Multigrid Methods — An Overview

Lecture 1: Basics

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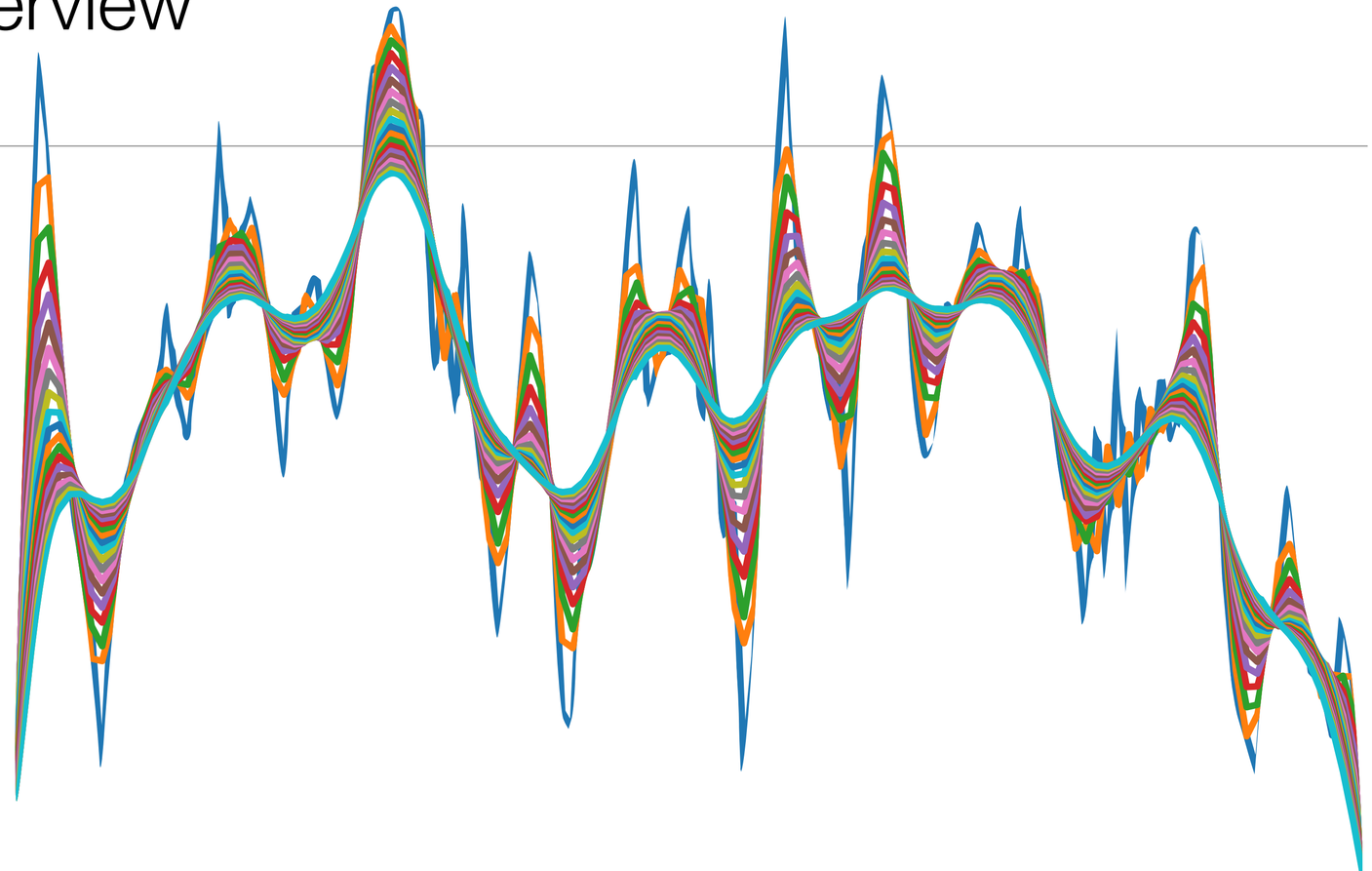


Announcements

- **Special thanks**

  Imperial College Department of Mathematics
  Colin Cotter @ IC
  David Ham @ IC

- **Upcoming conference on Multigrid Methods:**

  19th Copper Mountain Conference On Multigrid Methods
  Mar 28 - Apr 1, 2021
  https://grandmaster.colorado.edu/copper/2021/

What are we trying to do…

- Solve problems of the form

$$Ax = b$$

- Solve this problem **iteratively**:

$$x_1 \leftarrow x_0 + v$$

- Solve this problem **inexpensively:**
  - The update should be “good”
  - Finding the update should be “cheap”

Objectives — high level

- Construct a multigrid method for **your** problem
- Interpret the effectiveness of a multigrid method
- Identify *why* a method works — or — *why* a method does not
- Recognize different forms of multigrid, their pitfalls and their uses

Objectives — high level

1. **Lecture 1 - Basics**
   - Basic mechanics of a multigrid method
   - 1D, 2D, Poisson
2. **Lecture 2 - Extensions**
   - What can go wrong and how to fix it
3. **Lecture 3 - Algebraic**
   - What to do if we do not have a grid (hierarchy)
4. **Lecture 4 - Some Theory**
   - Why does any of this work

Objectives — today

1. **Lecture 1 - Basics**
   - Create a two level multigrid method
   - Illustrate the main components of a multigrid method
   - Calculate the effectiveness of a multigrid method

A cautionary example

Guess and look for and update

$$x_1 = x_0 + \text{update}$$

Updating with the error would be **ideal**

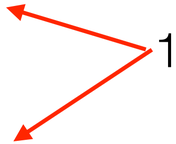
$$x_1 = x_0 + e_0$$

Or in another form

$$x_1 = x_0 + A^{-1} r_0$$

Not practical so…

$$x_1 = x_0 + D^{-1} r_0$$



1 iteration

x^* solution to $Ax = b$

$e_0 = x^* - x_0$ error

$r_0 = b - Ax_0$ residual

$= Ae_0$

$Ae_0 = r_0$ error equation

A cautionary example

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1 iteration

Jacobi ~ 1 SpMV

x^* solution to $Ax = b$

$e_0 = x^* - x_0$ error

$r_0 = b - Ax_0$ residual

$= Ae_0$

$Ae_0 = r_0$ error equation

A reminder — projection methods

- Take a guess

$$x_0$$

- Look for an update that is the “best”:

$$x_1 \leftarrow x_0 + u$$

- Minimize over a smaller space

$$\min_{u \in span\{V\}} \|x^* - x_1\|$$

- Then $u = Vy$

$$V^T V y = V^T e_0$$

- So the update looks like

$$x_1 = x_0 + V(V^T V)^{-1} V^T e_0$$

A reminder — projection methods

- Instead, look at the A-norm:

$$\min_{u \in span\{V\}} \|x^* - x_1\|_A$$

- Then $u = Vy$

$$V^T A V y = V^T A e_0$$

$$V^T A V y = V^T r_0$$

- So that

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

A reminder — projection methods

$$x_1 = x_0 + u$$

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

- What about the error

$$x^* - x_1 = x^* - x_0 - V(V^T A V)^{-1} V^T r_0$$

$$e_1 = e_0 - V(V^T A V)^{-1} V^T A e_0$$

$$= \left(I - V(V^T A V)^{-1} V^T A\right) e_0$$

A-orthogonal projection onto the range of V

Model Problem

- A model problem

$$-u_{xx} = f$$
$$u(0) = u(1) = 0$$

- Finite differences

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i \quad i = 1, \ldots, n \quad u_0 = u_{n+1} = 0$$

- A model matrix problem

$$A = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & & \end{bmatrix}$$

Model Problem

- A special matrix:

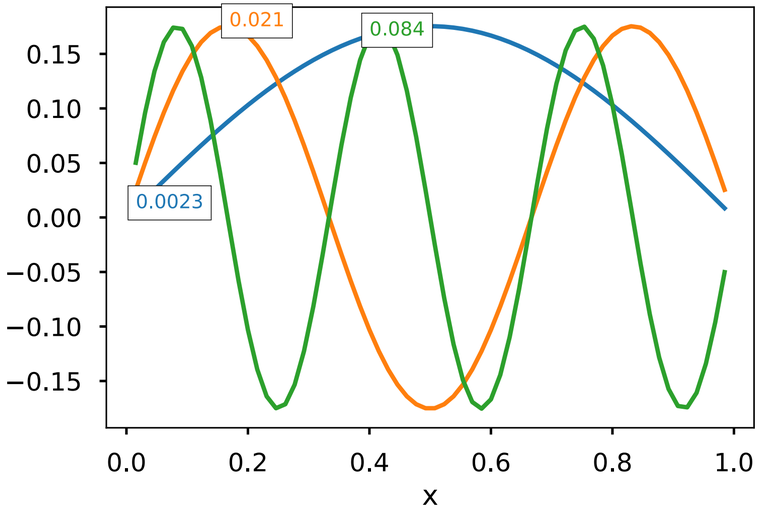
$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & & \\ & & & -1 & 2 \end{bmatrix}$$

