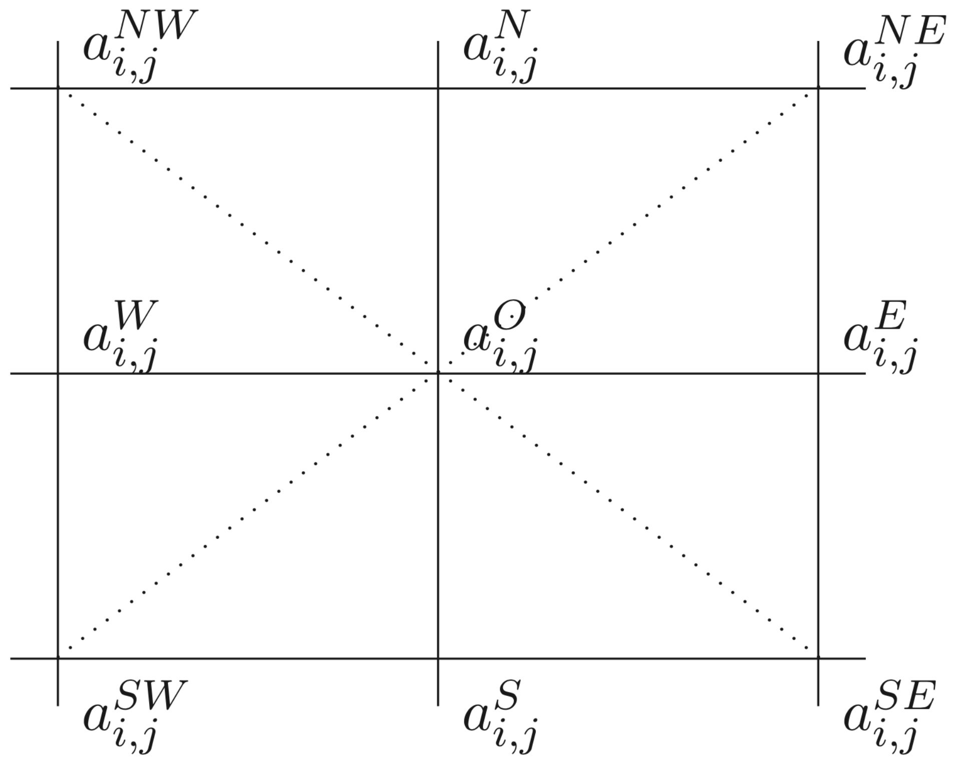
J. E. Dendy, Black box multigrid, J. Comput. Phys., 1982

J. E. Dendy and J. D. Moulton, Black box multigrid with coarsening by a factor of three, J. Numer. Lin. Alg. App., 2010

BoxMG

Math and figures from:
Robust and adaptive multigrid methods: comparing structured and algebraic approaches, MacLachlan, Moulton, Chartier

- Node (i,j) stencil:

