3.1 Exponential Functions and Their Graphs

▶ **What you should learn**

- How to recognize and evaluate exponential functions with base a
- How to graph exponential functions
- How to recognize and evaluate exponential functions with base e
- How to use exponential functions to model and solve real-life applications

▶ **Why you should learn it**

Exponential functions can be used to model and solve real-life problems. For instance, in Exercise 59 on page 207, an exponential function is used to model the amount of defoliation caused by the gypsy moth.

Jenny Hager/The Image Works

Exponential Functions

So far, this book has dealt only with **algebraic functions,** which include polynomial functions and rational functions. In this chapter you will study two types of nonalgebraic functions—*exponential* functions and *logarithmic* functions. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function f with base a** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

The base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.

You already know how to evaluate a^x for integer and rational values of x. For example, you know that $4^3 = 64$ and $4^{1/2} = 2$. However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this book, it is sufficient to think of

$$a^{\sqrt{2}} \quad (\text{where } \sqrt{2} \approx 1.41421356)$$

as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \ldots .$$

Example 1 shows how to use a calculator to evaluate exponential expressions.

Example 1 ▶ **Evaluating Exponential Functions**

Use a calculator to evaluate each function at the indicated value of x.

	Function	*Value*
a.	$f(x) = 2^x$	$x = -3.1$
b.	$f(x) = 2^{-x}$	$x = \pi$
c.	$f(x) = 12^x$	$x = \frac{5}{7}$
d.	$f(x) = 0.6^x$	$x = \frac{3}{2}$

Solution

	Function Value	*Graphing Calculator Keystrokes*	*Display*
a.	$f(-3.1) = 2^{-3.1}$	2 [^] [(-)] 3.1 [ENTER]	0.1166291
b.	$f(\pi) = 2^{-\pi}$	2 [^] [(-)] π [ENTER]	0.1133147
c.	$f(\frac{5}{7}) = 12^{5/7}$	12 [^] [(] 5 [÷] 7 [)] [ENTER]	5.8998877
d.	$f(\frac{3}{2}) = (0.6)^{3/2}$	.6 [^] [(] 3 [÷] 2 [)] [ENTER]	0.4647580

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

Example 2 ▶ **Graphs of $y = a^x$**

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$ b. $g(x) = 4^x$

Solution

The table below lists some values for each function, and Figure 3.1 shows the graphs of these two functions. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	2^x	4^x
-2	$\frac{1}{4}$	$\frac{1}{16}$
-1	$\frac{1}{2}$	$\frac{1}{4}$
0	1	1
1	2	4
2	4	16
3	8	64

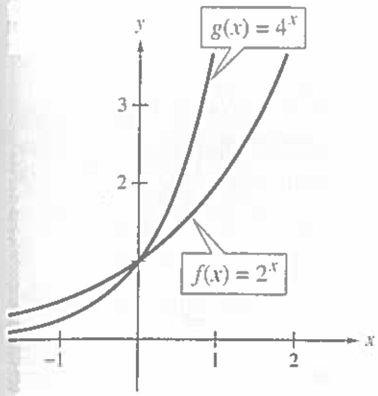

FIGURE 3.1

STUDY TIP

The table in Example 2 was evaluated by hand. You could, of course, use a graphing utility to construct tables with even more values.

Example 3 ▶ **Graphs of $y = a^{-x}$**

In the same coordinate plane, sketch the graph of each function.

a. $F(x) = 2^{-x}$ b. $G(x) = 4^{-x}$

Solution

The table below lists some values for each function, and Figure 3.2 shows the graphs of the two functions. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	-3	-2	-1	0	1	2
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
4^{-x}	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

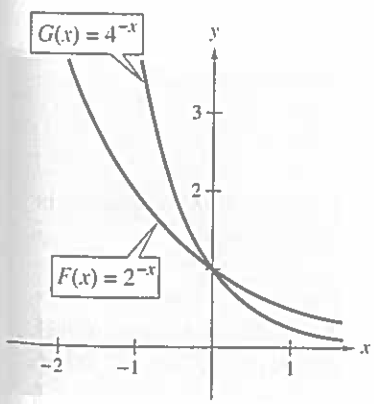

FIGURE 3.2

In Example 3, note that the functions $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents.

$$F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x \quad \text{and} \quad G(x) = 4^{-x} = \left(\frac{1}{4}\right)^x$$

The icon identifies examples and concepts related to features of the Learning Tools CD-ROM and the *Interactive* and *Internet* versions of this text. For more details see the chart on pages xxi–xxv.

Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \qquad \text{and} \qquad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y-axis) of the graph of f. The graphs of G and g have the same relationship. The graphs in Figures 3.1 and 3.2 are typical of the exponential functions $y = a^x$ and $y = a^{-x}$. They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous. The basic characteristics of these exponential functions are summarized in Figures 3.3 and 3.4.

STUDY TIP

Notice that the range of an exponential function is $(0, \infty)$, which means that $a^x > 0$ for all values of x.

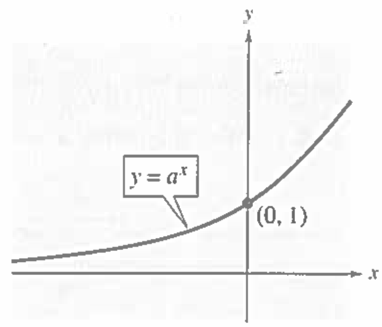

FIGURE 3.3

Graph of $y = a^x, a > 1$
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Increasing
- x-Axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$)
- Continuous

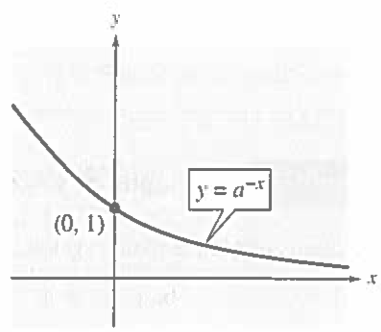

FIGURE 3.4

Graph of $y = a^{-x}, a > 1$
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Intercept: $(0, 1)$
- Decreasing
- x-Axis is a horizontal asymptote ($a^{-x} \to 0$ as $x \to \infty$)
- Continuous

The *Interactive* CD-ROM and *Internet* versions of this text offer a Try It for each example in the text.

Exploration

Use a graphing utility to graph

$$y = a^x$$

for $a = 3, 5$, and 7 in the same viewing window. (Use a viewing window in which $-2 \le x \le 1$ and $0 \le y \le 2$.) For instance, the graph of

$$y = 3^x$$

is shown in Figure 3.5. How do the graphs compare with each other? Which graph is on the top in the interval $(-\infty, 0)$? Which is on the bottom? Which graph is on the top in the interval $(0, \infty)$? Which is on the bottom?

Repeat this experiment with the graphs of $y = b^x$ for $b = \frac{1}{3}, \frac{1}{5}$, and $\frac{1}{7}$. (Use a viewing window in which $-1 \le x \le 2$ and $0 \le y \le 2$.) What can you conclude about the shape of the graph of $y = b^x$ and the value of b?

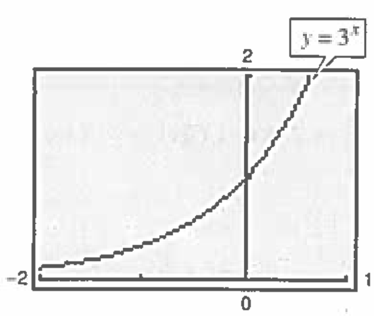

FIGURE 3.5

In the following example, notice how the graph of $y = a^x$ can be used to sketch the graphs of functions of the form $f(x) = b \pm a^{x+c}$.

Example 4 ▶ Transformations of Graphs of Exponential Functions

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*, as shown in Figure 3.6.

b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *downward* two units, as shown in Figure 3.7.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x-axis, as shown in Figure 3.8.

d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y-axis, as shown in Figure 3.9.

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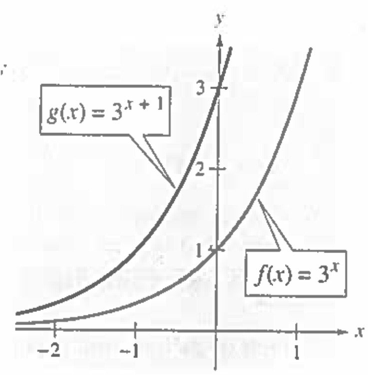


FIGURE 3.6 *Horizontal shift*

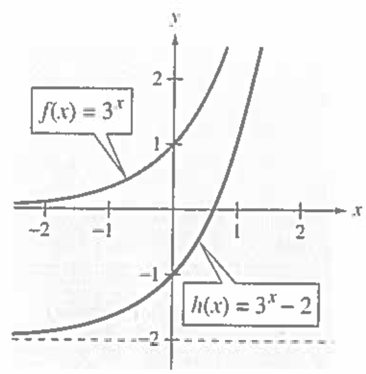


FIGURE 3.7 *Vertical shift*

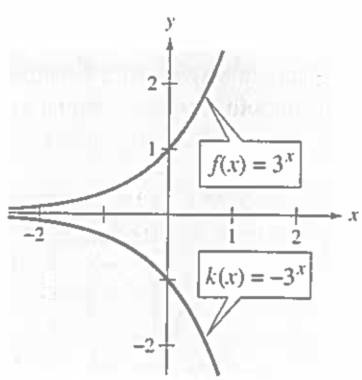


FIGURE 3.8 *Reflection in x-axis*

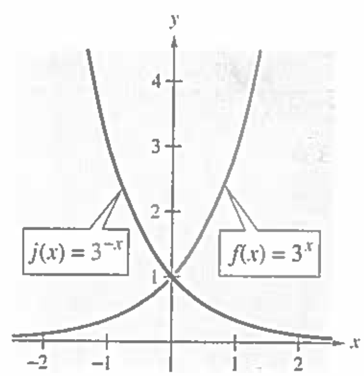


FIGURE 3.9 *Reflection in y-axis*

Notice that the transformations in Figures 3.6, 3.8, and 3.9 keep the x-axis as a horizontal asymptote, but the transformation in Figure 3.7 yields a new horizontal asymptote of $y = -2$. Also, be sure to note how the y-intercept is affected by each transformation.

The Natural Base e

In many applications, the most convenient choice for a base is the irrational number

$$e \approx 2.718281828 \ldots.$$

This number is called the **natural base.** The function $f(x) = e^x$ is called the **natural exponential function.** Its graph is shown in Figure 3.10. Be sure you see that for the exponential function $f(x) = e^x$, e is the constant $2.718281828 \ldots$, whereas x is the variable.

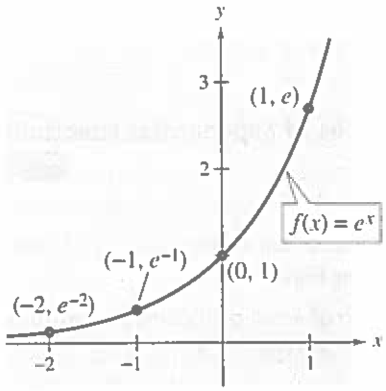


FIGURE 3.10

Example 5 ▶ Evaluating the Natural Exponential Function

Use a calculator to evaluate the function $f(x) = e^x$ at each indicated value of x.

a. $x = -2$ **b.** $x = -1$ **c.** $x = 0.25$ **d.** $x = -0.3$

Solution

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(-2) = e^{-2}$	e^x (−) 2 ENTER	0.1353353
b. $f(-1) = e^{-1}$	e^x (−) 1 ENTER	0.3678794
c. $f(0.25) = e^{0.25}$	e^x 0.25 ENTER	1.2840254
d. $f(-0.3) = e^{-0.3}$	e^x (−) 0.3 ENTER	0.7408182

Example 6 ▶ Graphing Natural Exponential Functions

Sketch the graph of each natural exponential function.

a. $f(x) = 2e^{0.24x}$ **b.** $g(x) = \frac{1}{2}e^{-0.58x}$

Solution

To sketch these two graphs, you can use a graphing utility to construct a table of values, as shown below. After constructing the table, plot the points and connect them with smooth curves, as shown in Figures 3.11 and 3.12. Note that the graph in Figure 3.11 is increasing whereas the graph in Figure 3.12 is decreasing.

x	$f(x)$	$g(x)$
-3	0.974	2.849
-2	1.238	1.595
-1	1.573	0.893
0	2.000	0.500
1	2.542	0.280
2	3.232	0.157
3	4.109	0.088

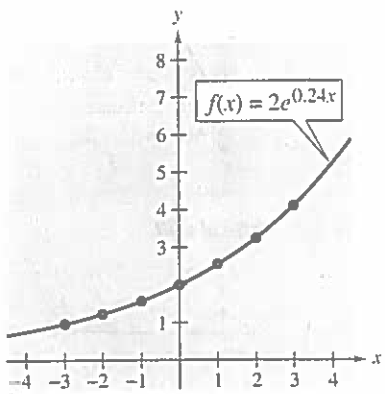


FIGURE 3.11

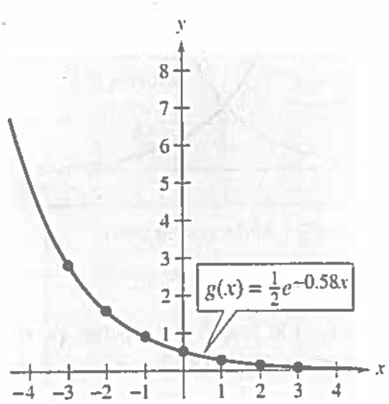


FIGURE 3.12

Exploration

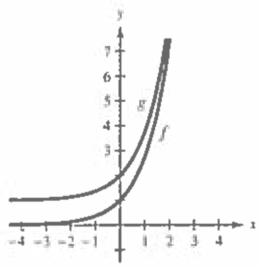
Use the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

to calculate the amount in an account when $P = \$3000$, $r = 6\%$, $t = 10$ years, and compounding is done (1) by the day, (2) by the hour, (3) by the minute, and (4) by the second. Does increasing the number of compoundings per year result in unlimited growth of the amount in the account? Explain.

Activities

1. Sketch the graphs of the functions $f(x) = e^x$ and $g(x) = 1 + e^x$ on the same coordinate system.



2. Determine the balance A at the end of 20 years if \$1500 is invested at 6.5% interest and the interest is compounded (a) quarterly and (b) continuously.

   Answer: (a) \$5446.73 (b) \$5503.95

3. Determine the amount of money that should be invested at 9% interest, compounded monthly, to produce a final balance of \$30,000 in 15 years.

   Answer: \$7816.48

Applications

One of the most familiar examples of exponential growth is that of an investment earning *continuously compounded interest*. Using exponential functions, you can now *develop* that formula and show how it leads to continuous compounding.

Suppose a principal P is invested at an annual interest rate r, compounded once a year. If the interest is added to the principal at the end of the year, the new balance P_1 is

$$\begin{aligned} P_1 &= P + Pr \\ &= P(1 + r). \end{aligned}$$

This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown below.

Year	*Balance After Each Compounding*
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
3	$P_3 = P_2(1 + r) = P(1 + r)^2(1 + r) = P(1 + r)^3$
	$\vdots$
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. Then the rate per compounding is r/n and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}. \qquad \text{Amount (balance) with } n \text{ compoundings per year}$$

If you let the number of compoundings n increase without bound, the process approaches what is called **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Amount with } n \text{ compoundings per year} \\ &= P\left(1 + \frac{r}{mr}\right)^{mrt} && \text{Substitute } mr \text{ for } n. \\ &= P\left(1 + \frac{1}{m}\right)^{mrt} && \text{Simplify.} \\ &= P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}. && \text{Property of exponents} \end{aligned}$$

As m increases without bound, it can be shown that $[1 + (1/m)]^m$ approaches e. (Try the values $m = 10$, 10,000, and 10,000,000.) From this, you can conclude that the formula for continuous compounding is

$$A = Pe^{rt}. \qquad \text{Substitute } e \text{ for } (1 + 1/m)^m.$$

STUDY TIP

Be sure you see that the annual interest rate must be expressed in decimal form. For instance, 6% should be expressed as 0.06.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$

Example 7 ▶ Compound Interest

A total of $12,000 is invested at an annual interest rate of 9%. Find the balance after 5 years if it is compounded

a. quarterly.

b. monthly.

c. continuously.

Solution

a. For quarterly compoundings, you have $n = 4$. So, in 5 years at 9%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 12{,}000\left(1 + \frac{0.09}{4}\right)^{4(5)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$\approx \$18{,}726.11. \qquad \text{Use a calculator.}$$

b. For monthly compoundings, you have $n = 12$. So, in 5 years at 9%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 12{,}000\left(1 + \frac{0.09}{12}\right)^{12(5)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$\approx \$18{,}788.17. \qquad \text{Use a calculator.}$$

c. For continuous compounding, the balance is

$$A = Pe^{rt} \qquad \text{Formula for continuous compounding}$$

$$= 12{,}000e^{0.09(5)} \qquad \text{Substitute for } P, r, \text{ and } t.$$

$$\approx \$18{,}819.75. \qquad \text{Use a calculator.}$$

In Example 7, note that continuous compounding yields more than quarterly or monthly compounding. This is typical of the two types of compounding. That is, for a given principal, interest rate, and time, continuous compounding will always yield a larger balance than compounding n times a year.

Additional Example

The number of fruit flies in an experimental population after t hours is given by $Q(t) = 20e^{0.03t}$, $t \geq 0$.

a. Find the initial number of fruit flies in the population.

b. How large is the population of fruit flies after 72 hours?

Solution

a. To find the initial population, evaluate $Q(t)$ at $t = 0$.
$Q(0) = 20e^{0.03(0)} = 20e^0 = 20(1) = 20$ flies.

b. After 72 hours, the population size is $Q(72) = 20e^{0.03(72)} = 20e^{2.16} \approx 173$ flies.

Group Activity

The sequence $3, 6, 9, 12, 15, \ldots$ is given by $f(n) = 3n$ and is an example of linear growth. The sequence $3, 9, 27, 81, 243, \ldots$ is given by $f(n) = 3^n$ and is an example of exponential growth. Explain the difference between these two types of growth. For each of the following sequences, indicate whether the sequence represents linear growth or exponential growth, and find a linear or exponential function that represents the sequence. Give several other examples of linear and exponential growth.

a. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

b. $4, 8, 12, 16, 20, \ldots$

c. $\frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}, \frac{10}{3}, 4, \ldots$

d. $5, 25, 125, 625, \ldots$

Example 8 ▶ Radioactive Decay

In 1986, a nuclear reactor accident occurred in Chernobyl in what was then the Soviet Union. The explosion spread highly toxic radioactive chemicals, such as plutonium, over hundreds of square miles, and the government evacuated the city and the surrounding area. To see why the city is now uninhabited, consider the model

$$P = 10\left(\frac{1}{2}\right)^{t/24{,}360}$$

which represents the amount of plutonium P that remains (from an initial amount of 10 pounds) after t years. Sketch the graph of this function over the interval from $t = 0$ to $t = 100{,}000$, where $t = 0$ represents 1986. How much of the 10 pounds will remain in the year 2005? How much of the 10 pounds will remain after 100,000 years?

