

# Learning Energy Minimization on Moduli Space of Connections

## 1 Overview

In gauge theory, the moduli space of connections results from quotienting the space of principal connections of a fiber or vector bundle by the structure group, generating a gauge equivariant space. A moduli space of connections can be used to cover the activity of a collection of deep neural networks, which are represented as trajectories on vector bundles. On this upper bounding space, a top-down energy-based attention mechanism can be trained from sampled activity of the bottom-up trajectories of the underlying networks through a composition of energies. Dynamics of attention are computed by minimizing the energy functional of the space. In particular, the Yang-Mills moduli space, a subset of the total connection space, can be constructed to be a smooth, compact, and oriented manifold in 4 dimensions with critical points known as Yang-Mills connections (or instantons). These connections minimize curvature between bundles, which can be thought of as minimizing relative entropy or divergence. Examining research in ergodic and hyperbolic dynamics of moduli spaces allows the ergodic free-energy minimizing model described in previous works to be developed further. An encoding mechanism using the cobordism property of the Yang-mills moduli space is considered.

### 1.1 Prerequisite Review

A fiber bundle serves as a useful mathematical object to analyse both the recursive construction of artificial and biological neurons and neural networks, as well as their application in the geometry of latent information which uses manifold representations for information processing and statistical learning. Finer details can be found in my notebooks on differential geometry, Lie groups and algebras, and gauge theory. A fiber bundle makes precise the idea of one topological space (called a fiber) being parameterized by another topological space (called a base). The bundle also comes with a group action on the fiber that represents the different ways the fiber can be viewed as equivalent. The fiber bundle has a property, known as local trivialization, that allows neighborhoods of the bundle to be computed as simple, oriented product spaces, despite the global space possibly being unoriented or twisted.

A family of fibers associated to a base can be described by defining a template standard fiber which all other fibers are isomorphic to. This is formalized by defining a projection mapping, which may be diffeomorphic or homotopic, that connects positional data from the entirety of fibers to a base and implicitly from one fiber to another. When the template fiber is a vector space, the bundle is called a vector bundle. The standard connections between fibers, known as a principal Ehresmann connection, can intuitively be understood as a covariant directional derivative on the tangent spaces of the manifolds and will always exist. An interesting type of connection occurs when using a fiber bundle equipped with a Lie group action, which has a unique recursive nature as a result of the Lie group being itself a differentiable manifold. In particular, the bundle structure can be used to represent both the original fiber or vector bundle as well as the higher level collections of tangent spaces between bundles in what's known as a bundle of connections. As will be explained in the section on moduli spaces, The Yang-Mills or instanton connections of this bundle correspond to those connections that minimize their curvature.

### 1.2 A Priori Structure Groups

A biological first principle of covariance arises naturally from analysis of neuronal activity, which appears to favour functional localization and Hebbian learning. Moreover, human brain networks appear to flow in connectome-specific smooth diffusive waves along gyrification paths which are

theorized to be caused by differential tangential growth. Recall, covariance is a measure of the joint variability of two random variables and is increasingly positive when a pair show similar behavior and is negative when dissimilar. Covariance finds a direct description in differential geometry as a principal Ehressmann connection, which represents the covariant derivative between fibers of a bundle. To analyze dynamics on a bundle using gauge theory, it becomes necessary to impose a generalized a priori covariance principle to maintain integrity of information being transported in bilateral and hierarchical directions. Yet for a standard learning model, covariance of functionality is an a posteriori feature since the joint variability is unknown until individual modules are fully trained. Covariance of fibers can be achieved by imposing the structure group to be a Lie Group, but can also be achieved by imposing restrictions on the projection map of Riemannian manifolds without explicitly defining the structure group beforehand.

### 1.3 Moduli Spaces

With covariance established on connections, it becomes possible to perform inference using higher levels of abstraction on gauge fields. This is done by constructing a gauge equivariant bundle of connections known as a moduli space of connections which can be further reduced into a Yang-Mills moduli space defined to be a finite dimensional manifold. This reduced space has local and global minima being connections with minimized energy known as Yang-Mills connections or instantons. Yang-Mills connections serve as a natural choice of connection on principal and vector bundles since they minimize their curvature. From an information geometric perspective, this can be thought of as minimizing relative entropy between sampled trajectories of two manifolds which happen to be gauge equivariant. The gauge field strength is the curvature  $F_A$  of the connection, and the energy of the gauge field is given by the Yang–Mills action functional:

$$\text{YM}(A) = \int_X \|F_A\|^2 d\text{vol}_g.$$

In search of zero or vanishing curvature, one aims for a bundle with curvature as small as possible. The Yang–Mills action functional described above is the square of the  $L^2$ -norm of the curvature, and its Euler–Lagrange equations describe the critical points of this functional, either the absolute minima or local minima, i.e. Yang–Mills connections are those that minimize their curvature.

Moduli of Yang–Mills connections have been most studied when the dimension of the base manifold  $X$  is four. Here the Yang–Mills equations admit a simplification from a second-order PDE to a first-order PDE, the anti-self-duality equations. Additionally, these manifolds demonstrate a cobordism scheme where in specific circumstances (when the intersection form is definite) the moduli space of ASD instantons on a smooth, compact, oriented, simply-connected four-manifold  $X$  gives a cobordism between a copy of the manifold itself, and a disjoint union of copies of the complex projective plane  $\mathbb{CP}^2$ . This provides a naturally occurring symmetry (gauge invariance) of hierarchical encoding.

### References

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