

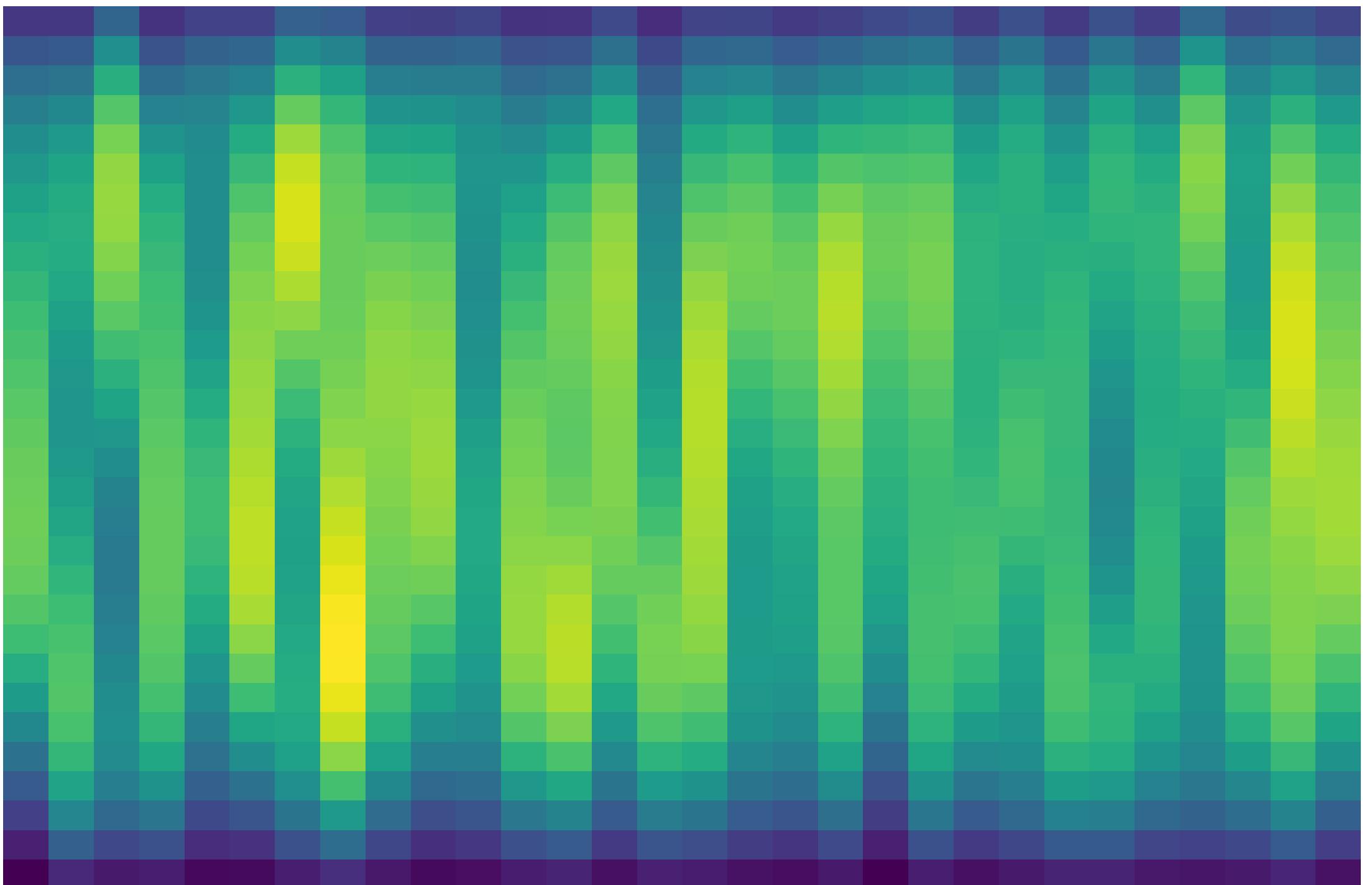
# Multigrid Methods – An Overview

Lecture 2: extensions

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# Objectives – high level

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## **1. Lecture 1 - Basics**

- Basic mechanics of a multigrid method
- 1D, 2D, Poisson

## **2. Lecture 2 - Extensions**

- What can go wrong and how to fix it

## **3. Lecture 3 - Algebraic**

- What to do if we do not have a grid (hierarchy)

## **4. Lecture 4 - Some Theory**

- Why does any of this work

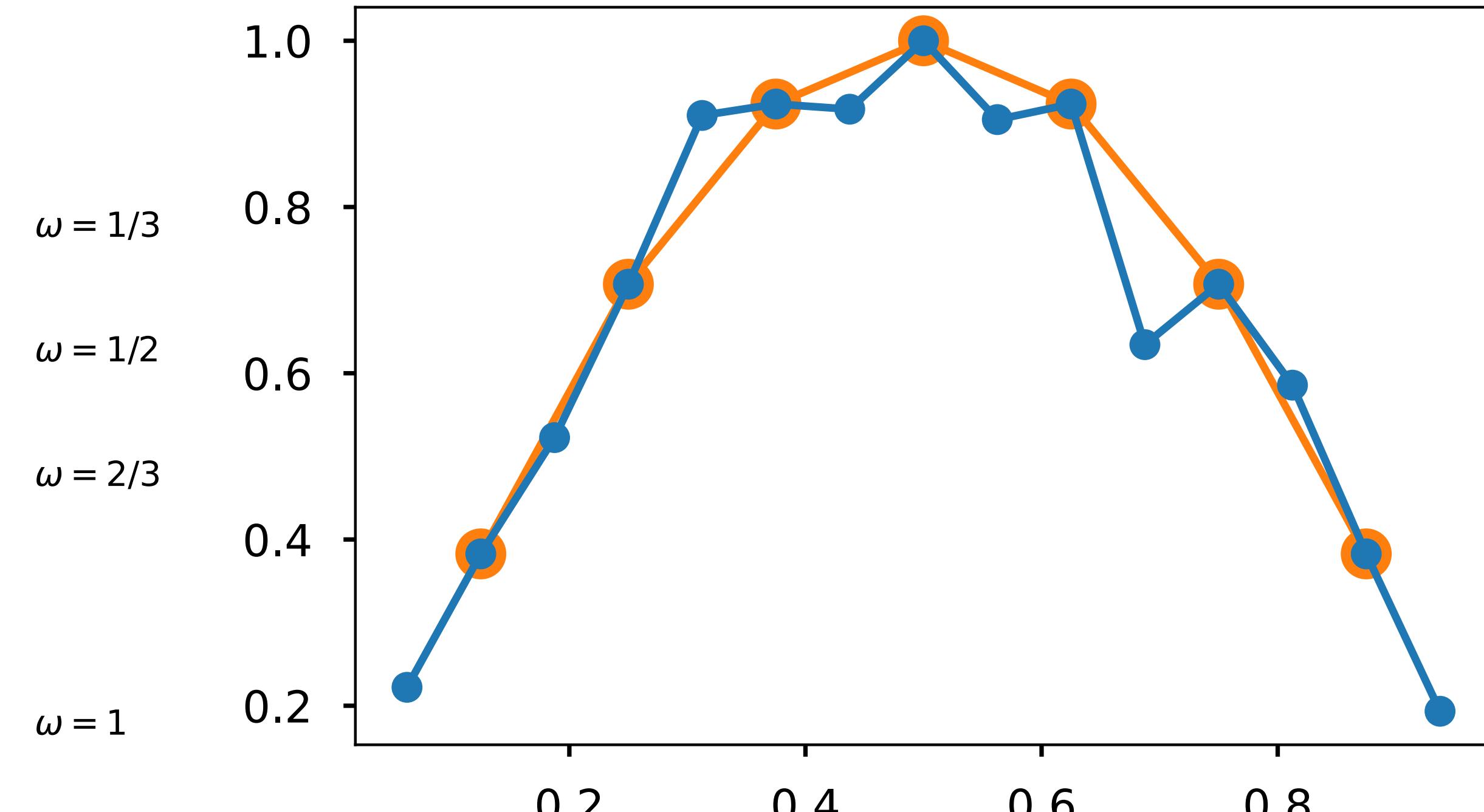
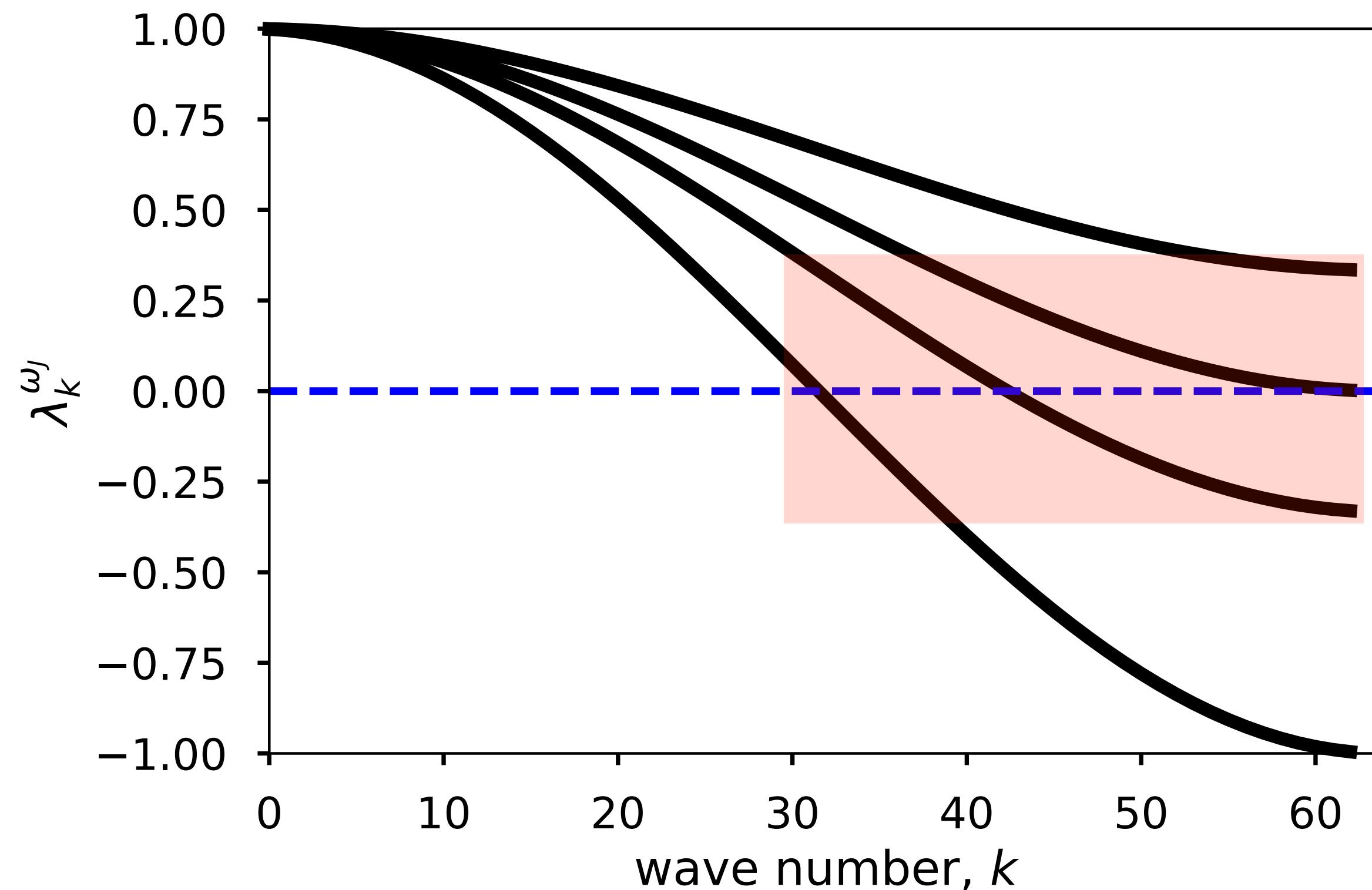
# Objectives – today

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## **1. Lecture 2 - Extensions to multigrid**

- Illustrate some limitations of multigrid
- Outline some approaches to fixing multigrid (and why)
- Identify key pieces in moving to unstructured problems

From last time ...



- Smoothing:  
Reduce high frequency error

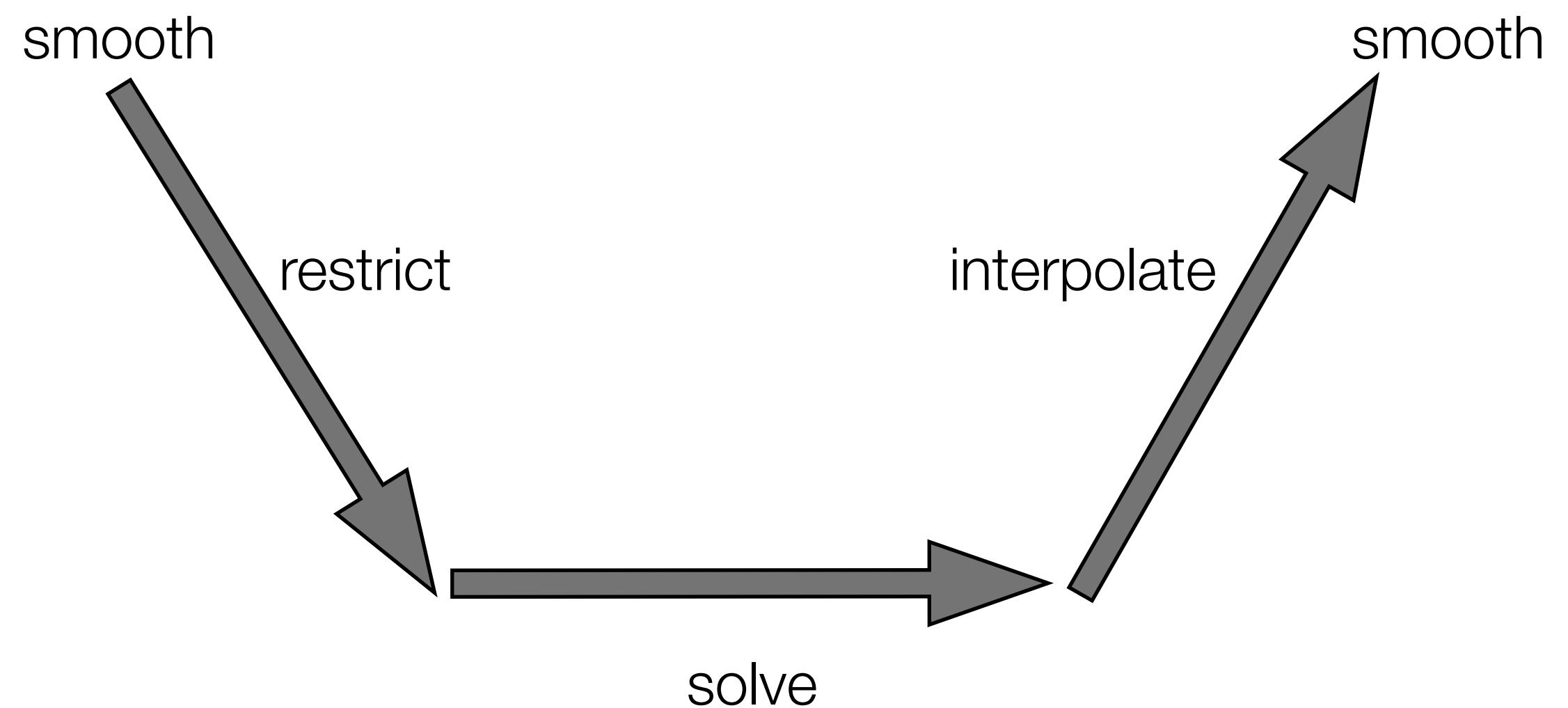
- Coarse-grid correction:  
Reduce smooth-ish things in the range of interpolation