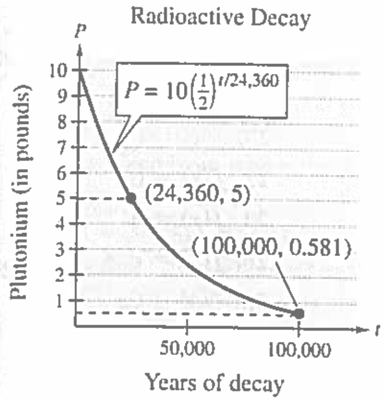


FIGURE 3.13

Solution

The graph of this function is shown in Figure 3.13. Note from this graph that plutonium has a *half-life* of about 24,360 years. That is, after 24,360 years, *half* of the original amount will remain. After another 24,360 years, one-quarter of the original amount will remain, and so on. In the year 2005 ($t = 19$), there will still be

$$P = 10\left(\frac{1}{2}\right)^{19/24{,}360} \approx 10\left(\frac{1}{2}\right)^{0.0007800} \approx 9.995 \text{ pounds}$$

of plutonium remaining. After 100,000 years, there will still be

$$P = 10\left(\frac{1}{2}\right)^{100{,}000/24{,}360} \approx 10\left(\frac{1}{2}\right)^{4.105} \approx 0.581 \text{ pound}$$

of plutonium remaining.

One way your students might approach this problem is to create a table, covering x-values from -2 through 3, for each of the functions and compare this table with the given tables. If this method is used, you might consider dividing your class into groups of three or six and having the groups assign one or two functions to each member. They should then pool their results and work cooperatively to determine that each function has a y-intercept of $(0, 8)$.

Another approach is a graphical one: the groups can create scatter plots of the data shown in the table and compare them with sketches of the graphs of the given functions. Consider assigning students to groups of four and giving the responsibility for sketching three graphs to each group member.

Writing ABOUT MATHEMATICS

Identifying Exponential Functions Which of the following functions generated the two tables below? Discuss how you were able to decide. What do these functions have in common? Are any of them the same? If so, explain why.

a. $f_1(x) = 2^{(x+3)}$ b. $f_2(x) = 8\left(\frac{1}{2}\right)^x$ c. $f_3(x) = \left(\frac{1}{2}\right)^{(x-3)}$

d. $f_4(x) = \left(\frac{1}{2}\right)^x + 7$ e. $f_5(x) = 7 + 2^x$ f. $f_6(x) = (8)2^x$

x	-1	0	1	2	3
$g(x)$	7.5	8	9	11	15

x	-2	-1	0	1	2
$h(x)$	32	16	8	4	2

Create two different exponential functions of the forms $y = a(b)^x$ and $y = c^x + d$ with y-intercepts of $(0, -3)$.

3.1 Exercises

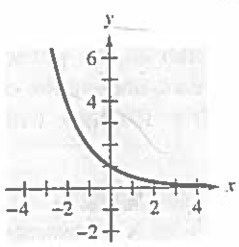
The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

In Exercises 1–6, evaluate the function at the indicated value of *x*. Round your result to three decimal places.

Function	*Value*
1. $f(x) = 3.4^x$	$x = 5.6$
2. $f(x) = 2.3^x$	$x = \frac{3}{2}$
3. $f(x) = 5^x$	$x = -\pi$
4. $g(x) = 5000(2^x)$	$x = -1.5$
5. $h(x) = e^{-x}$	$x = \frac{3}{4}$
6. $f(x) = e^x$	$x = 3.2$

In Exercises 7–10, match the exponential function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)



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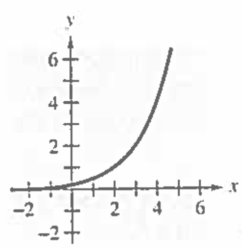

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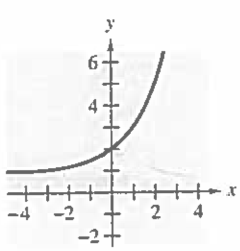

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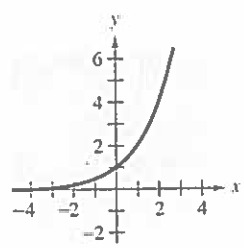


7. $f(x) = 2^x$
8. $f(x) = 2^x + 1$
9. $f(x) = 2^{-x}$
10. $f(x) = 2^{x-2}$

In Exercises 11–18, use the graph of *f* to describe the transformation that yields the graph of *g*.

11. $f(x) = 3^x$, $g(x) = 3^{x-4}$
12. $f(x) = 4^x$, $g(x) = 4^x + 1$
13. $f(x) = -2^x$, $g(x) = 5 - 2^x$
14. $f(x) = 10^x$, $g(x) = 10^{-x+3}$
15. $f(x) = \left(\frac{3}{5}\right)^x$, $g(x) = -\left(\frac{3}{5}\right)^{x+4}$
16. $f(x) = \left(\frac{7}{2}\right)^x$, $g(x) = -\left(\frac{7}{2}\right)^{-x+6}$
17. $f(x) = 0.3^x$, $g(x) = -0.3^x + 5$
18. $f(x) = 3.6^x$, $g(x) = -3.6^{-x} + 8$

In Exercises 19–32, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.

19. $f(x) = \left(\frac{1}{2}\right)^x$
20. $f(x) = \left(\frac{1}{2}\right)^{-x}$
21. $f(x) = 6^{-x}$
22. $f(x) = 6^x$
23. $f(x) = 2^{x-1}$
24. $f(x) = 3^{x+2}$
25. $f(x) = e^x$
26. $f(x) = e^{-x}$
27. $f(x) = 3e^{x+4}$
28. $f(x) = 2e^{-0.5x}$
29. $f(x) = 2e^{x-2} + 4$
30. $f(x) = 2 + e^{x-5}$
31. $f(x) = 4^{x-3} + 3$
32. $f(x) = -4^{x-3} - 3$

In Exercises 33–42, use a graphing utility to graph the exponential function.

33. $y = 2^{-x^2}$
34. $y = 3^{-|x|}$
35. $y = 3^{x-2} + 1$
36. $y = 4^{x+1} - 2$
37. $y = 1.08^{-5x}$
38. $y = 1.08^{5x}$
39. $s(t) = 2e^{0.12t}$
40. $s(t) = 3e^{-0.2t}$
41. $g(x) = 1 + e^{-x}$
42. $h(x) = e^{x-2}$

Compound Interest **In Exercises 43–46, complete the table to determine the balance *A* for *P* dollars invested at rate *r* for *t* years and compounded *n* times per year.**

n	1	2	4	12	365	Continuous
A						

43. $P = \$2500$, $r = 8\%$, $t = 10$ years
44. $P = \$1000$, $r = 6\%$, $t = 10$ years
45. $P = \$2500$, $r = 8\%$, $t = 20$ years
46. $P = \$1000$, $r = 6\%$, $t = 40$ years

Compound Interest **In Exercises 47–50, complete the table to determine the balance *A* for \$12,000 invested at rate *r* for *t* years, compounded continuously.**

t	10	20	30	40	50
A					

47. $r = 8\%$
48. $r = 6\%$
49. $r = 6.5\%$
50. $r = 7.5\%$

1-10, 11-17, 19, 20, 25, 26, 31 27-30, 51-53, 56, 58

51. ***Trust Fund*** On the day of a child's birth, a deposit of \$25,000 is made in a trust fund that pays 8.75% interest, compounded continuously. Determine the balance in this account on the child's 25th birthday.

52. ***Trust Fund*** A deposit of \$5000 is made in a trust fund that pays 7.5% interest, compounded continuously. It is specified that the balance will be given to the college from which the donor graduated after the money has earned interest for 50 years. How much will the college receive?

53. ***Inflation*** If the annual rate of inflation averages 4% over the next 10 years, the approximate cost C of goods or services during any year in that decade will be modeled by $C(t) = P(1.04)^t$, where t is the time in years and P is the present cost. The price of an oil change for your car is presently \$23.95. Estimate the price 10 years from now.

54. ***Demand*** The demand equation for a product is

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right).$$

(a) Use a graphing utility to graph the demand function for $x > 0$ and $p > 0$.

(b) Find the price p for a demand of $x = 500$ units.

(c) Use the graph in part (a) to approximate the greatest price that will still yield a demand of at least 600 units.

55. ***Population Growth*** A certain type of bacterium increases according to the model $P(t) = 100e^{0.2197t}$, where t is the time in hours. Find (a) $P(0)$, (b) $P(5)$, and (c) $P(10)$.

56. ***Population Growth*** The population of a town increases according to the model $P(t) = 2500e^{0.0293t}$, where t is the time in years, with $t = 0$ corresponding to 2000. Use the model to estimate the population in (a) 2010 and (b) 2020.

57. ***Radioactive Decay*** Let Q represent a mass of radioactive radium (^{226}Ra) (in grams), whose half-life is 1620 years. The quantity of radium present after t years is

$$Q = 25\left(\frac{1}{2}\right)^{t/1620}.$$

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 1000 years.

(c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 5000$.

58. ***Radioactive Decay*** Let Q represent a mass of carbon 14 (^{14}C) (in grams), whose half-life is 5730 years. The quantity of carbon 14 present after t years is

$$Q = 10\left(\frac{1}{2}\right)^{t/5730}.$$

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 2000 years.

(c) Sketch the graph of this function over the interval $t = 0$ to $t = 10{,}000$.

▶ Model It

59. ***Data Analysis*** To estimate the amount of defoliation caused by the gypsy moth during a given year, a forester counts the number x of egg masses on $\frac{1}{40}$ of an acre (circle of radius 18.6 feet) in the fall. The percent of defoliation y the next spring is shown in the table. (Source: USDA, Forest Service)

Egg masses, x	Percent of defoliation, y
0	12
25	44
50	81
75	96
100	99

A model for the data is

$$y = \frac{100}{1 + 7e^{-0.069x}}.$$

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window.

(b) Create a table that compares the model with the sample data.

(c) Estimate the percent of defoliation if 36 egg masses are counted on $\frac{1}{40}$ acre.

(d) You observe that $\frac{2}{3}$ of a forest is defoliated the following spring. Use the graph in part (a) to estimate the number of egg masses per $\frac{1}{40}$ acre.

60. ***Data Analysis*** A meteorologist measures the atmospheric pressure P (in pascals) at altitude h (in kilometers). The data is shown in the table.

Altitude, h	Pressure, P
0	101,293
5	54,735
10	23,294
15	12,157
20	5,069

A model for the data is given by

$P = 102{,}303e^{-0.137h}$.

(a) Sketch a scatter plot of the data and graph the model on the same set of axes.

(b) Estimate the atmospheric pressure at a height of 8 kilometers.

Synthesis

True or False? **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

61. The line $y = -2$ is an asymptote for the graph of $f(x) = 10^x - 2$.

62. $e = \dfrac{271{,}801}{99{,}990}$.

Think About It **In Exercises 63–66, use properties of exponents to determine which functions (if any) are the same.**

63. $f(x) = 3^{x-2}$
$g(x) = 3^x - 9$
$h(x) = \frac{1}{9}(3^x)$

64. $f(x) = 4^x + 12$
$g(x) = 2^{2x+6}$
$h(x) = 64(4^x)$

65. $f(x) = 16(4^{-x})$
$g(x) = \left(\frac{1}{4}\right)^{x-2}$
$h(x) = 16(2^{-2x})$

66. $f(x) = 5^{-x} + 3$
$g(x) = 5^{3-x}$
$h(x) = -5^{x-3}$

67. Graph the functions $y = 3^x$ and $y = 4^x$ and use the graphs to solve the inequalities.

(a) $4^x < 3^x$ (b) $4^x > 3^x$

68. Graph the functions $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{4}\right)^x$ and use the graphs to solve the inequalities.

(a) $\left(\frac{1}{4}\right)^x < \left(\frac{1}{2}\right)^x$ (b) $\left(\frac{1}{4}\right)^x > \left(\frac{1}{2}\right)^x$

69. Use a graphing utility to graph each function. Use the graph to find any asymptotes of the function.

(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ (b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

70. Use a graphing utility to graph each function. Use the graph to find where the function is increasing and decreasing, and approximate any relative maximum or minimum values.

(a) $f(x) = x^2e^{-x}$ (b) $g(x) = x2^{3-x}$

71. ***Graphical Analysis*** Use a graphing utility to graph

$$f(x) = \left(1 + \frac{0.5}{x}\right)^x \quad \text{and} \quad g(x) = e^{0.5}$$

in the same viewing window. What is the relationship between f and g as x increases and decreases without bound?

72. ***Conjecture*** Use the result of Exercise 71 to make a conjecture about the value of $[1 + (r/x)]^x$ as x increases without bound. Create a table that illustrates your conjecture for $r = 1$.

73. ***Think About It*** Which functions are exponential?

(a) $3x$ (b) $3x^2$
(c) 3^x (d) 2^{-x}

74. ***Writing*** Explain why $2^{\sqrt{2}}$ is greater than 2, but less than 4.

Review

In Exercises 75–78, solve for y.

75. $2x - 7y + 14 = 0$ **76.** $x^2 + 3y = 4$
77. $x^2 + y^2 = 25$ **78.** $x - |y| = 2$

In Exercises 79–82, sketch the graph of the rational function.

79. $f(x) = \dfrac{2}{9 + x}$ **80.** $f(x) = \dfrac{4x - 3}{x}$

81. $f(x) = \dfrac{6}{x^2 + 5x - 24}$ **82.** $f(x) = \dfrac{x^2 - 7x + 12}{x + 2}$

3.2 Logarithmic Functions and Their Graphs

▶ **What you should learn**

- How to recognize and evaluate logarithmic functions with base *a*
- How to graph logarithmic functions
- How to recognize and evaluate natural logarithmic functions
- How to use logarithmic functions to model and solve real-life applications

▶ **Why you should learn it**

You can use logarithmic functions to model and solve real-life problems. For instance, in Exercise 57 on page 217, you can use a logarithmic function to approximate the length of a home mortgage.

Ryan McVay/Getty Images

Logarithmic Functions

In Section 1.8, you studied the concept of an inverse function. There, you learned that if a function is one-to-one—that is, if the function has the property that no horizontal line intersects the graph of the function more than once—the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 3.1, you will see that every function of the form

$$f(x) = a^x$$

passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a*.**

> **Definition of Logarithmic Function with Base *a***
>
> For $x > 0$ and $0 < a \neq 1$,
>
> $y = \log_a x$ if and only if $x = a^y$.
>
> The function given by
>
> $$f(x) = \log_a x$$
>
> is called the **logarithmic function with base *a*.**

The equations

$$y = \log_a x \qquad \text{and} \qquad x = a^y$$

are equivalent. The first equation is in logarithmic form and the second is in exponential form.

When evaluating logarithms, remember that *a logarithm is an exponent.* This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to get 8.

Example 1 ▶ **Evaluating Logarithms**

Use the definition of logarithmic function to evaluate each logarithm at the indicated value of x.

a. $f(x) = \log_2 x, \quad x = 32$ **b.** $f(x) = \log_3 x, \quad x = 1$

c. $f(x) = \log_4 x, \quad x = 2$ **d.** $f(x) = \log_{10} x, \quad x = \frac{1}{100}$

Solution

a. $f(32) = \log_2 32 = 5$ because $2^5 = 32$.

b. $f(1) = \log_3 1 = 0$ because $3^0 = 1$.

c. $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.

d. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

STUDY TIP

Remember that a logarithm is an exponent. So, to evaluate the logarithmic expression $\log_a x$, you need to ask the question, "To what power must a be raised to obtain x?"

Exploration

Complete the table for $f(x) = 10^x$.

x	-2	-1	0	1	2
$f(x)$					

Complete the table for $f(x) = \log_{10} x$.

x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100
$f(x)$					

Compare the two tables. What is the relationship between $f(x) = 10^x$ and $f(x) = \log_{10} x$?

The logarithmic function with base 10 is called the **common logarithmic function.** On most calculators, this function is denoted by LOG. Example 2 shows how to use a calculator to evaluate common logarithmic functions. You will learn how to use a calculator to calculate logarithms to any base in the next section.

Example 2 ▶ Evaluating Common Logarithms on a Calculator

Use a calculator to evaluate the function $f(x) = \log_{10} x$ at each value of x.

a. $x = 10$ **b.** $x = \frac{1}{3}$

c. $x = 2.5$ **d.** $x = -2$

Solution

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(10) = \log_{10} 10$	LOG 10 ENTER	1
b. $f(\frac{1}{3}) = \log_{10}\frac{1}{3}$	LOG (1 ÷ 3) ENTER	-0.4771213
c. $f(2.5) = \log_{10} 2.5$	LOG 2.5 ENTER	0.3979400
d. $f(-2) = \log_{10}(-2)$	LOG (−) 2 ENTER	ERROR

Note that the calculator displays an error message (or a complex number) when you try to evaluate $\log_{10}(-2)$. The reason for this is that there is no real number power to which 10 can be raised to obtain -2.

The following properties follow directly from the definition of the logarithmic function with base a.

The logarithmic function can be one of the most difficult concepts for students to understand. Remind students that a logarithm is an exponent. Converting back and forth from logarithmic form to exponential form supports this concept.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ — Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$. — One-to-One Property

Example 3 ▶ Using Properties of Logarithms

a. Solve for x: $\log_2 x = \log_2 3$ **b.** Solve for x: $\log_4 4 = x$

c. Simplify: $\log_5 5^x$ **d.** Simplify: $6^{\log_6 20}$

Solution

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.

b. Using Property 2, you can conclude that $x = 1$.

c. Using the Inverse Property (Property 3), it follows that $\log_5 5^x = x$.

d. Using the Inverse Property (Property 3), it follows that $6^{\log_6 20} = 20$.

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 4 ▶ Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$ **b.** $g(x) = \log_2 x$

Solution

a. For $f(x) = 2^x$, construct a table of values.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

By plotting these points and connecting them with a smooth curve, you obtain the graph shown in Figure 3.14.

b. Because $g(x) = \log_2 x$ is the inverse of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 3.14.

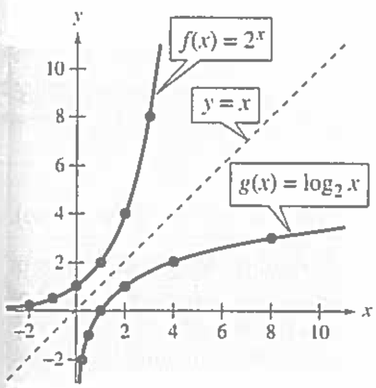


FIGURE 3.14

Example 5 ▶ Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function $f(x) = \log_{10} x$. Identify the x-intercept and the vertical asymptote.

Solution

Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the Inverse Property of Logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 3.15. The x-intercept of the graph is $(1, 0)$ and the vertical asymptote is $x = 0$ (y-axis).

	Without calculator				With calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$\log_{10} x$	-2	-1	0	1	0.301	0.699	0.903

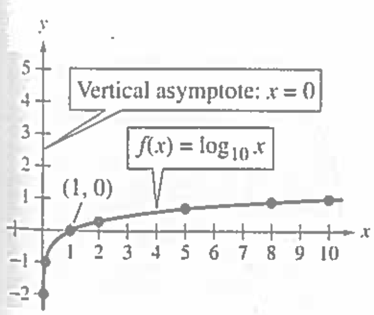


FIGURE 3.15

The nature of the graph in Figure 3.15 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$. The basic characteristics of logarithmic graphs are summarized in Figure 3.16.