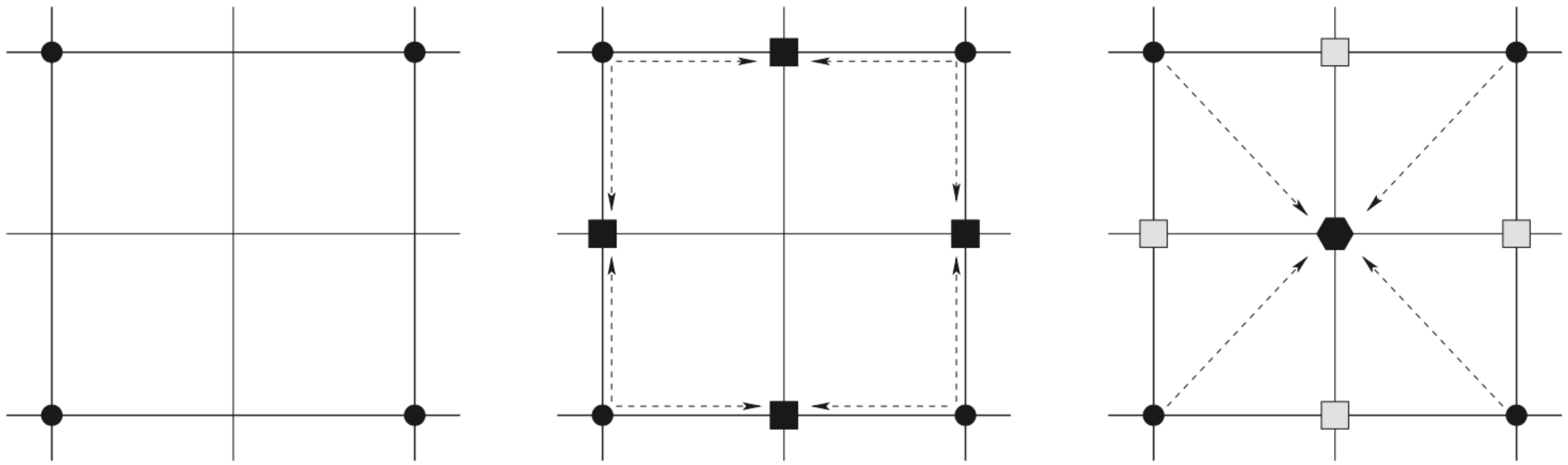


$$Ax = b$$

$$a_{i,j}^{SW} x_{i-1,j-1} + a_{i,j}^{S} x_{i,j-1} + a_{i,j}^{SE} x_{i+1,j-1} + a_{i,j}^{W} x_{i-1,j} + a_{i,j}^{O} x_{i,j}$$
$$+ a_{i,j}^{E} x_{i+1,j} + a_{i,j}^{NW} x_{i-1,j+1} + a_{i,j}^{N} x_{i,j+1} + a_{i,j}^{NE} x_{i+1,j+1} = b_{i,j}$$

BoxMG

Math and figures from:
Robust and adaptive multigrid methods: comparing structured and algebraic approaches, MacLachlan, Moulton, Chartier



1. Inject coarse points (left)
2. Assume the error is constant along x-lines (and y-lines)
3. Infer interpolation from the edges (right)

Example: Assuming Ae=0 and contant y-lines:

$$(a^S + a^O + a^N)e = -(a^{SW} + a^W + a^{NW})e^c - (a^{SE} + a^E + a^{NW})e^c$$