$$e_1 = e_0 - P(P^T A P)^{-1} P^T A G e$$

# Algorithm: two-level multigrid

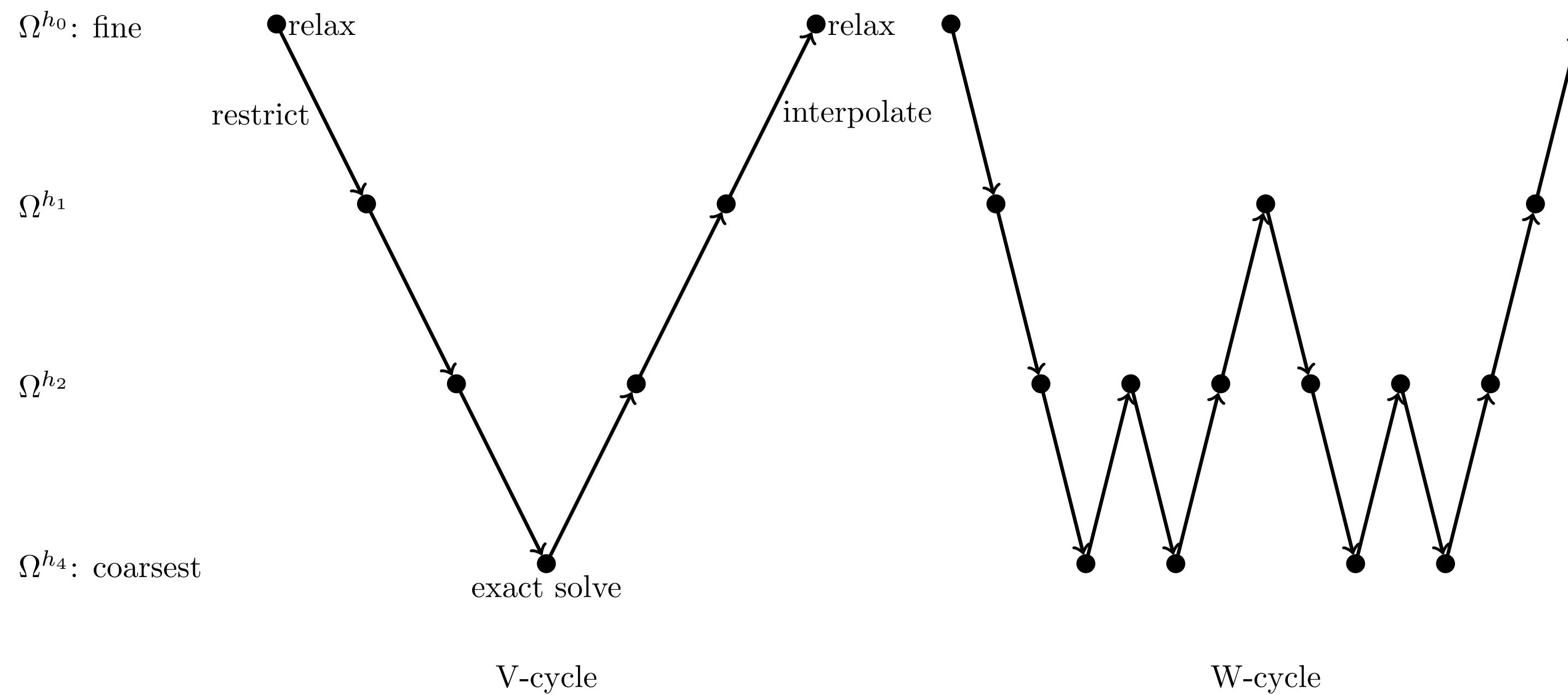
Input: initial guess

1. Smooth  $\nu_{pre}$  times on  $Au = f$
2. Compute  $r = f - Au$
3. Compute  $r_c = Rr$
4. Solve  $A_c e_c = r_c$
5. Interpolate  $\hat{e} = Pe_c$
6. Correct  $u \leftarrow u + \hat{e}$
7. Smooth  $\nu_{post}$  times on  $Au = f$



A two-level “V” cycle

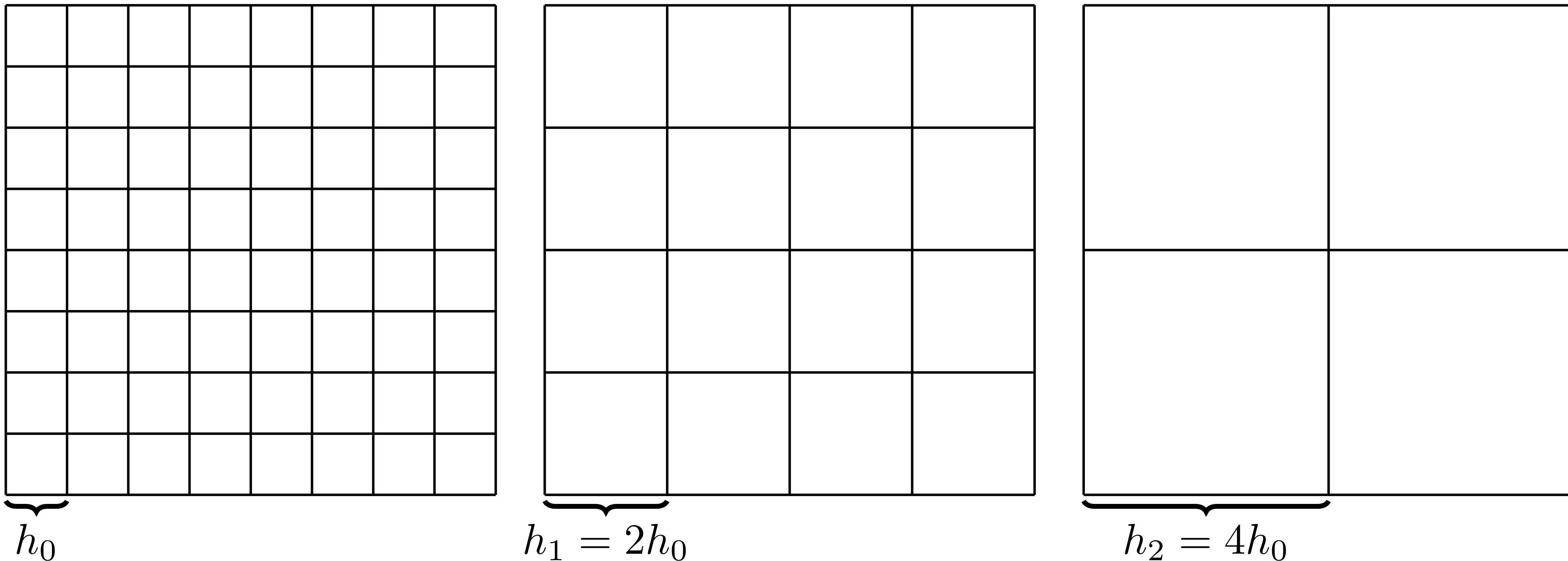
# The Multigrid V-Cycle and W-Cycle



- Two-grid cycle can expose issues with coarser interpolation
- W-Cycle can account for inadequate coarser level solves
- **Exact solve?** Usually a pseudo-inverse

# Multigrid in 2D

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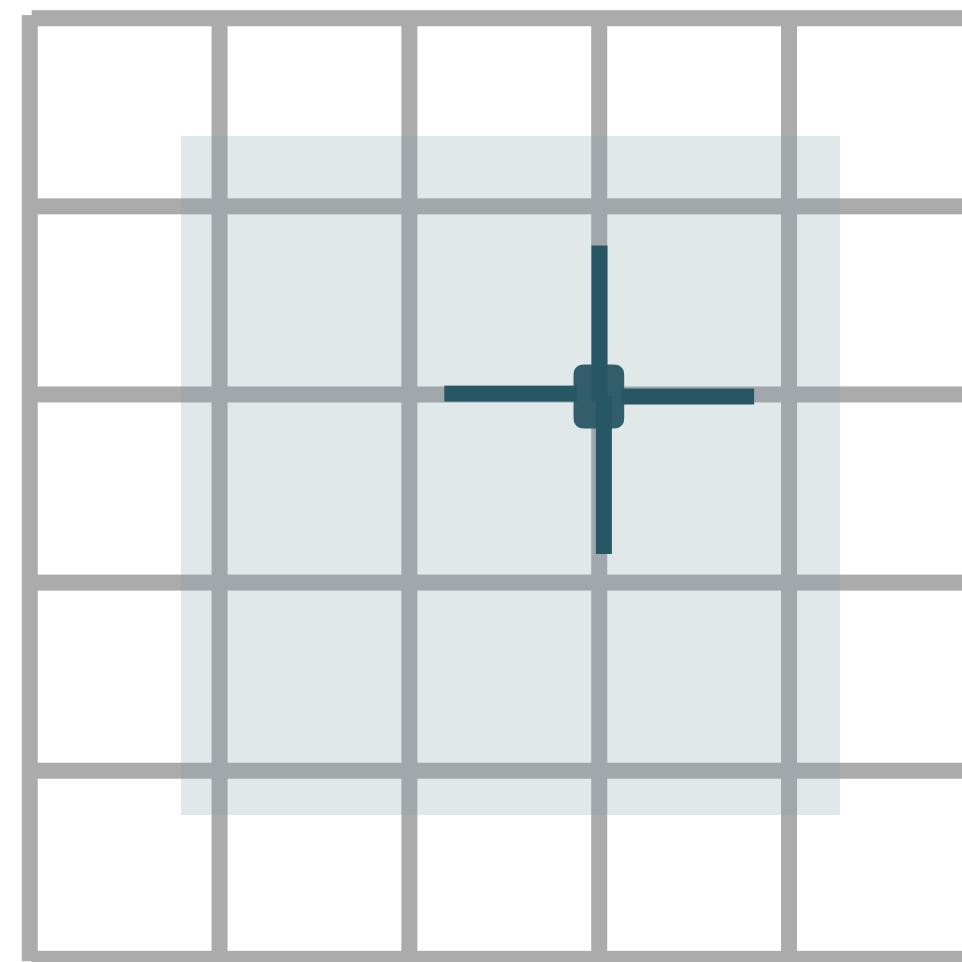
- Again, assume we have a series of uniform grids
- Relaxation remains the same (what is  $\omega$ ?)

# Multigrid in 2D

- Model problem

$$\begin{aligned} -u_{xx} - u_{yy} &= f \\ u &= 0 \quad \text{on boundary} \end{aligned}$$

- Results in the stencil / matrix



$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

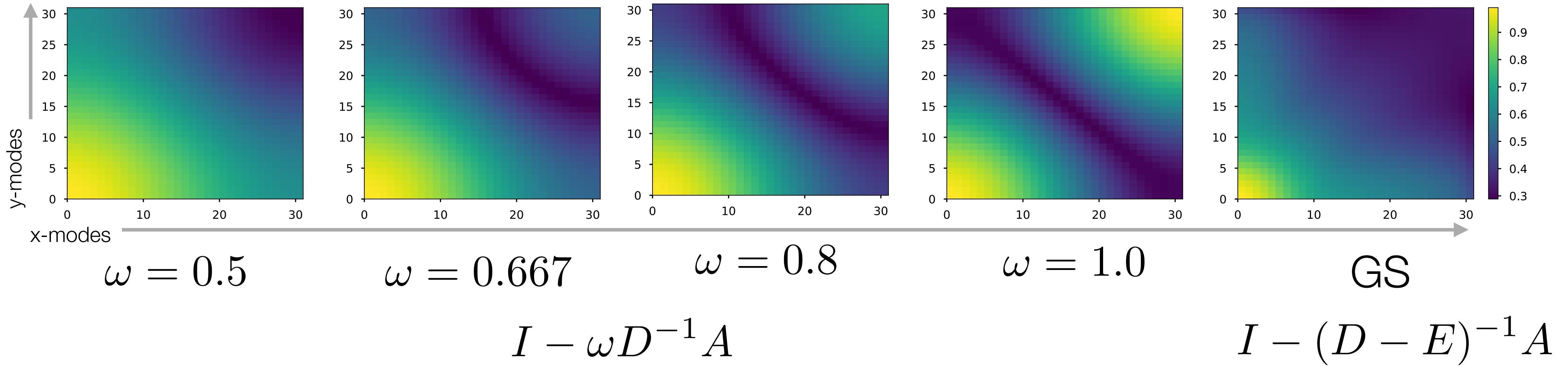


$$\frac{1}{h^2} \begin{bmatrix} 4 & -1 & & & -1 & & & \\ -1 & 4 & -1 & & & -1 & & \\ & -1 & 4 & -1 & & & -1 & \\ & & -1 & 4 & -1 & & & \\ & & & -1 & 4 & -1 & & \\ & & & & -1 & 4 & -1 & \\ & & & & & -1 & 4 & -1 \\ & & & & & & -1 & 4 \\ & & & & & & & -1 \\ & & & & & & & & \ddots \end{bmatrix}$$

# Relaxation in 2D

$$\sin\left(\frac{k_x i \pi}{n+1}\right) \sin\left(\frac{k_y j \pi}{n+1}\right)$$

Convergence factor  
over 10 sweeps



- weighted Jacobi: Same issue – need to select a parameter
- Gauss-Seidel improved
- Red-Black Gauss-Seidel and other schemes even more effective

# Interpolation in 2D

- Bilinear interpolation, tensor of 1D interpolation

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

- Example  $3 \times 3 \rightarrow 7 \times 7$

$$\begin{bmatrix} 0.5 \\ 1 \\ 0.5 & 0.5 \\ 1 \\ 0.5 & 0.5 \\ 1 \\ 0.5 \end{bmatrix}$$



$$\begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 0.5 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

# Notebook

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- 9-multigrid-2d.ipynb

## A few observations so far: **One**

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- Let's consider a V(1,1) cycle – weighted Jacobi, etc.
- The **error** propagation for this looks like

$$e_1 = \frac{G(I - P(P^T A P)^{-1} P^T A) G e_0}{M}$$

$$G = I - \omega D^{-1} A$$

$$M e_k \leftarrow 0?$$

- One thing we can do, is consider bounds on each operation.

# A few observations so far: **One**

---

- Take the operator

$$M = G(I - P(P^T A P)^{-1} P^T A)G$$

- And makes some bounds

$$\|I - P(P^T A P)^{-1} P^T A\| \|G\|^2$$

The diagram shows the expression  $\|I - P(P^T A P)^{-1} P^T A\| \|G\|^2$  decomposed into two parts. A horizontal blue bar spans the width of  $P(P^T A P)^{-1}$ , and a green bar spans the width of  $P^T A$ . Below the blue bar is the text "approximation property" and below the green bar is the text "smoothing property".