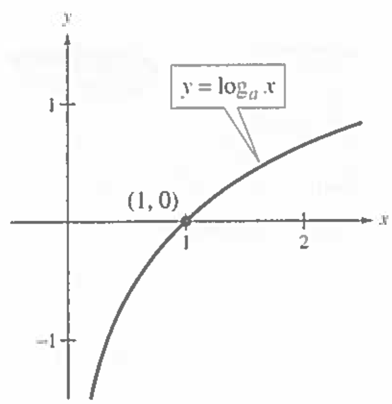


FIGURE 3.16

Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- x-Intercept: $(1, 0)$
- Increasing
- One-to-one, therefore has an inverse function
- y-Axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$).
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$

The basic characteristics of the graph of $f(x) = a^x$ are shown below to illustrate the inverse relation between the functions $f(x) = a^x$ and $g(x) = \log_a x$.

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-Intercept: $(0, 1)$
- x-Axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$).

In the next example, the graph of $y = \log_a x$ is used to sketch the graphs of functions of the form $f(x) = b \pm \log_a(x + c)$. Notice how a horizontal shift of the graph results in a horizontal shift of the vertical asymptote.

STUDY TIP

You can use your understanding of transformations to identify vertical asymptotes of logarithmic functions. For instance, in Example 6(a) the graph of $g(x) = f(x - 1)$ shifts the graph of $f(x)$ one unit to the right. So the vertical asymptote of $g(x)$ is $x = 1$, one unit to the right of the asymptote of the graph of $f(x)$.

Example 6 ▶ Shifting Graphs of Logarithmic Functions

The graph of each of the functions is similar to the graph of $f(x) = \log_{10} x$.

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the right, as shown in Figure 3.17.

b. Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units upward, as shown in Figure 3.18.

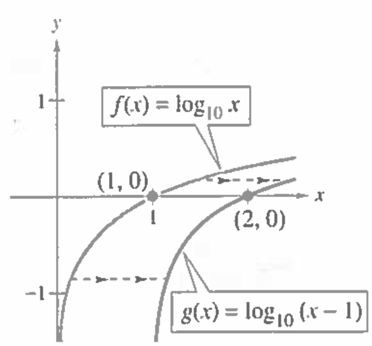


FIGURE 3.17

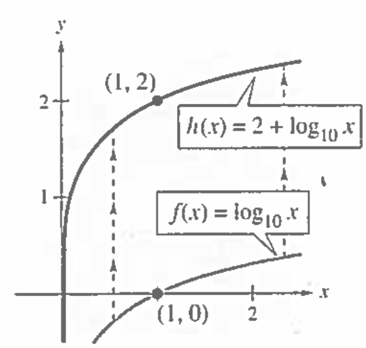
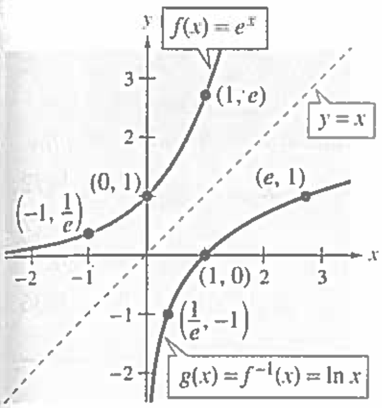


FIGURE 3.18

The Natural Logarithmic Function

By looking back at the graph of the natural exponential function introduced in Section 3.1, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as "the natural log of x" or "el en of x."



Reflection of graph of $f(x) = e^x$ about the line $y = x$
FIGURE 3.19

The Natural Logarithmic Function

The function defined by

$$f(x) = \log_e x = \ln x, \quad x > 0$$

is called the **natural logarithmic function.**

The definition above implies that the natural logarithmic function and the natural exponential function are inverse functions of each other. So, every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form. That is, $y = \ln x$ and $x = e^y$ are equivalent equations.

Because the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 3.19.

The four properties of logarithms listed on page 210 are also valid for natural logarithms.

STUDY TIP

Notice that as with every other logarithmic function, the domain of the natural logarithmic function is the set of *positive real numbers*—be sure you see that $\ln x$ is not defined for zero or for negative numbers.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$ — Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. — One-to-One Property

Example 7 ▶ **Using Properties of Natural Logarithms**

Use the properties of natural logarithms to simplify each expression.

a. $\ln \frac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $\frac{\ln 1}{3}$ **d.** $2 \ln e$

Solution

a. $\ln \frac{1}{e} = \ln e^{-1} = -1$ — Inverse Property

b. $e^{\ln 5} = 5$ — Inverse Property

c. $\frac{\ln 1}{3} = \frac{0}{3} = 0$ — Property 1

d. $2 \ln e = 2(1) = 2$ — Property 2

STUDY TIP

Some graphing utilities display a complex number instead of an ERROR message when evaluating an expression such as $\ln(-1)$.

On most calculators, the natural logarithm is denoted by LN, as illustrated in Example 8.

Example 8 ▶ Evaluating the Natural Logarithmic Function

Use a calculator to evaluate the function $f(x) = \ln x$ for each value of x.

a. $x = 2$ **b.** $x = 0.3$ **c.** $x = -1$ **d.** $x = 1 + \sqrt{2}$

Solution

	Function Value	*Graphing Calculator Keystrokes*	*Display*
a.	$f(2) = \ln 2$	LN 2 ENTER	0.6931472
b.	$f(0.3) = \ln 0.3$	LN .3 ENTER	−1.2039728
c.	$f(-1) = \ln(-1)$	LN (−) 1 ENTER	ERROR
d.	$f(1 + \sqrt{2}) = \ln(1 + \sqrt{2})$	LN (1 + √ 2) ENTER	0.8813736

In Example 8, be sure you see that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of $\ln x$ is the set of positive real numbers (see Figure 3.19). So, $\ln(-1)$ is undefined.

Example 9 ▶ Finding the Domains of Logarithmic Functions

Find the domain of each function.

a. $f(x) = \ln(x - 2)$

b. $g(x) = \ln(2 - x)$

c. $h(x) = \ln x^2$

Solution

a. Because $\ln(x - 2)$ is defined only if $x - 2 > 0$, it follows that the domain of f is $(2, \infty)$. The graph of f is shown in Figure 3.20.

b. Because $\ln(2 - x)$ is defined only if $2 - x > 0$, it follows that the domain of g is $(-\infty, 2)$. The graph of g is shown in Figure 3.21.

c. Because $\ln x^2$ is defined only if $x^2 > 0$, it follows that the domain of h is all real numbers except $x = 0$. The graph of h is shown in Figure 3.22.

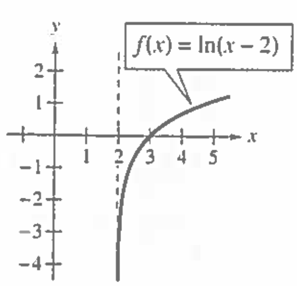


FIGURE 3.20

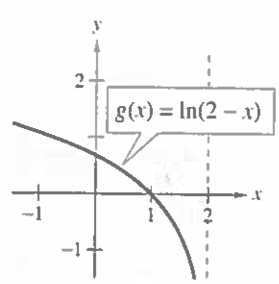


FIGURE 3.21

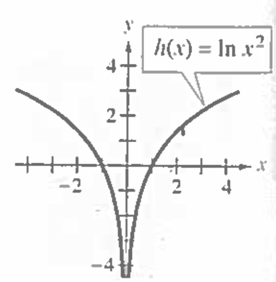


FIGURE 3.22

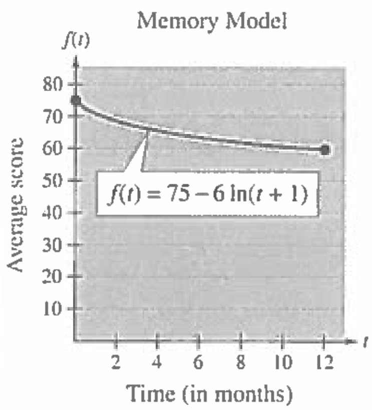


FIGURE 3.23

Application

Example 10 ▶ **Human Memory Model**

Students participating in a psychology experiment attended several lectures on a subject and were given an exam. Every month for a year after the exam, the students were retested to see how much of the material they remembered. The average scores for the group are given by the *human memory model*

$$f(t) = 75 - 6\ln(t+1), \quad 0 \le t \le 12$$

where t is the time in months. The graph of f is shown in Figure 3.23.

a. What was the average score on the original ($t = 0$) exam?

b. What was the average score at the end of $t = 2$ months?

c. What was the average score at the end of $t = 6$ months?

Solution

a. The original average score was

$$\begin{aligned} f(0) &= 75 - 6\ln(0+1) && \text{Substitute 0 for } t. \\ &= 75 - 6\ln 1 && \text{Simplify.} \\ &= 75 - 6(0) && \text{Property of natural logarithms} \\ &= 75. && \text{Solution} \end{aligned}$$

b. After 2 months, the average score was

$$\begin{aligned} f(2) &= 75 - 6\ln(2+1) && \text{Substitute 2 for } t. \\ &= 75 - 6\ln 3 && \text{Simplify.} \\ &\approx 75 - 6(1.0986) && \text{Use a calculator.} \\ &\approx 68.4. && \text{Solution} \end{aligned}$$

c. After 6 months, the average score was

$$\begin{aligned} f(6) &= 75 - 6\ln(6+1) && \text{Substitute 6 for } t. \\ &= 75 - 6\ln 7 && \text{Simplify.} \\ &\approx 75 - 6(1.9459) && \text{Use a calculator.} \\ &\approx 63.3. && \text{Solution} \end{aligned}$$

Alternative Writing About Mathematics

Use a graphing utility to graph $f(x) = \ln x$. How will the graphs of $h(x) = \ln x + 5$, $j(x) = \ln(x - 3)$, and $l(x) = \ln x - 4$ differ from the graph of f?

How will the basic graph of f be affected when a constant c is introduced: $g(x) = c\ln x$? Use a graphing utility to graph g with several different positive values of c, and summarize the effect of c.

Writing ABOUT MATHEMATICS

Analyzing a Human Memory Model Use a graphing utility to determine the time in months when the average score in Example 10 was 60. Explain your method of solving the problem. Describe another way that you can use a graphing utility to determine the answer.

3.2 Exercises

In Exercises 1–8, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 25 = 2$ is $5^2 = 25$.

1. $\log_4 64 = 3$
2. $\log_3 81 = 4$
3. $\log_7 \frac{1}{49} = -2$
4. $\log_{10} \frac{1}{1000} = -3$
5. $\log_{32} 4 = \frac{2}{5}$
6. $\log_{16} 8 = \frac{3}{4}$
7. $\ln \frac{1}{2} = -0.693\ldots$
8. $\ln 4 = 1.386\ldots$

In Exercises 9–18, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.

9. $5^3 = 125$
10. $8^2 = 64$
11. $81^{1/4} = 3$
12. $9^{3/2} = 27$
13. $6^{-2} = \frac{1}{36}$
14. $10^{-3} = 0.001$
15. $e^3 = 20.0855\ldots$
16. $e^{1/2} = 1.6487\ldots$
17. $e^x = 4$
18. $u^v = w$

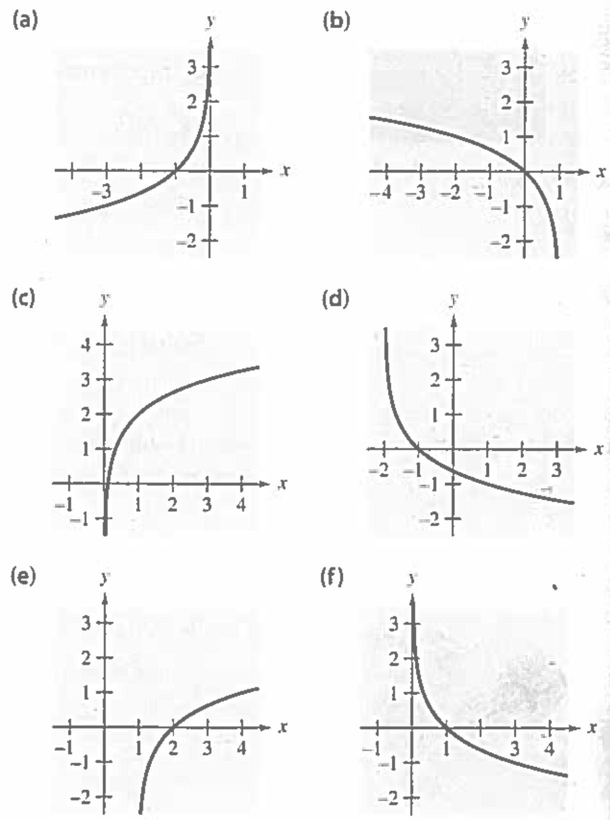
In Exercises 19–26, evaluate the function at the indicated value of x without using a calculator.

	Function	*Value*
19.	$f(x) = \log_2 x$	$x = 16$
20.	$f(x) = \log_{16} x$	$x = 4$
21.	$f(x) = \log_7 x$	$x = 1$
22.	$f(x) = \log_{10} x$	$x = 10$
23.	$g(x) = \ln x$	$x = e^3$
24.	$g(x) = \ln x$	$x = e^{-2}$
25.	$g(x) = \log_a x$	$x = a^2$
26.	$g(x) = \log_b x$	$x = b^{-3}$

In Exercises 27–32, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.

	Function	*Value*
27.	$f(x) = \log_{10} x$	$x = \frac{4}{5}$
28.	$f(x) = \log_{10} x$	$x = 12.5$
29.	$f(x) = \ln x$	$x = 18.42$
30.	$f(x) = 3 \ln x$	$x = 0.32$
31.	$g(x) = 2 \ln x$	$x = 0.75$
32.	$g(x) = -\ln x$	$x = \frac{1}{2}$

In Exercises 33–38, use the graph of $y = \log_3 x$ to match the given function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



33. $f(x) = \log_3 x + 2$
34. $f(x) = -\log_3 x$
35. $f(x) = -\log_3(x + 2)$
36. $f(x) = \log_3(x - 1)$
37. $f(x) = \log_3(1 - x)$
38. $f(x) = -\log_3(-x)$

In Exercises 39–50, find the domain, x-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

39. $f(x) = \log_4 x$
40. $g(x) = \log_6 x$
41. $y = -\log_3 x + 2$
42. $h(x) = \log_4(x - 3)$
43. $f(x) = -\log_6(x + 2)$
44. $y = \log_5(x - 1) + 4$
45. $y = \log_{10}\left(\frac{x}{5}\right)$
46. $y = \log_{10}(-x)$
47. $f(x) = \ln(x - 2)$
48. $h(x) = \ln(x + 1)$
49. $g(x) = \ln(-x)$
50. $f(x) = \ln(3 - x)$

p 216 #s 1-26, 27

In Exercises 51–56, use a graphing utility to graph the function. Be sure to use an appropriate viewing window.

51. $f(x) = \log_{10}(x + 1)$ **52.** $f(x) = \log_{10}(x - 1)$

53. $f(x) = \ln(x - 1)$ **54.** $f(x) = \ln(x + 2)$

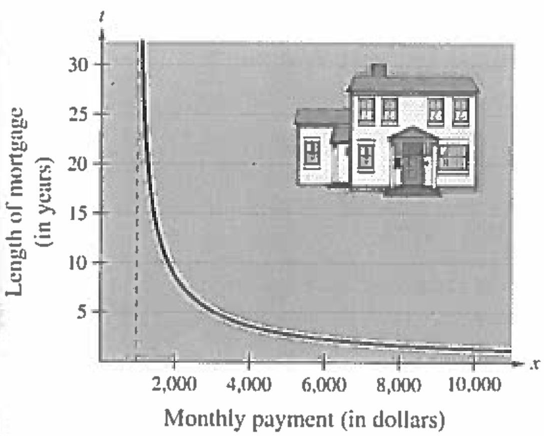
55. $f(x) = \ln x + 2$ **56.** $f(x) = 3 \ln x - 1$

▶ Model It

57. ***Monthly Payment*** The model

$$t = 12.542 \ln\left(\frac{x}{x - 1000}\right), \quad x > 1000$$

approximates the length of a home mortgage of \$150,000 at 8% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars (see figure).



(a) Use the model to approximate the length of a \$150,000 mortgage at 8% when the monthly payment is \$1100.65 and when the monthly payment is \$1254.68.

(b) Approximate the total amount paid over the term of the mortgage with a monthly payment of \$1100.65 and with a monthly payment of \$1254.68.

(c) Approximate the total interest charge for a monthly payment of \$1100.65 and for a monthly payment of \$1254.68.

(d) What is the vertical asymptote for the model? Interpret its meaning in the context of the problem.

58. ***Compound Interest*** A principal P, invested at $9\frac{1}{2}\%$ and compounded continuously, increases to an amount K times the original principal after t years, where t is given by

$$t = \frac{\ln K}{0.095}.$$

(a) Complete the table and interpret your results.

K	1	2	4	6	8	10	12
t							

(b) Sketch a graph of the function.

59. ***Population*** The time t in years for the world population to double if it is increasing at a continuous rate of r is given by

$$t = \frac{\ln 2}{r}.$$

(a) Complete the table.

r	0.005	0.01	0.015	0.02	0.025	0.03
t						

(b) Sketch a graph of the function.

(c) Use a reference source to decide which value of r best approximates the actual rate of growth for the world population.

60. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Determine the number of decibels of a sound with an intensity of 1 watt per square meter.

(b) Determine the number of decibels of a sound with an intensity of 10^{-2} watt per square meter.

(c) The intensity of the sound in part (a) is 100 times as great as that in part (b). Is the number of decibels 100 times as great? Explain.

61. ***Human Memory Model*** Students in a mathematics class were given an exam and then retested monthly with an equivalent exam. The average scores for the class are given by the human memory model

$$f(t) = 80 - 17\log_{10}(t+1), \qquad 0 \le t \le 12$$

where t is the time in months.

(a) Use a graphing utility to graph the model over the specified domain.

(b) What was the average score on the original exam $(t = 0)$?

(c) What was the average score after 4 months?

(d) What was the average score after 10 months?

62. (a) Complete the table for the function

$$f(x) = \frac{\ln x}{x}.$$

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(b) Use the table in part (a) to determine what value $f(x)$ approaches as x increases without bound.

(c) Use a graphing utility to confirm the result of part (b).

Synthesis

True or False? **In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.**

63. You can determine the graph of $f(x) = \log_6 x$ by graphing $g(x) = 6^x$ and reflecting it about the x-axis.

64. The graph of $f(x) = \log_3 x$ contains the point $(27, 3)$.

In Exercises 65–68, describe the relationship between the graphs of f and g. What is the relationship between the functions f and g?

65. $f(x) = 3^x$
$g(x) = \log_3 x$

66. $f(x) = 5^x$
$g(x) = \log_5 x$

67. $f(x) = e^x$
$g(x) = \ln x$

68. $f(x) = 10^x$
$g(x) = \log_{10} x$

69. ***Graphical Analysis*** Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the greater rate as x approaches $+\infty$. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x, \quad g(x) = \sqrt{x}$

(b) $f(x) = \ln x, \quad g(x) = \sqrt[4]{x}$

70. ***Think About It*** The table of values was obtained by evaluating a function. Determine which of the statements may be true and which must be false.

x	y
1	0
2	1
8	3

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

In Exercises 71–73, answer the question for the function $f(x) = \log_{10} x$. Do not use a calculator.

71. What is the domain of f?

72. What is f^{-1}?

73. If x is a real number between 1000 and 10,000, in which interval will $f(x)$ be found?

74. ***Writing*** Explain why $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

In Exercises 75 and 76, (a) use a graphing utility to graph the function, (b) use the graph to determine the intervals in which the function is increasing and decreasing, and (c) approximate any relative maximum or minimum values of the function.

