

CSC498 Formula Sheet

Probability Review

Identities

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$

$$f(x) = \frac{dF(x)}{dx}, \int_{-\infty}^{\infty} f(x)dx = 1 \quad P(a < X < b) = \int_a^b f(x)dx$$

$$F(y) = \int_{-\infty}^y f(x)dx \quad P(X < x) = F(x)$$

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f(x)dx \quad Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Uniform Distribution: $X \sim uniform(a, b)$

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise} \end{cases} \quad (\text{pdf})$$

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x > b \end{cases} \quad (\text{cdf})$$

$$\mathbb{E}[X] = \frac{a+b}{2} \quad Var[X] = \frac{(b-a)^2}{12}$$

Exponential Distribution: $X \sim exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \quad (\text{pdf})$$

$$F(x) = 1 - \lambda e^{-\lambda x} \quad (\text{cdf})$$

$$\mathbb{E}[X] = \frac{1}{\lambda} \quad Var[X] = \frac{1}{\lambda^2}$$

$$P(X > s + t | X > s) = P(X > t) \quad [\text{Memoryless}]$$

Poisson Distribution: $X \sim pois(\lambda)$

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad [\text{pmf}]$$

$$\mathbb{E}[X] = Var(X)$$

Poisson Process

$$N(0) = 0$$

$$f(x) = \lambda e^{-\lambda x} \quad [\text{Arrival See Time Average}]$$

$$P(X > s + t | X > s) = P(X > t) \quad [\text{Memoryless}]$$

$$\lambda = \sum_{i=1}^n \lambda_i, Y = \left(\sum_{i=1}^n X_i \right) \sim pois(\lambda) \quad [\text{Merge Poisson Processes}]$$

$$X \sim pois(\lambda), X = [X_1, X_2]$$

$$\implies X_{1,2} \sim pois\left(\frac{\lambda}{2}\right) \quad [\text{Split Poisson Processes}]$$

Performance Analysis

Identities

$$A_i(t) \text{ [Number of arrivals]}$$

$$C_i(t) \text{ [Completions]}$$

$$B_i(t) \text{ [Busy time]}$$

$$S_i(t) = \frac{B_i(t)}{C_i(t)} \text{ [Avg process time]}$$

$$D_i \text{ [Processing time of cycle]}$$

$$V_i(t) \text{ [Visits to device]}$$

$$\lim_{t \rightarrow \infty} \frac{A_i(t)}{t} = \lim_{t \rightarrow \infty} \frac{C_i(t)}{t}$$

$$N(t) = A(t) - C(t)$$

$$R(t) \approx \int_0^t \frac{A(s) - C(s)}{A(t)} ds$$

$$\bar{N}(t) \approx \int_0^t \frac{A(s) - C(s)}{t} ds$$

$$\bar{N}(t) = \frac{R(t)A(t)}{t}$$

$$Z$$

$$\mathbb{E}[N] = N, \lambda = X, R = R + Z$$

Operation Laws

$$\rho_i = \mathbb{E}[S_i]X_i = \frac{\lambda_i}{\rho_i}$$

$$\rho_i = \mathbb{E}[S_i]\mathbb{E}[V_i]X = \mathbb{E}[D_i]X$$

$$X_i = \mathbb{E}[V_i]X$$

$$\mathbb{E}[N] = \lambda \mathbb{E}[R]$$

$$\mathbb{E}[N_i] = \lambda_i \mathbb{E}[R_i]$$

$$\mathbb{E}[R] = \frac{N}{X} - \mathbb{E}[Z]$$

Bottleneck Analysis

$$D_{max} \text{ [Bottleneck Device]}$$

$$\mathbb{E}[R] \geq D$$

$$\mathbb{E}[R] \geq max(D, ND_{max} - \mathbb{E}[Z])$$

$$N^* = \frac{D + \mathbb{E}[Z]}{D_{max}}$$

$$\lambda_i(t) = \frac{A_i(t)}{t} \text{ [Arrival Rate]}$$

$$X_i(t) = \frac{C_i(t)}{t} \text{ [Throughput]}$$

$$\rho_i(t) = \frac{B_i(t)}{t} \text{ [Utilization]}$$

$$S_i(t) = \mathbb{E}[S]$$

$$\mathbb{E}[D_i] = \mathbb{E}[S_i]\mathbb{E}[V_i]$$

$$V_{user} = V_0 = 1$$

$$\lambda_i = X_i \text{ [Steady state]}$$

$$\text{[Number of jobs in system]}$$

$$\text{[Avg response time]}$$

$$\text{[Avg number of jobs in system]}$$

$$\text{[Think time]}$$

$$\text{[Closed System]}$$

$$\text{[Utilization Law]}$$

$$\text{[Bottleneck Law]}$$

$$\text{[Forced Flow Law]}$$

$$\text{[Little's Law]}$$

$$\text{[Closed System Response Time Law]}$$

$$D = \sum D_i$$

$$X = \frac{\rho_{max}}{D_{max}}$$

$$X \leq \min\left(\frac{1}{D_{max}}, \frac{N}{D + \mathbb{E}[Z]}\right)$$

$$\implies \text{optimal } X \text{ and } \mathbb{E}[R]$$

Queuing Models

(Arrivals / Service Times / Number of servers / Room in system)

