CSC498 Formula Sheet

Probability Review

Identities

$P(A \cap B) = P(A)P(B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\qquad P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$

$f(x) = \frac{dF(x)}{dx}, \int_{-\infty}^{\infty} f(x) = 1$ $\qquad P(a < X < b) = \int_a^b f(x)dx$

$F(y) = \int_{-\infty}^{y} f(x)dx$ $\qquad P(X < x) = F(X)$

$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f(x)dx$ $\qquad Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Uniform Distribution: $X \sim uniform(a, b)$

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise} \end{cases} \qquad \text{(pdf)}$$

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x > b \end{cases} \qquad \text{(cdf)}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ $\qquad Var[X] = \frac{(b-a)^2}{12}$

Exponential Distribution: $X \sim exp(\lambda)$

$f(x) = \lambda e^{-\lambda x}$ (pdf)

$F(x) = 1 - \lambda e^{-\lambda x}$ (cdf)

$\mathbb{E}[X] = \frac{1}{\lambda}$ $\qquad Var[X] = \frac{1}{\lambda^2}$

$P(X > s + t | X > s) = P(X > t)$ [Memoryless]

Poisson Distribution: $X \sim pois(\lambda)$

$P(N(t) = n) = \frac{(\lambda t)^t}{n!} e^{-\lambda t}$ [pmf]

$\mathbb{E}[X] = Var(X)$

Poisson Process

$N(0) = 0$

$f(x) = \lambda e^{-\lambda x}$ [Arrival See Time Average]

$P(X > s + t | X > s) = P(X > t)$ [Memoryless]

$\lambda = \sum_{i=1}^{n} \lambda_i, Y = \left(\sum_{i=1}^{n} X_i \right) \sim \text{pois}(\lambda)$ [Merge Poisson Processes]

$X \sim pois(\lambda), X = [X_1, X_2]$

$\implies X_{1,2} \sim pois(\frac{\lambda}{2})$ [Split Poisson Processes]

Performance Analysis

Identities

$A_i(t)$ [Number of arrivals] $\qquad \lambda_i(t) = \frac{A_i(t)}{t}$ [Arrival Rate]

$C_i(t)$ [Completions] $\qquad X_i(t) = \frac{C_i(t)}{t}$ [Throughput]

$B_i(t)$ [Busy time] $\qquad \rho_i(t) = \frac{B_i(t)}{t}$ [Utilization]

$S_i(t) = \frac{B_i(t)}{C_i(t)}$ [Avg process time] $\qquad S_i(t) = \mathbb{E}[S]$

D_i [Processing time of cycle] $\qquad \mathbb{E}[D_i] = \mathbb{E}[S_i]\mathbb{E}[Vi]$

$V_i(t)$ [Visits to device] $\qquad V_{user} = V_0 = 1$

$\lim_{t \to \infty} \frac{A_i(t)}{t} = \lim_{t \to \infty} \frac{C_i(t)}{t}$ $\qquad \lambda_i = X_i$ [Steady state]

$N(t) = A(t) - C(t)$ [Number of jobs in system]

$R(t) \approx \int_0^t \frac{A(s) - C(s)}{A(t)} ds$ [Avg response time]

$\bar{N}(t) \approx \int_0^t \frac{A(s) - C(s)}{t} ds$ [Avg number of jobs in system]

$\bar{N}(t) = \frac{R(t)A(t)}{t}$

Z [Think time]

$\mathbb{E}[N] = N, \lambda = X, R = R + Z$ [Closed System]

Operation Laws

$\rho_i = \mathbb{E}[S_i]X_i = \frac{\lambda_i}{\rho_i}$ [Utilization Law]

$\rho_i = \mathbb{E}[S_i]\mathbb{E}[Vi]X = \mathbb{E}[D_i]X$ [Bottleneck Law]

$X_i = \mathbb{E}[V_i]X$ [Forced Flow Law]

$\mathbb{E}[N] = \lambda \mathbb{E}[R]$ [Little's Law]

$\mathbb{E}[N_i] = \lambda_i \mathbb{E}[R_i]$

$\mathbb{E}[R] = \frac{N}{X} - \mathbb{E}[Z]$ [Closed System Response Time Law]

Bottleneck Analysis

D_{max} [Bottleneck Device] $\qquad D = \sum D_i$

$\mathbb{E}[R] \ge D$ $\qquad X = \frac{\rho_{max}}{D_{max}}$

$\mathbb{E}[R] \ge max(D, ND_{max} - \mathbb{E}[Z])$ $\qquad X \le min(\frac{1}{D_{max}}, \frac{N}{D + \mathbb{E}[Z]})$