75. $f(x) = |\ln x|$ **76.** $h(x) = \ln(x^2 + 1)$

Review

In Exercises 77 and 78, translate the statement into an algebraic expression.

77. The total cost for auto repairs if the cost of parts was \$83.95 and there were t hours of labor at \$37.50 per hour

78. The area of a rectangle if the length is 10 units more than the width w

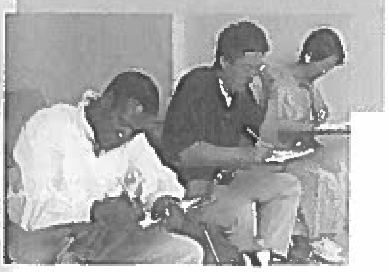
3.3 Properties of Logarithms

▶ What you should learn

- How to use the change-of-base formula to rewrite and evaluate logarithmic expressions
- How to use properties of logarithms to evaluate or rewrite logarithmic expressions
- How to use properties of logarithms to expand or condense logarithmic expressions
- How to use logarithmic functions to model and solve real-life applications

▶ Why you should learn it

Logarithmic functions are often used to model scientific observations. For instance, in Exercise 81 on page 224, a logarithmic function is used to model human memory.



Gary Conner/PhotoEdit

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms to other bases. To do this, you can use the following **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b. The constant multiplier is $1/(\log_b a)$.

Example 1 ▶ Changing Bases Using Common Logarithms

a. $\log_4 30 = \dfrac{\log_{10} 30}{\log_{10} 4}$ $\qquad \log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

$\approx \dfrac{1.47712}{0.60206}$ $\qquad$ Use a calculator.

≈ 2.4534 $\qquad$ Simplify.

b. $\log_2 14 = \dfrac{\log_{10} 14}{\log_{10} 2} \approx \dfrac{1.14613}{0.30103} \approx 3.8074$

Example 2 ▶ Changing Bases Using Natural Logarithms

a. $\log_4 30 = \dfrac{\ln 30}{\ln 4}$ $\qquad \log_a x = \dfrac{\ln x}{\ln a}$

$\approx \dfrac{3.40120}{1.38629}$ $\qquad$ Use a calculator.

≈ 2.4535 $\qquad$ Simplify.

b. $\log_2 14 = \dfrac{\ln 14}{\ln 2} \approx \dfrac{2.63906}{0.69315} \approx 3.8073$

Encourage your students to know these properties well. They will be used for solving logarithmic and exponential equations, as well as in calculus.

STUDY TIP

There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is not equal to $\log_a x + \log_a y$.

Remind your students to note the domain when applying properties of logarithms to a logarithmic function. For example, the domain of $f(x) = \ln x^2$ is all real $x \neq 0$, whereas the domain of $g(x) = 2 \ln x$ is all real $x > 0$.

Historical Note

John Napier, a Scottish mathematician, developed logarithms as a way to simplify some of the tedious calculations of his day. Beginning in 1594, Napier worked about 20 years on the invention of logarithms. Napier was only partially successful in his quest to simplify tedious calculations. Nonetheless, the development of logarithms was a step forward and received immediate recognition.

Properties of Logarithms

You know from the preceding section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents should have corresponding properties involving logarithms. For instance, the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true.

Logarithm with Base a	*Natural Logarithm*
1. $\log_a(uv) = \log_a u + \log_a v$	1. $\ln(uv) = \ln u + \ln v$
2. $\log_a \frac{u}{v} = \log_a u - \log_a v$	2. $\ln \frac{u}{v} = \ln u - \ln v$
3. $\log_a u^n = n \log_a u$	3. $\ln u^n = n \ln u$

For a proof of the properties listed above, see Proofs in Mathematics on page 257.

Example 3 ▶ Using Properties of Logarithms

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$ b. $\ln \frac{2}{27}$

Solution

a. $\ln 6 = \ln(2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ Property 1

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ Property 2

$= \ln 2 - \ln 3^3$ Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ Property 3

Example 4 ▶ Using Properties of Logarithms

Use the properties of logarithms to verify that $-\log_{10} \frac{1}{100} = \log_{10} 100$.

Solution

$-\log_{10} \frac{1}{100} = -\log_{10}(100^{-1})$ Rewrite $\frac{1}{100}$ as 100^{-1}.

$= -(-1)\log_{10} 100$ Property 3

$= \log_{10} 100$ Simplify.

Try checking this result on your calculator.

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because these properties convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

A common error made in expanding logarithmic expressions is to rewrite $\log ax^n$ as $n \log ax$ instead of as $\log a + n \log x$.

Exploration

Use a graphing utility to graph the functions

$$y_1 = \ln x - \ln(x - 3)$$

and

$$y_2 = \ln \frac{x}{x - 3}$$

in the same viewing window. Does the graphing utility show the functions with the same domain? If so, should it? Explain your reasoning.

A common error made in condensing logarithmic expressions is to rewrite $\log x - \log y$ as $\frac{\log x}{\log y}$ instead of as $\log \frac{x}{y}$.

Example 5 ▶ **Expanding Logarithmic Expressions**

Expand each logarithmic expression.

a. $\log_4 5x^3y$ b. $\ln \frac{\sqrt{3x - 5}}{7}$

Solution

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Property 1

$= \log_4 5 + 3 \log_4 x + \log_4 y$ Property 3

b. $\ln \frac{\sqrt{3x - 5}}{7} = \ln \frac{(3x - 5)^{1/2}}{7}$ Rewrite using rational exponent.

$= \ln(3x - 5)^{1/2} - \ln 7$ Property 2

$= \frac{1}{2} \ln(3x - 5) - \ln 7$ Property 3

In Example 5, the properties of logarithms were used to *expand* logarithmic expressions. In Example 6, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

Example 6 ▶ **Condensing Logarithmic Expressions**

Condense each logarithmic expression.

a. $\frac{1}{2} \log_{10} x + 3 \log_{10}(x + 1)$ b. $2 \ln(x + 2) - \ln x$

c. $\frac{1}{3}[\log_2 x + \log_2(x + 1)]$

Solution

a. $\frac{1}{2} \log_{10} x + 3 \log_{10}(x + 1) = \log_{10} x^{1/2} + \log_{10}(x + 1)^3$ Property 3

$= \log_{10}\left[\sqrt{x}(x + 1)^3\right]$ Property 1

b. $2 \ln(x + 2) - \ln x = \ln(x + 2)^2 - \ln x$ Property 3

$= \ln \frac{(x + 2)^2}{x}$ Property 2

c. $\frac{1}{3}[\log_2 x + \log_2(x + 1)] = \frac{1}{3}\{\log_2[x(x + 1)]\}$ Property 1

$= \log_2[x(x + 1)]^{1/3}$ Property 3

$= \log_2 \sqrt[3]{x(x + 1)}$ Rewrite with a radical.

Application

One method of determining how the x- and y-values for a set of nonlinear data are related begins by taking the natural log of each of the x- and y-values. If the points are graphed and fall on a straight line, then you can determine that the x- and y-values are related by the equation

$$\ln y = m \ln x$$

where m is the slope of the straight line.

Example 7 ▶ Finding a Mathematical Model

The table shows the mean distance x and the period (the time it takes a planet to orbit the sun) y for each of the six planets that are closest to the sun. In the table, the mean distance is given in terms of astronomical units (where Earth's mean distance is defined as 1.0), and the period is given in terms of years. Find an equation that relates y and x.

Planet	Mean distance, x	Period, y
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.862
Saturn	9.555	29.458

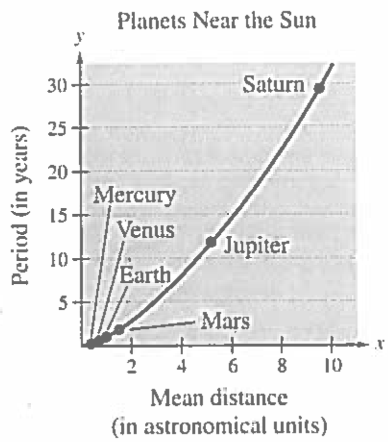


FIGURE 3.24

Solution

The points in the table are plotted in Figure 3.24. From this figure it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values in the table. This produces the following results.

Planet	$\ln x$	$\ln y$
Mercury	−0.949	−1.423
Venus	−0.324	−0.486
Earth	0.000	0.000
Mars	0.421	0.632
Jupiter	1.649	2.473
Saturn	2.257	3.383

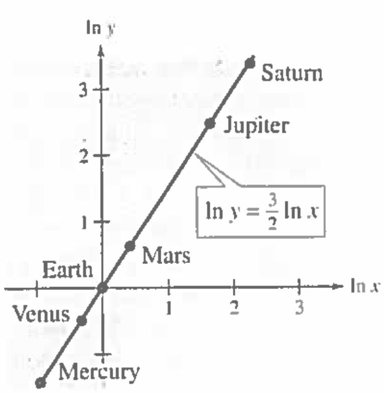


FIGURE 3.25

Now, by plotting the points in the second table, you can see that all six of the points appear to lie in a line (see Figure 3.25). You can use a graphical approach or the algebraic approach discussed in Section 1.9 to find that the slope of this line is $\frac{3}{2}$. You can therefore conclude that $\ln y = \frac{3}{2} \ln x$.

p. 223 HS 1-15 odds; 18-54 (4x), 59-74

3.3 Exercises

In Exercises 1–8, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.

1. $\log_5 x$
2. $\log_3 x$
3. $\log_{1/5} x$
4. $\log_{1/3} x$
5. $\log_x \frac{3}{10}$
6. $\log_x \frac{3}{4}$
7. $\log_{2.6} x$
8. $\log_{7.1} x$

In Exercises 9–16, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

9. $\log_3 7$
10. $\log_7 4$
11. $\log_{1/2} 4$
12. $\log_{1/4} 5$
13. $\log_9 0.4$
14. $\log_{20} 0.125$
15. $\log_{15} 1250$
16. $\log_3 0.015$

In Exercises 17–38, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

17. $\log_4 5x$
18. $\log_3 10z$
19. $\log_8 x^4$
20. $\log_{10} \frac{y}{2}$
21. $\log_5 \frac{5}{x}$
22. $\log_6 \frac{1}{z^3}$
23. $\ln \sqrt{z}$
24. $\ln \sqrt[3]{t}$
25. $\ln xyz^2$
26. $\log_{10} 4x^2 y$
27. $\ln z(z-1)^2,\ z > 1$
28. $\ln\left(\frac{x^2-1}{x^3}\right),\ x > 1$
29. $\log_2 \frac{\sqrt{a-1}}{9},\ a > 1$
30. $\ln \frac{6}{\sqrt{x^2+1}}$
31. $\ln \sqrt[3]{\frac{x}{y}}$
32. $\ln \sqrt{\frac{x^2}{y^3}}$
33. $\ln \frac{x^4\sqrt{y}}{z^5}$
34. $\log_2 \frac{\sqrt{x}\,y^4}{z^4}$
35. $\log_5 \frac{x^2}{y^2z^3}$
36. $\log_{10} \frac{xy^4}{z^5}$
37. $\ln \sqrt[4]{x^3(x^2+3)}$
38. $\ln \sqrt{x^2(x+2)}$

In Exercises 39–56, condense the expression to the logarithm of a single quantity.

39. $\ln x + \ln 3$
40. $\ln y + \ln t$
41. $\log_4 z - \log_4 y$
42. $\log_5 8 - \log_5 t$
43. $2\log_2(x+4)$
44. $\frac{2}{3}\log_7(z-2)$
45. $\frac{1}{4}\log_3 5x$
46. $-4\log_6 2x$
47. $\ln x - 3\ln(x+1)$
48. $2\ln 8 + 5\ln(z-4)$
49. $\log_{10} x - 2\log_{10} y + 3\log_{10} z$
50. $3\log_3 x + 4\log_3 y - 4\log_3 z$
51. $\ln x - 4[\ln(x+2) + \ln(x-2)]$
52. $4[\ln z + \ln(z+5)] - 2\ln(z-5)$
53. $\frac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)]$
54. $2[3\ln x - \ln(x+1) - \ln(x-1)]$
55. $\frac{1}{3}[\log_8 y + 2\log_8(y+4)] - \log_8(y-1)$
56. $\frac{1}{2}[\log_4(x+1) + 2\log_4(x-1)] + 6\log_4 x$

In Exercises 57 and 58, compare the logarithmic quantities. If two are equal, explain why.

57. $\frac{\log_2 32}{\log_2 4},\quad \log_2 \frac{32}{4},\quad \log_2 32 - \log_2 4$
58. $\log_7 \sqrt{70},\quad \log_7 35,\quad \frac{1}{2} + \log_7 \sqrt{10}$

In Exercises 59–74, find the exact value of the logarithm without using a calculator. (If this is not possible, state the reason.)

59. $\log_3 9$
60. $\log_5 \frac{1}{125}$
61. $\log_2 \sqrt[4]{8}$
62. $\log_6 \sqrt[3]{6}$
63. $\log_4 16^{1.2}$
64. $\log_3 81^{-0.2}$
65. $\log_3(-9)$
66. $\log_2(-16)$
67. $\ln e^{4.5}$
68. $3\ln e^4$
69. $\ln \frac{1}{\sqrt{e}}$
70. $\ln \sqrt[4]{e^3}$
71. $\ln e^2 + \ln e^5$
72. $2\ln e^6 - \ln e^5$
73. $\log_5 75 - \log_5 3$
74. $\log_4 2 + \log_4 32$

In Exercises 75–80, use the properties of logarithms to rewrite and simplify the logarithmic expression.

75. $\log_4 8$

76. $\log_2(4^2 \cdot 3^4)$

77. $\log_5 \frac{1}{250}$

78. $\log_{10} \frac{9}{300}$

79. $\ln(5e^6)$

80. $\ln \dfrac{6}{e^2}$

▶ Model It

81. ***Human Memory Model*** Students participating in a psychology experiment attended several lectures and were given an exam. Every month for a year after the exam, the students were retested to see how much of the material they remembered. The average scores for the group can be modeled by the human memory model

$$f(t) = 90 - 15 \log_{10}(t + 1), \qquad 0 \le t \le 12$$

where t is the time in months.

(a) What was the average score on the original exam ($t = 0$)?

(b) What was the average score after 6 months?

(c) What was the average score after 12 months?

(d) When will the average score decrease to 75?

(e) Use the properties of logarithms to write the function in another form.

(f) Sketch the graph of the function over the specified domain.

82. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of 10^{-6} watt per square meter.

Synthesis

True or False? **In Exercises 83–88, determine whether the statement is true or false given that $f(x) = \ln x$. Justify your answer.**

83. $f(0) = 0$

84. $f(ax) = f(a) + f(x), \quad a > 0, x > 0$

85. $f(x - 2) = f(x) - f(2), \quad x > 2$

86. $\sqrt{f(x)} = \frac{1}{2}f(x)$

87. If $f(u) = 2f(v)$, then $v = u^2$.

88. If $f(x) < 0$, then $0 < x < 1$.

89. ***Proof*** Prove that $\log_b \dfrac{u}{v} = \log_b u - \log_b v$.

90. ***Proof*** Prove that $\log_b u^n = n \log_b u$.

In Exercises 91–96, use the change-of-base formula to rewrite the logarithm as a ratio of logarithms. Then use a graphing utility to graph both functions in the same viewing window to verify that the functions are equivalent.

91. $f(x) = \log_2 x$

92. $f(x) = \log_4 x$

93. $f(x) = \log_{1/2} x$

94. $f(x) = \log_{1/4} x$

95. $f(x) = \log_{11.8} x$

96. $f(x) = \log_{12.4} x$

97. ***Think About It*** Consider the functions below.

$$f(x) = \ln \frac{x}{2}, \quad g(x) = \frac{\ln x}{\ln 2}, \quad h(x) = \ln x - \ln 2$$

Which two functions should have identical graphs? Verify your answer by sketching the graphs of all three functions on the same set of coordinate axes.

98. ***Exploration*** For how many integers between 1 and 20 can the natural logarithms be approximated given that $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms (do not use a calculator).

Review

In Exercises 99–102, simplify the expression.

99. $\dfrac{24xy^{-2}}{16x^{-3}y}$

100. $\left(\dfrac{2x^2}{3y}\right)^{-3}$

101. $(18x^3y^4)^{-3}(18x^3y^4)^3$

102. $xy(x^{-1} + y^{-1})^{-1}$

In Exercises 103–106, solve the equation.

103. $3x^2 + 2x - 1 = 0$

104. $4x^2 - 5x + 1 = 0$

105. $\dfrac{2}{3x + 1} = \dfrac{x}{4}$

106. $\dfrac{5}{x - 1} = \dfrac{2x}{3}$

3.4 Exponential and Logarithmic Equations

▶ **What you should learn**

- How to solve simple exponential and logarithmic equations
- How to solve more complicated exponential equations
- How to solve more complicated logarithmic equations
- How to use exponential and logarithmic equations to model and solve real-life applications

▶ **Why you should learn it**

Applications of exponential and logarithmic equations are found in consumer safety testing. For instance, in Exercise 119, on page 234, a logarithmic function is used to model crumple zones for automobile crash tests.

David Woods/The Stock Market

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving these exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$a^{\log_a x} = x$

$\log_a a^x = x$

Example 1 ▶ Solving Simple Equations

	Original Equation	*Rewritten Equation*	*Solution*	*Property*
a.	$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b.	$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c.	$\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
d.	$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
e.	$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
f.	$\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Exponential Equations

Example 2 ▶ Solving Exponential Equations

Solve each equation and approximate the result to three decimal places.

a. $4^x = 72$ **b.** $3(2^x) = 42$

Solution

a.

$$\begin{aligned} 4^x &= 72 && \text{Write original equation.} \\ \log_4 4^x &= \log_4 72 && \text{Take logarithm (base 4) of each side.} \\ x &= \log_4 72 && \text{Inverse Property} \\ x &= \frac{\ln 72}{\ln 4} && \text{Change-of-base formula} \\ x &\approx 3.085 && \text{Use a calculator.} \end{aligned}$$

or $\frac{\log_{10} 72}{\log_{10} 4}$

The solution is $x = \log_4 72 \approx 3.085$. Check this in the original equation.

b.

$$\begin{aligned} 3(2^x) &= 42 && \text{Write original equation.} \\ 2^x &= 14 && \text{Divide each side by 3.} \\ \log_2 2^x &= \log_2 14 && \text{Take log (base 2) of each side.} \\ x &= \log_2 14 && \text{Inverse Property} \\ x &= \frac{\ln 14}{\ln 2} && \text{Change-of-base formula} \\ x &\approx 3.807 && \text{Use a calculator.} \end{aligned}$$

or $\frac{\log_{10} 14}{\log_{10} 2}$

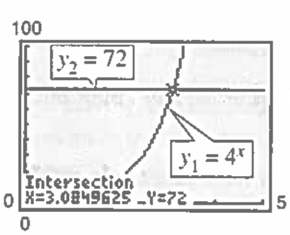
The solution is $x = \log_2 14 \approx 3.807$. Check this in the original equation.