M/M/1

$$\rho = \lambda / \mu \quad \mu > \lambda \text{ [Stability condition]}$$

$$\pi_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho \quad \pi_i = \pi_0 \left(\frac{\lambda}{\mu}\right)^i = (1 - \rho)\rho^i$$

$$\mathbb{E}[N] = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho} \quad \mathbb{E}[N_Q] = \mathbb{E}[N] - \rho$$

$$\mathbb{E}[R] = \frac{1}{\mu - \lambda} \quad \mathbb{E}[R_Q] = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$



M/M/c

$$\rho = \frac{\lambda}{c\mu} \quad c\mu > \lambda \text{ [Stability condition]}$$

$$\pi_0 = \left(\frac{\lambda}{\mu}\right)^c \frac{1}{1 - \rho} \quad \pi_i = \begin{cases} \frac{\lambda^i}{i! \mu^i} \pi_0, & \text{if } i < c \\ \frac{\lambda^i}{c! \mu^i c^{i-c}} \pi_0, & \text{if } i \geq c \end{cases}$$

$$\mathbb{E}[N] = \lambda \mathbb{E}[R]$$

$$\mathbb{E}[R] = \mathbb{E}[R_Q] + \mathbb{E}[S] = \mathbb{E}[R_Q] + \frac{1}{\mu}$$

$$\mathbb{E}[R_Q] = \frac{\left(\frac{\lambda}{\mu}\right)^c \mu}{(c-1)!(c\mu - \lambda)^2}$$

$$P(\text{job is queued}) = \sum_{i=0}^{\infty} \pi = \frac{1}{c!} \left(\frac{\lambda}{\mu}\right)^c \frac{1}{1 - \rho} \pi_0 \quad [\text{Erlang C Formula}]$$



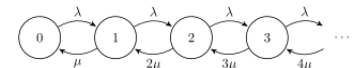
M/M/∞

$$\rho = \lambda / \mu \quad \mu > \lambda \text{ [Always Stable]}$$

$$\pi_0 = e^{-\frac{\lambda}{\mu}} = e^{-\rho} \quad \pi_i = \frac{(\lambda/\mu)^i}{i!} e^{-\frac{\lambda}{\mu}} = \frac{\rho^i}{i!} e^{-\rho}$$

$$\mathbb{E}[N] = \frac{\lambda}{\mu} = \rho \quad \mathbb{E}[N_Q] = 0$$

$$\mathbb{E}[R] = \frac{1}{\mu} = \mathbb{E}[S] \quad \mathbb{E}[R_Q] = 0$$



Birth-Death Process

CTMC where state transitions increase or decrease by a constant factor.

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^k \frac{\lambda_{i-1}}{\mu_i}}$$
$$\pi_i = \frac{\prod_{j=0}^{i-1} \lambda_j}{\prod_{j=1}^i \mu_j} \pi_0$$

Threshold System

$T > 0$, Arrival rate s , processing rate s . If $r > s, N \rightarrow 0$. If $s > r, N \rightarrow \infty$.

$$\pi_0 = \frac{1}{1 - \frac{r}{s}} \left(\frac{s}{r}\right)^T - 1$$
$$\pi_i = \begin{cases} \left(\frac{s}{r}\right)^i \pi_0, & \text{if } i < T \\ \left(\frac{s}{r}\right)^{i-T} \left(\frac{r}{s}\right)^2 \pi_0, & \text{if } i \geq T \end{cases}$$

Misc

- $\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}, |\alpha| < 1$.
- $h = \frac{f}{g} \implies h' = \frac{f'g - fg'}{g^2}$
- Blocking system structure $\implies X_1 = X_2 = \dots = X_n$
- Max system utilization \implies only bottleneck utilization is 100%
- Want to minimize $\mathbb{E}[R]$ and maximize X .
- Operation Laws work regardless of distributions of random variables
- exponential distributions are a very good assumption for modeling arrivals, but only moderately good for modelling processing times