The eigenvalues

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2(n+1)}\right)$$

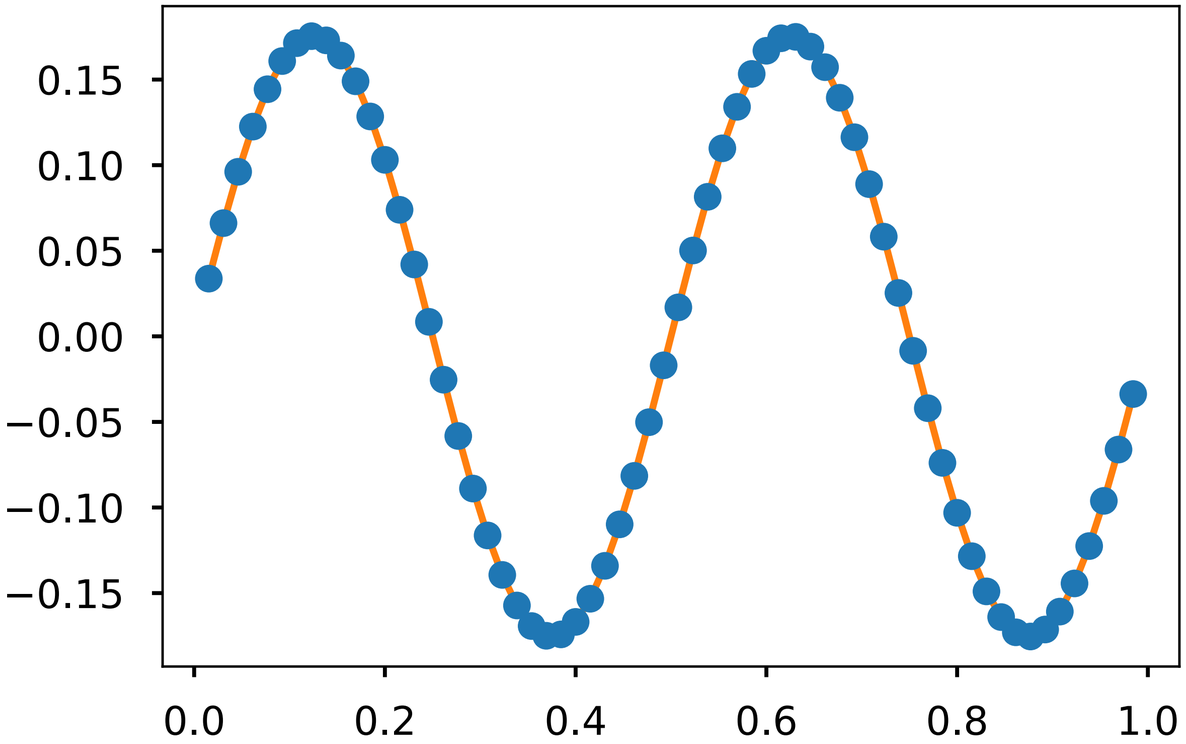
range from (0,4]
(or from h^2 to 4)
The eigenvectors are Fourier modes:

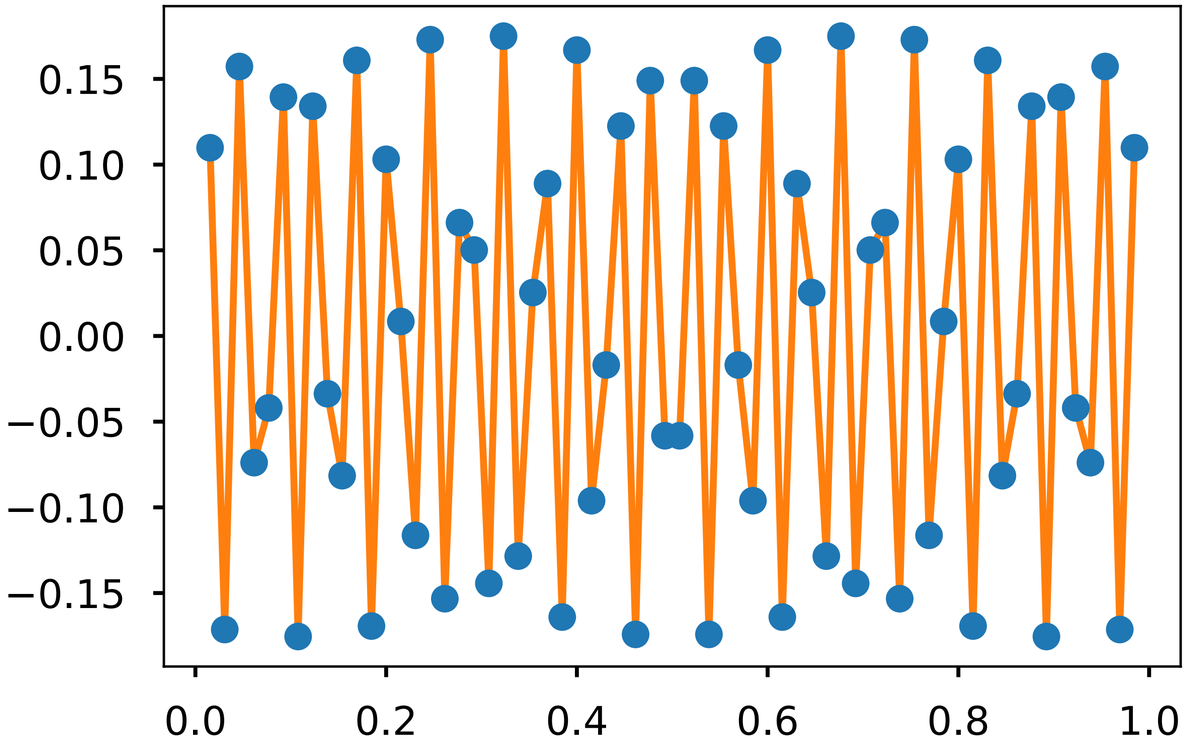
$$(v_k)_j = \sin\left(\frac{(j+1)*k\pi}{n+1}\right)$$



Smooth Mode ~ low Fourier Modes

- We will talk a lot about “smoothness” — how much variation there is **per** grid point.





First relaxation scheme: Jacobi

- The discretization at point i:

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i$$

- Solving for the variable at this point (eliminating the residual):

$$u_i \leftarrow \frac{1}{2}\left(u_{i-1} + u_{i+1} + h^2 f_i\right)$$

- In matrix form:

$$u \leftarrow (I - D^{-1}A)u + h^2 D^{-1} f$$

$$u \leftarrow u + D^{-1} r$$

- And the error:

$$e \leftarrow Ge \qquad G = I - D^{-1}A$$

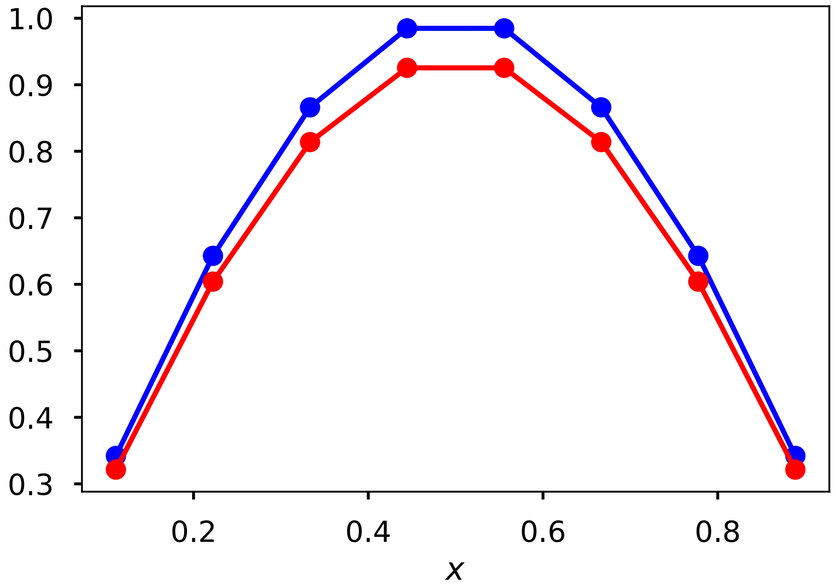
The “error propagation operator”

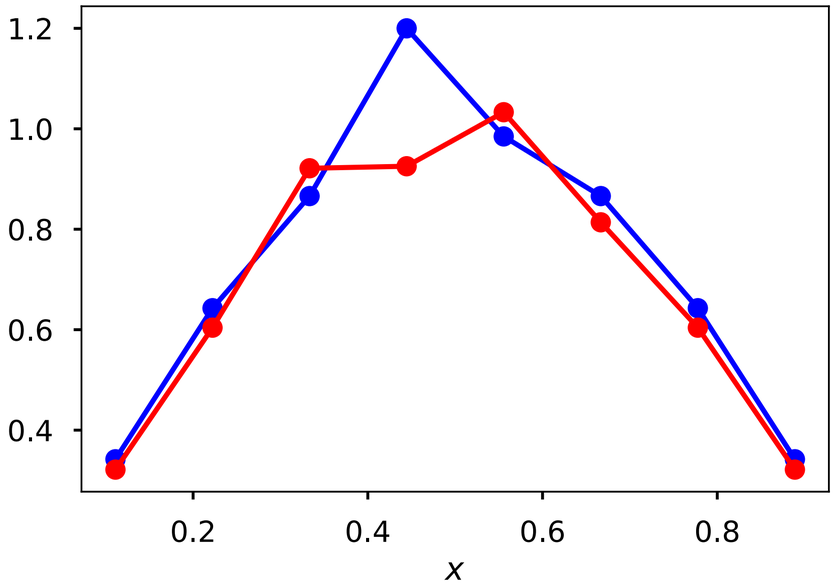
What does Jacobi do to error?