Technology

When solving an exponential or logarithmic equation, remember that you can check your solution graphically by "graphing the left and right sides separately" and using the *intersect* feature of your graphing utility to determine the point of intersection. For instance, to check the solution of the equation in Example 2(a), you can graph

$$y_1 = 4^x \quad \text{and} \quad y_2 = 72$$

in the same viewing window, as shown below. Using the *intersect* feature of your graphing utility, you can determine that the graphs intersect when $x \approx 3.085$, which confirms the solution found in Example 2(a).



In Example 2(a), the exact solution is $x = \log_4 72$ and the approximate solution is $x \approx 3.085$. An exact answer is preferred when the solution is an intermediate step in a larger problem. For a final answer, an approximate solution is easier to comprehend.

Example 3 ▶ Solving an Exponential Equation

Solve $e^x + 5 = 60$ and approximate the result to three decimal places.

Solution

$$\begin{aligned} e^x + 5 &= 60 && \text{Write original equation.} \\ e^x &= 55 && \text{Subtract 5 from each side.} \\ \ln e^x &= \ln 55 && \text{Take natural log of each side.} \\ x &= \ln 55 && \text{Inverse Property} \\ x &\approx 4.007 && \text{Use a calculator.} \end{aligned}$$

The solution is $x = \ln 55 \approx 4.007$. Check this in the original equation.

Additional Example

$2^x = 10$

$\ln 2^x = \ln 10$

$x \ln 2 = \ln 10$

$x = \dfrac{\ln 10}{\ln 2} \approx 3.322$

Note: Using the change-of-base formula or the definition of a logarithmic function, you could write this solution as $x = \log_2 10$.

STUDY TIP

Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula.

$$\log_3 7.5 = \frac{\ln 7.5}{\ln 3} \approx 1.834$$

Example 4 ▶ Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$ and approximate the result to three decimal places.

Solution

$2(3^{2t-5}) - 4 = 11$ Write original equation.

$2(3^{2t-5}) = 15$ Add 4 to each side.

$3^{2t-5} = \dfrac{15}{2}$ Divide each side by 2.

$\log_3 3^{2t-5} = \log_3 \dfrac{15}{2}$ Take log (base 3) of each side.

$2t - 5 = \log_3 \dfrac{15}{2}$ Inverse Property

$2t = 5 + \log_3 7.5$ Add 5 to each side.

$t = \dfrac{5}{2} + \dfrac{1}{2}\log_3 7.5$ Divide each side by 2. $\dfrac{\log_{10} 7.5}{\log_{10} 3}$ or $\dfrac{\ln 7.5}{\ln 3}$

$t \approx 3.417$ Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.417$. Check this in the original equation.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in Examples 2, 3, and 4. However, the algebra is a bit more complicated.

Example 5 ▶ Solving an Exponential Equation of Quadratic Type

Solve $e^{2x} - 3e^x + 2 = 0$.

Solution

$e^{2x} - 3e^x + 2 = 0$ Write original equation.

$(e^x)^2 - 3e^x + 2 = 0$ Write in quadratic form.

$(e^x - 2)(e^x - 1) = 0$ Factor.

$e^x - 2 = 0$ Set 1st factor equal to 0.

$e^x = 2$

$x = \ln 2$ Solution

$\ln e^x = \ln 2$

$e^x - 1 = 0$ Set 2nd factor equal to 0.

$e^x = 1$

$x = 0$ Solution

The solutions are $x = \ln 2$ and $x = 0$. Check these in the original equation.

To ensure that students first solve for the unknown variable algebraically and then use their calculators, you can require both exact algebraic solutions and approximate numerical answers.

In Example 5, use a graphing utility to graph $y = e^{2x} - 3e^x + 2$. The graph should have two x-intercepts: one at $x = \ln 2 \approx 0.693$ and one at $x = 0$.

Solving Logarithmic Equations

To solve a logarithmic equation such as

$\ln x = 3$ Logarithmic form

$e^3 = x$

write the equation in exponential form as follows.

$e^{\ln x} = e^3$ Exponentiate each side.

$x = e^3$ Exponential form

This procedure is called *exponentiating* each side of an equation.

Example 6 ▶ Solving a Logarithmic Equation

a. Solve $\ln x = 2$.

b. Solve $\log_3(5x - 1) = \log_3(x + 7)$.

Solution

$e^2 = x$

a. $\ln x = 2$ Write original equation.

$e^{\ln x} = e^2$ Exponentiate each side.

$x = e^2$ Inverse Property

The solution is $x = e^2$. Check this in the original equation.

b. $\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.

$5x - 1 = x + 7$ One-to-One Property

$4x = 8$ Add $-x$ and 1 to each side.

$x = 2$ Divide each side by 4.

The solution is $x = 2$. Check this in the original equation.

Example 7 ▶ Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$ and approximate the result to three decimal places.

Solution

$5 + 2 \ln x = 4$ Write original equation.

$2 \ln x = -1$ Subtract 5 from each side.

$e^{-\frac{1}{2}} = x$

$\ln x = -\frac{1}{2}$ Divide each side by 2.

$e^{\ln x} = e^{-1/2}$ Exponentiate each side.

$x = e^{-1/2}$ Inverse Property

$x \approx 0.607$ Use a calculator.

The solution is $x = e^{-1/2} \approx 0.607$. Check this in the original equation.

Activities

1. Solve for x: $7^x = 3$.

   Answer: $x = \frac{\ln 3}{\ln 7} \approx 0.5646$

2. Solve for x: $\log_{10}(x + 4) + \log_{10}(x + 1) = 1$.

   Answer: $x = 1$ ($x = -6$ is not in the domain.)

Example 8 ▶ Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$2 \log_5 3x = 4$	Write original equation.
$\log_5 3x = 2$ $\quad < 5^2 = 3x$	Divide each side by 2.
$5^{\log_5 3x} = 5^2$ $\quad 25 = 3x$	Exponentiate each side (base 5).
$3x = 25$ $\quad \frac{25}{3} = x$	Inverse Property
$x = \dfrac{25}{3}$	Divide each side by 3.

The solution is $x = \frac{25}{3}$. Check this in the original equation.

STUDY TIP

Notice in Example 9 that the logarithmic part of the equation is condensed into a single logarithm before exponentiating each side of the equation.

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations.

Example 9 ▶ Checking for Extraneous Solutions

Solve $\log_{10} 5x + \log_{10}(x - 1) = 2$.

Solution

$\log_{10} 5x + \log_{10}(x - 1) = 2$	Write original equation.
$\log_{10}[5x(x - 1)] = 2$	Product Property of Logarithms
$10^{\log_{10}(5x^2 - 5x)} = 10^2$	Exponentiate each side (base 10).
$5x^2 - 5x = 100$	Inverse Property
$x^2 - x = 20$	
$x^2 - x - 20 = 0$	Write in general form.
$(x - 5)(x + 4) = 0$	Factor.
$x - 5 = 0$	Set 1st factor equal to 0.
$x = 5$	Solution
$x + 4 = 0$	Set 2nd factor equal to 0.
$x = -4$	Solution

The solutions appear to be $x = 5$ and $x = -4$. However, when you check these in the original equation, you can see that $x = 5$ is the only solution.

Technology

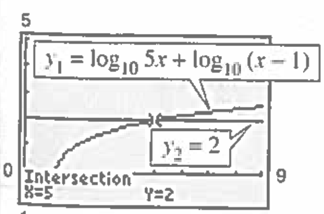
You can use a graphing utility to verify that the equation in Example 9 has $x = 5$ as its only solution. Graph

$$y_1 = \log_{10} 5x + \log_{10}(x - 1)$$

and

$$y_2 = 2$$

in the same viewing window. From the graph shown below, it appears that the graphs of the two equations intersect at one point. Use the *intersect* feature or the *zoom* and *trace* features to determine that $x = 5$ is an approximate solution. You can verify this algebraically by substituting $x = 5$ into the original equation.



In Example 9, the domain of $\log_{10} 5x$ is $x > 0$ and the domain of $\log_{10}(x - 1)$ is $x > 1$, so the domain of the original equation is $x > 1$. Because the domain is all real numbers greater than 1, the solution $x = -4$ is extraneous.

Applications

Activity

Determine the amount of time it would take \$1000 to double in an account that pays 6.75% interest, compounded continuously. How does this compare to Example 10?

Answer: ≈ 10.27 years; it takes the same amount of time.

Example 10 ▶ Doubling an Investment

You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution

Using the formula for continuous compounding, you can find that the balance in the account is

$$A = Pe^{rt}$$

$$A = 500e^{0.0675t}.$$

To find the time required for the balance to double, let $A = 1000$ and solve the resulting equation for t.

$500e^{0.0675t} = 1000$	Let $A = 1000$.
$e^{0.0675t} = 2$	Divide each side by 500.
$\ln e^{0.0675t} = \ln 2$	Take natural log of each side.
$0.0675t = \ln 2$	Inverse Property
$t = \dfrac{\ln 2}{0.0675}$	Divide each side by 0.0675.
$t \approx 10.27$	Use a calculator.

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically in Figure 3.26.

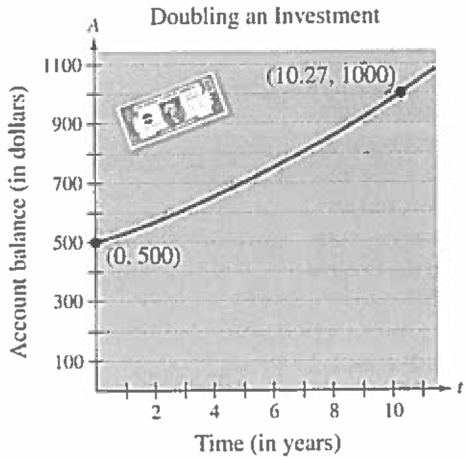


FIGURE 3.26

Exploration

The *effective yield* of a savings plan is the percent increase in the balance after 1 year. Find the effective yield for each savings plan when \$1000 is deposited in a savings account.

a. 7% annual interest rate, compounded annually

b. 7% annual interest rate, compounded continuously

c. 7% annual interest rate, compounded quarterly

d. 7.25% annual interest rate, compounded quarterly

Which savings plan has the greatest effective yield? Which savings plan will have the highest balance after 5 years?

In Example 10, an approximate answer of 10.27 years is given. Within the context of the problem, the exact solution, $(\ln 2)/0.0675$ years, does not make sense as an answer.

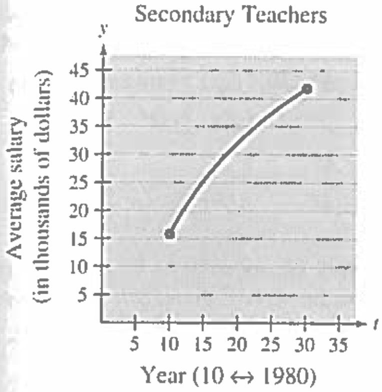


FIGURE 3.27

Example 11 ▶ Average Salary for Secondary Teachers

For selected years from 1980 to 2000, the average salary for secondary teachers y (in thousands of dollars) for the year t can be modeled by the equation

$$y = -38.8 + 23.7 \ln t, \quad 10 \le t \le 30$$

where $t = 10$ represents 1980 (see Figure 3.27). During which year did the average salary for secondary teachers reach 2.5 times its 1980 level of \$16.5 thousand? (Source: National Education Association)

Solution

$-38.8 + 23.7 \ln t = y$	Write original equation.
$-38.8 + 23.7 \ln t = 41.25$	Let $y = (2.5)(16.5) = 41.25$.
$23.7 \ln t = 80.05$	Add 38.8 to each side.
$\ln t \approx 3.378$	Divide each side by 23.7.
$e^{\ln t} \approx e^{3.378}$	Exponentiate each side.
$t \approx e^{3.378}$	Inverse Property
$t \approx 29$	Use a calculator.

The solution is $t \approx 29$ years. Because $t = 10$ represents 1980, it follows that the average salary for secondary teachers reached 2.5 times its 1980 level in 1999.

Writing ABOUT MATHEMATICS

Comparing Mathematical Models The table shows the U.S. Postal Service rates y for sending an express mail package for selected years from 1985 through 2001, where $x = 5$ represents 1985. (Source: U.S. Postal Service)

Year, x	Rate, y
5	10.75
8	12.00
11	13.95
15	15.00
19	15.75
21	16.00

a. Create a scatter plot of the data. Find a linear model for the data, and add its graph to your scatter plot. According to this model, when will the rate for sending an express mail package reach \$17.50?

b. Create a new table showing values for $\ln x$ and $\ln y$ and create a scatter plot of this transformed data. Use the method illustrated in Example 7 in Section 3.3 to find a model for the transformed data, and add its graph to your scatter plot. According to this model, when will the rate for sending an express mail package reach \$17.50?

c. Solve the model in part (b) for y, and add its graph to your scatter plot in part (a). Which model better fits the original data? Which model will better predict future shipments? Explain.

3.4 Exercises

In Exercises 1–6, determine whether each *x*-value is a solution (or an approximate solution) of the equation.

1. $4^{2x-7} = 64$
   (a) $x = 5$
   (b) $x = 2$

2. $2^{3x+1} = 32$
   (a) $x = -1$
   (b) $x = 2$

3. $3e^{x+2} = 75$
   (a) $x = -2 + e^{25}$
   (b) $x = -2 + \ln 25$
   (c) $x \approx 1.219$

4. $5^{2x+3} = 812$
   (a) $x = -\frac{3}{2} + \log_5 \sqrt{812}$
   (b) $x \approx 0.581$
   (c) $x = -1.5 + \dfrac{\ln 812}{\ln 5}$

5. $\log_4(3x) = 3$
   (a) $x \approx 20.356$
   (b) $x = -4$
   (c) $x = \frac{64}{3}$

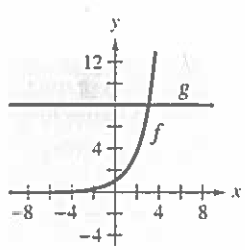
6. $\ln(x - 1) = 3.8$
   (a) $x = 1 + e^{3.8}$
   (b) $x \approx 45.701$
   (c) $x = 1 + \ln 3.8$

In Exercises 7–26, solve for *x*.

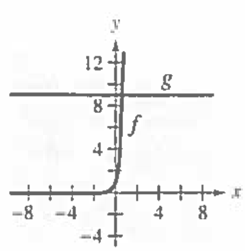
7. $4^x = 16$
8. $3^x = 243$
9. $5^x = 625$
10. $3^x = 729$
11. $7^x = \frac{1}{49}$
12. $8^x = 4$
13. $\left(\frac{1}{2}\right)^x = 32$
14. $\left(\frac{1}{4}\right)^x = 64$
15. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$
16. $\left(\frac{2}{3}\right)^x = \frac{4}{9}$
17. $\ln x - \ln 2 = 0$
18. $\ln x - \ln 5 = 0$
19. $e^x = 2$
20. $e^x = 4$
21. $\ln x = -1$
22. $\ln x = -7$
23. $\log_4 x = 3$
24. $\log_5 x = -3$
25. $\log_{10} x = -1$
26. $\log_{10} x - 2 = 0$

In Exercises 27–30, approximate the point of intersection of the graphs of *f* and *g*. Then solve the equation *f*(*x*) = *g*(*x*) algebraically.

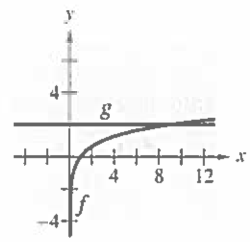
27. $f(x) = 2^x$
    $g(x) = 8$



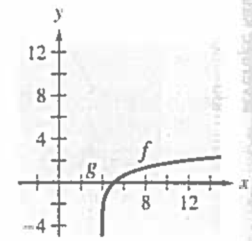
28. $f(x) = 27^x$
    $g(x) = 9$



29. $f(x) = \log_3 x$
    $g(x) = 2$



30. $f(x) = \ln(x - 4)$
    $g(x) = 0$



In Exercises 31–68, solve the exponential equation algebraically. Approximate the result to three decimal places.

31. $4(3^x) = 20$
32. $2(5^x) = 32$
33. $2e^x = 10$
34. $4e^x = 91$
35. $e^x - 9 = 19$
36. $6^x + 10 = 47$
37. $3^{2x} = 80$
38. $6^{5x} = 3000$
39. $5^{-t/2} = 0.20$
40. $4^{-3t} = 0.10$
41. $3^{x-1} = 27$
42. $2^{x-3} = 32$
43. $2^{3-x} = 565$
44. $8^{-2-x} = 431$
45. $8(10^{3x}) = 12$
46. $5(10^{x-6}) = 7$
47. $3(5^{x-1}) = 21$
48. $8(3^{6-x}) = 40$
49. $e^{3x} = 12$
50. $e^{2x} = 50$
51. $500e^{-x} = 300$
52. $1000e^{-4x} = 75$
53. $7 - 2e^x = 5$
54. $-14 + 3e^x = 11$
55. $6(2^{3x-1}) - 7 = 9$
56. $8(4^{6-2x}) + 13 = 41$
57. $e^{2x} - 4e^x - 5 = 0$
58. $e^{2x} - 5e^x + 6 = 0$
59. $e^{2x} - 3e^x - 4 = 0$
60. $e^{2x} + 9e^x + 36 = 0$
61. $\dfrac{500}{100 - e^{x/2}} = 20$
62. $\dfrac{400}{1 + e^{-x}} = 350$
63. $\dfrac{3000}{2 + e^{2x}} = 2$
64. $\dfrac{119}{e^{6x} - 14} = 7$
65. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$
66. $\left(4 - \dfrac{2.471}{40}\right)^{9t} = 21$
67. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$
68. $\left(16 - \dfrac{0.878}{26}\right)^{3t} = 30$

p. 232 #s 7-25 (3x), 29, 31-58 (3x), 77-101 (4x)
3 3 4

In Exercises 69–76, use a graphing utility to solve the equation. Approximate the result to three decimal places. Verify your result algebraically.

69. $6e^{1-x} = 25$
70. $-4e^{-x-1} + 15 = 0$
71. $3e^{3x/2} = 962$
72. $8e^{-2x/3} = 11$
73. $e^{0.09t} = 3$
74. $-e^{1.8x} + 7 = 0$
75. $e^{0.125t} - 8 = 0$
76. $e^{2.724x} = 29$

In Exercises 77–104, solve the logarithmic equation algebraically. Approximate the result to three decimal places.

77. $\ln x = -3$
78. $\ln x = 2$
79. $\ln 2x = 2.4$
80. $\ln 4x = 1$
81. $\log_{10} x = 6$
82. $\log_{10} 3z = 2$
83. $6 \log_3(0.5x) = 11$
84. $5 \log_{10}(x - 2) = 11$
85. $3 \ln 5x = 10$
86. $2 \ln x = 7$
87. $\ln\sqrt{x + 2} = 1$
88. $\ln\sqrt{x - 8} = 5$
89. $7 + 3 \ln x = 5$
90. $2 - 6 \ln x = 10$
91. $\ln x - \ln(x + 1) = 2$
92. $\ln x + \ln(x + 1) = 1$
93. $\ln x + \ln(x - 2) = 1$
94. $\ln x + \ln(x + 3) = 1$
95. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$
96. $\ln(x + 1) - \ln(x - 2) = \ln x$
97. $\log_2(2x - 3) = \log_2(x + 4)$
98. $\log_{10}(x - 6) = \log_{10}(2x + 1)$
99. $\log_{10}(x + 4) - \log_{10} x = \log_{10}(x + 2)$
100. $\log_2 x + \log_2(x + 2) = \log_2(x + 6)$
101. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$
102. $\log_3 x + \log_3(x - 8) = 2$
103. $\log_{10} 8x - \log_{10}(1 + \sqrt{x}) = 2$
104. $\log_{10} 4x - \log_{10}(12 + \sqrt{x}) = 2$

In Exercises 105–108, use a graphing utility to solve the equation. Approximate the result to three decimal places. Verify your result algebraically.