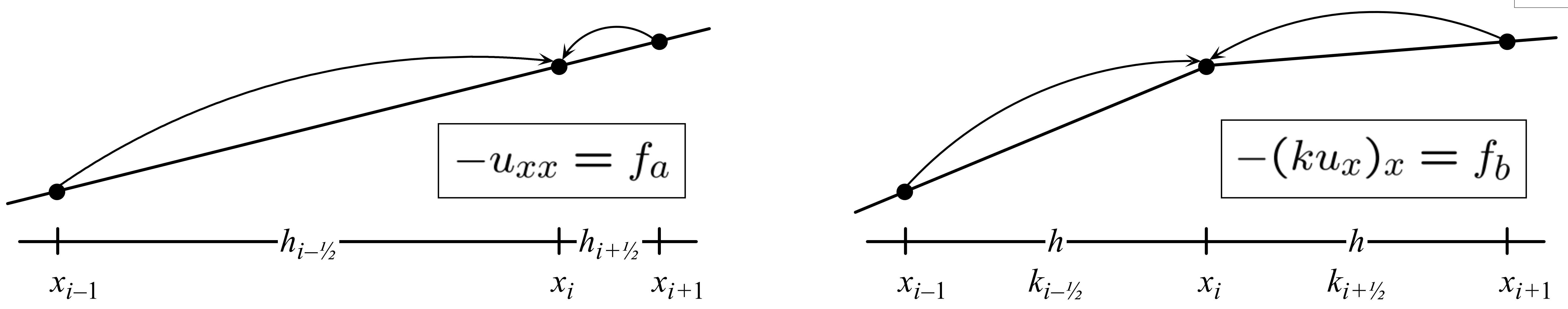
Interpolation based on entries in A

Algebraic Multigrid (AMG) uses matrix coefficients

- Geometric information alone is not sufficient

Linear Interpolation Operator-Dependent Interpolation

Slide credit: R. Falgout, 2/1/2016

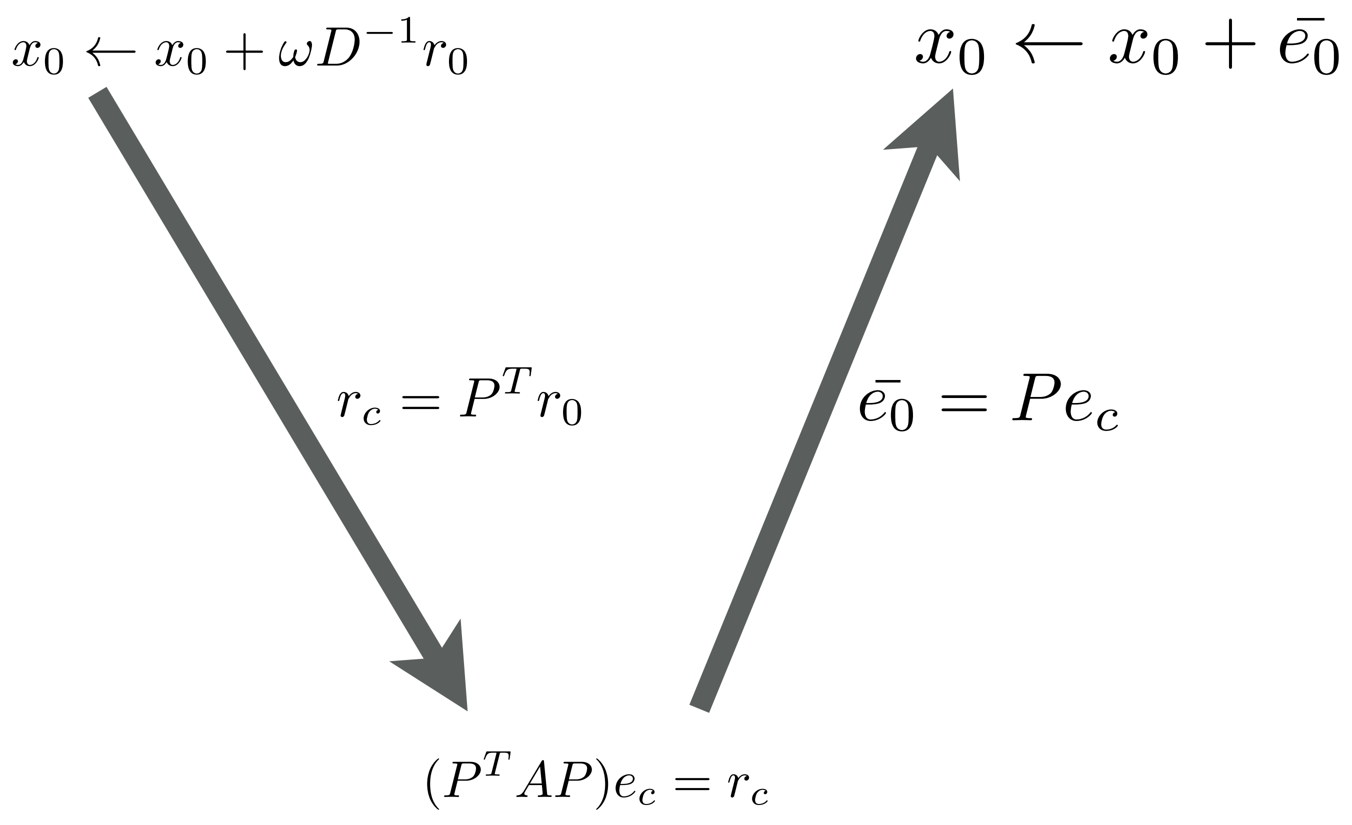


- AMG does not use geometric information, but captures both linear & operator-dep interpolation

$$(A\mathbf{u})_i = a_{i,i-1}u_{i-1} + a_{i,i}u_i + a_{i,i+1}u_{i+1}$$

$$u_i = \left(-\frac{a_{i,i-1}}{a_{i,i}}\right)u_{i-1} + \left(-\frac{a_{i,i+1}}{a_{i,i}}\right)u_{i+1}$$