$$e^{new} \leftarrow (I - D^{-1}A)e^{old}$$

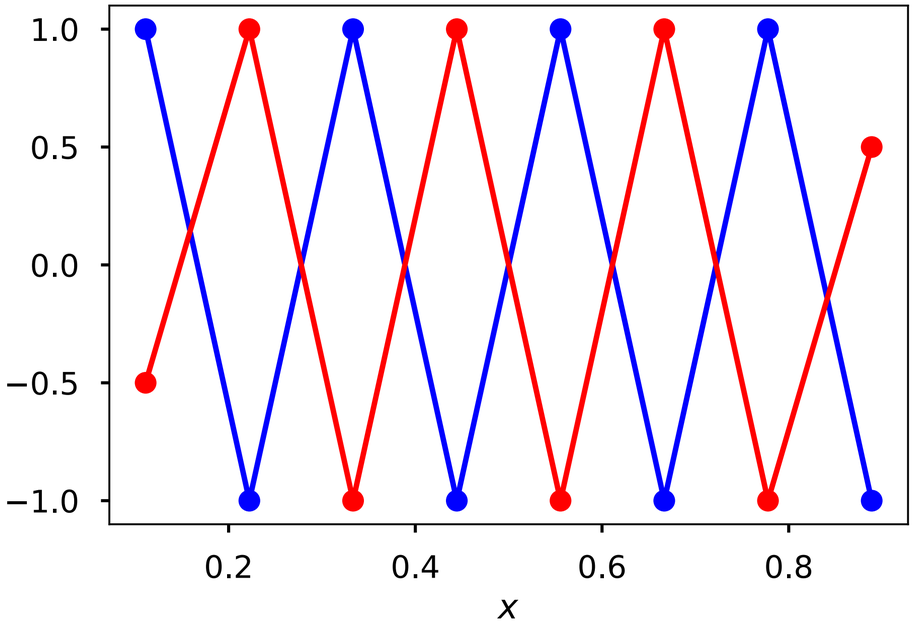
$$e_i^{new} \leftarrow \frac{1}{2}\left(e_{i-1}^{old} + e_{i+1}^{old}\right)$$

- It averages…
- Consider smooth and oscillatory error:





But…

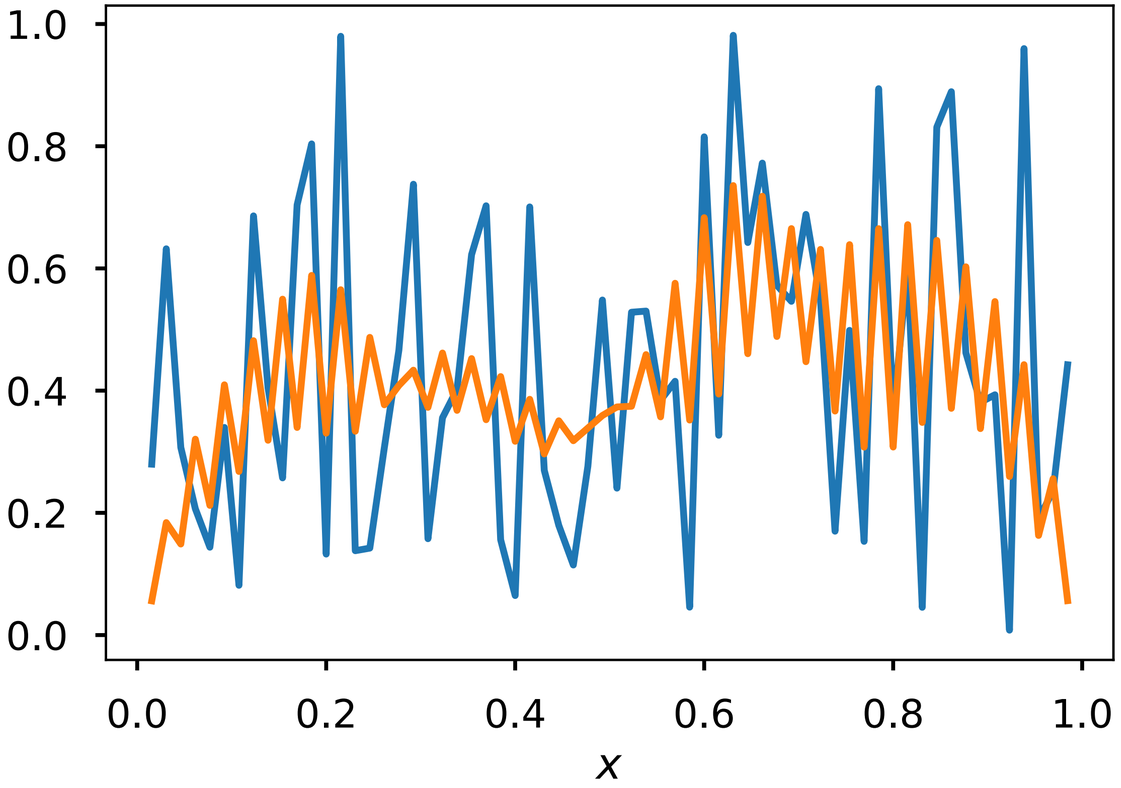


- Jacobi “averages out” certain error very quickly.
- But stagnates on very smooth error or very oscillatory error

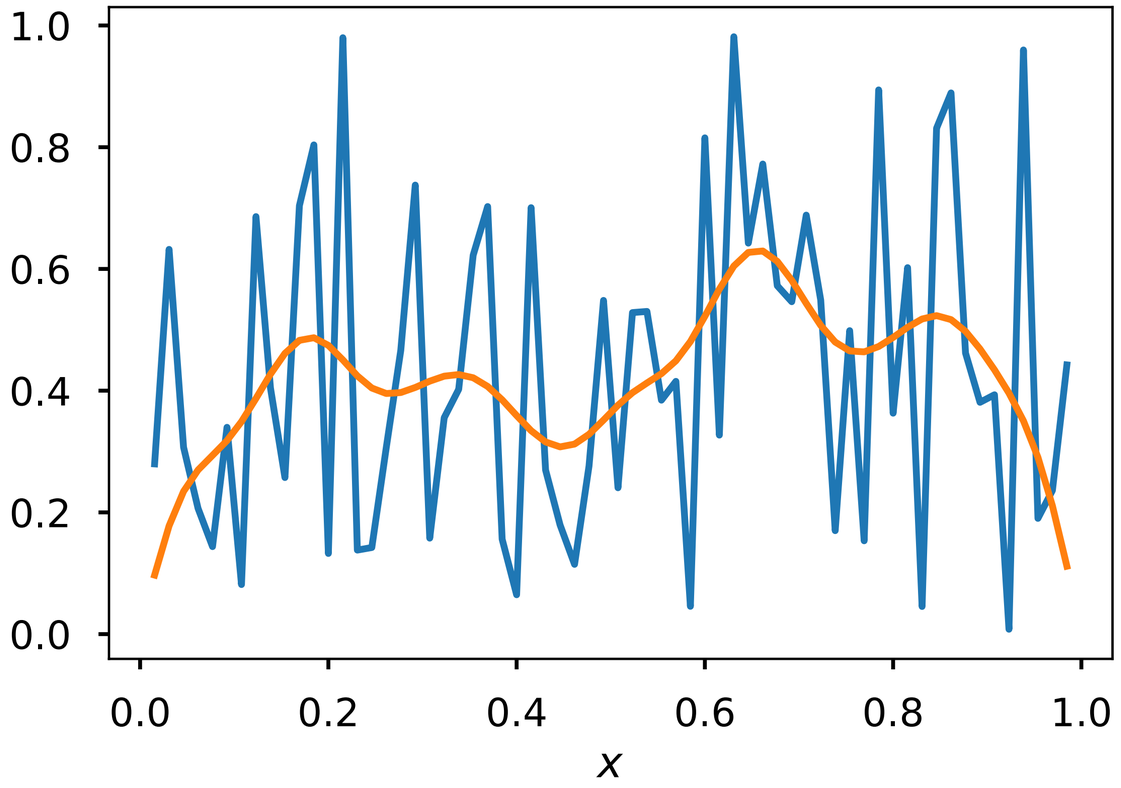
From Jacobi to weighted-Jacobi

- Random initial guess (random error):