105. $7 = 2^x$
106. $500 = 1500e^{-x/2}$
107. $3 - \ln x = 0$
108. $10 - 4 \ln(x - 2) = 0$

Compound Interest In Exercises 109 and 110, find the time required for a \$1000 investment to double at interest rate r, compounded continuously.

109. $r = 0.085$
110. $r = 0.12$

Compound Interest In Exercises 111 and 112, find the time required for a \$1000 investment to triple at interest rate r, compounded continuously.

111. $r = 0.085$
112. $r = 0.12$

113. ***Demand*** The demand equation for a microwave oven is

$$p = 500 - 0.5(e^{0.004x}).$$

Find the demand x for a price of (a) $p = \$350$ and (b) $p = \$300$.

114. ***Demand*** The demand equation for a hand-held electronic organizer is

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right).$$

Find the demand x for a price of (a) $p = \$600$ and (b) $p = \$400$.

115. ***Forest Yield*** The yield V (in millions of cubic feet per acre) for a forest at age t years is

$$V = 6.7e^{-48.1/t}.$$

(a) Use a graphing utility to graph the function.

(b) Determine the horizontal asymptote of the function. Interpret its meaning in the context of the problem.

(c) Find the time necessary to obtain a yield of 1.3 million cubic feet.

116. ***Trees per Acre*** The number of trees per acre N of a species is approximated by the model

$$N = 68(10^{-0.04x}), \quad 5 \le x \le 40$$

where x is the average diameter of the trees 3 feet above the ground. Use the model to approximate the average diameter of the trees in a test plot when $N = 21$.

117. ***Average Heights*** The percent of American males between the ages of 18 and 24 who are no more than x inches tall is

$$m(x) = \frac{100}{1 + e^{-0.6114(x-69.71)}}$$

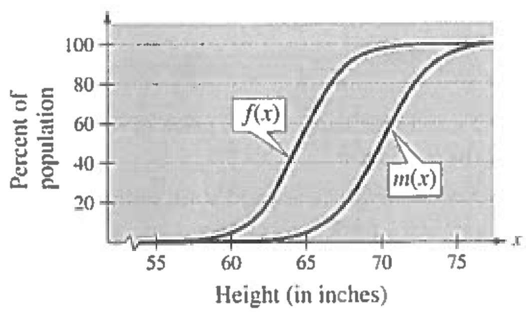
and the percent of American females between the ages of 18 and 24 who are no more than x inches tall is

$$f(x) = \frac{100}{1 + e^{-0.66607(x-64.51)}}$$

where m and f are the percents and x is the height in inches. (Source: U.S. National Center for Health Statistics)

(a) Use the graph to determine any horizontal asymptotes of the functions. Interpret the meaning in the context of the problem.

(b) What is the average height of each sex?



118. ***Learning Curve*** In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

(a) Use a graphing utility to graph the function.

(b) Use the graph to determine any horizontal asymptotes of the function. Interpret the meaning of the upper asymptote in the context of this problem.

(c) After how many trials will 60% of the responses be correct?

▶ Model It

119. ***Automobiles*** Automobiles are designed with crumple zones that help protect their occupants in crashes. The crumple zones allow the occupants to move short distances when the automobiles come to abrupt stops. The greater the distance moved, the fewer g's the crash victims experience. (One g is equal to the acceleration due to gravity. For very short periods of time, humans have withstood as much as 40 g's.) In crash tests with vehicles moving at 90 kilometers per hour, analysts measured the numbers of g's experienced during deceleration by crash dummies that were permitted to move x meters during impact. The data is shown in the table.

x	g's
0.2	158
0.4	80
0.6	53
0.8	40
1.0	32

A model for this data is

$$y = -3.00 + 11.88 \ln x + \frac{36.94}{x}$$

where y is the number of g's.

(a) Complete the table using the model.

x	0.2	0.4	0.6	0.8	1.0
y					

(b) Use a graphing utility to graph the data points and the model in the same viewing window. How do they compare?

(c) Use the model to estimate the distance traveled during impact if the passenger deceleration must not exceed 30 g's.

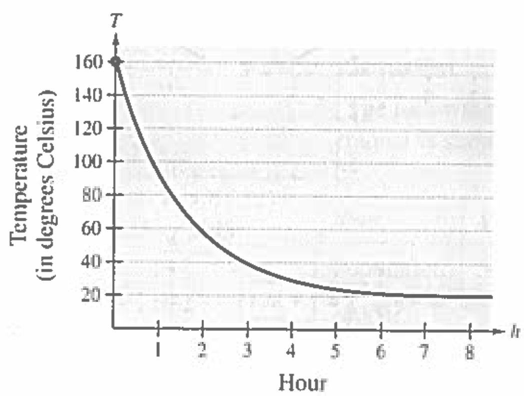
(d) Do you think it is practical to lower the number of g's experienced during impact to fewer than 23? Explain your reasoning.

120. ***Data Analysis*** An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C. The temperature T of the object was measured each hour h and recorded in the table. A model for this data is $T = 20[1 + 7(2^{-h})]$. The graph of this model is shown in the figure.

Hour, h	Temperature, T
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

(a) Use the graph to identify the horizontal asymptote of the model and interpret the asymptote in the context of the problem.

(b) Use the model to approximate the time when the temperature of the object was 100°C.



Synthesis

True or False? **In Exercises 121–124, rewrite each verbal statement as an equation. Then decide whether the statement is true or false. Justify your answer.**

121. The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

122. The logarithm of the sum of two numbers is equal to the product of the logarithms of the numbers.

123. The logarithm of the difference of two numbers is equal to the difference of the logarithms of the numbers.

124. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

125. ***Think About It*** Is it possible for a logarithmic equation to have more than one extraneous solution? Explain.

126. ***Finance*** You are investing P dollars at an annual interest rate of r, compounded continuously, for t years. Which of the following would result in the highest value of the investment? Explain your reasoning.

(a) Double the amount you invest.

(b) Double your interest rate.

(c) Double the number of years.

127. ***Think About It*** Are the times required for the investments in Exercises 109 and 110 to quadruple twice as long as the times for them to double? Give a reason for your answer and verify your answer algebraically.

128. ***Writing*** Write two or three sentences stating the general guidelines that you follow when solving (a) exponential equations and (b) logarithmic equations.

Review

In Exercises 129–132, simplify the expression.

129. $\sqrt{48x^2y^5}$ 130. $\sqrt{32} - 2\sqrt{25}$

131. $\sqrt[3]{25} \cdot \sqrt[3]{15}$ 132. $\dfrac{3}{\sqrt{10} - 2}$

In Exercises 133–136, find a mathematical model for the verbal statement.

133. M varies directly as the cube of p.

134. t varies inversely as the cube of s.

135. d varies jointly as a and b.

136. x is inversely proportional to $b - 3$.

In Exercises 137–140, evaluate the logarithm using the change-of-base formula. Approximate your result to three decimal places.

137. $\log_6 9$ 138. $\log_3 4$

139. $\log_{3/4} 5$ 140. $\log_8 22$

3.5 Exponential and Logarithmic Models

▶ What you should learn

- How to recognize the five most common types of models involving exponential and logarithmic functions
- How to use exponential growth and decay functions to model and solve real-life problems
- How to use Gaussian functions to model and solve real-life problems
- How to use logistic growth functions to model and solve real-life problems
- How to use logarithmic functions to model and solve real-life problems

▶ Why you should learn it

Exponential growth and decay models are often used to model the population of a country. For instance, in Exercise 36 on page 244, you will use exponential growth and decay models to compare the populations of several countries.

Vittoriano Rastelli/Corbis

Introduction

The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b \ln x, \quad y = a + b \log_{10} x$

The graphs of the basic forms of these functions are shown in Figure 3.28.

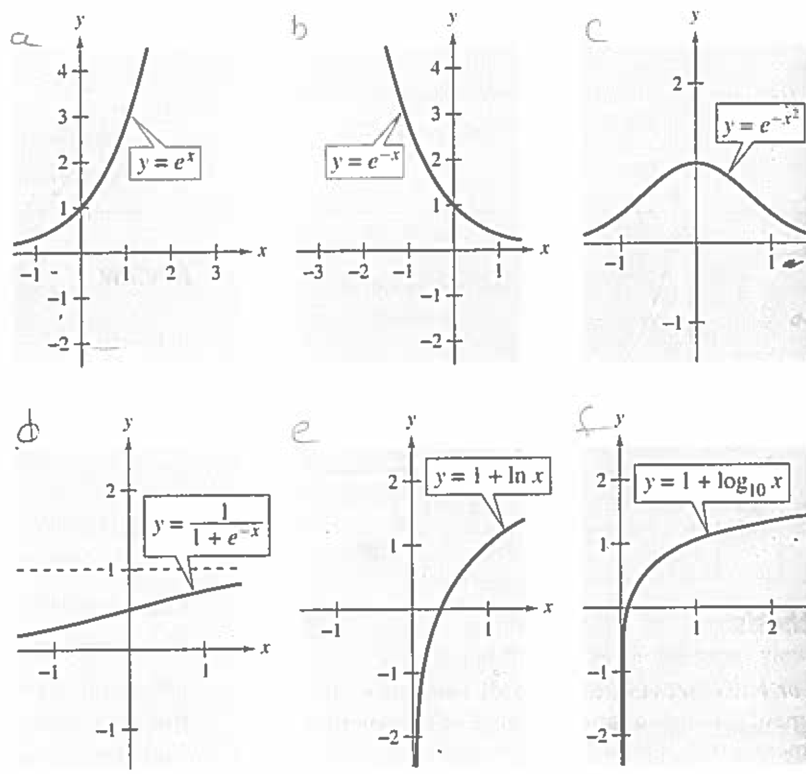


FIGURE 3.28

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes. Use the graphs in Figure 3.28 to identify the asymptotes of each function.

This section shows students real-world applications for logarithmic and exponential functions.

Exponential Growth and Decay

Example 1 ▶ Population Increase

Estimates of the world population (in millions) from 1995 through 2003 are shown in the table. The scatter plot of the data is shown in Figure 3.29. (Source: U.S. Census Bureau)

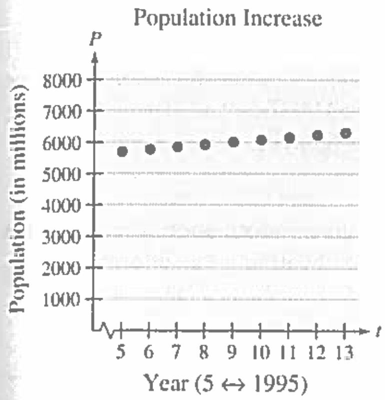


FIGURE 3.29

Year	Population
1995	5691
1996	5769
1997	5847
1998	5925
1999	6003

Year	Population
2000	6080
2001	6157
2002	6234
2003	6311

An exponential growth model that approximates this data is

$$P = 5340e^{0.012922t}, \quad 5 \le t \le 13$$

where P is the population (in millions) and $t = 5$ represents 1995. Compare the values given by the model with the estimates given by the U.S. Census Bureau. According to this model, when will the world population reach 6.8 billion?

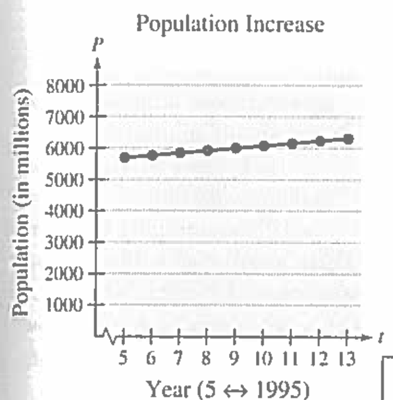


FIGURE 3.30

Solution

The following table compares the two sets of population figures. The graph of the model is shown in Figure 3.30.

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003
Population	5691	5769	5847	5925	6003	6080	6157	6234	6311
Model	5696	5771	5846	5922	5999	6077	6156	6236	6317

To find when the world population will reach 6.8 billion, let $P = 6800$ in the model and solve for t.

$$5340e^{0.012922t} = P \qquad \text{Write original model.}$$

$$5340e^{0.012922t} = 6800 \qquad \text{Let } P = 6800.$$

$$e^{0.012922t} \approx 1.27341 \qquad \text{Divide each side by 5340.}$$

$$\ln e^{0.012922t} \approx \ln 1.27341 \qquad \text{Take natural log of each side.}$$

$$0.012922t \approx 0.241698 \qquad \text{Inverse Property}$$

$$t \approx 18.7 \qquad \text{Divide each side by 0.012922.}$$

According to the model, the world population will reach 6.8 billion in 2008.

Technology

Some graphing utilities have curve-fitting capabilities that can be used to find models that represent data. If you have such a graphing utility, try using it to find a model for the data given in Example 1. How does your model compare with the model given in Example 1?

Additional Example

Radioactive iodine is a by-product of some types of nuclear reactors. Its half-life is 60 days. That is, after 60 days, a given amount of radioactive iodine will have decayed to half the original amount. Suppose a contained nuclear accident occurs and gives off an initial amount C of radioactive iodine.

a. Write an equation for the amount of radioactive iodine present at any time t following the accident.

b. How long will it take for the radioactive iodine to decay to a level of 20% of the original amount?

Solution

a. Knowing that half the original amount remains after 60 days, you can use the exponential decay model $y = ae^{-bt}$ to obtain

$$\frac{1}{2}C = Ce^{-b(60)}$$

$$\frac{1}{2} = e^{-60b} \quad \ln\tfrac{1}{2} = -60b$$

$$-\ln 2 = -60b$$

$$b = \frac{\ln 2}{60} \approx 0.0116.$$

So, $y = Ce^{-0.0116t}$.

b. The time required for the radioactive iodine to decay to 20% of the original amount is

$$Ce^{-0.0116t} = (0.2)C$$

$$e^{-0.0116t} = 0.2$$

$$-0.0116t = \ln 0.2$$

$$t = \frac{\ln 0.2}{-0.0116} \approx 139 \text{ days.}$$

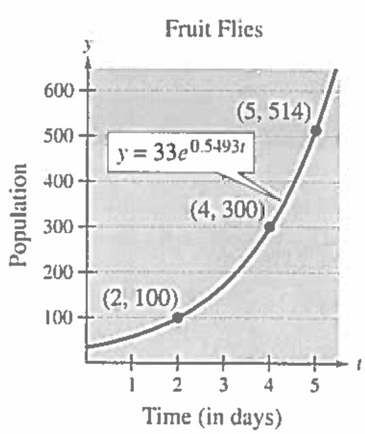


FIGURE 3.31

In Example 1, you were given the exponential growth model. But suppose this model were not given; how could you find such a model? One technique for doing this is demonstrated in Example 2.

Example 2 ▶ Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

Solution

Let y be the number of flies at time t. From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \qquad \text{and} \qquad 300 = ae^{4b}.$$

To solve for b, solve for a in the first equation.

$$100 = ae^{2b} \quad \Longrightarrow \quad a = \frac{100}{e^{2b}} \qquad \text{Solve for } a \text{ in the first equation.}$$

Then substitute the result into the second equation.

$$300 = ae^{4b} \qquad \text{Write second equation.}$$

$$300 = \left(\frac{100}{e^{2b}}\right)e^{4b} \qquad \text{Substitute } 100/e^{2b} \text{ for } a.$$

$$\frac{300}{100} = e^{2b} \qquad \text{Divide each side by 100.}$$

$$\ln 3 = 2b \qquad \text{Take natural log of each side.}$$

$$\frac{1}{2}\ln 3 = b \qquad \text{Solve for } b.$$

Using $b = \frac{1}{2}\ln 3$ and the equation you found for a, you can determine that

$$a = \frac{100}{e^{2[(1/2)\ln 3]}} \qquad \text{Substitute } (1/2)\ln 3 \text{ for } b.$$

$$= \frac{100}{e^{\ln 3}} \qquad \text{Simplify.}$$

$$= \frac{100}{3} \qquad \text{Inverse Property}$$

$$\approx 33. \qquad \text{Simplify.}$$

So, with $a \approx 33$ and $b = \frac{1}{2}\ln 3 \approx 0.5493$, the exponential growth model is

$$y = 33e^{0.5493t}$$

as shown in Figure 3.31. This implies that, after 5 days, the population will be

$$y = 33e^{0.5493(5)} \approx 514 \text{ flies.}$$

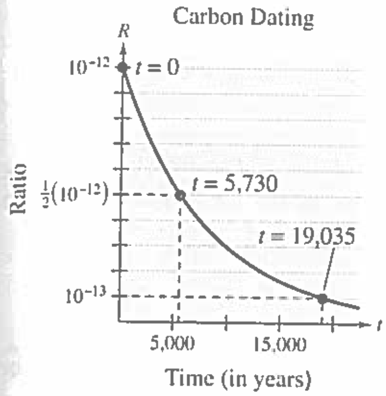


FIGURE 3.32

In living organic material, the ratio of the number of radioactive carbon isotopes (carbon 14) to the number of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of 5730 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time t (in years).

$$R = \frac{1}{10^{12}}e^{-t/8267} \qquad \text{Carbon dating model}$$

The graph of R is shown in Figure 3.32. Note that R decreases as t increases.

Example 3 ▶ Carbon Dating

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}.$$

Estimate the age of the fossil.

Solution

In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}}e^{-t/8267} = R \qquad \text{Write original model.}$$

$$\frac{e^{-t/8267}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Let } R = \frac{1}{10^{13}}.$$

$$e^{-t/8267} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8267} = \ln\frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8267} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 19{,}036 \qquad \text{Multiply each side by } -8267.$$

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

The carbon dating model in Example 3 assumed that the carbon 14/carbon 12 ratio was one part in 10,000,000,000,000. Suppose an error in measurement occurred and the actual ratio was only one part in 8,000,000,000,000. The fossil age corresponding to the actual ratio would then be approximately 17,000 years. Try checking this result.

STUDY TIP

An exponential model can be used to determine the *decay* of radioactive isotopes. For instance, to find how much of an initial 10 grams of ^{226}Ra isotope with a half-life of 1620 years is left after 500 years, you would use the exponential decay model.

$$y = ae^{-bt}$$

$$\frac{1}{2}(10) = 10e^{-b(1620)}$$

$$b = -\ln\left(\frac{1}{2}\right)/1620$$

Using the value of b found above and $a = 10$, the amount left is

$$y = 10e^{-[-\ln(1/2)/1620](500)}$$

$$y \approx 8.07 \text{ grams.}$$

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** One model for this situation takes the form

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/(2\sigma^2)}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). The graph of a Gaussian model is called a **bell-shaped curve.**

The average value for a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable—in this case, x.

Example 4 ▶ SAT Scores

In 2001, the Scholastic Aptitude Test (SAT) math scores for college-bound seniors roughly followed a normal distribution

$$y = 0.0035e^{-(x-514)^2/25{,}538}, \quad 200 \le x \le 800$$

where x is the SAT score for mathematics. Sketch the graph of this function. From the graph, estimate the average SAT score. (Source: College Board)

Solution

The graph of the function is shown in Figure 3.33. From the graph, you can see that the average mathematics score for college-bound seniors in 2001 was 514.