approximation property    smoothing property

- General  $\|G\| \leq 1$

1D over  $[n/2, n]$ :

$$\|G\| \leq \frac{1}{3}$$

2D over  $[n/2, n]$ :

$$\|G\| \leq \frac{3}{5}$$

# A few observations so far: **One**

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$$\|I - P(P^T A P)^{-1} P^T A\| \|G\|^2$$

approximation property    smoothing property

- General  $\|G\| \leq 1$

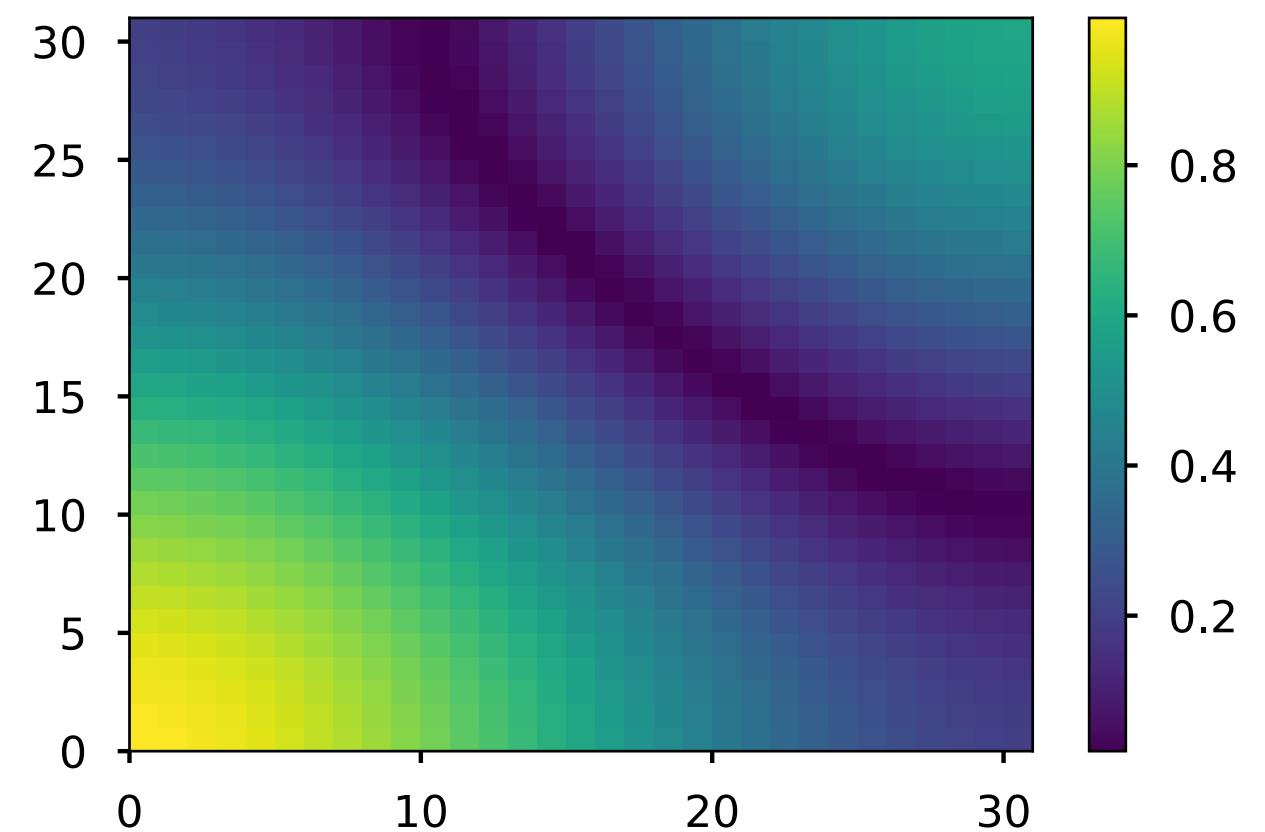
$$\text{1D over } [n/2, n]: \quad \|G\| \leq \frac{1}{3}$$

$$\text{2D over } [n/2, n]: \quad \|G\| \leq \frac{3}{5}$$

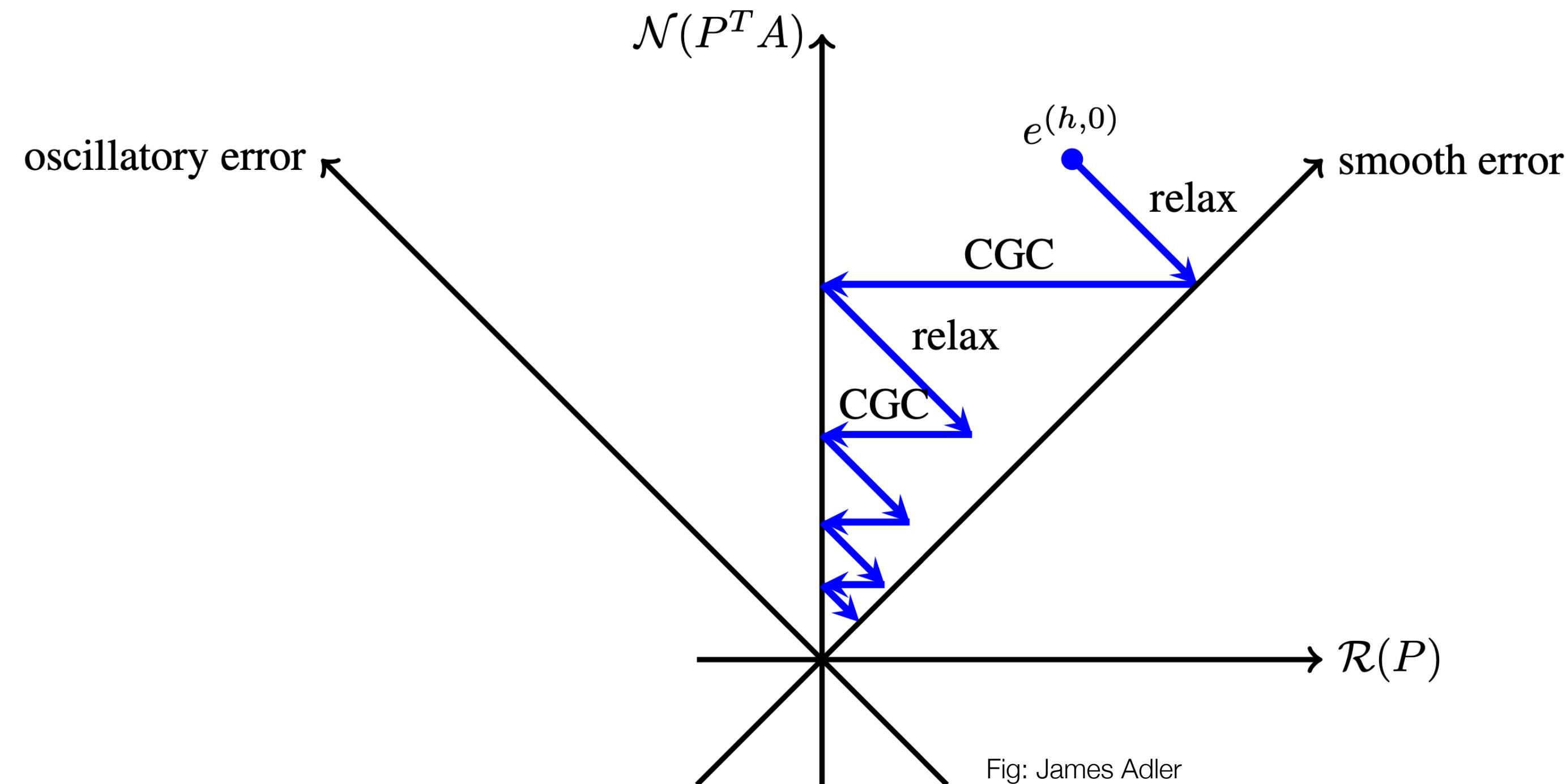
- Also, if  $w$  s.t.  $Aw \in \mathcal{N}(P^T)$

then  $(I - P(P^T A P)^{-1} P^T A)w = w$

$$\|\cdot\| \geq 1$$

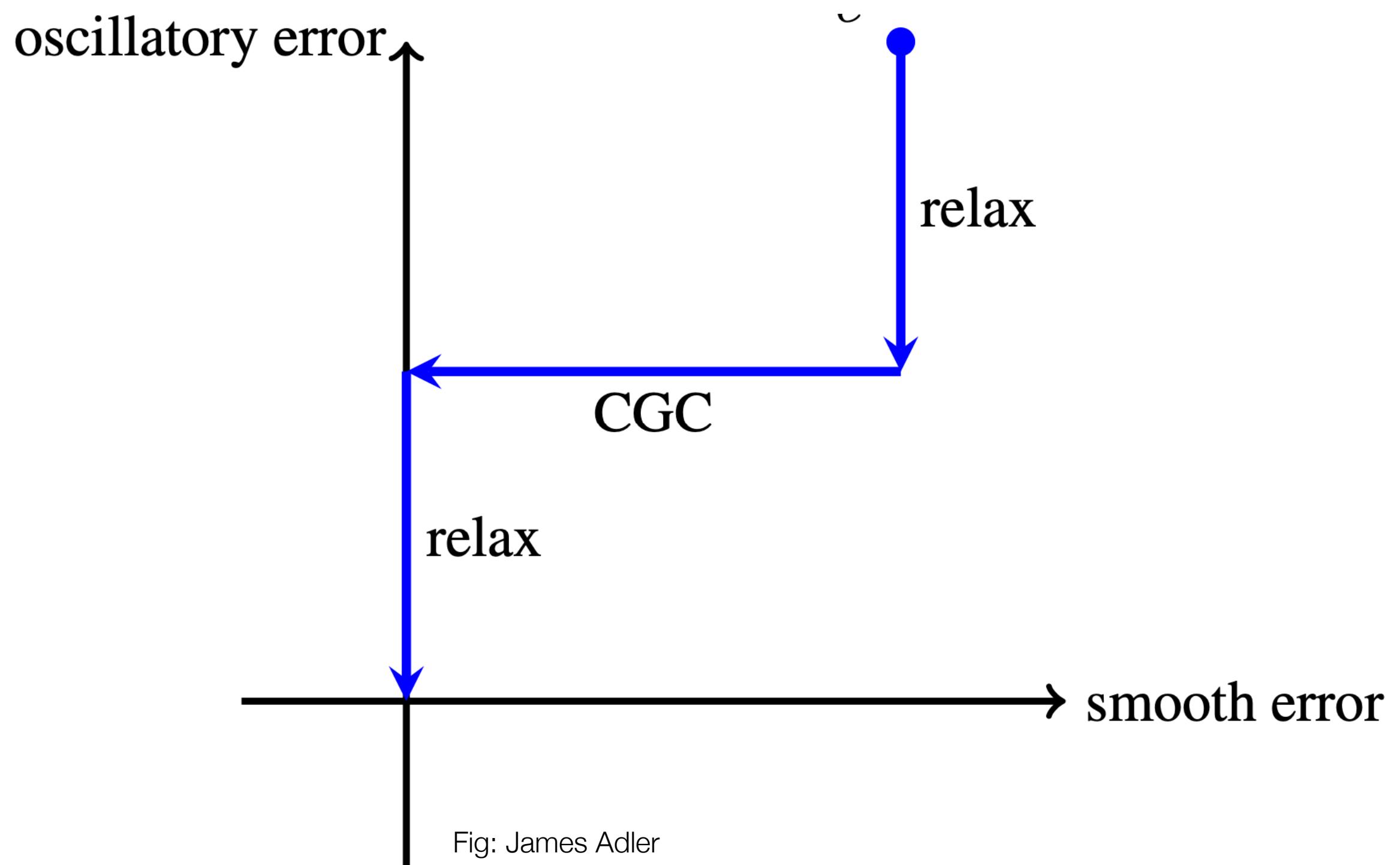


# A few observations so far: **Two**



- Complementary processes:
  - **relaxation:** targets for (Fourier) smoothing
  - **coarse grid correction:** targets things in the range of interpolation

# A few observations so far: **Three**



$$e_1 \leftarrow (I - P(P^T A P)^{-1} P^T A) G e_0$$

$$G e_0 \in \mathcal{R}(P) \Rightarrow e_1 = 0$$

interpolation should capture what relaxation misses

Fig: James Adler

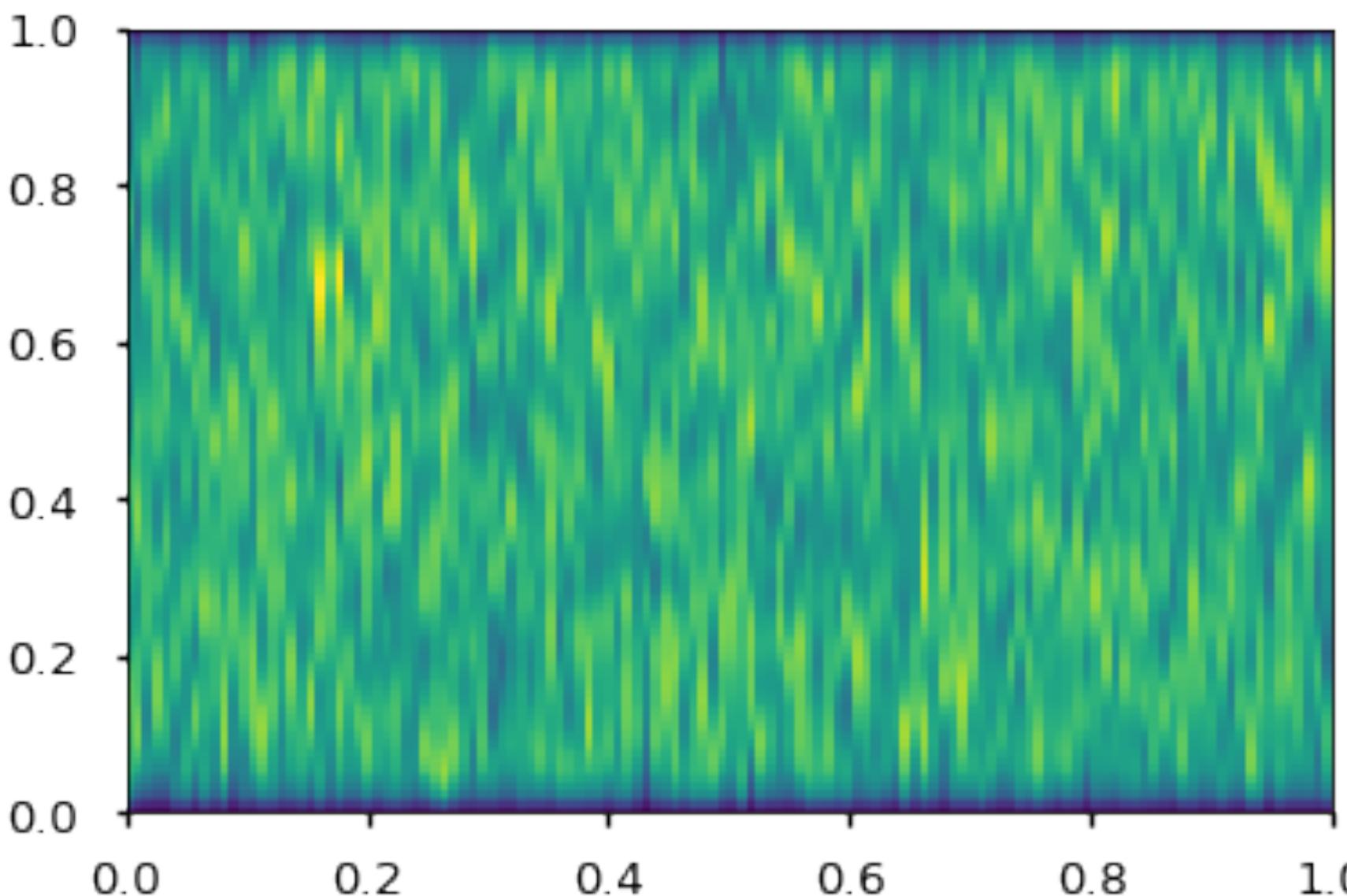
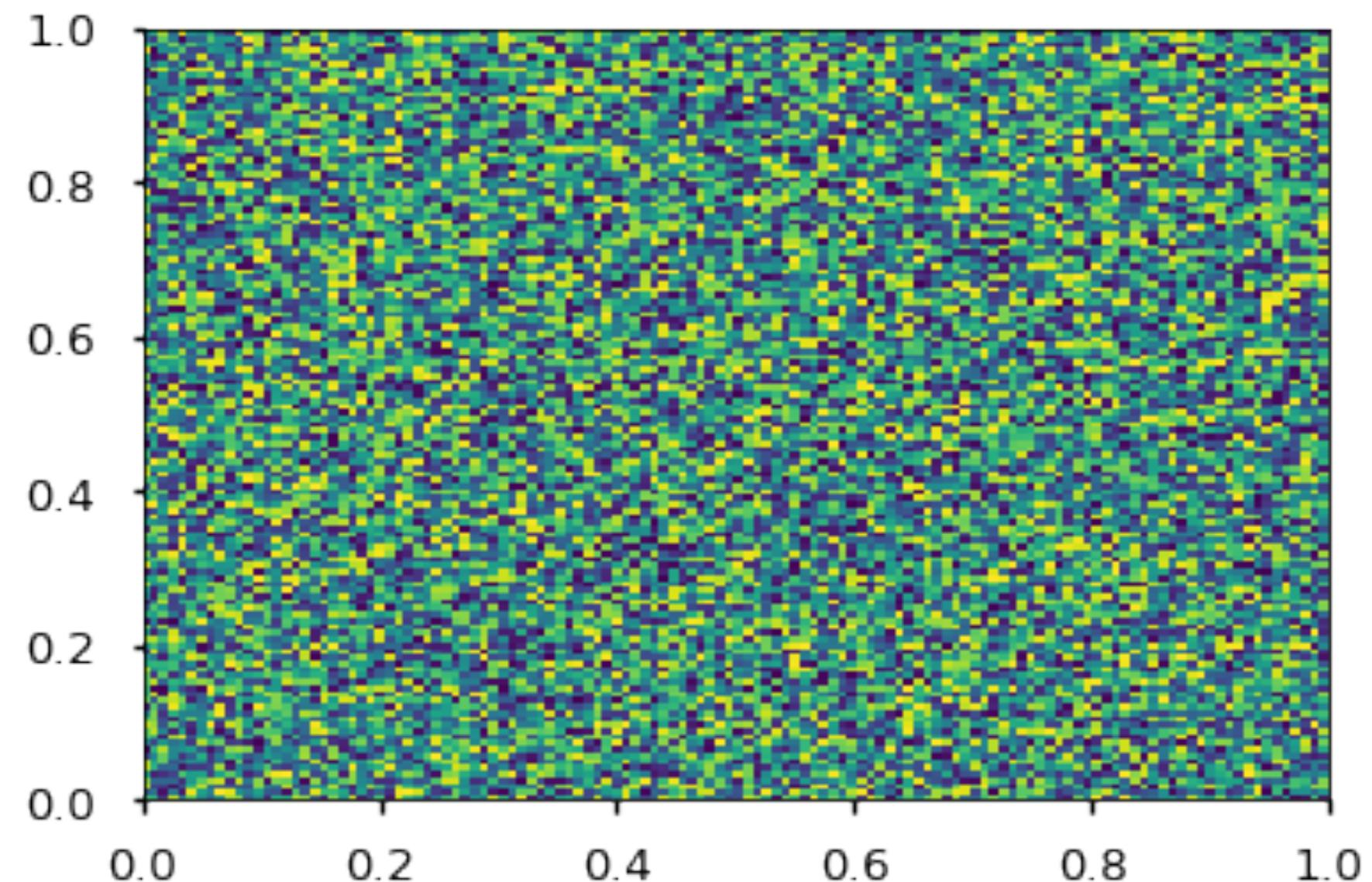
# What can go wrong?!