$$u \leftarrow u + D^{-1}r$$



$$u \leftarrow u + (2/3)D^{-1}r$$



Weighted Jacobi

- Weighted Jacobi

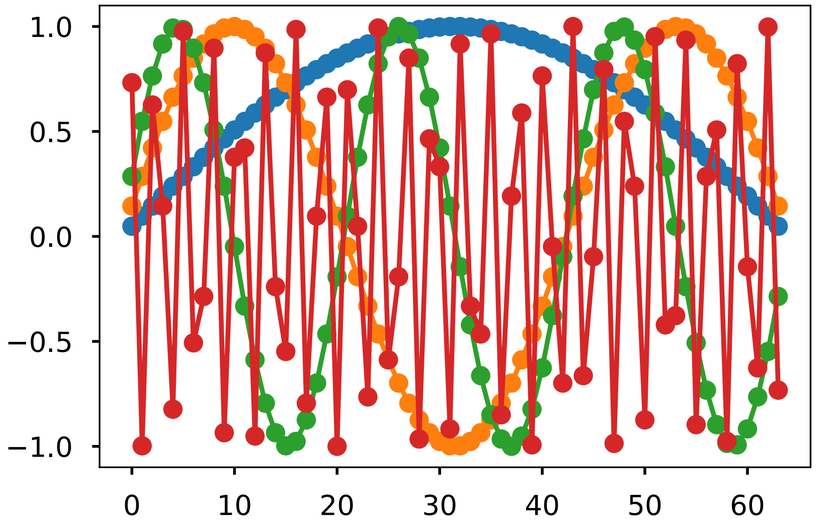
$$u \leftarrow u + (2/3)D^{-1}r$$

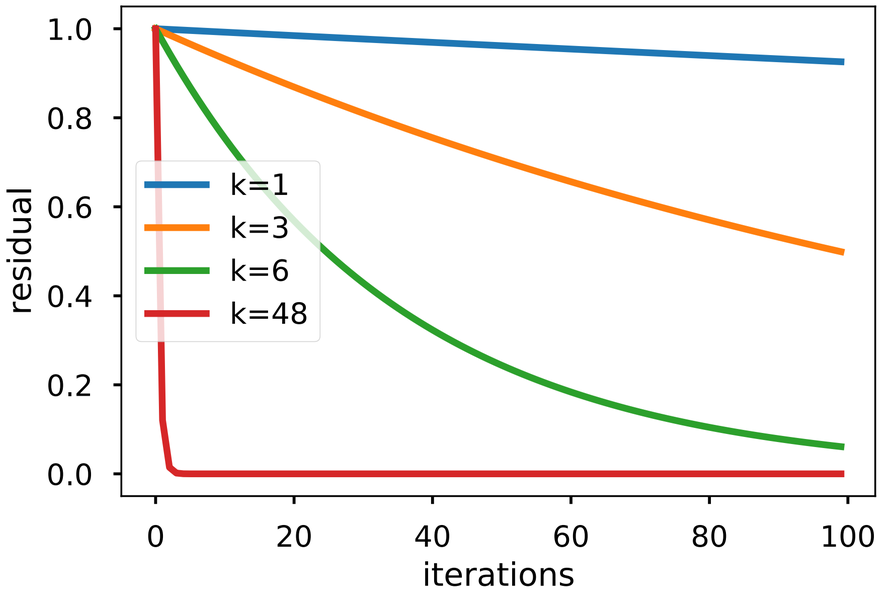
- What does this do to other modes?
  - Are smooth modes damed slowly?
  - What about oscillatory modes?

- Why did we pick 2/3 — that seemed like a luck guess?!

Weighted Jacobi

- If we picked 4 modes, 1, 3, 6, and 48
- Then smooth modes still dampen less quickly than higher ones





Error Propagation

- Let's consider an initial error

$$e_0 = \sum_{k=1}^{n} c_k v_k$$

- And the weight Jacobi iteration matrix

$$e \leftarrow (I - \omega D^{-1} A)e = Ge$$

- From ν iterations we have

$$G^\nu e_0 = \sum_{k=1}^{n} c_k G^\nu v_k$$

$$= \sum_{k=1}^{n} c_k \lambda_k^\nu v_k$$

- As a result, mode **k** is reduced by the magnitude of λ_k in every pass

Fundamental Theorem of Iteration

$$G = I - M^{-1}A$$

- Convergent ($G^n \to 0$) if and only if $\rho(G) \leq 1$

- Suppose *A* is s.p.d. $$\frac{\|e_n\|}{\|e_0\|} \leq \|G^n\| \leq \|G\|^n \approx 10^{-d}$$

- How many iterations do we need to guarantee the reduction of the error by *d* digits?

$$n \approx -\frac{d}{\log_{10} \rho(G)}$$

Relaxation

- Convergence **factor**

$$\|G\| \text{ or } \rho(G)$$

- Convergence **rate**

$$-\log_{10} \rho(G)$$

- For Jacobi:

$$G = I - D^{-1}A \longrightarrow \lambda_k = 1 - \frac{1}{2} \cdot 4 \cdot \sin^2\left(\frac{k\pi}{2(n+1)}\right)$$

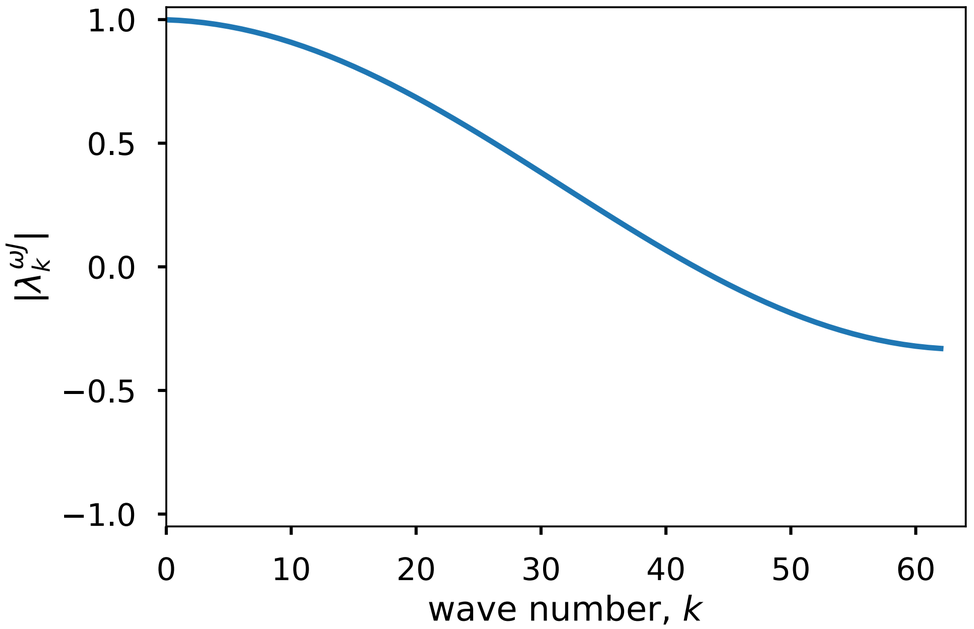
$$G = I - (2/3)D^{-1}A \longrightarrow \lambda_k = 1 - \frac{2}{3} \cdot \frac{1}{2} \cdot 4 \cdot \sin^2\left(\frac{k\pi}{2(n+1)}\right)$$

The spectral radius is the same, but…

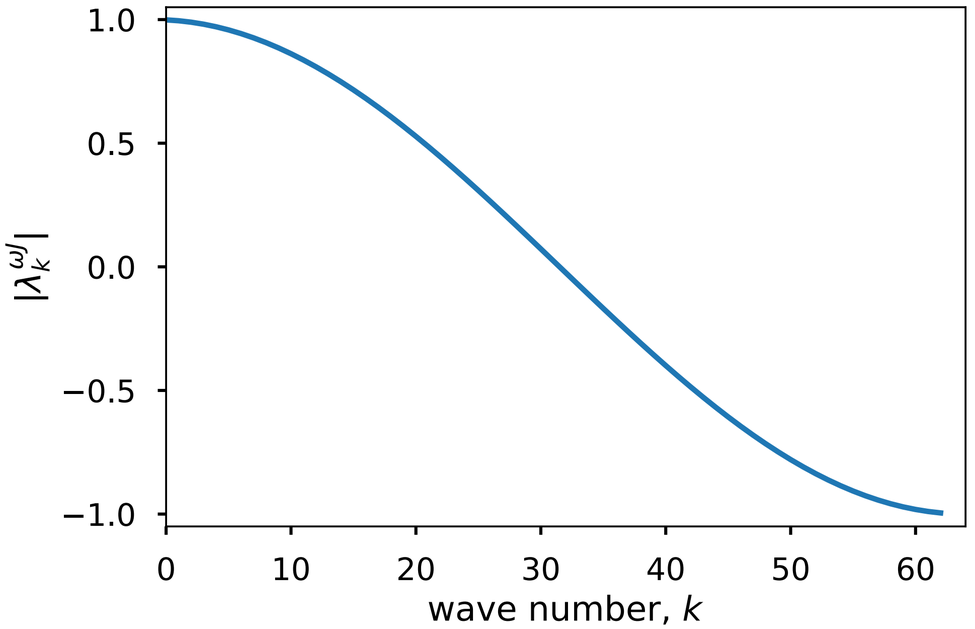
Weighted Jacobi

- If we look at the spectrum:
  - Weighted Jacobi dampens modes that are highly oscillatory