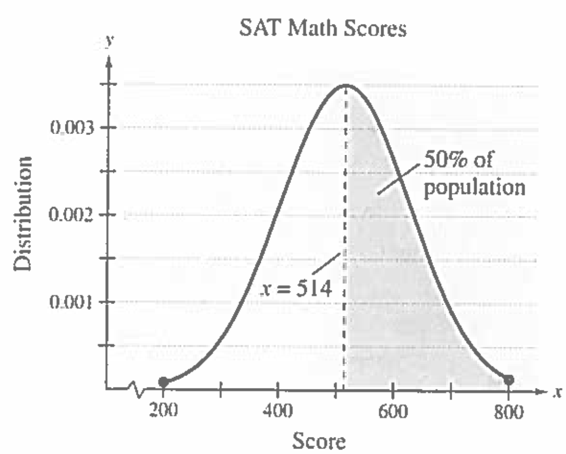


FIGURE 3.33

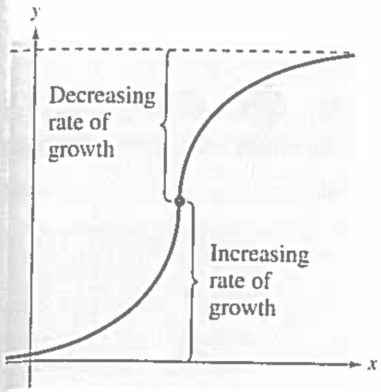


FIGURE 3.34

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 3.34. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

Example 5 ▶ Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious and long-lasting flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where y is the total number of students infected after t days. The college will cancel classes when 40% or more of the students are infected.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

Solution

a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

b. Classes are canceled when the number infected is $(0.40)(5000) = 2000$.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$\ln e^{-0.8t} = \ln \frac{1.5}{4999}$$

$$-0.8t = \ln \frac{1.5}{4999}$$

$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999}$$

$$t \approx 10.1$$

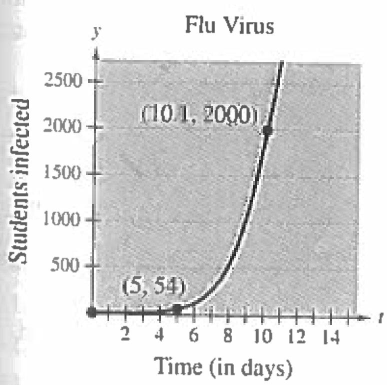


FIGURE 3.35

So, after 10 days, at least 40% of the students will be infected, and classes will be canceled. The graph of the function is shown in Figure 3.35.

AFP/Corbis

On January 13, 2001 an earthquake of magnitude 7.7 in El Salvador caused 185 landslides and killed over 800 people.

Logarithmic Models

Example 6 ▶ **Magnitude of Earthquakes**

On the Richter scale, the magnitude R of an earthquake of intensity I is

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensities per unit of area for the following earthquakes. (Intensity is a measure of the wave energy of an earthquake.)

a. Tokyo and Yokohama, Japan in 1923: $R = 8.3$.
b. El Salvador in 2001: $R = 7.7$.

Solution

a. Because $I_0 = 1$ and $R = 8.3$, you have

$$8.3 = \log_{10} \frac{I}{1}$$ Substitute 1 for I_0 and 8.3 for R.

$$10^{8.3} = 10^{\log_{10} I}$$ Exponentiate each side.

$$I = 10^{8.3} \approx 199{,}526{,}000.$$ Inverse property of exponents and logs

b. For $R = 7.7$, you have

$$7.7 = \log_{10} \frac{I}{1}$$ Substitute 1 for I_0 and 7.7 for R.

$$10^{7.7} = 10^{\log_{10} I}$$ Exponentiate each side.

$$I = 10^{7.7} \approx 50{,}119{,}000.$$ Inverse property of exponents and logs

Note that an increase of 0.6 unit on the Richter scale (from 7.7 to 8.3) represents an increase in intensity by a factor of

$$\frac{199{,}526{,}000}{50{,}119{,}000} \approx 4.$$

In other words, the earthquake in 1923 had an intensity about 4 times greater than that of the 2001 earthquake.

Alternative Writing About Mathematics

Use your school's library, the Internet, or some other reference source to find an application that fits one of the five models discussed in this section. After you have collected data for the model, plot the corresponding points and find an equation that describes the points you have plotted.

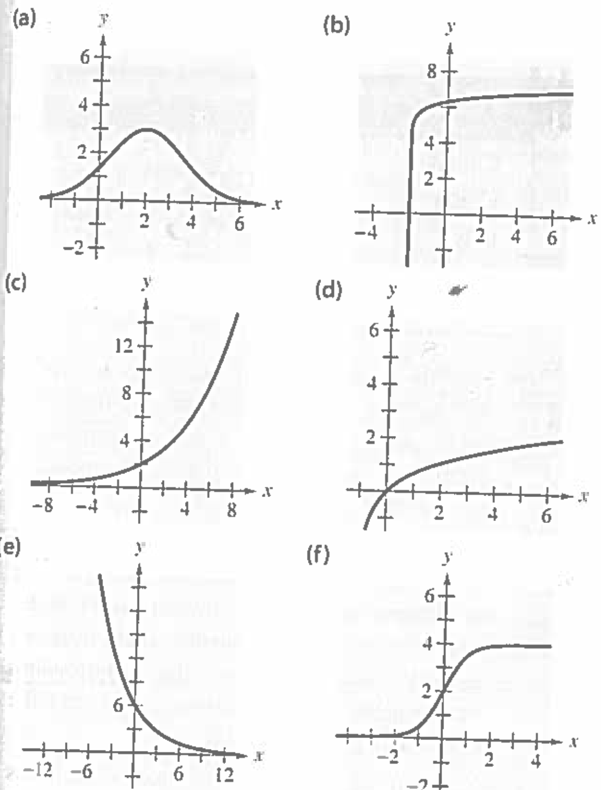
Writing ABOUT MATHEMATICS

Comparing Population Models The population (in millions) of the United States from 1910 to 2000 is shown in the table at the left. (Source: U.S. Census Bureau) Least squares regression analysis gives the best quadratic model for this data as $P = 1.0317t^2 + 9.668t + 81.38$ and the best exponential model for this data as $P = 82.367e^{0.125t}$. Which model better fits the data? Describe the method you used to reach your conclusion.

t	Year	Population
1	1910	91.97
2	1920	105.71
3	1930	122.78
4	1940	131.67
5	1950	151.33
6	1960	179.32
7	1970	203.30
8	1980	226.54
9	1990	248.72
10	2000	281.42

3.5 Exercises

In Exercises 1–6, match the function with its graph. (The graphs are labeled (a), (b), (c), (d), (e), and (f).)



1. $y = 2e^{x/4}$ c (a)
2. $y = 6e^{-x/4}$ e (b)
3. $y = 6 + \log_{10}(x + 2)$ b(f)
4. $y = 3e^{-(x-2)^2/5}$ a (c)
5. $y = \ln(x + 1)$ d (e)
6. $y = \dfrac{4}{1 + e^{-2x}}$ f (d)

p. 236

Compound Interest In Exercises 7–14, complete the table for a savings account in which interest is compounded continuously.

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
7.	\$1000	12%		
8.	\$20,000	$10\frac{1}{2}$%		
9.	\$750		$7\frac{3}{4}$ yr	
10.	\$10,000		12 yr	
11.	\$500			\$1505.00
12.	\$600			\$19,205.00
13.		4.5%		\$10,000.00
14.		8%		\$20,000.00

Compound Interest In Exercises 15 and 16, determine the principal P that must be invested at rate r, compounded monthly, so that \$500,000 will be available for retirement in t years.

15. $r = 7\frac{1}{2}\%,\ t = 20$
16. $r = 12\%,\ t = 40$

Compound Interest In Exercises 17 and 18, determine the time necessary for \$1000 to double if it is invested at interest rate r compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.

17. $r = 11\%$
18. $r = 10\frac{1}{2}\%$

19. ***Compound Interest*** Complete the table for the time t necessary for P dollars to triple if interest is compounded continuously at rate r.

r	2%	4%	6%	8%	10%	12%
t						

20. ***Modeling Data*** Draw a scatter plot of the data in Exercise 19. Use the *regression* feature of a graphing utility to find a model for the data.

21. ***Compound Interest*** Complete the table for the time t necessary for P dollars to triple if interest is compounded annually at rate r.

r	2%	4%	6%	8%	10%	12%
t						

22. ***Modeling Data*** Draw a scatter plot of the data in Exercise 21. Use the *regression* feature of a graphing utility to find a model for the data.

23. ***Comparing Models*** If \$1 is invested in an account over a 10-year period, the amount in the account, where t represents the time in years, is

$$A = 1 + 0.075[\![t]\!] \quad \text{or} \quad A = e^{0.07t}$$

depending on whether the account pays simple interest at $7\frac{1}{2}\%$ or continuous compound interest at 7%. Graph each function on the same set of axes. Which grows at the faster rate? (Remember that $[\![t]\!]$ is the greatest integer function discussed in Section 1.5.)

24. ***Comparing Models*** If \$1 is invested in an account over a 10-year period, the amount in the account, where t represents the time in years, is

$$A = 1 + 0.06[\![t]\!] \quad \text{or} \quad A = \left(1 + \frac{0.055}{365}\right)^{[\![365t]\!]}$$

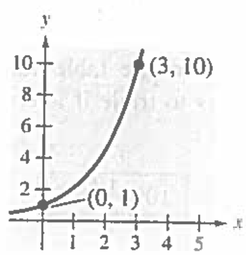
depending on whether the account pays simple interest at 6% or compound interest at $5\frac{1}{2}\%$ compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at the faster rate?

Radioactive Decay **In Exercises 25–30, complete the table for the radioactive isotope.**

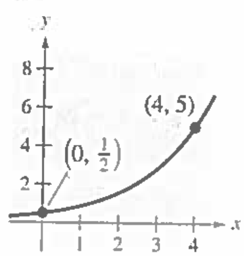
	Isotope	*Half-life (years)*	*Initial Quantity*	*Amount After 1000 Years*
25.	^{226}Ra	1620	10 g	
26.	^{226}Ra	1620		1.5 g
27.	^{14}C	5730		2 g
28.	^{14}C	5730	3 g	
29.	^{239}Pu	24,360		2.1 g
30.	^{239}Pu	24,360		0.4 g

In Exercises 31–34, find the exponential model $y = ae^{bx}$ that fits the points in the graph or table.

31.



32.



33.

x	0	4
y	5	1

34.

x	0	3
y	1	$\frac{1}{4}$

35. ***Population*** The population P of Texas (in thousands) from 1991 through 2000 can be modeled by

$$P = 16{,}968e^{0.019t}$$

where $t = 1$ represents the year 1991. According to this model, when will the population reach 22 million? (Source: U.S. Census Bureau)

▶ Model It

36. ***Population*** The table shows the populations (in millions) of five countries in 2000 and the projected populations (in millions) for the year 2010. (Source: U.S. Census Bureau)

Country	2000	2010
Canada	31.3	34.3
China	1261.8	1359.1
Italy	57.6	57.4
United Kingdom	59.5	60.6
United States	275.6	300.1

(a) Find the exponential growth or decay model $y = ae^{bt}$ or $y = ae^{-bt}$ for the population in each country by letting $t = 0$ correspond to 2000. Use the model to predict the population of each country in 2030.

(b) You can see that the populations of the United States and the United Kingdom are growing at different rates. What constant in the equation $y = ae^{bt}$ is determined by these different growth rates? Discuss the relationship between the different growth rates and the magnitude of the constant.

(c) You can see that the population of China is increasing while the population of Italy is decreasing. What constant in the equation $y = ae^{bt}$ reflects this difference? Explain.

37. ***Population*** The population P of Charlotte, North Carolina (in thousands) is

$$P = 548e^{kt}$$

where $t = 0$ represents the year 2000. In 1970, the population was 241,000. Find the value of k, and use this result to predict the population in the year 2010. (Source: U.S. Census Bureau)

38. ***Population*** The population P of Lincoln, Nebraska (in thousands) is

$$P = 224e^{kt}$$

where $t = 0$ represents the year 2000. In 1980, the population was 172,000. Find the value of k, and use this result to predict the population in the year 2020. (Source: U.S. Census Bureau)

39. ***Bacteria Growth*** The number N of bacteria in a culture is modeled by

$$N = 100e^{kt}$$

where t is the time in hours. If $N = 300$ when $t = 5$, estimate the time required for the population to double in size.

40. ***Bacteria Growth*** The number N of bacteria in a culture is modeled by $N = 250e^{kt}$, where t is the time in hours. If $N = 280$ when $t = 10$, estimate the time required for the population to double in size.

41. ***Radioactive Decay*** The half-life of radioactive radium (^{226}Ra) is 1620 years. What percent of a present amount of radioactive radium will remain after 100 years?

42. ***Radioactive Decay*** Carbon 14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal if the half-life of ^{14}C is 5730 years?

43. ***Depreciation*** A car that cost \$22,000 new has a book value of \$13,000 after 2 years.
   (a) Find the straight-line model $V = mt + b$.
   (b) Find the exponential model $V = ae^{kt}$.
   (c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first 2 years?
   (d) Find the book values of the car after 1 year and after 3 years using each model.
   (e) Interpret the slope of the straight-line model.

44. ***Depreciation*** A computer that costs \$2000 new has a book value of \$500 after 2 years.
   (a) Find the straight-line model $V = mt + b$.
   (b) Find the exponential model $V = ae^{kt}$.
   X (c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first 2 years?
   (d) Find the book values of the computer after 1 year and after 3 years using each model.
   (e) Interpret the slope of the straight-line model.

45. ***Sales*** The sales S (in thousands of units) of a new CD burner after it has been on the market t years are modeled by $S(t) = 100(1 - e^{kt})$. Fifteen thousand units of the new product were sold the first year.
   (a) Complete the model by solving for k.
   (b) Sketch the graph of the model.
   (c) Use the model to estimate the number of units sold after 5 years.

46. ***Sales*** After discontinuing all advertising for a tool kit in 1998, the manufacturer noted that sales began to drop according to the model

$$S = \frac{500,000}{1 + 0.6e^{kt}}$$

where S represents the number of units sold and $t = 0$ represents 1998. In 2000, the company sold 300,000 units.
   (a) Complete the model by solving for k.
   (b) Estimate sales in 2005.

47. ***Sales*** The sales S (in thousands of units) of a cleaning solution after x hundred dollars is spent on advertising are modeled by $S = 10(1 - e^{kx})$. When \$500 is spent on advertising, 2500 units are sold.
   (a) Complete the model by solving for k.
   (b) Estimate the number of units that will be sold if advertising expenditures are raised to \$700.

48. ***Profit*** Because of a slump in the economy, a department store finds that its annual profits have dropped from \$742,000 in 2000 to \$632,000 in 2002. The profit follows an exponential pattern of decline. What is the expected profit for 2005? (Let $t = 0$ represent 2000.)

49. ***Learning Curve*** The management at a plastics factory has found that the maximum number of units a worker can produce in a day is 30. The learning curve for the number N of units produced per day after a new employee has worked t days is

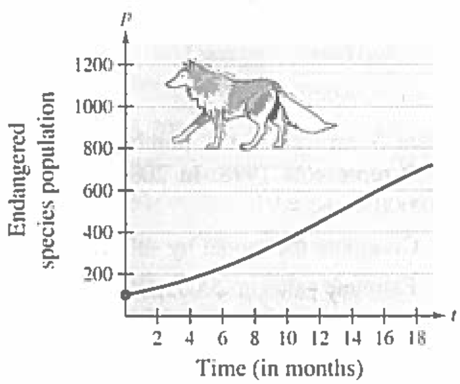
$$N = 30(1 - e^{kt}).$$

After 20 days on the job, a new employee produces 19 units.
   (a) Find the learning curve for this employee (first, find the value of k).
   (b) How many days should pass before this employee is producing 25 units per day?

50. *Population Growth* A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will be modeled by the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where t is measured in months (see figure).



(a) Estimate the population after 5 months.

(b) After how many months will the population be 500?

(c) Use a graphing utility to graph the function. Use the graph to determine the horizontal asymptotes, and interpret the meaning of the larger p-value in the context of the problem.

Geology **In Exercises 51 and 52, use the Richter scale for measuring the magnitudes of earthquakes.**

51. Find the magnitude R of an earthquake of intensity I (let $I_0 = 1$).

(a) $I = 80{,}500{,}000$

(b) $I = 48{,}275{,}000$

(c) $I = 251{,}200$

52. Find the intensity I of an earthquake measuring R on the Richter scale (let $I_0 = 1$).

(a) Chile in 1906, $R = 8.2$

(b) Los Angeles in 1971, $R = 6.7$

(c) India in 2001, $R = 7.7$

Intensity of Sound **In Exercises 53–56, use the following information for determining sound intensity. The level of sound β, in decibels, with an intensity of I is**

$$\beta = 10 \log_{10} \frac{I}{I_0}$$

where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 53 and 54, find the level of sound, β.

53. (a) $I = 10^{-10}$ watt per m^2 (faint whisper)

(b) $I = 10^{-5}$ watt per m^2 (busy street corner)

(c) $I = 10^{-2.5}$ watt per m^2 (air hammer)

(d) $I = 10^{0}$ watt per m^2 (threshold of pain)

54. (a) $I = 10^{-9}$ watt per m^2 (whisper)

(b) $I = 10^{-3.5}$ watt per m^2 (jet 4 miles from takeoff)

(c) $I = 10^{-3}$ watt per m^2 (diesel truck at 25 feet)

(d) $I = 10^{-0.5}$ watt per m^2 (auto horn at 3 feet)

55. Due to the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

56. Due to the installation of a muffler, the noise level of an engine was reduced from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise as a result of the installation of the muffler.

pH Levels **In Exercises 57–62, use the acidity model given by $\text{pH} = -\log_{10}[H^+]$, where acidity (pH) is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution.**

57. Find the pH if $[H^+] = 2.3 \times 10^{-5}$.

58. Find the pH if $[H^+] = 11.3 \times 10^{-6}$.

59. Compute $[H^+]$ for a solution in which pH = 5.8.

60. Compute $[H^+]$ for a solution in which pH = 3.2.

61. A fruit has a pH of 2.5 and an antacid tablet has a pH of 9.5. The hydrogen ion concentration of the fruit is how many times the concentration of the tablet?

62. The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

63. ***Forensics*** At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperatures, the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time in hours elapsed since the person died and T is the temperature (in degrees Fahrenheit) of the person's body. Assume that the person had a normal body temperature of 98.6°F at death, and that the room temperature was a constant 70°F. (This formula is derived from a general cooling principle called Newton's Law of Cooling.) Use the formula to estimate the time of death of the person.

64. ***Home Mortgage*** A \$120,000 home mortgage for 35 years at $7\frac{1}{2}\%$ has a monthly payment of \$809.39. Part of the monthly payment goes for the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that goes toward the interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward the reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the size of the mortgage, r is the interest rate, M is the monthly payment, and t is the time in years.

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 35 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$966.71$). What can you conclude?

65. ***Home Mortgage*** The total interest u paid on a home mortgage of P dollars at interest rate r for t years is

$$u = P\left[\frac{rt}{1 - \left(\dfrac{1}{1 + r/12}\right)^{12t}} - 1\right].$$

Consider a \$120,000 home mortgage at $7\frac{1}{2}\%$.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage for which the total interest paid is the same as the size of the mortgage. Is it possible that some people are paying twice as much in interest charges as the size of the mortgage?