Algebraic Observation



$$e_1 \leftarrow (I - P(P^T A P)^{-1} P^T A) G e_0$$

coarse grid correction **relax**

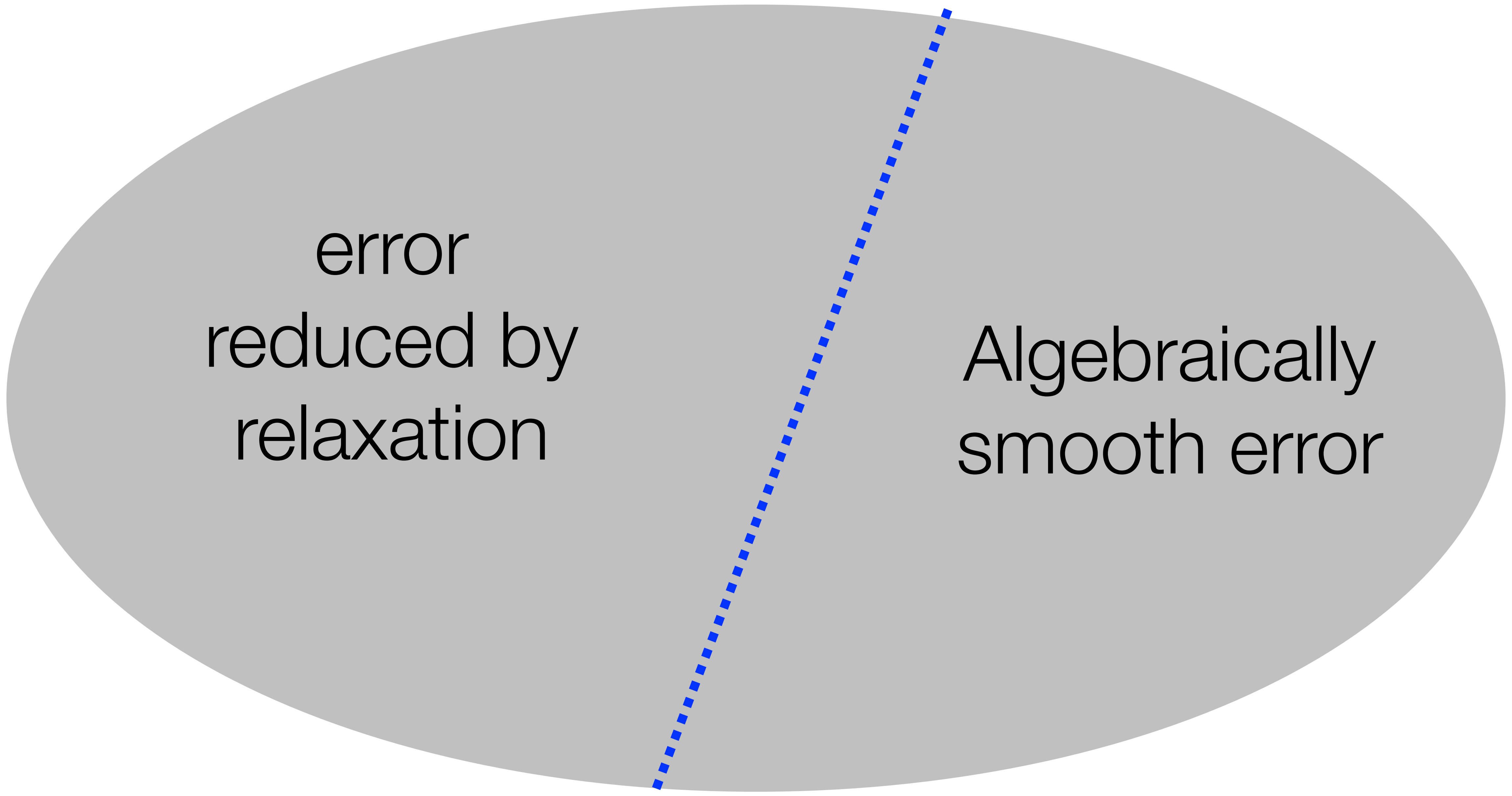
relax
residual
restrict
coarse solve
interpolate
correct