- Demo: 10-multigrid-anisotropy.ipynb

Consider an *anisotropic problem*

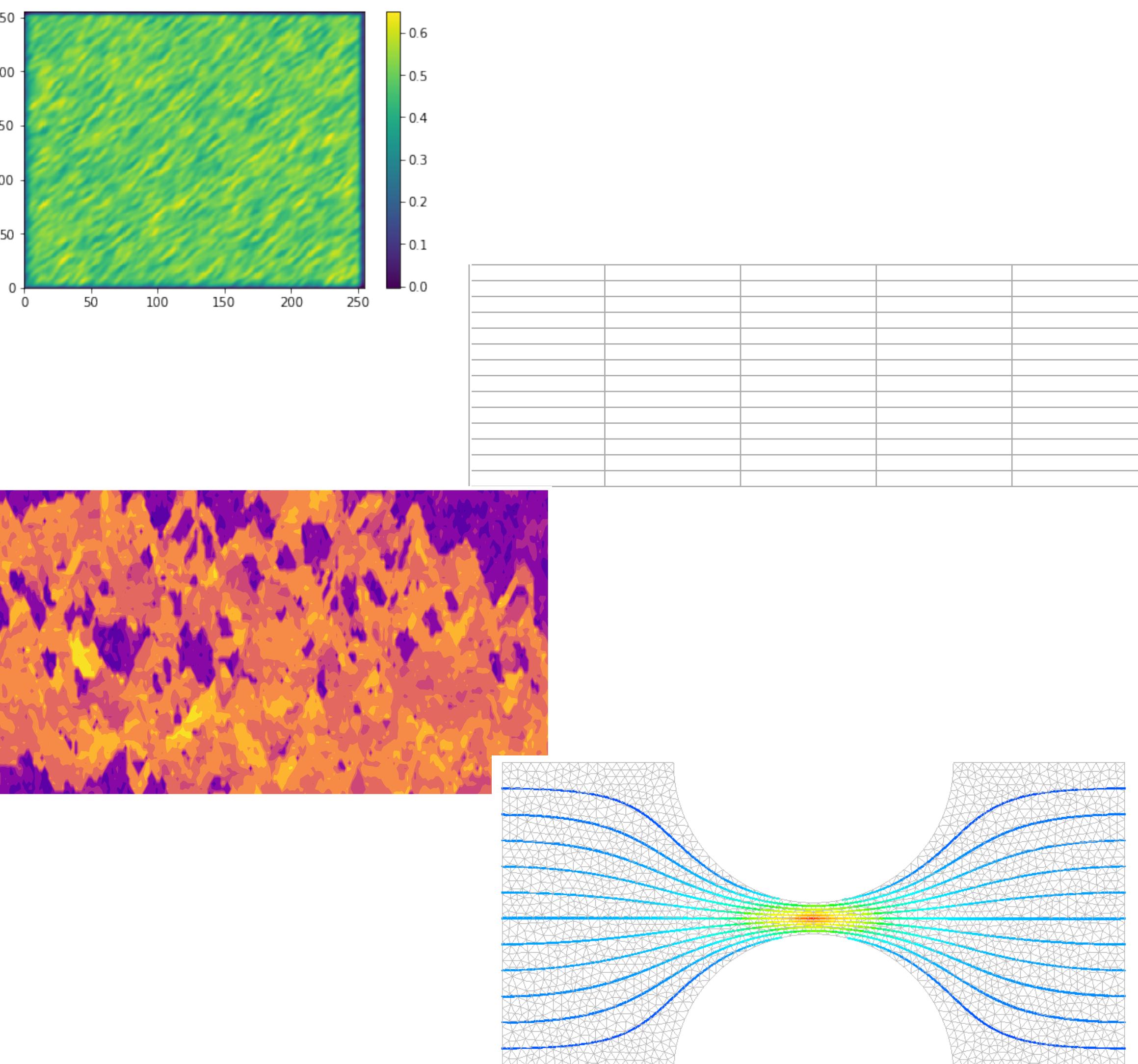
$$\begin{aligned} -\varepsilon u_{xx} - u_{yy} &= f \\ u &= 0 \quad \text{on boundary} \end{aligned}$$

$$\begin{bmatrix} & & -1 \\ -\varepsilon & 2 + 2\varepsilon & -\varepsilon \\ & -1 & \end{bmatrix}$$



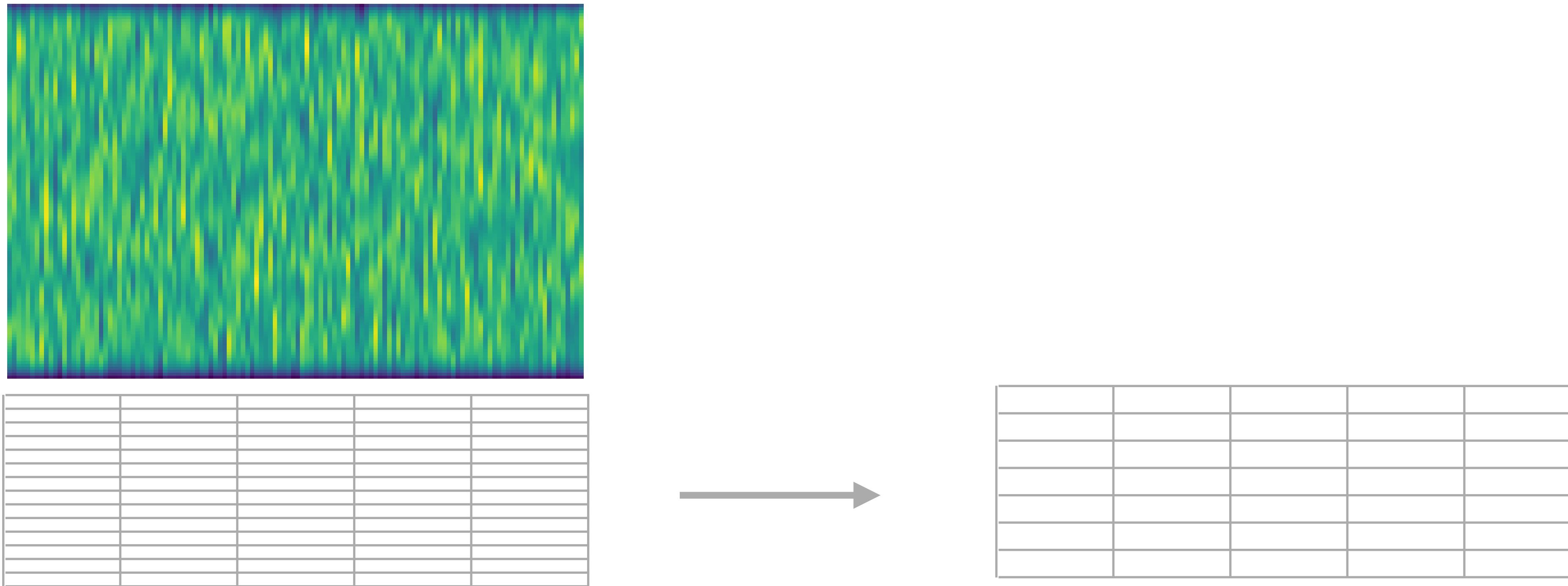
# What can go wrong?!

- Anisotropy
- Mesh stretching
- Jumping coefficients
- Non-elliptic



# Options for more robust Multigrid

- **Semicoarsening:** Coarsen in the direction of smoothness



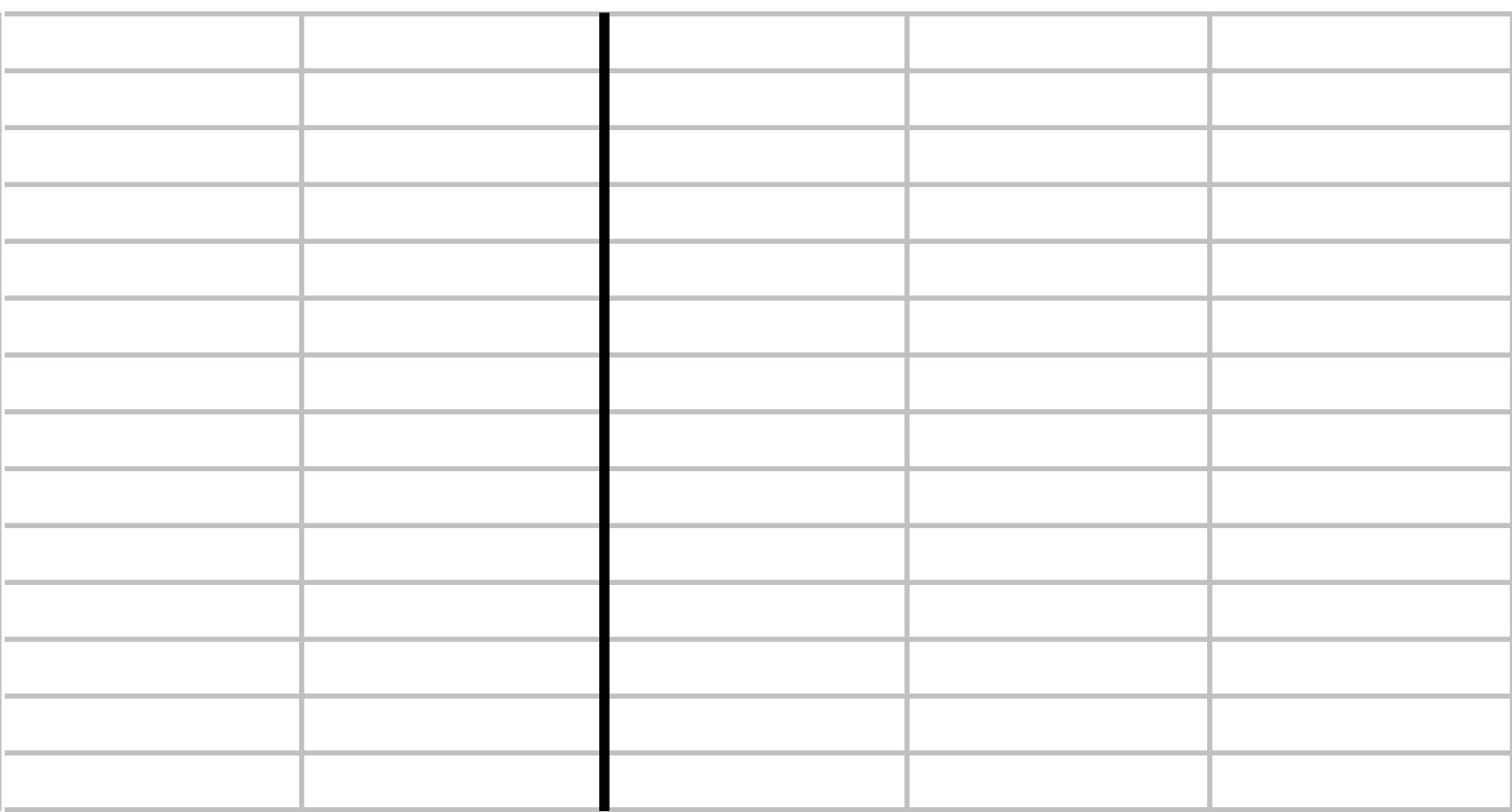
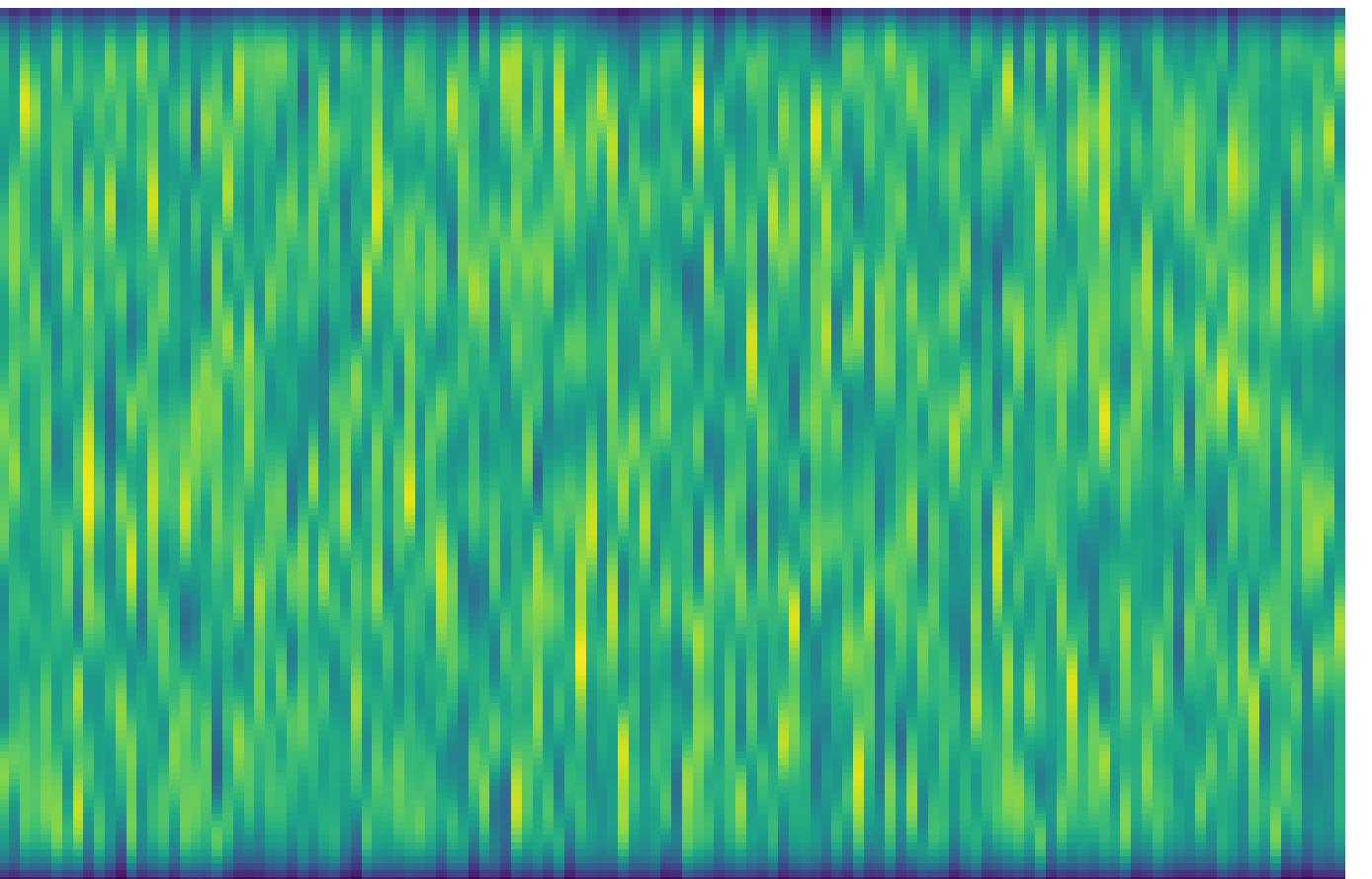
- Downside: if anisotropy varies in another direction, we need a different grid

# Options for more robust Multigrid

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- **Line/plane relaxation**

Perform relaxation in groups (in a line)