$\omega = 2/3$

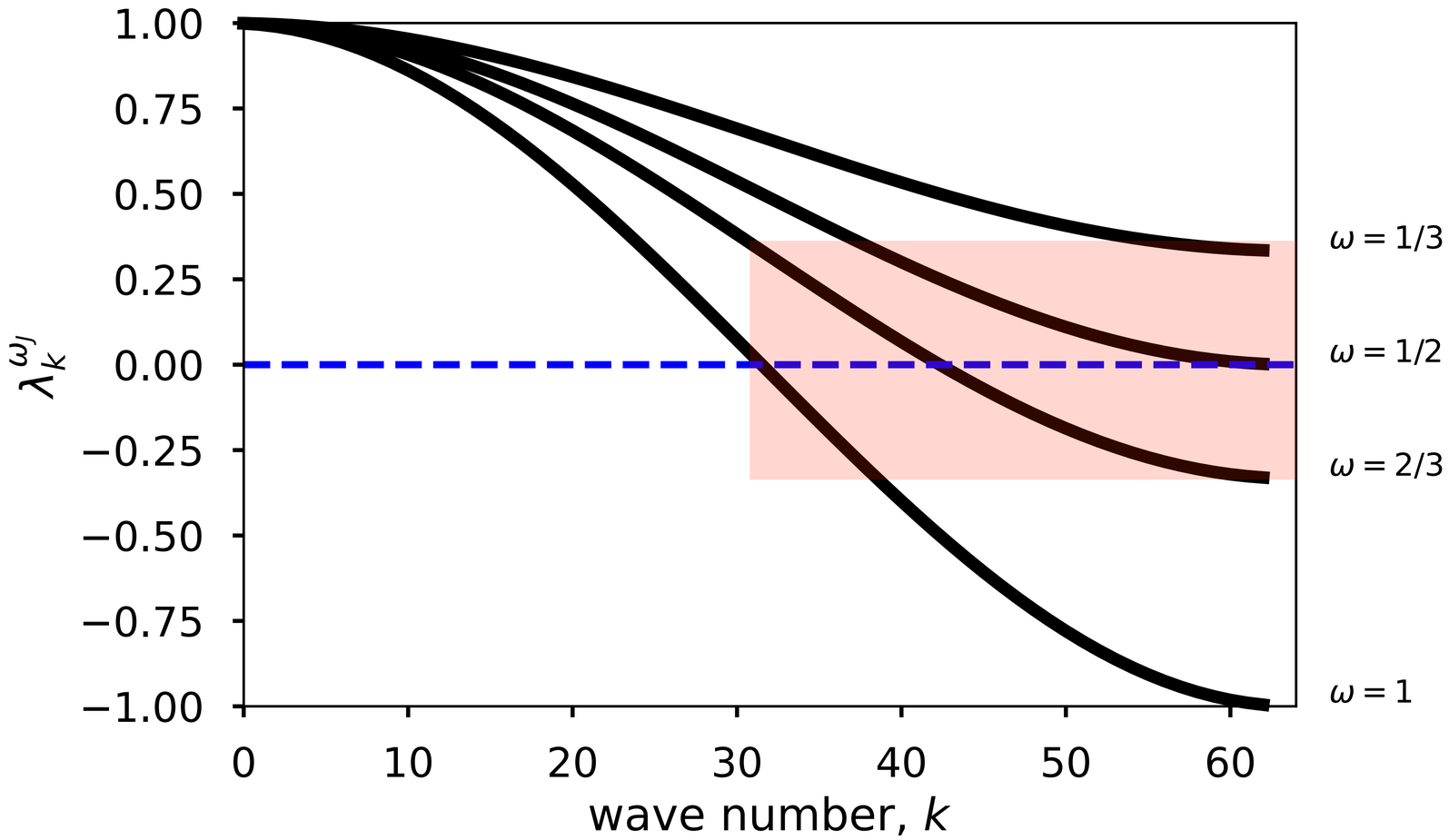


$\omega = 1$



Weighted Jacobi

- Selecting 2/3 balances the reduction in error in the **high** modes



The multigrid smoothing factor

- The **smoothing factor** of relaxation method G is the maximum magnitude of the upper half of the spectrum:

$$\max_{k \in [n/2, n]} |\lambda_k^G|$$

- A common feature:

Oscillatory modes are quick to converge
Smooth modes are slow to converge

Multigrid Step #1: pick a smoother

- For $\omega = 2/3$

$$|\lambda_{n/2}| = |\lambda_n| = 1/3$$

- For $\omega = 1$

$$|\lambda_{n/2}| = |\lambda_n| = 1$$

- Jacobi is not a smoother (weighted *is*)

An important observation on “smoothness”

- So far, we’ve mainly looked at

$$Au = 0$$

- In general we need to consider

$$Au = f$$

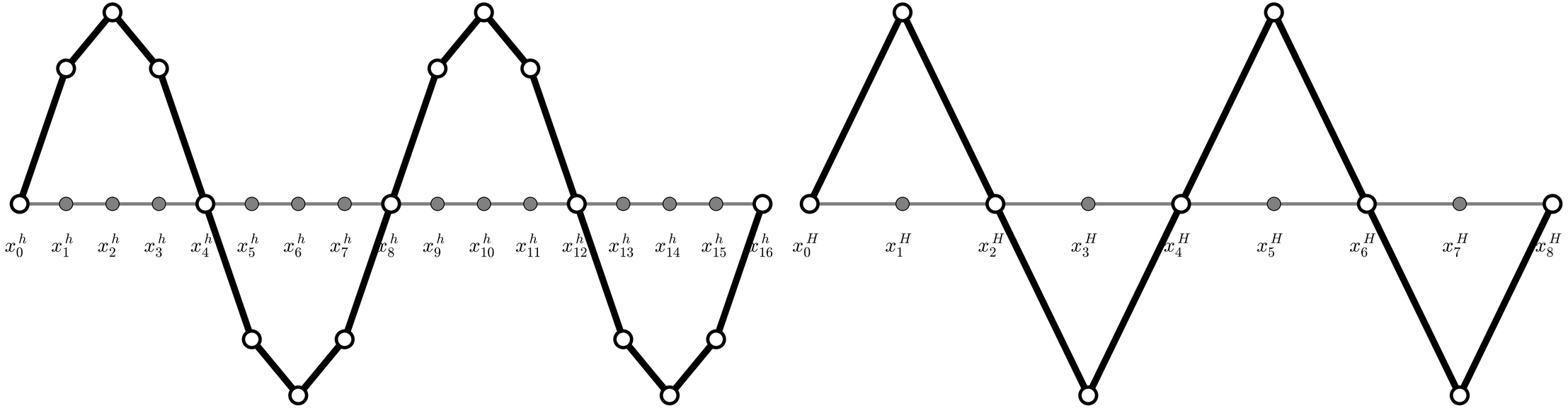
- If we **smooth** with

$$u \leftarrow u + \omega D^{-1} r$$

then the **error** is smooth, not (necessarily the solution).

Coarse Grids

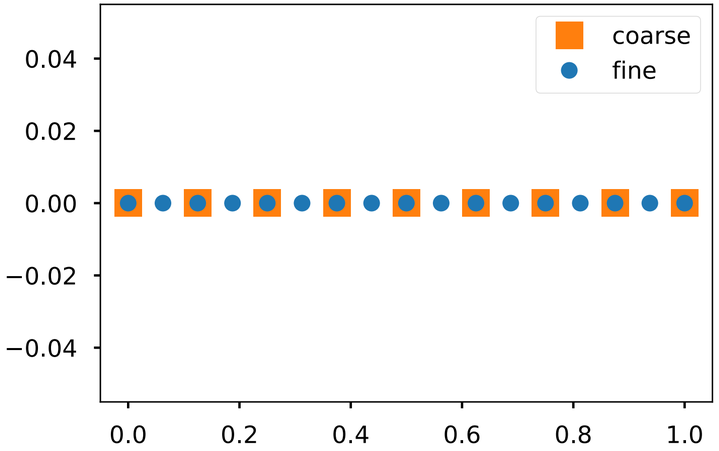
- Smooth modes look like oscillatory modes when sampled on a coarse grid
- 4-mode of 15 **versus** 4-mode of 7