$$G e_0 \in \mathcal{R}(P) \quad \Rightarrow \quad e_1 = 0$$

interpolation should capture what relaxation misses

Algebraically Smooth Error

- **“Algebraically smooth” error may not be geometrically smooth**

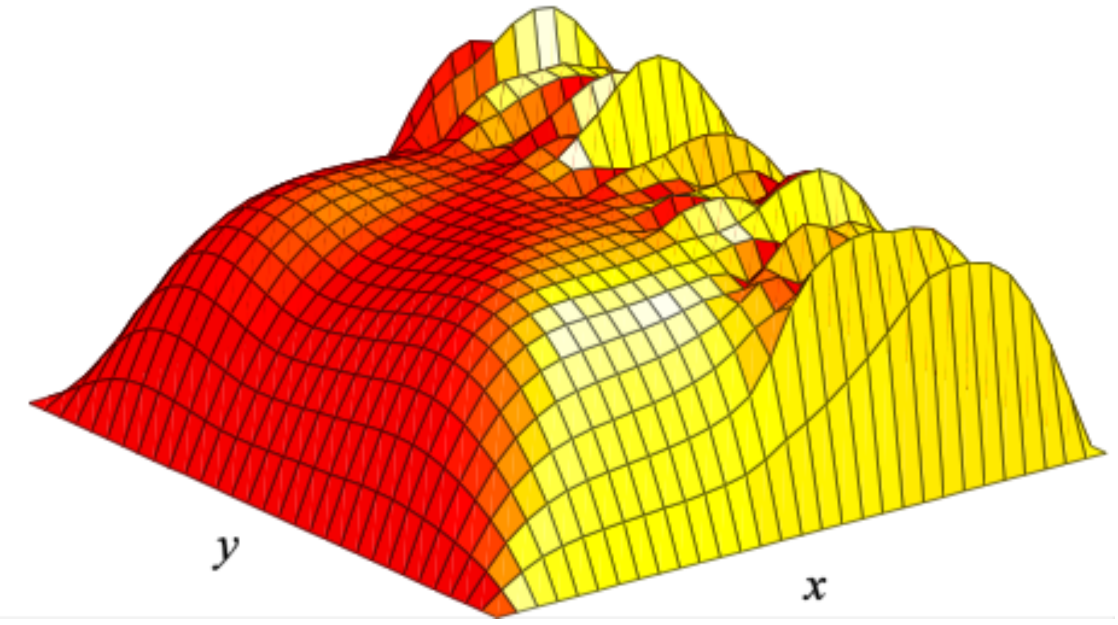


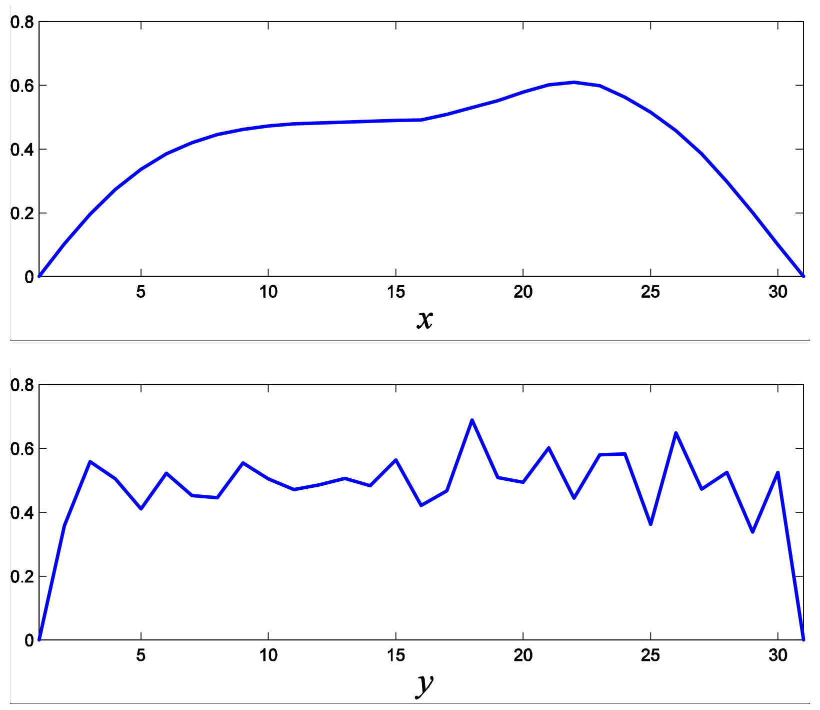
Error left by relaxation can be geometrically oscillatory