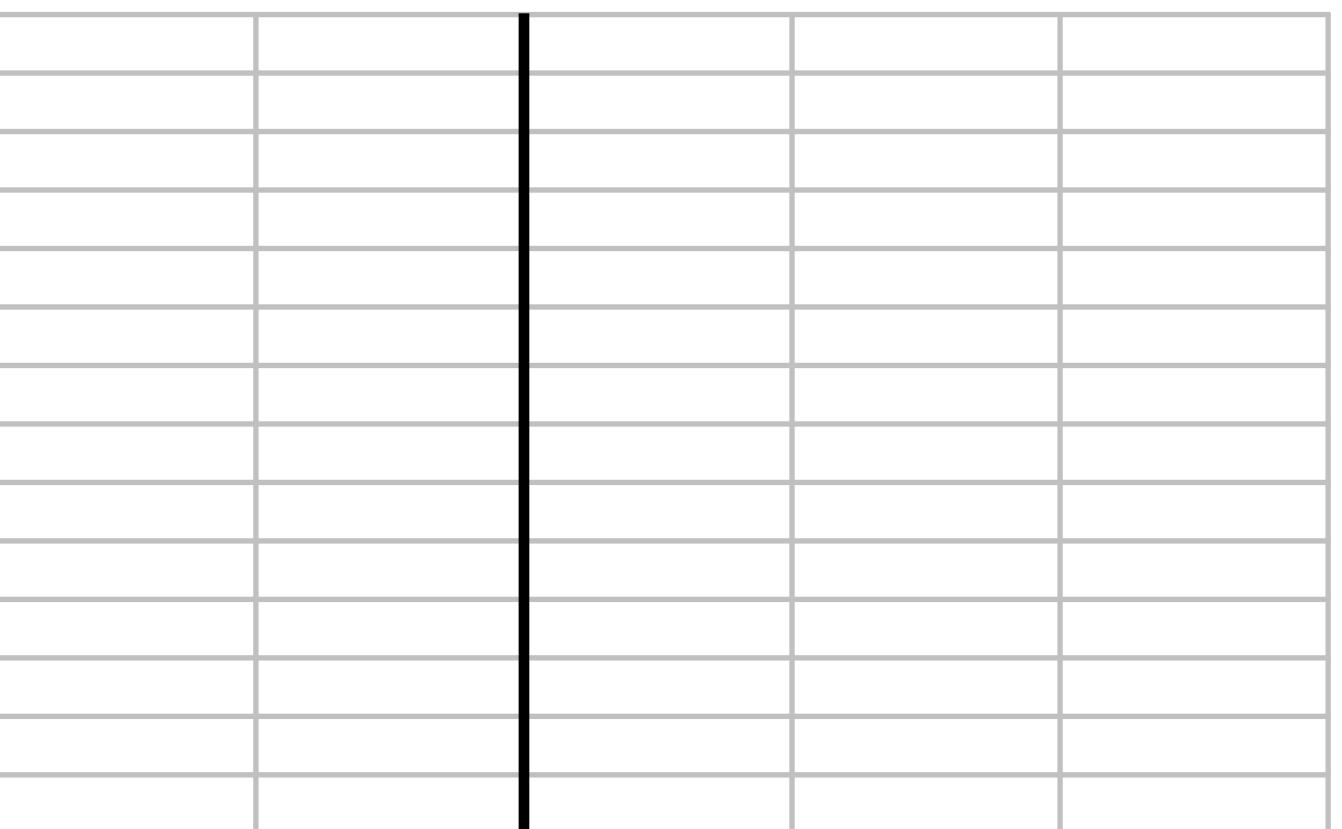
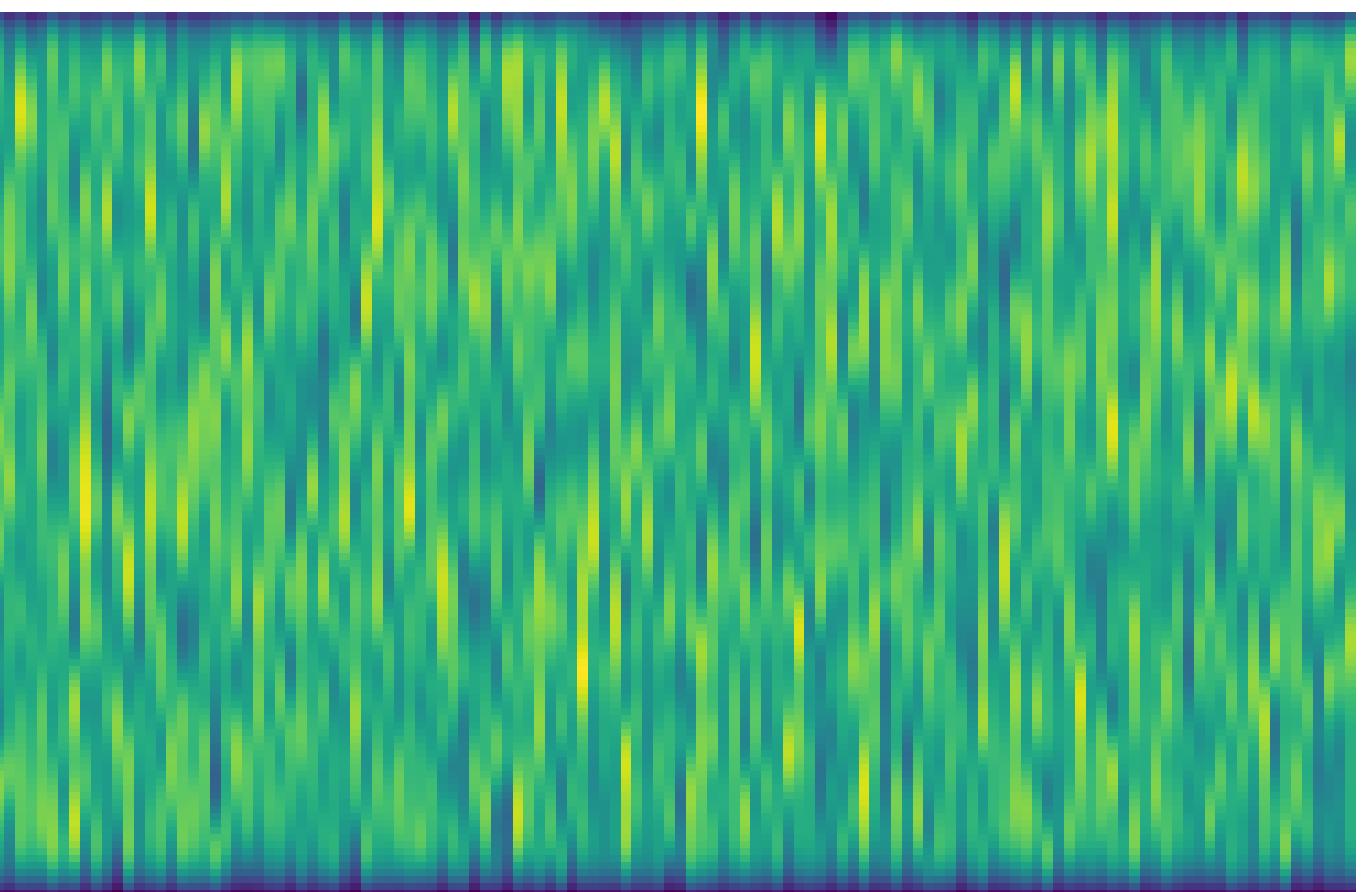


- Example:
  - smooth error in the y-direction (x-lines)
  - no smoothing in the x-direction (y-lines)

# Options for more robust Multigrid

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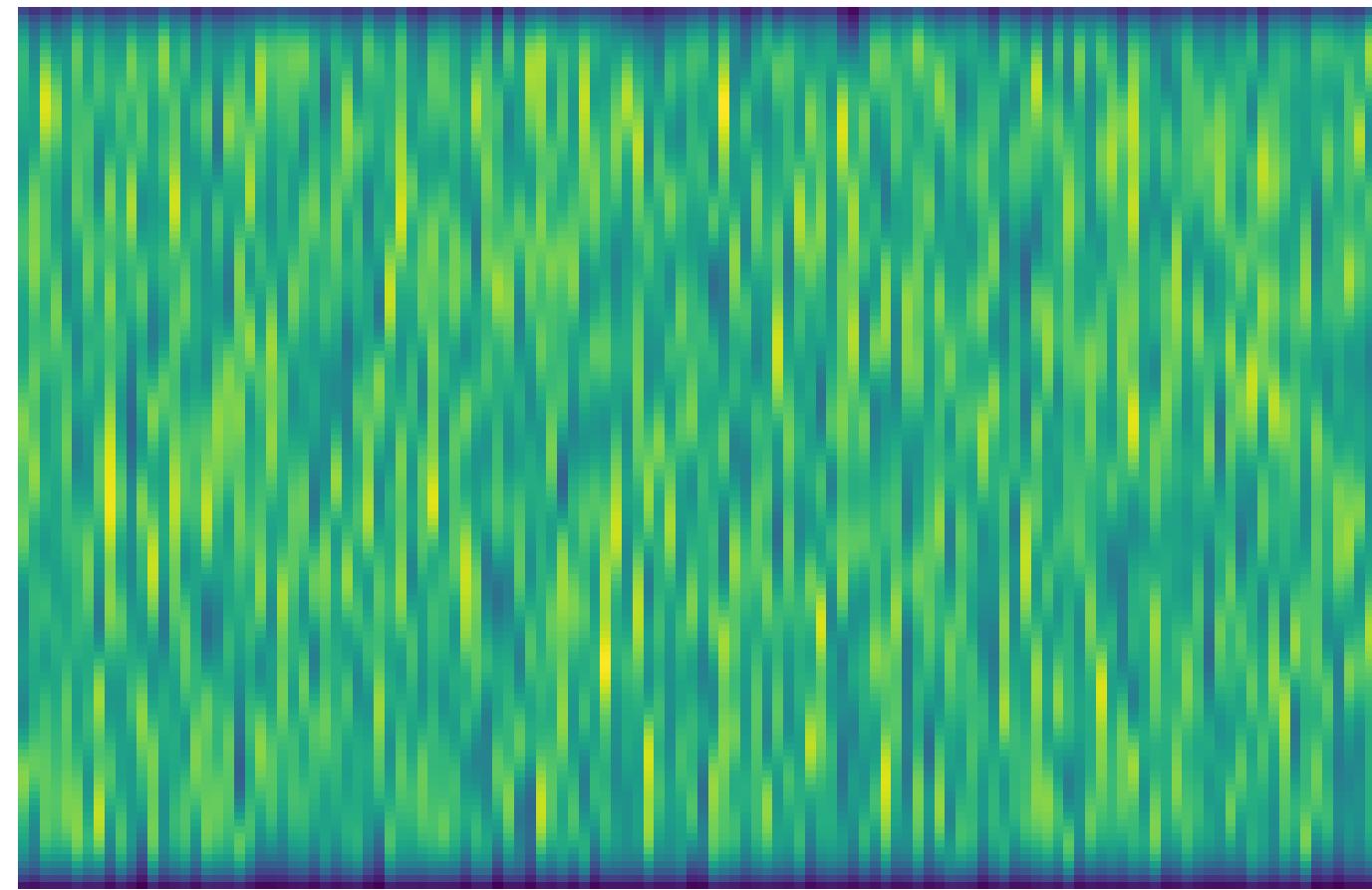
- **Line/plane relaxation**
- For each x-line (lines of strong anisotropy):
  - Eliminate the residual on the entire line
  - (Gauss-Siedel, by lines)



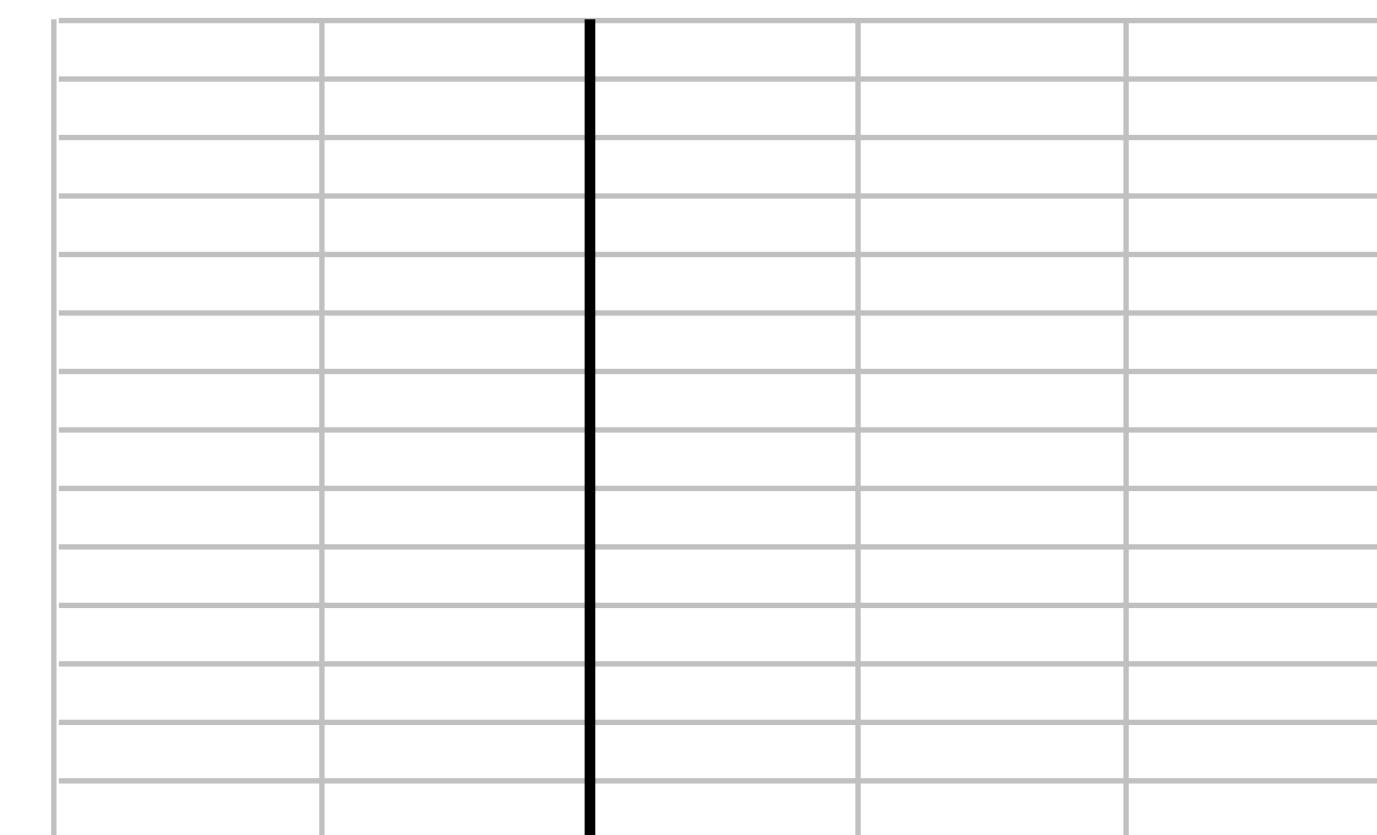
# Options for more robust Multigrid

- **Line/plane relaxation**

- $$A = (n + 1)^2 \begin{bmatrix} A_{1D} & -\varepsilon I & & \\ -\varepsilon I & A_{1D} & -\varepsilon I & \\ & -\varepsilon I & A_{1D} & -\varepsilon I \\ & & \ddots & \\ & & & -\varepsilon I & A_{1D} \end{bmatrix}$$



$$A_{1D} = (n + 1)^2 \begin{bmatrix} 2 + 2\varepsilon & -1 & & \\ -1 & 2 + 2\varepsilon & -1 & \\ & \ddots & \ddots & \\ & & -1 & 2 + 2\varepsilon \end{bmatrix}$$