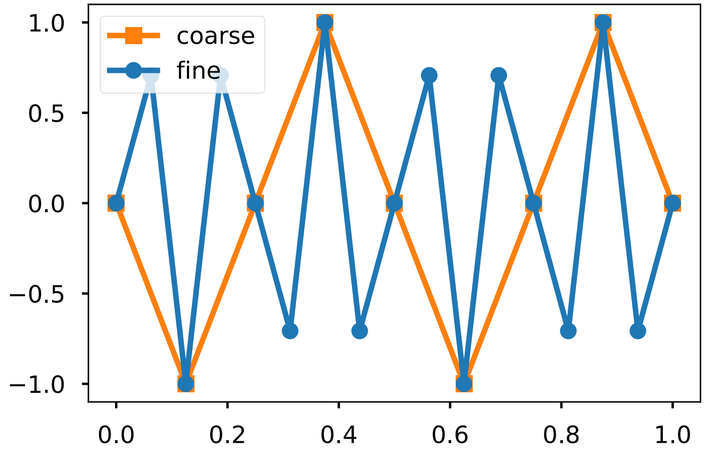


This looks like a “smooth” mode

This looks like an “oscillatory” mode

Coarse modes





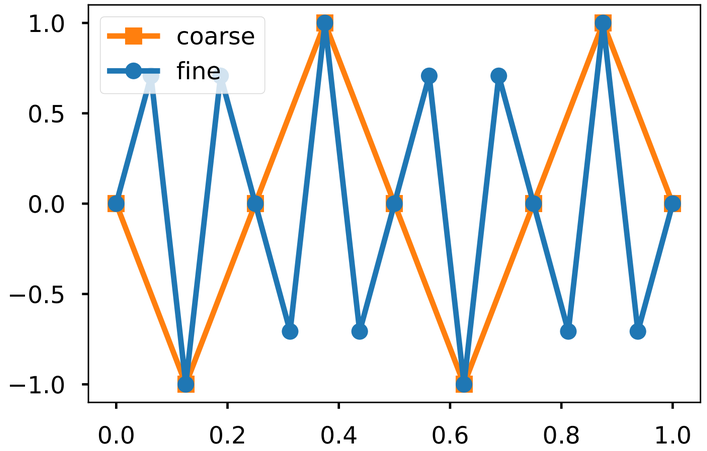
mode 4 of 15

$$(v_k)_j = \sin \frac{jk\pi}{n+1}$$ Fine mode

$$(v_k)_{2j} = \sin \frac{2jk\pi}{n+1}$$ Fine mode (every other)

$$= \sin \frac{jk\pi}{(n+1)/2}$$ Coarse mode

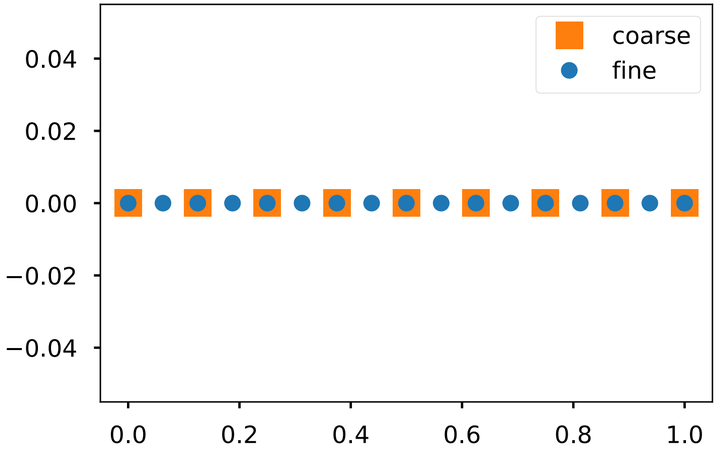
$$= (\hat{v}_k)_j$$

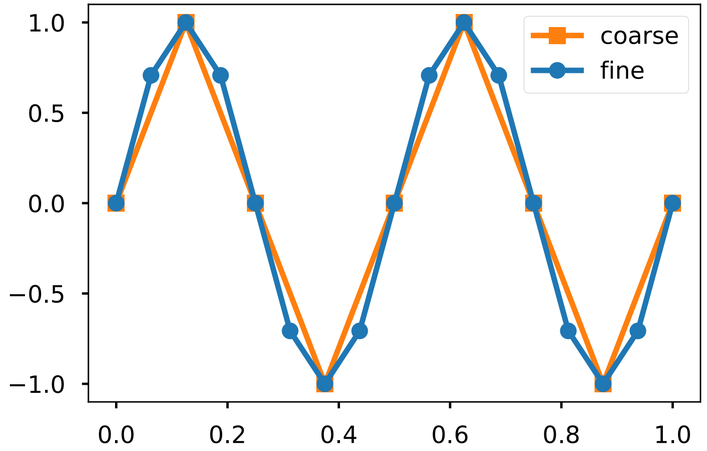


mode 12 of 15

For low modes, k-modes are preserved

Coarse modes





mode 4 of 15

$$(v_k)_{2j} = \sin \frac{2jk\pi}{n+1}$$

Fine mode

$$= -\sin \frac{2j(n-k)\pi}{n+1}$$

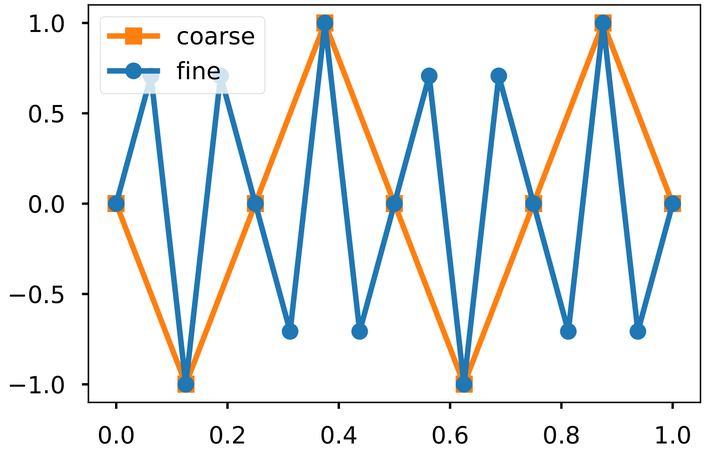
Fine mode

(every other)

$$= -\sin \frac{j(n-k)\pi}{(n+1)/2}$$

Coarse mode

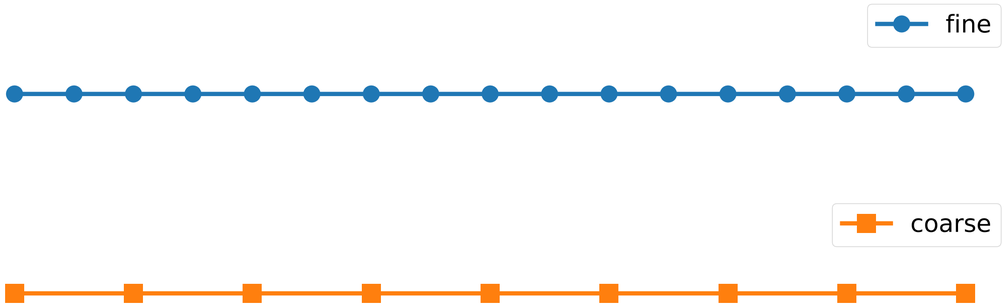
$$= -(\hat{v}_{n-k})_j$$



mode 12 of 15

For high modes, k-modes are *aliased*

Questions to resolve…



- How to transfer between **fine** and **coarse**?

- What do we do “solve” on a coarse grid?

What to use to transfer…

- Return to our projection problem:

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

- If we have a smoothing property, then smoothing a few times will leave smooth error.
- This is A-orthogonal projection onto V
- Let's construct V from say piecewise cont. linears

Interpolation

- Consider coarse grid

$$\Omega^{2h}$$

- and fine grid

$$\Omega^{h}$$

- Construct an operator

$$P : \Omega^{2h} \rightarrow \Omega^{h}$$

- Such that

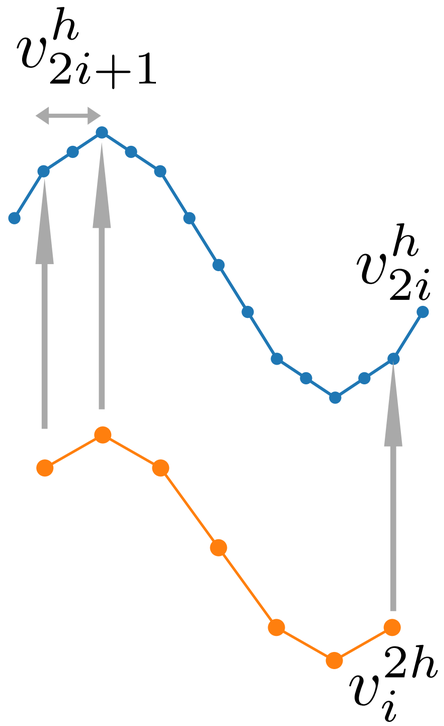
$$Pv^{2h}$$

  is continuous and piecewise linear

Interpolation

$$v_{2i}^h = v_i^{2h}$$ Injection

$$v_{2i+1}^h = \frac{1}{2}(v_i^{2h} + v_{i+1}^{2h})$$ Average
(linear interp)