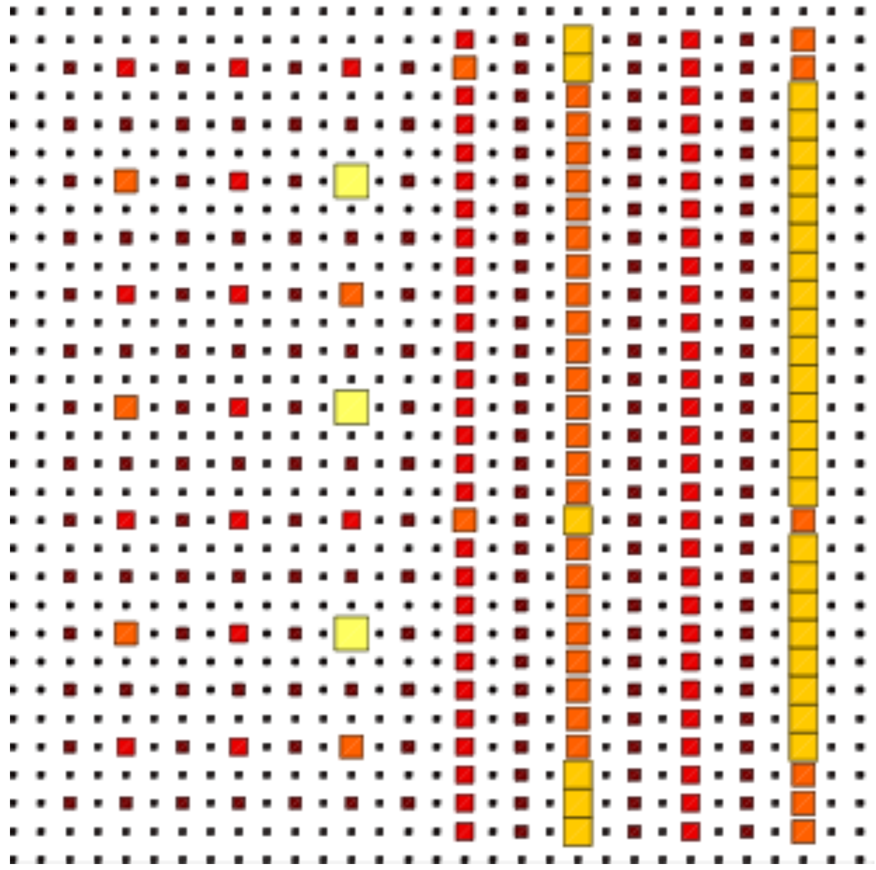
- 7 GS sweeps on

$$-au_{xx} - bu_{yy} = f$$

$a = b$	$a \gg b$







- Caution: this example
  - targets geometric smoothness
  - uses pointwise smoothers

Slide credit: R. Falgout, 2/1/2016

AMG coarsens grids in the direction
of geometric smoothness

Main idea: Algebraically smooth error

- Take a relaxation scheme such as w-Jacobi

$$e \leftarrow (I - M^{-1}A)e$$

- If relaxation stagnates, then the remaining error exhibits poor convergence, so

- Formally (characterized by small eigenvalues)

$$(I - M^{-1}A)e \approx e \Rightarrow M^{-1}Ae \approx 0 \Rightarrow r \approx 0$$

$$\langle Ae, e \rangle << 1$$

Main idea: Algebraically smooth error

- We then have

$$
\begin{aligned}
\langle Ae, e\rangle &= \sum_i e_i (A_{ii} e_i + \sum_{j\neq i} A_{ij} e_j) \\
&= \sum_i e_i \left(\sum_{j\neq i} -A_{ij}(e_i - e_j) \right) \\
&= \sum_{i<j} -A_{ij}\cdot e_i \cdot (e_i - e_j) + \sum_{i>j} -A_{ij}\cdot e_i \cdot (e_i - e_j) \\
&= \sum_{i<j} -A_{ij}\cdot e_i \cdot (e_i - e_j) - \sum_{i<j} -A_{ij}\cdot e_i \cdot (e_i - e_j) \\
&= \sum_{i<j} -A_{ij}\cdot (e_i - e_j)^2
\end{aligned}
$$

assume zero row sum

swap i, j

- Ok, so smooth error varies **slowly** in the direction of large matrix coefficients

Briggs, William L. and Henson, Van Emden and McCormick, Steve F., A Multigrid Tutorial (2Nd Ed., 20000