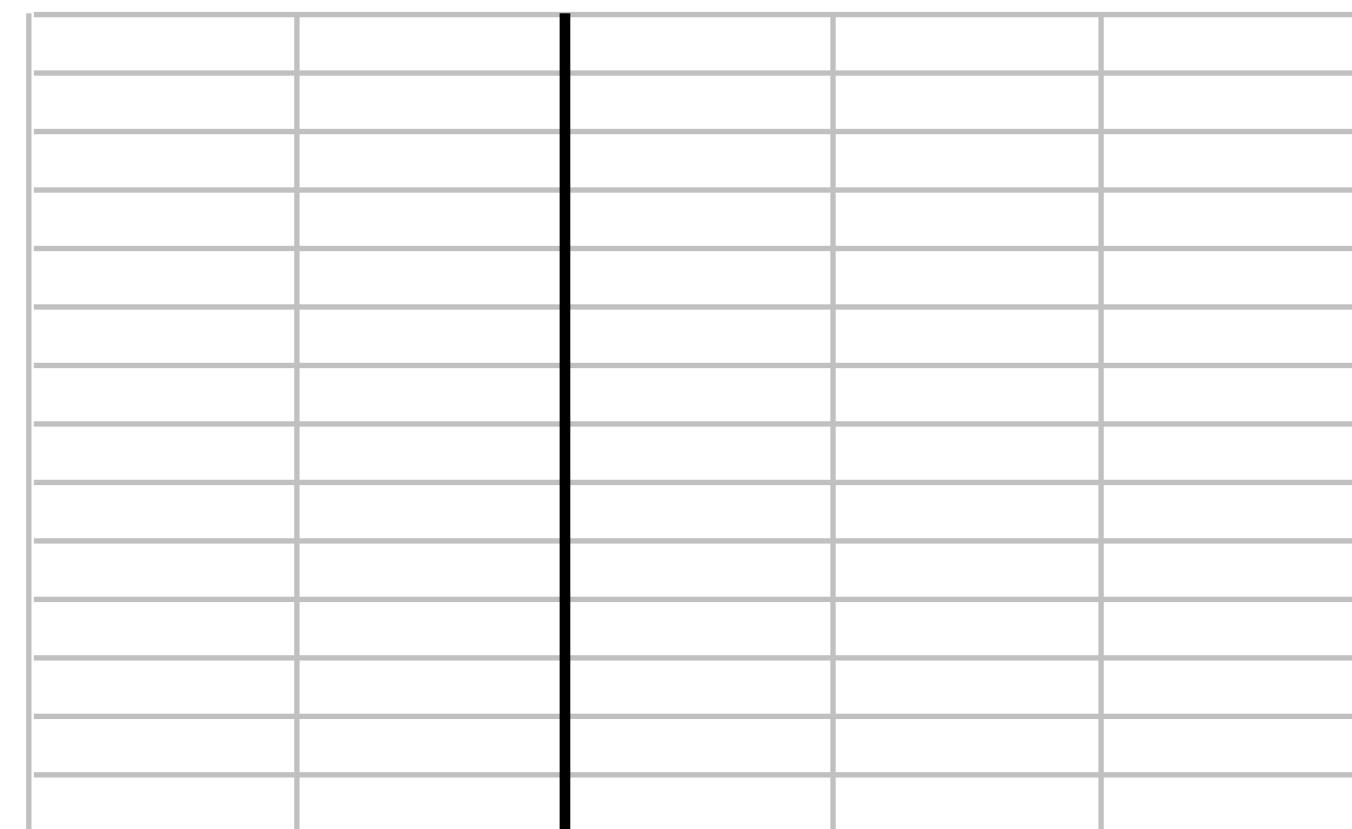
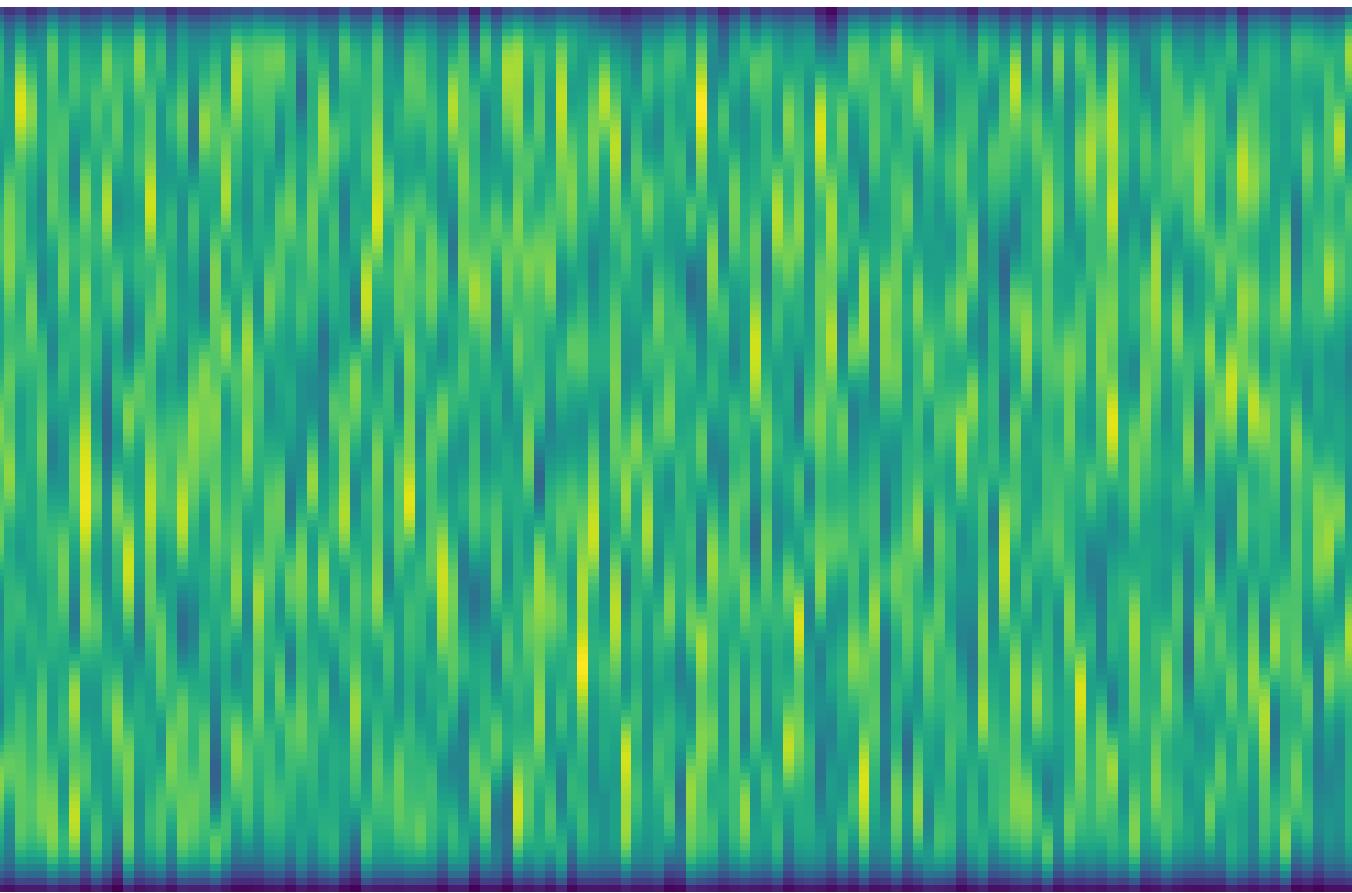
# Options for more robust Multigrid

- **Line/plane relaxation**
- For each line, solve

$$A_{1D}v_k = g_k$$

$$v_k = \begin{bmatrix} \vdots \\ v_{k,j-1} \\ v_{k,j} \\ v_{k,j+1} \\ \vdots \end{bmatrix} \quad g_k = \begin{bmatrix} \vdots \\ f_{k,j-1} + \varepsilon(v_{k-1,j-1} + v_{k+1,j-1}) \\ f_{k,j} + \varepsilon(v_{k-1,j} + v_{k+1,j}) \\ f_{k,j+1} + \varepsilon(v_{k-1,j+1} + v_{k+1,j+1}) \\ \vdots \end{bmatrix}$$

- In 3D, lines become planes...



# Notebook

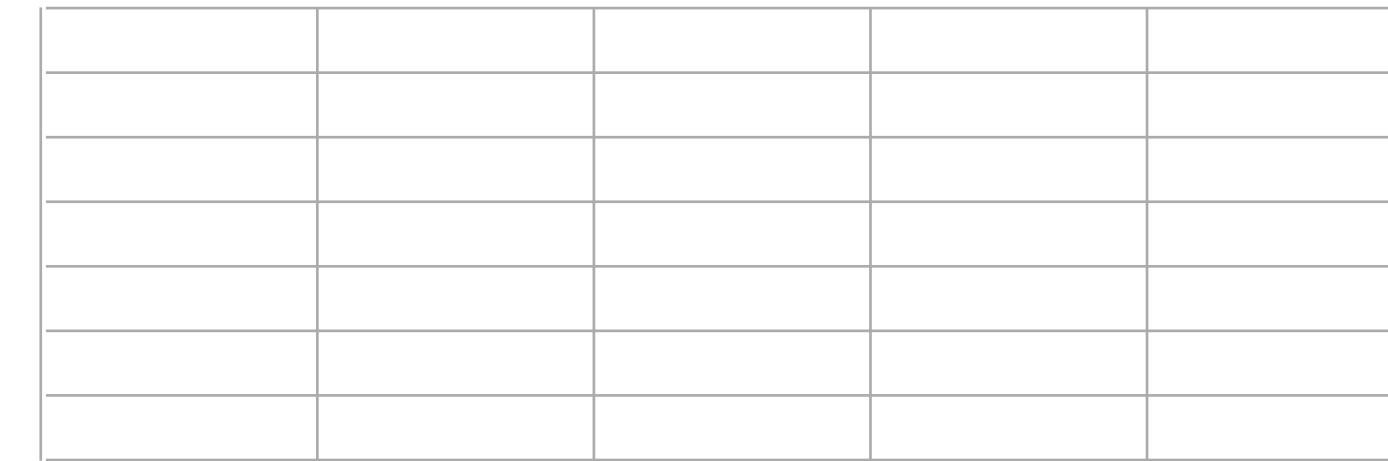
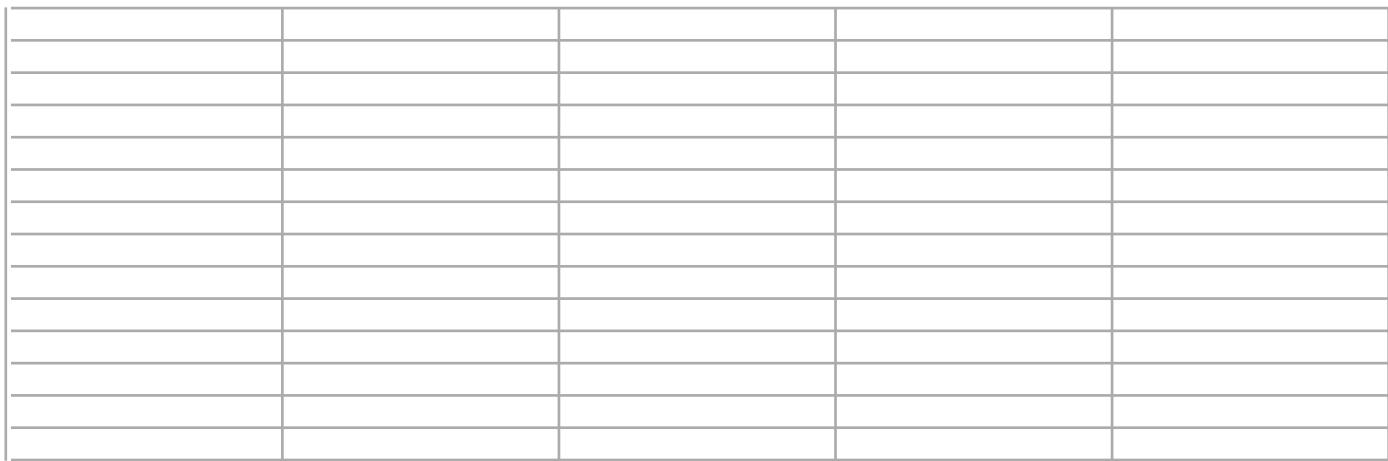
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- `11-multigrid-2d-line-relaxation.ipynb`

# Options for more robust Multigrid

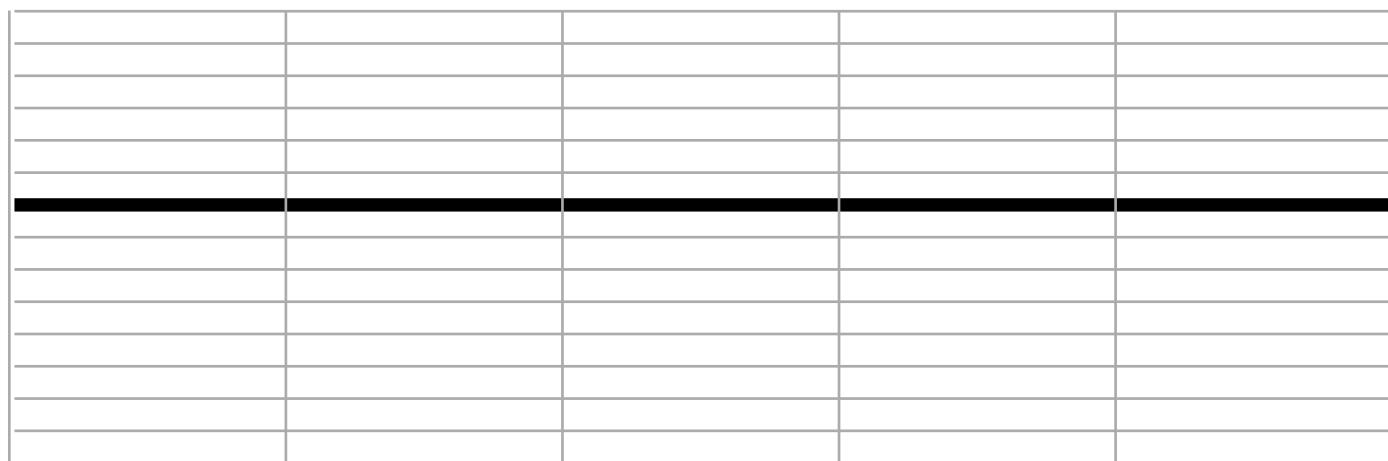
- **Semicoarsening**

Coarsen in the direction of smoothness



- **Line/plane relaxation**

Perform relaxation in groups (in a line)

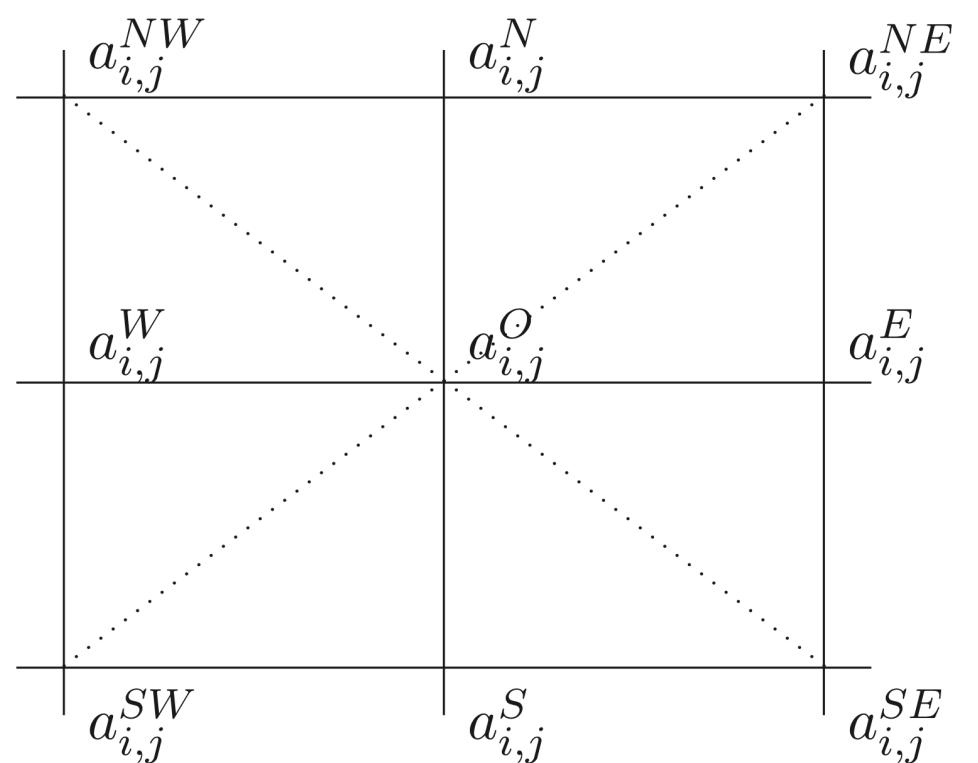


- **Operator based Interpolation** (e.g. BoxMG)  
 $Ae = 0$

J. E. Dendy, Black box multigrid, J. Comput. Phys., 1982

J. E. Dendy and J. D. Moulton, Black box multigrid with coarsening by a factor of three, J. Numer. Lin. Alg. App., 2010

- Node  $(i,j)$  stencil:

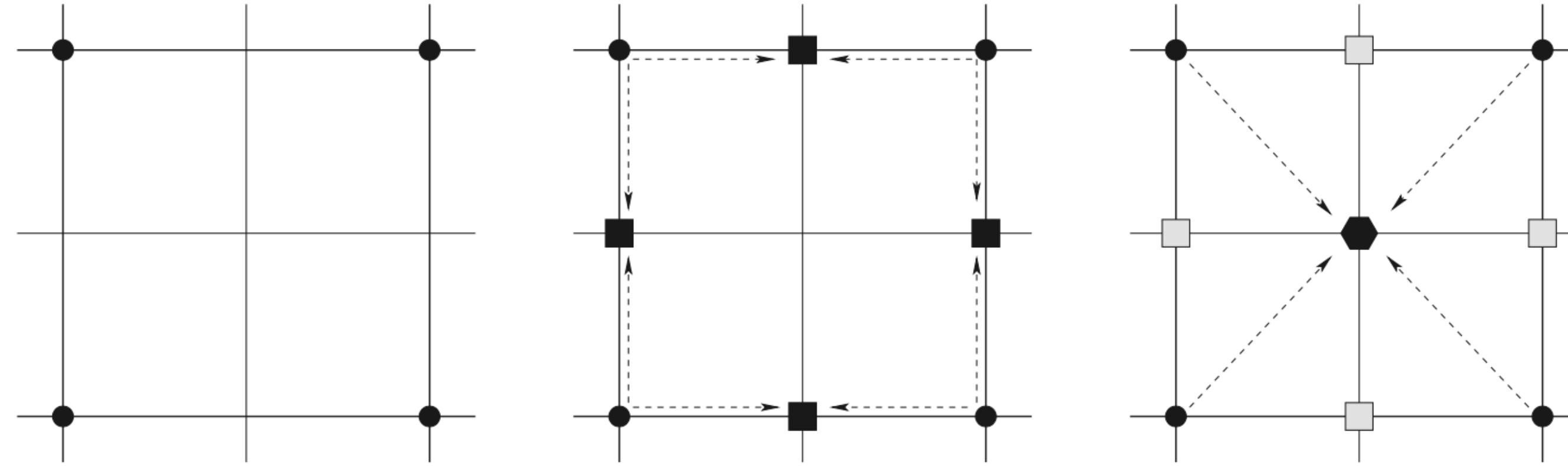


$$Ax = b$$

$$\begin{aligned}
 a_{i,j}^{SW} x_{i-1,j-1} + a_{i,j}^S x_{i,j-1} + a_{i,j}^{SE} x_{i+1,j-1} + a_{i,j}^W x_{i-1,j} + a_{i,j}^O x_{i,j} \\
 + a_{i,j}^E x_{i+1,j} + a_{i,j}^{NW} x_{i-1,j+1} + a_{i,j}^N x_{i,j+1} + a_{i,j}^{NE} x_{i+1,j+1} = b_{i,j}
 \end{aligned}$$

# BoxMG

Math and figures from:  
[Robust and adaptive multigrid methods: comparing structured and algebraic approaches](#), MacLachlan, Moulton, Chartier



1. Inject coarse points (left)
2. Assume the error is constant along x-lines (and y-lines)
3. Infer interpolation from the edges (right)

Example: Assuming  $Ae=0$  and constant y-lines:

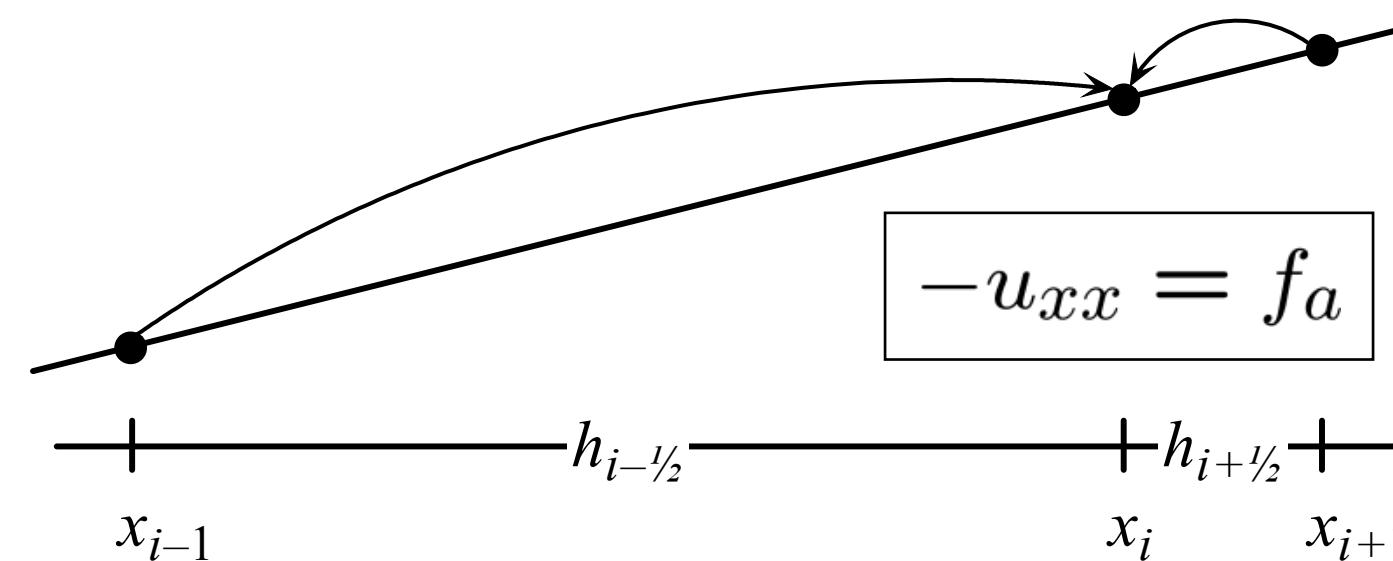
$$(a^S + a^O + a^N)e = -(a^{SW} + a^W + a^{NW})e^c - (a^{SE} + a^E + a^{NW})e^c$$

Interpolation based on entries in A

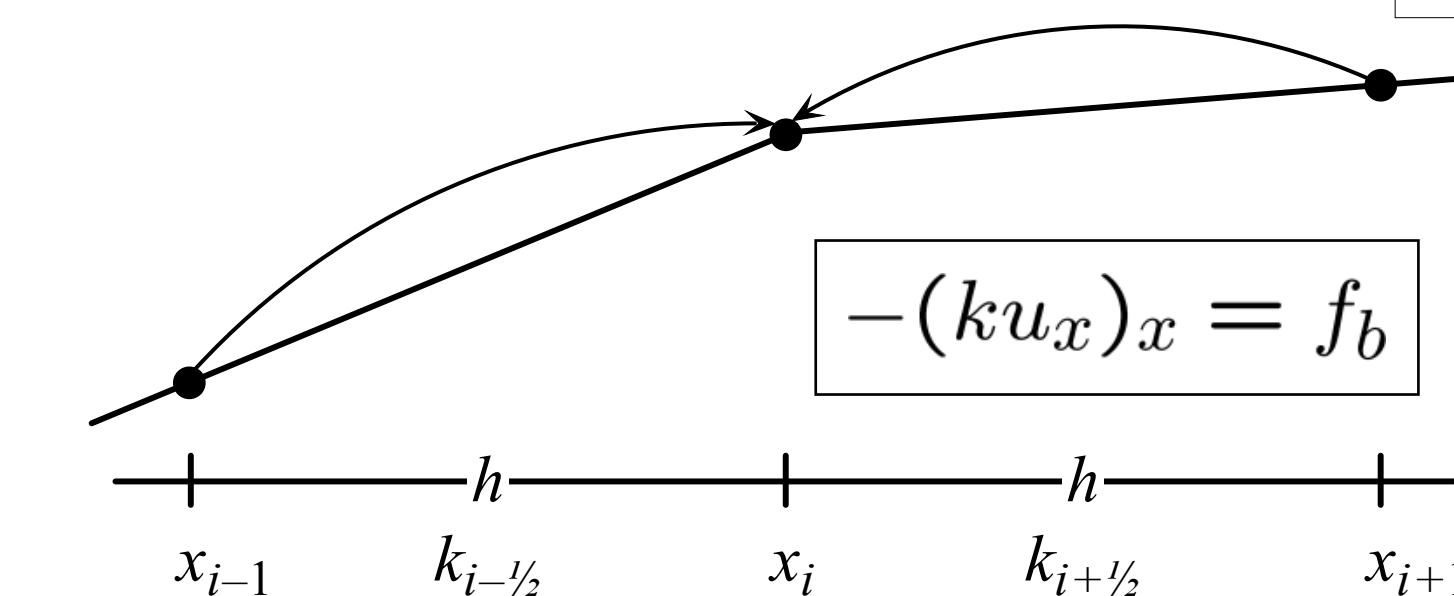
# Algebraic Multigrid (AMG) uses matrix coefficients

- Geometric information alone is not sufficient

Linear Interpolation



Operator-Dependent Interpolation

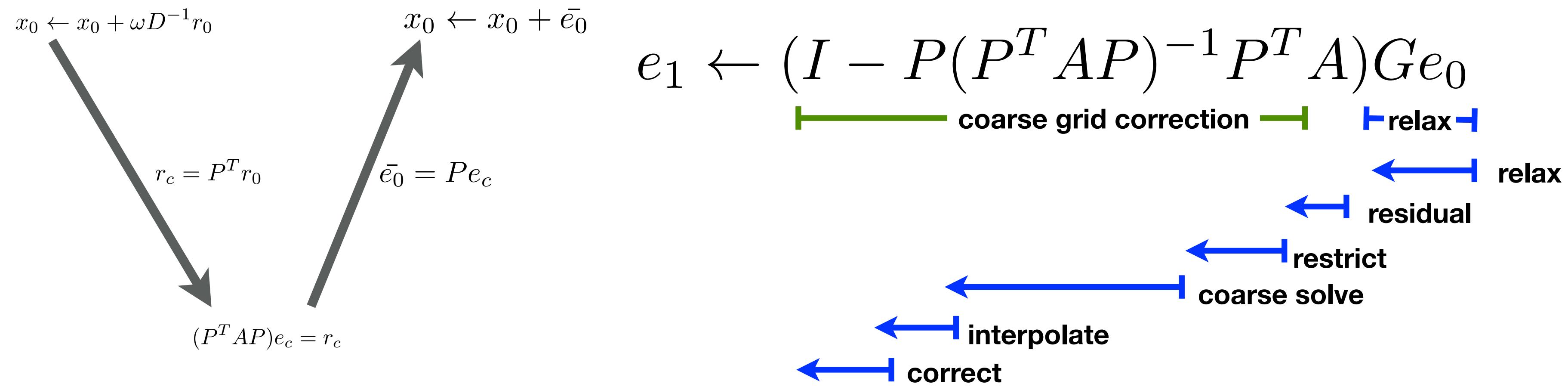


- AMG does not use geometric information, but captures both linear & operator-dep interpolation

$$(A\mathbf{u})_i = a_{i,i-1}u_{i-1} + a_{i,i}u_i + a_{i,i+1}u_{i+1}$$

$$u_i = \left( -\frac{a_{i,i-1}}{a_{i,i}} \right) u_{i-1} + \left( -\frac{a_{i,i+1}}{a_{i,i}} \right) u_{i+1}$$

# Algebraic Observation



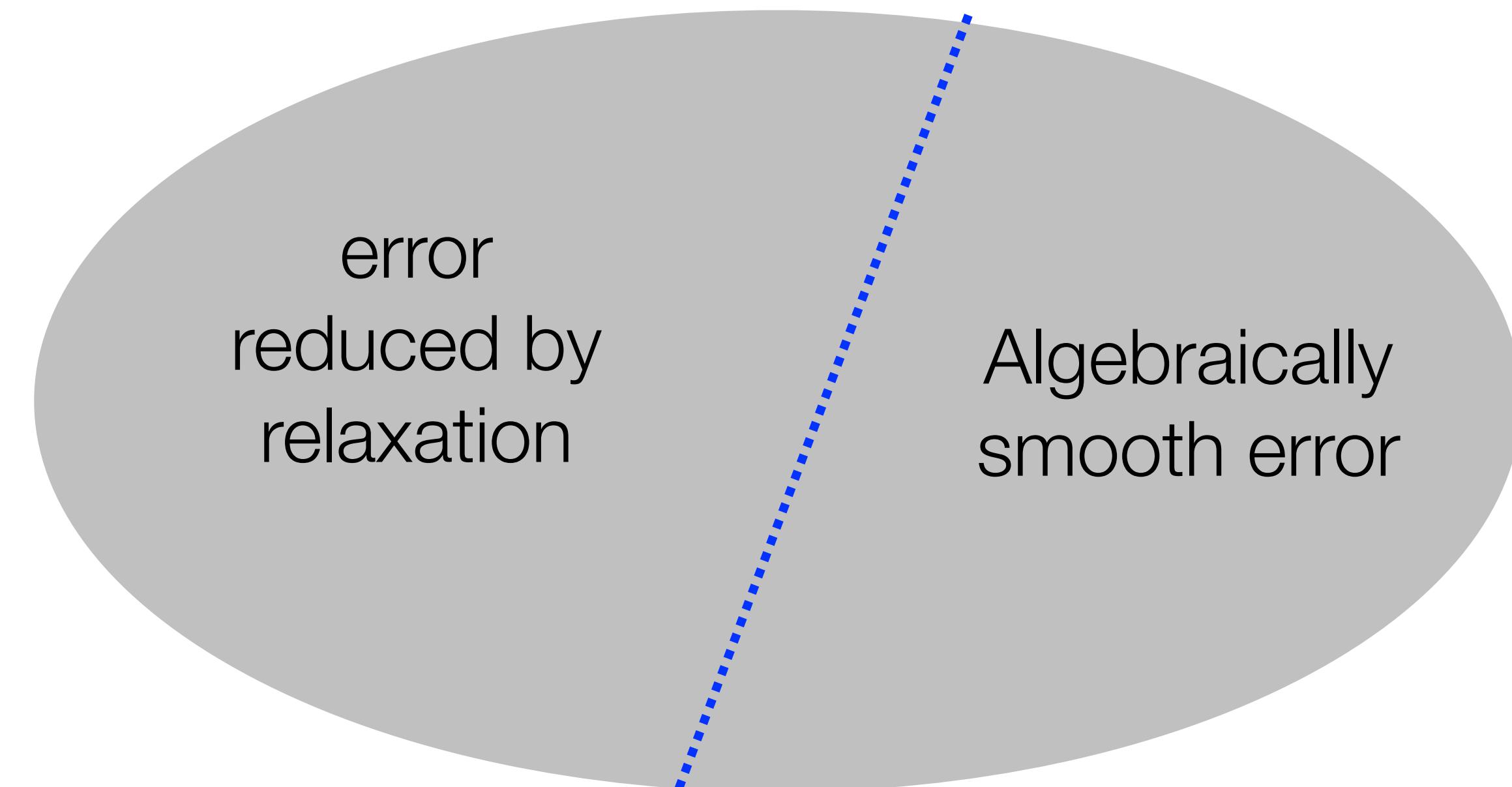
$$G e_0 \in \mathcal{R}(P) \Rightarrow e_1 = 0$$

**interpolation should capture what relaxation misses**

# Algebraically Smooth Error

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- “Algebraically smooth” error may not be geometrically smooth