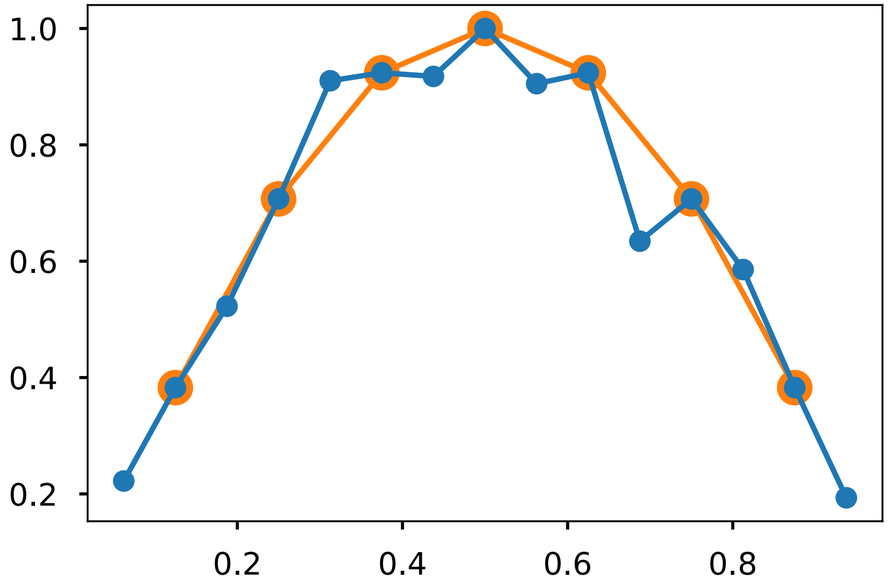


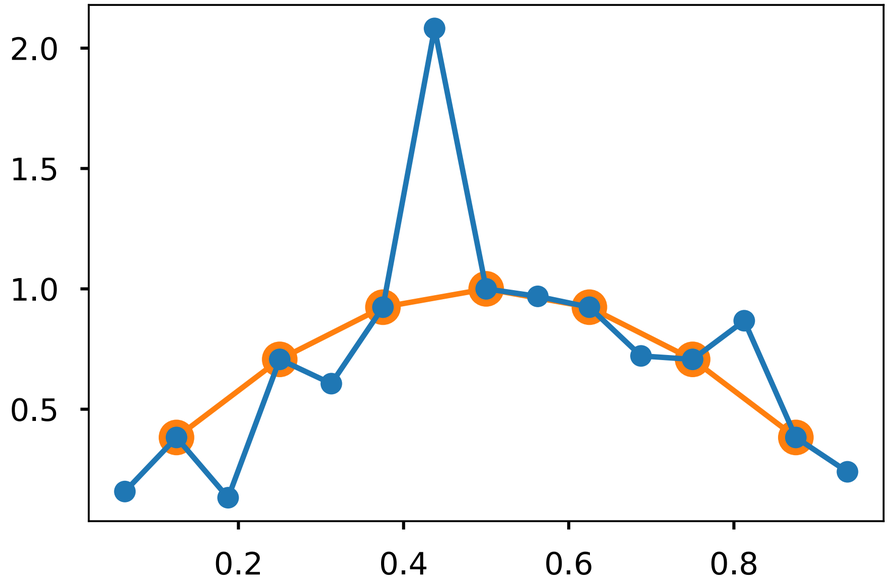
- Values at points common to both grids are reused (injected

Interpolation

- Things that are smooth are interpolated well (smooth error!)

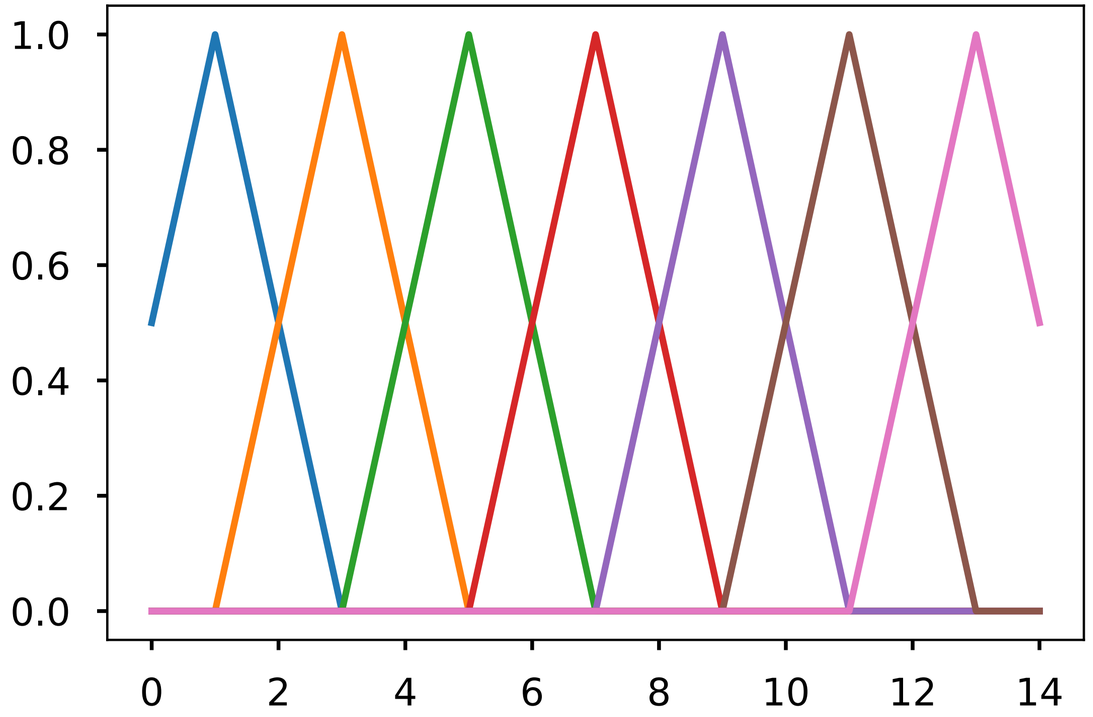
- Things that are not smooth are not interpolated well (un-smoothed error!)





In matrix form

$$P = \frac{1}{2}\begin{bmatrix}
1. & & & & & & & \\
2. & & & & & & & \\
1. & 1. & & & & & & \\
 & 2. & & & & & & \\
 & 1. & 1. & & & & & \\
 & & 2. & & & & & \\
 & & 1. & 1. & & & & \\
 & & & 2. & & & & \\
 & & & 1. & 1. & & & \\
 & & & & 2. & & & \\
 & & & & 1. & 1. & & \\
 & & & & & 2. & & \\
 & & & & & 1. & 1. & \\
 & & & & & & 2. & \\
 & & & & & & 1. &
\end{bmatrix}$$



- The columns of P are basis functions (right)
- The fine grid vectors, Pv, are linear combinations of these functions
- Notice: P is full rank!

The coarse grid operator

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

(sub)space defined by P

- If V is defined by $span\{P\}$ —or just P,
- Then V^Tdefines **restriction** as P^T
- And the **coarse level operator** is defined by $P^T A P$

Two level method

$$x_1 = x_0 + V(V^T A V)^{-1} V^T r_0$$

$$x_1 = x_0 + P(P^T A P)^{-1} P^T r_0$$

- Given x_0
- Smooth a few times $x_0 \leftarrow x_0 + \omega D^{-1} A r_0$
- Form residual $r_0 = b - A x_0$
- Restrict the residual $P^T r_0$
- Solve the coarse problem $P^T A P \hat{e}_c = P^T r_0$ $A_c \hat{e}_c = r_c$
- Interpolate the approx error $P \hat{e}_c$
- Correct the initial guess $x_0 + P \hat{e}_0$

- What should this be?
- How many times

- What does this mean?
- What are we interpolating?
- Are we done?

Notes on variants

- An alternative to restriction, is injection

$$\begin{bmatrix} 0 & 1 & 0 & & & & & & & & & \\ & & 0 & 1 & 0 & & & & & & & \\ & & & & 0 & 1 & 0 & & & & & \\ & & & & & & 0 & 1 & 0 & & & \\ & & & & & & & & 0 & 1 & 0 & \\ & & & & & & & & & & 0 & 1 \end{bmatrix}$$

- Or a weighted transpose of linear interpolation (to restrict constants exactly)

$$\frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & & & & & & \\ & & 1 & 2 & 1 & & & & & & & \\ & & & & 1 & 2 & 1 & & & & & \\ & & & & & & 1 & 2 & 1 & & & \\ & & & & & & & & 1 & 2 & 1 & \\ & & & & & & & & & & 1 & 2 & 1 \end{bmatrix}$$

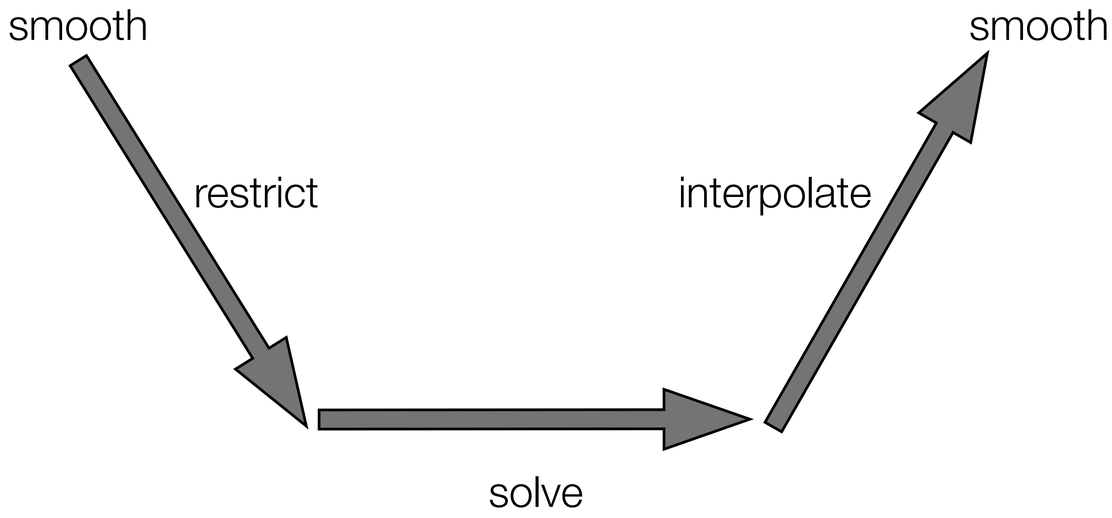
- An alternative to $A_c = P^T A_h P$ is to rediscretize $A_c = A_{2h}$

$$x_1 \leftarrow x_0 + P A_{2h}^{-1} R r_0$$

Algorithm: two-level multigrid

Input: initial guess

1. Smooth ν_{pre} times on $Au = f$
2. Compute $r = f - Au$
3. Compute $r_c = Rr$
4. Solve $A_c e_c = r_c$
5. Interpolate $\hat{e} = Pe_c$
6. Correct $u \leftarrow u + \hat{e}$
7. Smooth ν_{post} times on $Au = f$



A two-level “V” cycle

How Accurate is Multigrid?