Take a look: `12-AMG-coarse-mesh.ipynb`

Main idea: Algebraically smooth error

- We have assumed **geometric** smoothness to show

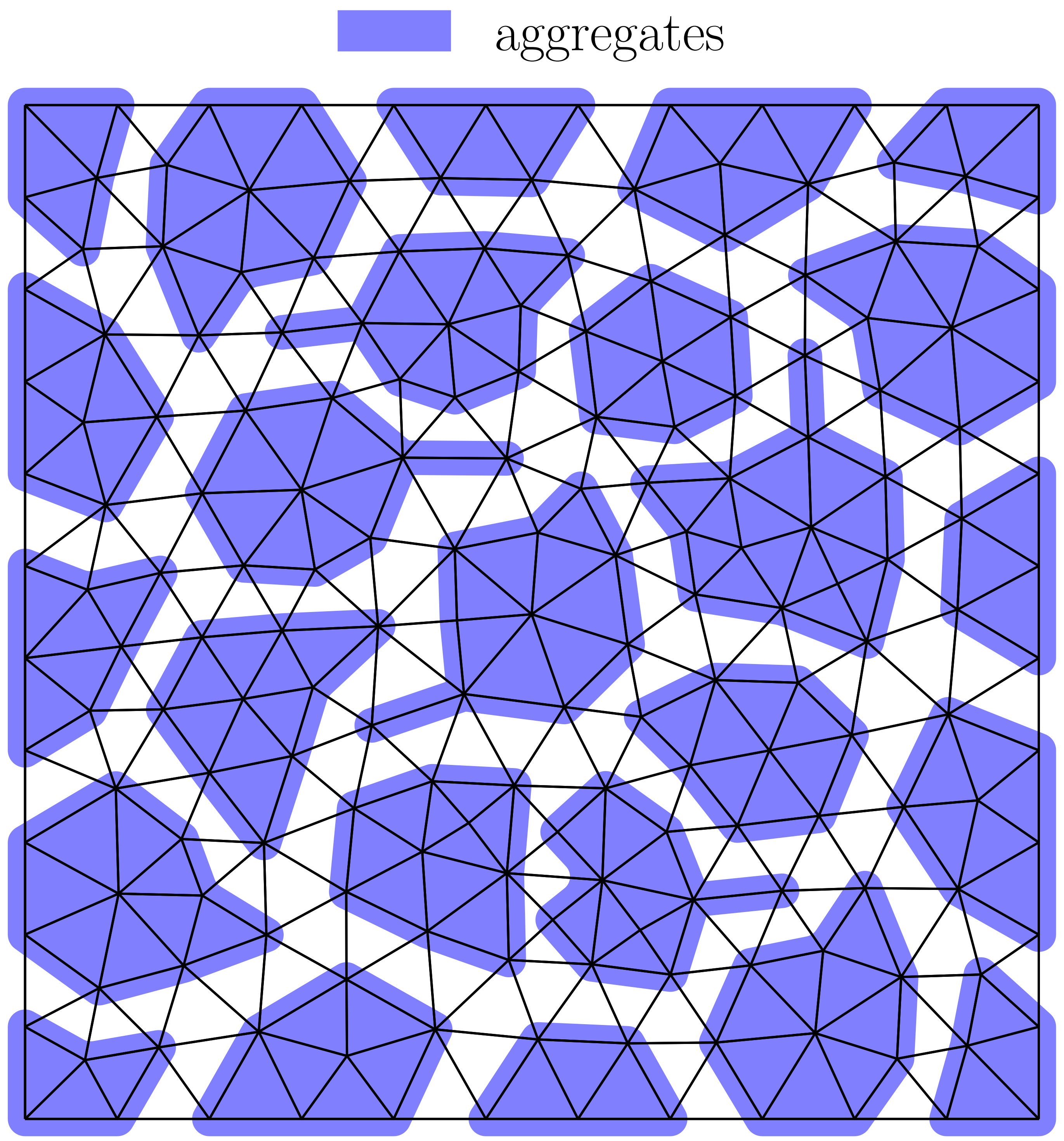
$$\mathbf{e}^T A\mathbf{e} = \sum_{i<j} (-a_{ij})(e_i - e_j)^2 \ll 1$$

- **CF AMG**: Smooth error varies slowly in the direction of "large" matrix coefficients
- **Strength of connection**: Given a threshold $0 < \theta \leq 1$, we say that variable u_i strongly depends on variable u_j if