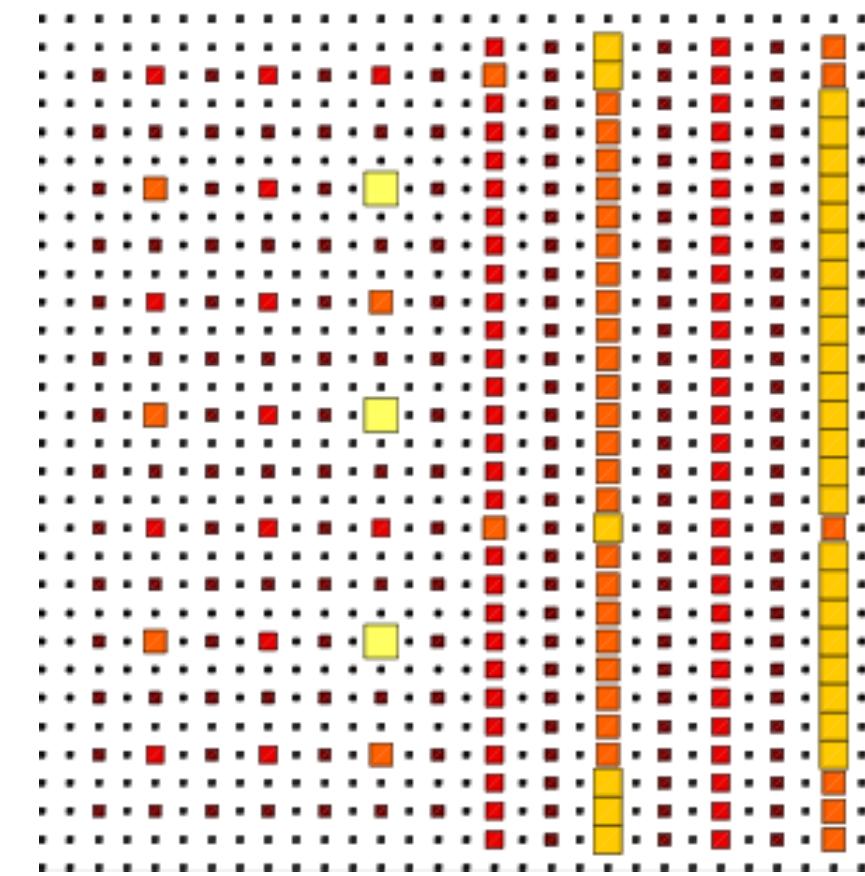
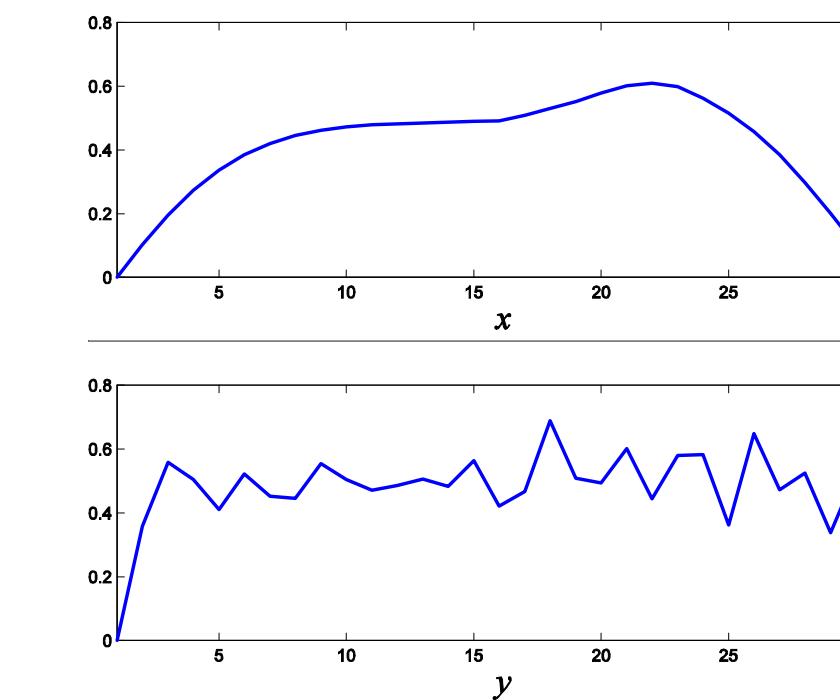
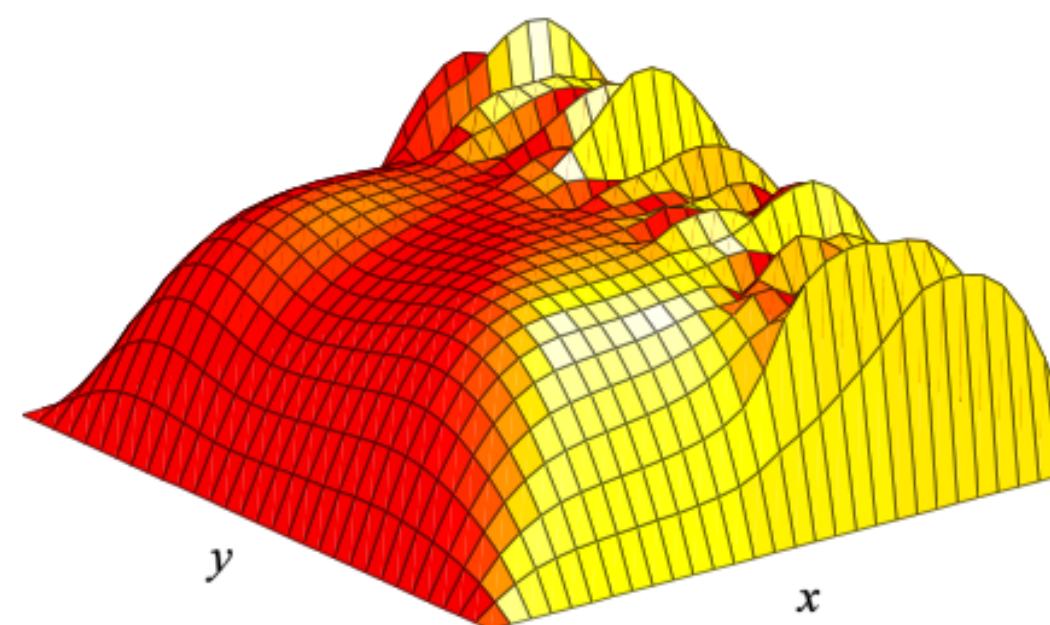


# Error left by relaxation can be geometrically oscillatory

- 7 GS sweeps on

$$-au_{xx} - bu_{yy} = f$$

$$\begin{array}{|c|c|} \hline a & b \\ \hline a \gg b & \\ \hline \end{array}$$



Slide credit: R. Falgout, 2/1/2016

- Caution: this example
  - targets **geometric smoothness**
  - uses **pointwise smoothers**

AMG coarsens grids in the direction  
of geometric smoothness

## Main idea: Algebraically smooth error

---

- Take a relaxation scheme such as w-Jacobi

$$e \leftarrow (I - M^{-1}A)e$$

- If relaxation stagnates, then the remaining error exhibits poor convergence, so
- Formally (characterized by small eigenvalues)

$$(I - M^{-1}A)e \approx e \Rightarrow M^{-1}Ae \approx 0 \Rightarrow r \approx 0$$

$$\langle Ae, e \rangle \ll 1$$

# Main idea: Algebraically smooth error

---

- We then have

$$\begin{aligned}\langle Ae, e \rangle &= \sum_i e_i (A_{ii}e_i + \sum_{j \neq i} A_{ij}e_j) && \text{assume zero row sum} \\ &= \sum_i e_i \left( \sum_{j \neq i} -A_{ij}(e_i - e_j) \right) \\ &= \sum_{i < j} -A_{ij} \cdot e_i \cdot (e_i - e_j) + \sum_{i > j} -A_{ij} \cdot e_i \cdot (e_i - e_j) && \text{swap } i, j \\ &= \sum_{i < j} -A_{ij} \cdot e_i \cdot (e_i - e_j) - \sum_{i < j} -A_{ij} \cdot e_i \cdot (e_i - e_j) \\ &= \sum_{i < j} -A_{ij} \cdot (e_i - e_j)^2\end{aligned}$$

- Ok, so smooth error varies **slowly** in the direction of large matrix coefficients

Briggs, William L. and Henson, Van Emden and  
McCormick, Steve F., A Multigrid Tutorial (2Nd Ed.,  
20000

# Main idea: Algebraically smooth error

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- We have assumed **geometric** smoothness to show

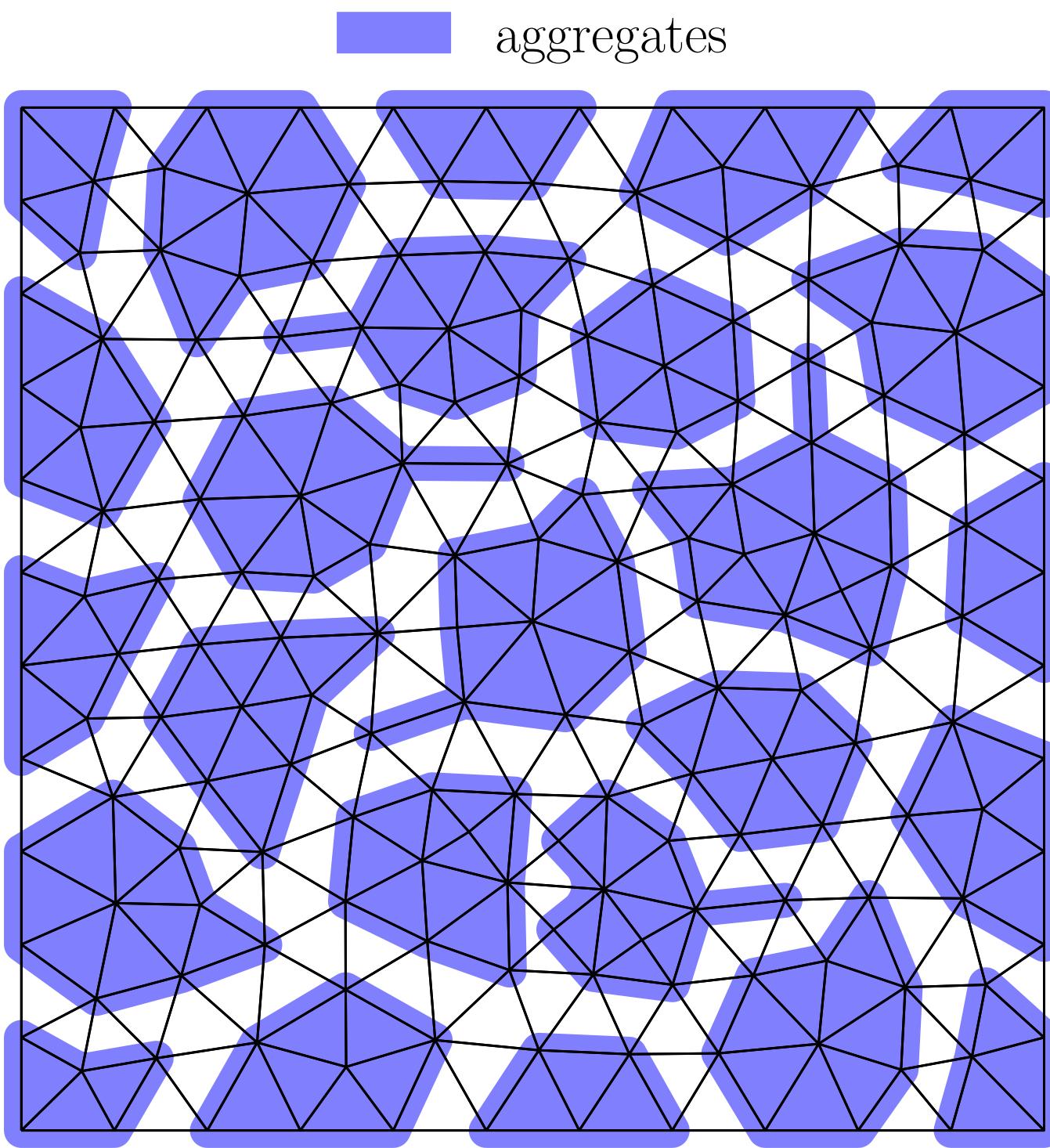
$$\mathbf{e}^T A \mathbf{e} = \sum_{i < j} (-a_{ij})(e_i - e_j)^2 \ll 1$$

- **CF AMG:** Smooth error varies slowly in the direction of “large” matrix coefficients
- **Strength of connection:** Given a threshold  $0 < \theta \leq 1$ , we say that variable  $u_i$  strongly depends on variable  $u_j$  if

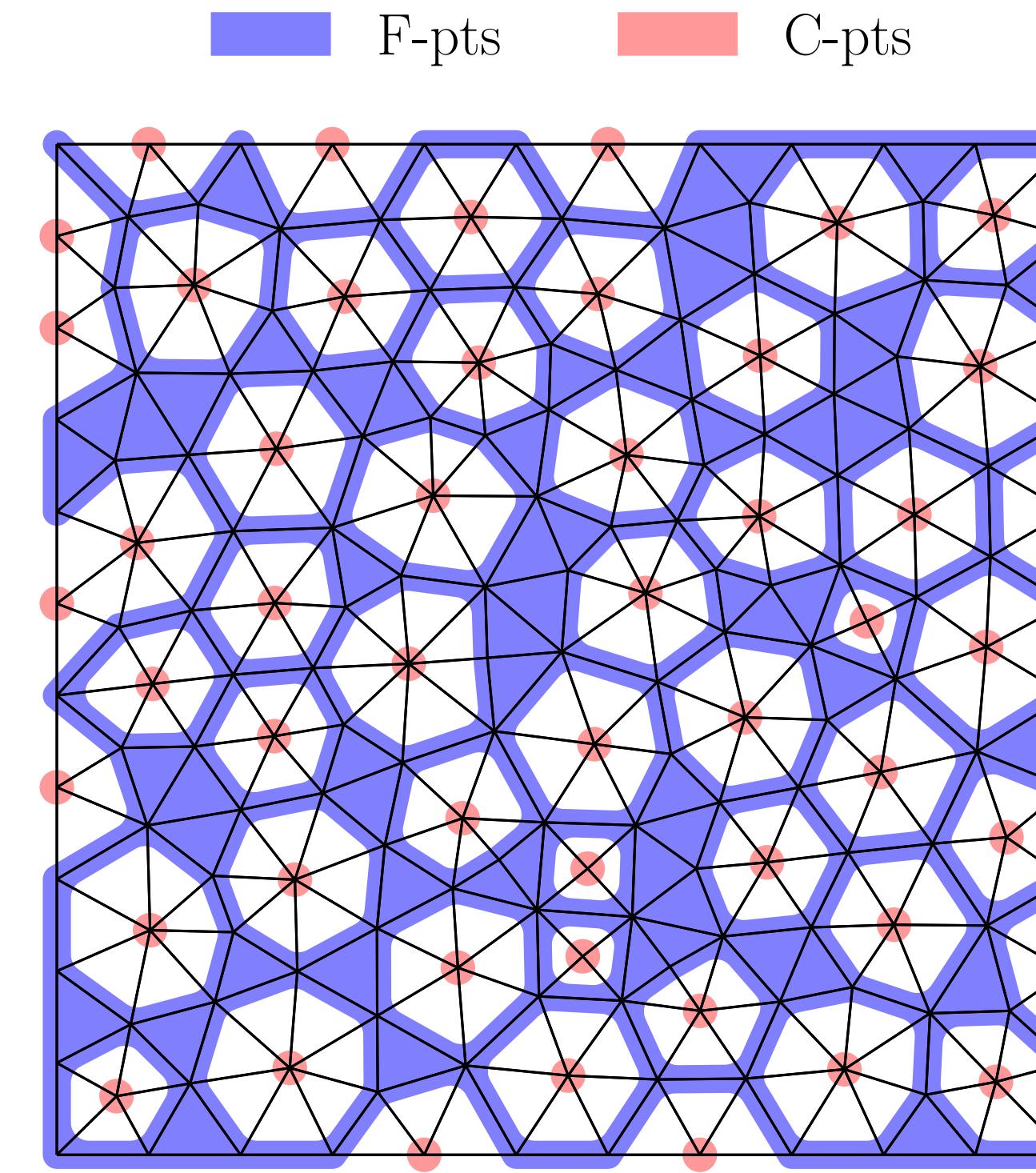
$$-a_{ij} \geq \theta \max_{k \neq i} \{-a_{ik}\}$$

- Often positive off-diagonals are treated as **weak**
- This definition of strength of connection is not symmetric

# Two (general) forms of AMG



- Smoothed Aggregation AMG (SA-AMG)
- Interpolation constructed from candidate vectors
- Clear approach to *optimize* interpolation



- Coarse-Fine AMG (CF-AMG) or Ruge-Stüben
- Coarse grid points are a subset of the fine grid points
- Edge-wise construction of interpolation, allowing straightforward control of sparsity
- Incorporating near-nullspace is not straightforward

## Up next...

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- What happens when we drop the notion of a *grid* ? *AMG* (Monday)
- What does this work at all? Theory. (next Friday)