- Consider the exact solution to the PDE u^*

$$-u'' = f$$

- The exact solution to the **discrete** problem u_h^*

$$Au = b$$

- The approximate **discrete** solution $u_h \approx u_h^*$

- Define

$u^* - u_h^*$ Discretization error

$u_h^* - u_h$ Algebraic error

How Accurate is Multigrid?

- Would like the error bounded

$$\begin{aligned}\|u^* - u_h\| &\leq \|u^* - u_h^*\| + \|u_h^* - u^h\| \\ &\leq \varepsilon\end{aligned}$$

- To achieve this,

  —force the discretization (grid space) so that

$$\|u^* - u_h^*\| \leq ch^2 \leq \frac{\varepsilon}{2}$$

  —and the algebraic error (convergence) up to the discretization error

$$\|u_h^* - u^h\| \leq \frac{\varepsilon}{2}$$

Multigrid convergence

- Convergence factor of a cycle — the factor by which the error (residual) is reduced (in some norm) in each iteration

$$\gamma$$

  …assume this is independent of n

- Wish to have m cycles such that

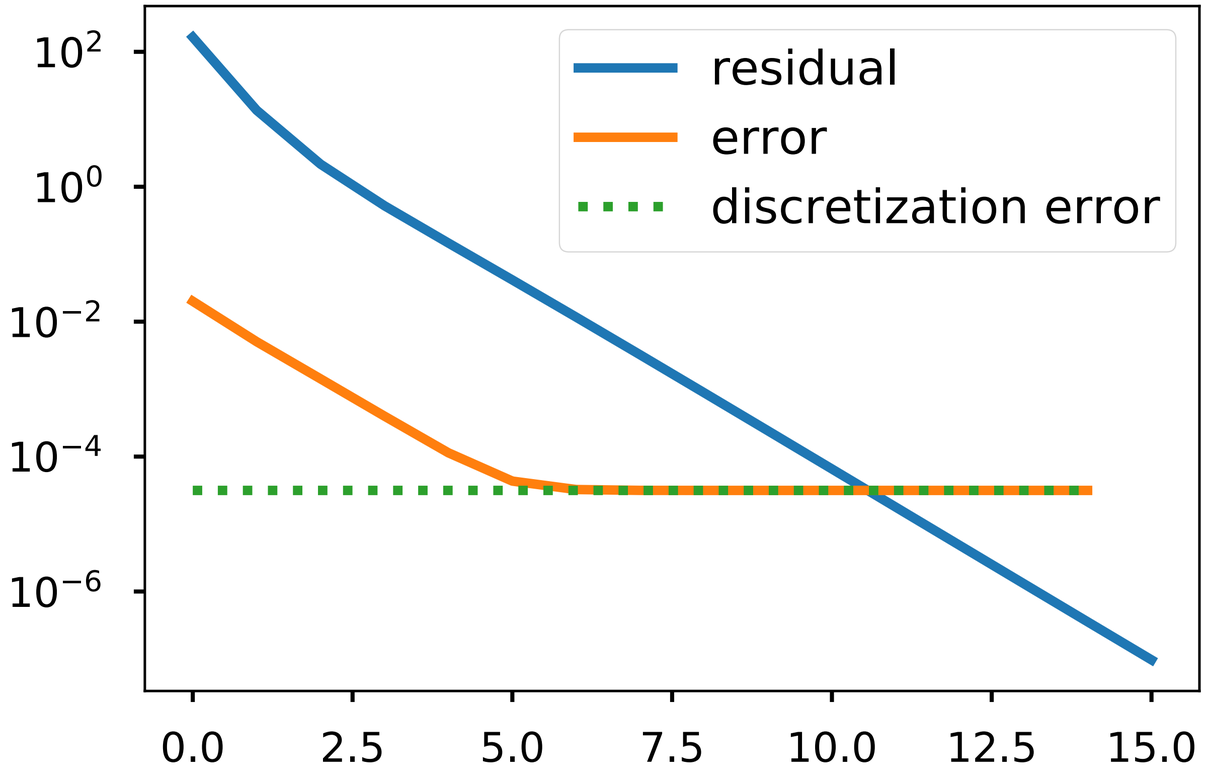
$$\gamma^m \sim \mathcal{O}(n^{-2}) \leq \frac{\varepsilon}{2}$$

- Then we need

$$m \sim \mathcal{O}(\log n)$$

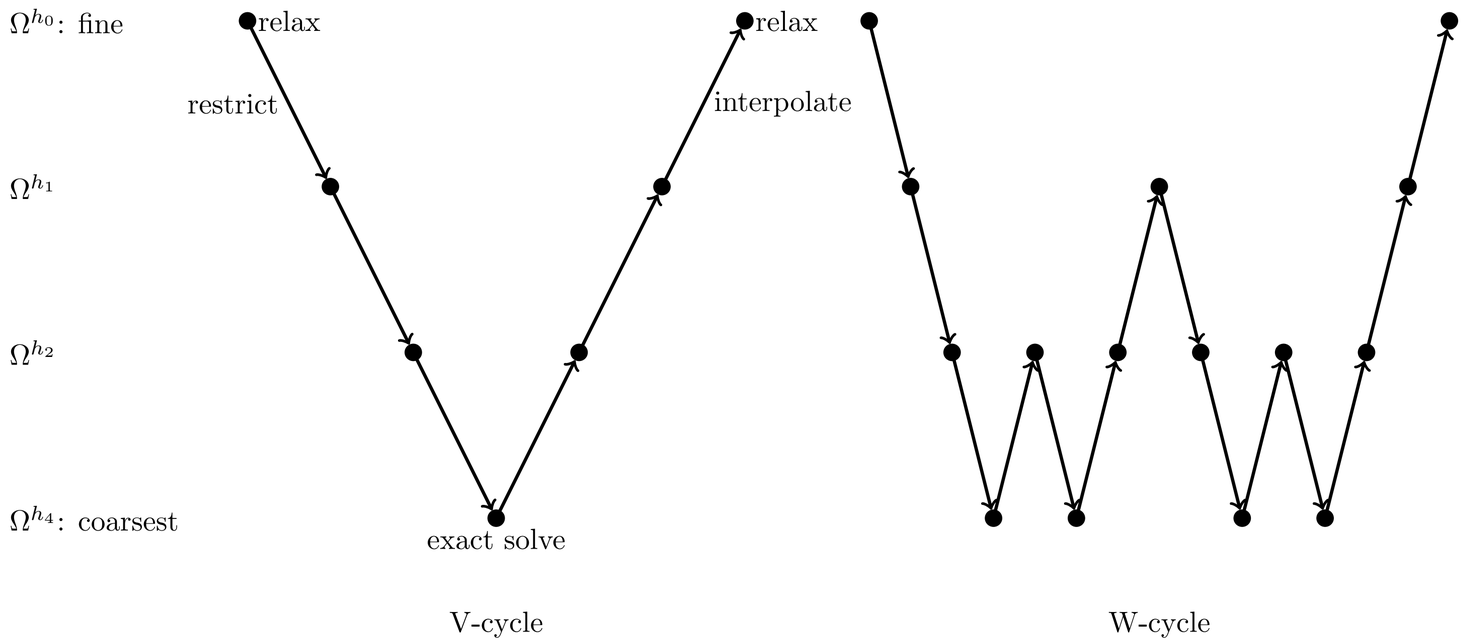
  cycles

Algebraic error



- The total error is limited by the discretization error

The Multigrid V-Cycle and W-Cycle



- Two-grid cycle can expose issues with coarser interpolation
- W-Cycle can account for inadequate coarser level solves
- **Exact solve?** Usually a pseudo-inverse

Up next…

- What can go wrong and what to do… (Friday)
- What happens when we drop the notion of a *grid ?* (Friday and Monday)
- What does this work at all? (the next Friday)