$N^* = \frac{D + \mathbb{E}[Z]}{D_{max}}$ $\qquad \implies$ optimal X and $\mathbb{E}[R]$

Queuing Models

(Arrivals / Service Times / Number of servers / Room in system)

M/M/1

$\rho = \lambda/\mu$ $\qquad \mu > \lambda$ [Stability condition]

$\pi_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$ $\qquad \pi_i = \pi_0(\frac{\lambda}{\mu})^i = (1-\rho)\rho^i$

$\mathbb{E}[N] = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$ $\qquad \mathbb{E}[N_Q] = \mathbb{E}[N] - \rho$

$\mathbb{E}[R] = \frac{1}{\mu - \lambda}$ $\qquad \mathbb{E}[R_Q] = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$

M/M/c

$\rho = \frac{\lambda}{c\mu}$ $\qquad c\mu > \lambda$ [Stability condition]

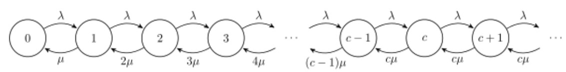
$$\pi_0 = (\frac{\lambda}{\mu})^c \frac{1}{1-\rho} \qquad \pi_i = \begin{cases} \frac{\lambda^i}{i!\mu^i}\pi_0, & \text{if } i < c \\ \frac{\lambda^i}{c!\mu^i c^{i-c}}\pi_0, & \text{if } i \ge c \end{cases}$$

$\mathbb{E}[N] = \lambda \mathbb{E}[R]$ $\qquad \mathbb{E}[N_Q] = \lambda \mathbb{E}[R_Q]$

$\mathbb{E}[R] = \mathbb{E}[R_Q] + \mathbb{E}[S] = \mathbb{E}[R_Q] + \frac{1}{\mu}$

$\mathbb{E}[R_Q] = \frac{(\frac{\lambda}{\mu})^c \mu}{(c-1)!(c\mu - \lambda)^2}$

$P(\text{job is queued}) = \sum_{i=0}^{\infty} \pi = \frac{1}{c!}(\frac{\lambda}{\mu})^c \frac{1}{1-\rho}\pi_0$ [Erlang C Formula]



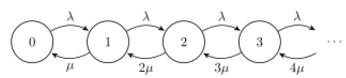
M/M/∞

$\rho = \lambda/\mu$ $\qquad \mu > \lambda$ [Always Stable]

$\pi_0 = e^{-\frac{\lambda}{\mu}} = e^{-\rho}$ $\qquad \pi_i = \frac{(\lambda/\mu)^i}{i!} e^{-\frac{\lambda}{\mu}} = \frac{\rho^i}{i!} e^{-\rho}$

$\mathbb{E}[N] = \frac{\lambda}{\mu} = \rho$ $\qquad \mathbb{E}[N_Q] = 0$

$\mathbb{E}[R] = \frac{1}{\mu} = \mathbb{E}[S]$ $\qquad \mathbb{E}[R_Q] = 0$



Birth-Death Process

CTMC where state transitions increase or decrease by a constant factor.

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}$$

$$\pi_i = \frac{\prod_{j=0}^{i-1} \lambda_j}{\prod_{j=1}^{i} \mu_j} \pi_0$$

Threshold System

$T > 0$, Arrival rate s, processing rate s. If $r > s, N \to 0$. If $s > r, N \to \infty$.

$$\pi_0 = \frac{1}{1 - \frac{r}{s}} (\frac{s}{r})^T - 1$$

$$\pi_i = \begin{cases} (\frac{s}{r})^i \pi_0, & \text{if } i < T \\ (\frac{s}{r})^{i-T} (\frac{r}{s})^2 \pi_0, & \text{if } i \geq T \end{cases}$$

Misc

- $\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}, |\alpha| < 1.$
- $h = \frac{f}{g} \implies h' = \frac{f'g - fg'}{g^2}$
- Blocking system structure $\implies X_1 = X_2 = ... = X_n$
- Max system utilization $\implies$ only bottleneck utilization is 100%
- Want to minimize $\mathbb{E}[R]$ and maximize X.
- Operation Laws work regardless of distributions of random variables
- exponential distributions are a very good assumption for modeling arrivals, but only moderately good for modelling processing times