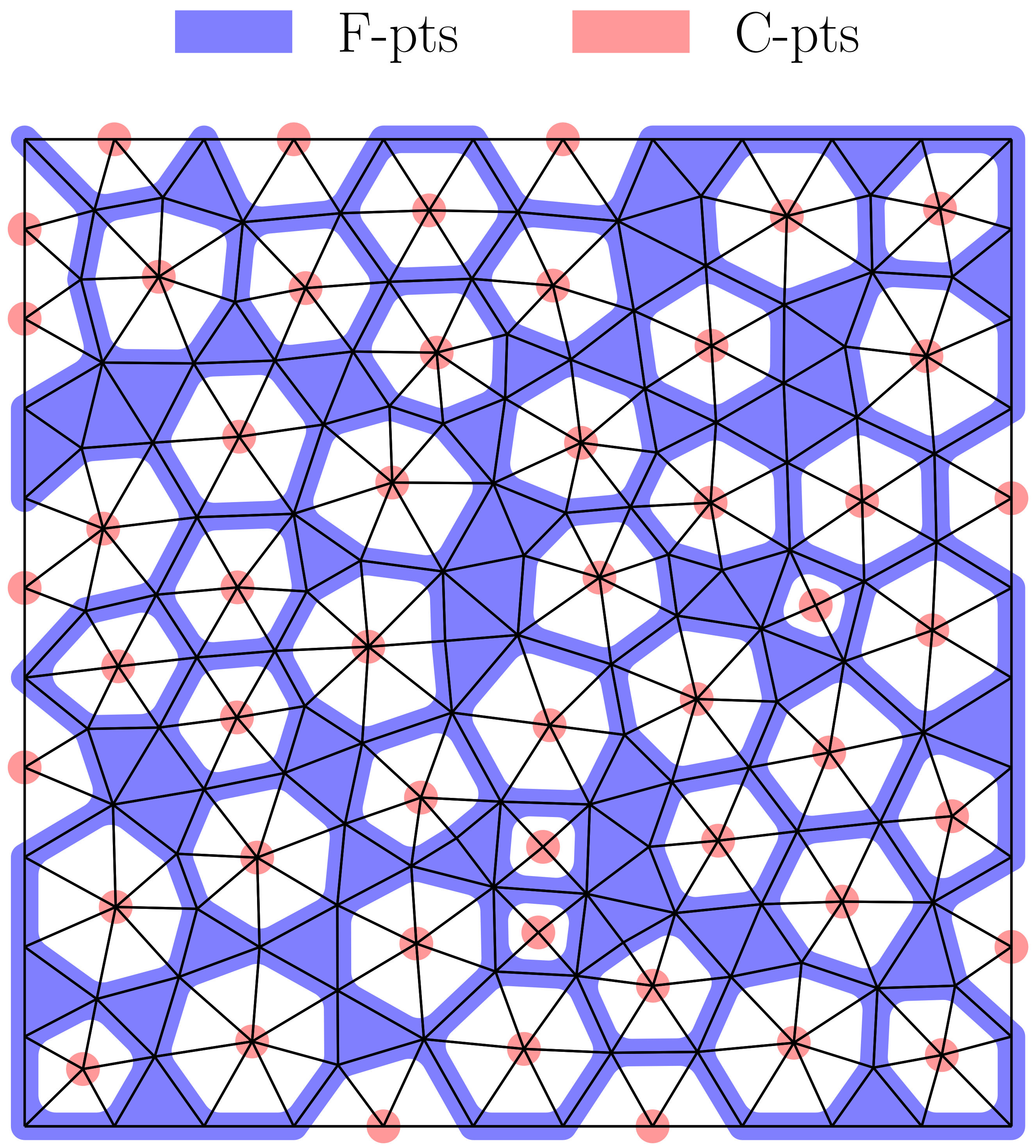
$$-a_{ij} \geq \theta \max_{k \neq i} \{-a_{ik}\}$$

- Often positive off-diagonals are treated as weak
- This definition of strength of connection is not symmetric

Two (general) forms of AMG



- Smoothed Aggregation AMG (SA-AMG)
- Interpolation constructed from candidate vectors
- Clear approach to *optimize* interpolation



- Coarse-Fine AMG (CF-AMG) or Ruge-Stüben
- Coarse grid points are a subset of the fine grid points
- Edge-wise construction of interpolation, allowing straightforward control of sparsity
- Incorporating near-nullspace is not straightforward