66. ***Data Analysis*** The table shows the time t (in seconds) required to attain a speed of s miles per hour from a standing start for a car.

Speed, s	Time, t
30	3.4
40	5.0
50	7.0
60	9.3
70	12.0
80	15.8
90	20.0

Two models for this data are as follows.

$$t_1 = 40.757 + 0.556s - 15.817 \ln s$$

$$t_2 = 1.2259 + 0.0023s^2$$

(a) Use a graphing utility to fit a linear model t_3 and an exponential model t_4 to the data.

(b) Use a graphing utility to graph the data points and each model in the same viewing window.

(c) Create a table comparing the data with estimates obtained from each model.

(d) Use the results of part (c) to find the sum of the absolute values of the differences between the data and estimated values given by each model. Based on the four sums, which model do you think better fits the data? Explain.

Synthesis

True or False? **In Exercises 67–70, determine whether the statement is true or false. Justify your answer.**

67. The domain of a logistic growth function cannot be the set of real numbers.

68. A logistic growth function will always have an x-intercept.

69. The graph of

$$f(x) = \frac{4}{1 + 6e^{-2x}} + 5$$

is the graph of

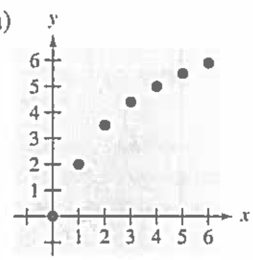
$$g(x) = \frac{4}{1 + 6e^{-2x}}$$

shifted to the right five units.

70. The graph of a Gaussian model will never have an x-intercept.

71. Identify each model as linear, logarithmic, exponential, logistic, or none of the above. Explain your reasoning.

(a)



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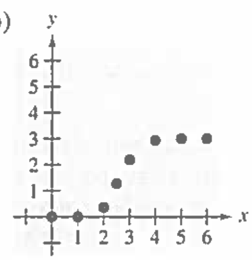

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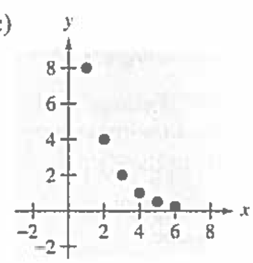

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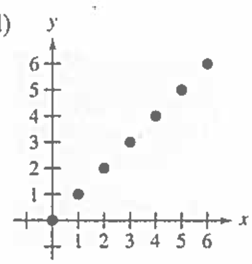

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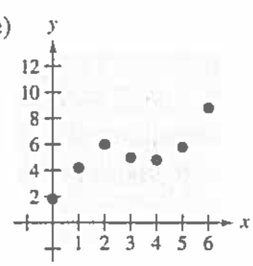
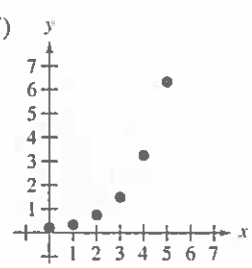


(f)



72. ***Writing*** Use your school's library, the Internet, or some other reference source to write a paper describing John Napier's work with logarithms.

Review

In Exercises 73–76, determine the right-hand and left-hand behavior of the polynomial function.

73. $f(x) = 2x^3 - 3x^2 + x - 1$

74. $f(x) = 5 - x^2 - 4x^4$

75. $g(x) = -1.6x^5 + 4x^2 - 2$

76. $g(x) = 7x^6 + 9.1x^5 - 3.2x^4 + 25x^3$

In Exercises 77–80, divide using synthetic division.

77. $\dfrac{4x^3 + 4x^2 - 39x + 36}{x + 4}$

78. $\dfrac{8x^3 - 36x^2 + 54x - 27}{x - \frac{3}{2}}$

79. $(2x^3 - 8x^2 + 3x - 9) \div (x - 4)$

80. $(x^4 - 3x + 1) \div (x + 5)$

In Exercises 81–90, sketch the graph of the equation.

81. $y = 10 - 3x$

82. $y = -4x - 1$

83. $y = -2x^2 - 3$

84. $y = 2x^2 - 7x - 30$

85. $3x^2 - 4y = 0$

86. $-x^2 - 8y = 0$

87. $y = \dfrac{4}{1 - 3x}$

88. $y = \dfrac{x^2}{-x - 2}$

89. $x^2 + (y - 8)^2 = 25$

90. $(x - 4)^2 + (y + 7) = 4$

In Exercises 91–94, graph the exponential function.

91. $f(x) = 2^{x-1} + 5$

92. $f(x) = -2^{-x-1} - 1$

93. $f(x) = 3^x - 4$

94. $f(x) = -3^x + 4$

Chapter Summary

▶ *What* did you learn?

	Review Exercises
Section 3.1	
☐ How to recognize and evaluate exponential functions with base *a*	1–10
☐ How to graph exponential functions	11–22, 27–30
☐ How to recognize and evaluate exponential functions with base *e*	23–26
☐ How to use exponential functions to model and solve real-life applications	31–36
Section 3.2	
☐ How to recognize and evaluate logarithmic functions with base *a*	37–42
☐ How to graph logarithmic functions	43–48, 55–58
☐ How to recognize and evaluate natural logarithmic functions	49–54
☐ How to use logarithmic functions to model and solve real-life applications	59, 60
Section 3.3	
☐ How to rewrite logarithmic functions with a different base	61–64
☐ How to use properties of logarithms to evaluate or rewrite and expand or condense logarithmic expressions	65–80
☐ How to use logarithmic functions to model and solve real-life applications	81, 82
Section 3.4	
☐ How to solve simple exponential and logarithmic equations	83–92
☐ How to solve more complicated exponential equations	93–106
☐ How to solve more complicated logarithmic equations	107–122
☐ How to use exponential and logarithmic equations to model and solve real-life applications	123, 124
Section 3.5	
☐ How to recognize the five most common types of models involving exponential and logarithmic functions	125–130
☐ How to use exponential growth and decay functions to model and solve real-life problems	131–136
☐ How to use Gaussian functions to model and solve real-life problems	137
☐ How to use logistic growth functions to model and solve real-life problems	138
☐ How to use logarithmic functions to model and solve real-life problems	139, 140

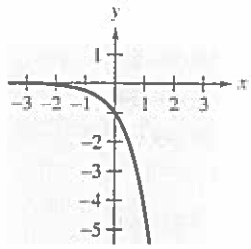
Review Exercises

3.1 **In Exercises 1–6, evaluate the expression. Approximate your result to three decimal places.**

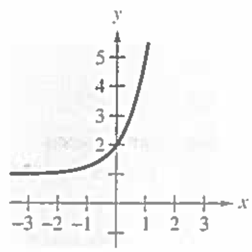
1. $(6.1)^{2.4}$
2. $-14(5^{-0.8})$
3. $2^{-0.5\pi}$
4. $\sqrt[5]{1278}$
5. $60^{\sqrt{3}}$
6. $7^{-\sqrt{11}}$

In Exercises 7–10, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

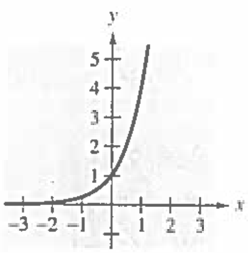
(a)



(b)



(c)



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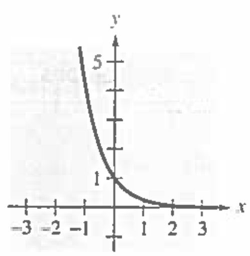


7. $f(x) = 4^x$
8. $f(x) = 4^{-x}$
9. $f(x) = -4^x$
10. $f(x) = 4^x + 1$

In Exercises 11–14, use the graph of f to describe the transformation that yields the graph of g.

11. $f(x) = 5^x, \quad g(x) = 5^{x-1}$
12. $f(x) = 4^x, \quad g(x) = 4^x - 3$
13. $f(x) = \left(\frac{1}{2}\right)^x, \quad g(x) = -\left(\frac{1}{2}\right)^{x+2}$
14. $f(x) = \left(\frac{2}{3}\right)^x, \quad g(x) = 8 - \left(\frac{2}{3}\right)^x$

In Exercises 15–22, use a graphing utility to construct a table of values. Then sketch the graph of the function.

15. $f(x) = 4^{-x} + 4$
16. $f(x) = -4^x - 3$
17. $f(x) = -2.65^{x+1}$
18. $f(x) = 2.65^{x-1}$
19. $f(x) = 5^{x-2} + 4$
20. $f(x) = 2^{x-6} - 5$
21. $f(x) = \left(\frac{1}{2}\right)^{-x} + 3$
22. $f(x) = \left(\frac{1}{8}\right)^{x+2} - 5$

In Exercises 23–26, evaluate the function $f(x) = e^x$ for the indicated value of x. Approximate your result to three decimal places.

23. $x = 8$
24. $x = \dfrac{5}{8}$
25. $x = -1.7$
26. $x = 0.278$

In Exercises 27–30, use a graphing utility to construct a table of values. Then sketch the graph of the function.

27. $h(x) = e^{-x/2}$
28. $h(x) = 2 - e^{-x/2}$
29. $f(x) = e^{x+2}$
30. $s(t) = 4e^{-2/t}, \quad t > 0$

Compound Interest **In Exercises 31 and 32, complete the table to determine the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous
A						

31. $P = \$3500, \ r = 6.5\%, \ t = 10$ years
32. $P = \$2000, \ r = 5\%, \ t = 30$ years
33. ***Waiting Times*** The average time between incoming calls at a switchboard is 3 minutes. The probability F of waiting less than t minutes until the next incoming call is approximated by the model $F(t) = 1 - e^{-t/3}$. A call has just come in. Find the probability that the next call will be within
    (a) $\frac{1}{2}$ minute. (b) 2 minutes. (c) 5 minutes.
34. ***Depreciation*** After t years, the value V of a car that cost \$14,000 is $V(t) = 14{,}000\left(\frac{3}{4}\right)^t$.
    (a) Use a graphing utility to graph the function.
    (b) Find the value of the car 2 years after it was purchased.
    (c) According to the model, when does the car depreciate most rapidly? Is this realistic? Explain.
35. ***Trust Fund*** On the day a person was born, a deposit of \$50,000 was made in a trust fund that pays 8.75% interest, compounded continuously.
    (a) Find the balance on the person's 35th birthday.
    (b) How much longer would the person have to wait to get twice as much?

9-19, 23-26, 35, 37-43, 46, 55, 56, 59 (in honor of ali Beiraes), 61, 64, 65-79 odds,
83-97 odds, 107-117 odds.

Party (T)

36. ***Radioactive Decay*** Let Q represent a mass of plutonium 241 (^{241}Pu) (in grams), whose half-life is 13 years. The quantity of plutonium 241 present after t years is

$$Q = 100\left(\frac{1}{2}\right)^{t/13}.$$

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 10 years.

(c) Sketch the graph of this function over the interval $t = 0$ to $t = 100$.

3.2 **In Exercises 37 and 38, write the exponential equation in logarithmic form.**

37. $4^3 = 64$

38. $25^{3/2} = 125$

In Exercises 39–42, evaluate the function at the indicated value of *x* without using a calculator.

Function	*Value*
39. $f(x) = \log_{10} x$	$x = 1000$
40. $g(x) = \log_9 x$	$x = 3$
41. $g(x) = \log_2 x$	$x = \frac{1}{8}$
42. $f(x) = \log_4 x$	$x = \frac{1}{4}$

In Exercises 43–48, find the domain, *x*-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

43. $g(x) = \log_7 x$

44. $g(x) = \log_5 x$

45. $f(x) = \log_{10}\left(\frac{x}{3}\right)$

46. $f(x) = 6 + \log_{10} x$

47. $f(x) = 4 - \log_{10}(x + 5)$

48. $f(x) = \log_{10}(x - 3) + 1$

In Exercises 49–54, use your calculator to evaluate the function $f(x) = \ln x$ for the indicated value of *x*. Approximate your result to three decimal places if necessary.

49. $x = 22.6$

50. $x = 0.98$

51. $x = e^{-12}$

52. $x = e^7$

53. $x = \sqrt{7} + 5$

54. $x = \dfrac{\sqrt{3}}{8}$

In Exercises 55–58, find the domain, *x*-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

55. $f(x) = \ln x + 3$

56. $f(x) = \ln(x - 3)$

57. $h(x) = \ln(x^2)$

58. $f(x) = \frac{1}{4} \ln x$

59. ***Antler Spread*** The antler spread a (in inches) and shoulder height h (in inches) of an adult male American elk are related by the model

$$h = 116 \log_{10}(a + 40) - 176.$$

Approximate the shoulder height of a male American elk with an antler spread of 55 inches.

60. ***Snow Removal*** The number of miles s of roads cleared of snow is approximated by the model

$$s = 25 - \frac{13 \ln(h/12)}{\ln 3}, \qquad 2 \le h \le 15$$

where h is the depth of the snow in inches. Use this model to find s when $h = 10$ inches.

3.3 **In Exercises 61–64, evaluate the logarithm using the change-of-base formula. Do each problem twice, once with common logarithms and once with natural logarithms. Approximate the results to three decimal places.**

61. $\log_4 9$

62. $\log_{12} 200$

63. $\log_{1/2} 5$

64. $\log_3 0.28$

In Exercises 65–72, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms.

65. $\log_5 5x^2$

66. $\log_{10} 7x^4$

67. $\log_3 \dfrac{6}{\sqrt[3]{x}}$

68. $\log_7 \dfrac{\sqrt{x}}{4}$

69. $\ln x^2y^2z$

70. $\ln 3xy^2$

71. $\ln\left(\dfrac{x + 3}{xy}\right)$

72. $\ln\left(\dfrac{y - 1}{4}\right)^2, \quad y > 1$

In Exercises 73–80, condense the expression to the logarithm of a single quantity.

73. $\log_2 5 + \log_2 x$

74. $\log_6 y - 2 \log_6 z$

75. $\ln x - \frac{1}{4} \ln y$

76. $3 \ln x + 2 \ln(x + 1)$

77. $\frac{1}{3} \log_8(x + 4) + 7 \log_8 y$

78. $-2 \log_{10} x - 5 \log_{10}(x + 6)$

79. $\frac{1}{2} \ln(2x - 1) - 2 \ln(x + 1)$

80. $5 \ln(x - 2) - \ln(x + 2) - 3 \ln x$

81. ***Climb Rate*** The time t (in minutes) for a small plane to climb to an altitude of h feet is modeled by

$$t = 50 \log_{10} \frac{18{,}000}{18{,}000 - h}$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the time function and identify any asymptotes.

(c) As the plane approaches its absolute ceiling, what can be said about the time required to increase its altitude further?

(d) Find the time for the plane to climb to an altitude of 4000 feet.

82. ***Human Memory Model*** Students in a sociology class were given an exam and then were retested monthly with an equivalent exam. The average score for the class was given by the human memory model

$$f(t) = 85 - 14 \log_{10}(t + 1), \quad 0 \le t \le 4$$

where t is the time in months. How did the average score change over the four-month period?

3.4 **In Exercises 83–92, solve for x.**

83. $8^x = 512$	**84.** $3^x = 729$
85. $6^x = \frac{1}{216}$	**86.** $5^x = \frac{1}{25}$
87. $e^x = 3$	**88.** $e^x = 6$
89. $\log_4 x = 2$	**90.** $\log_6 x = -1$
91. $\ln x = 4$	**92.** $\ln x = -3$

In Exercises 93–102, solve the exponential equation. Approximate your result to three decimal places.

93. $e^x = 12$	**94.** $e^{3x} = 25$
95. $3e^{-5x} = 132$	**96.** $14e^{3x+2} = 560$
97. $2^x + 13 = 35$	**98.** $6^x - 28 = -8$
99. $-4(5^x) = -68$	**100.** $2(12^x) = 190$
101. $e^{2x} - 7e^x + 10 = 0$	**102.** $e^{2x} - 6e^x + 8 = 0$

In Exercises 103–106, use a graphing utility to graph and solve the equation. Approximate the result to two decimal places.

103. $2^{0.6x} - 3x = 0$	**104.** $4^{-0.2x} + x = 0$
105. $25e^{-0.3x} = 12$	**106.** $4e^{1.2x} = 9$

In Exercises 107–118, solve the logarithmic equation. Approximate the result to three decimal places.

107. $\ln 3x = 8.2$	**108.** $\ln 5x = 7.2$
109. $2 \ln 4x = 15$	**110.** $4 \ln 3x = 15$
111. $\ln x - \ln 3 = 2$	**112.** $\ln \sqrt{x + 8} = 3$
113. $\ln \sqrt{x + 1} = 2$	**114.** $\ln x - \ln 5 = 4$

115. $\log_{10}(x - 1) = \log_{10}(x - 2) - \log_{10}(x + 2)$

116. $\log_{10}(x + 2) - \log_{10} x = \log_{10}(x + 5)$

117. $\log_{10}(1 - x) = -1$

118. $\log_{10}(-x - 4) = 2$

In Exercises 119–122, use a graphing utility to graph and solve the equation. Approximate the result to two decimal places.

119. $2 \ln(x + 3) + 3x = 8$

120. $6 \log_{10}(x^2 + 1) - x = 0$

121. $4 \ln(x + 5) - x = 10$

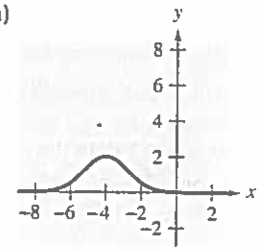
122. $x - 2 \log_{10}(x + 4) = 0$

123. ***Compound Interest*** \$7550 is deposited in an account that pays 7.25% interest, compounded continuously. How long will it take the money to triple?

124. ***Demand*** The demand equation for a 32-inch television is modeled by $p = 500 - 0.5e^{0.004x}$. Find the demand x for a price of (a) $p = \$450$ and (b) $p = \$400$.

3.5 **In Exercises 125–130, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

(a)



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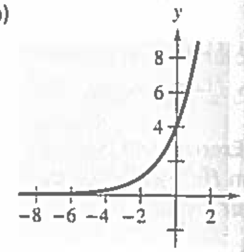

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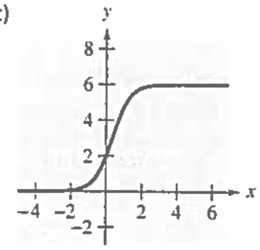

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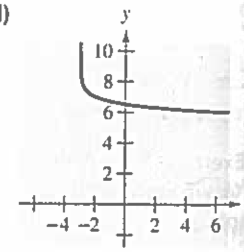

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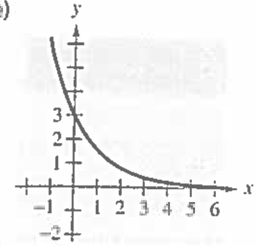
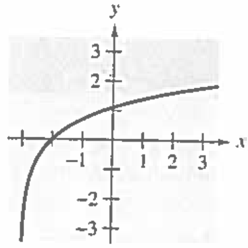


(f)



125. $y = 3e^{-2x/3}$

126. $y = 4e^{2x/3}$

127. $y = \ln(x + 3)$

128. $y = 7 - \log_{10}(x + 3)$

129. $y = 2e^{-(x+4)^2/3}$

130. $y = \dfrac{6}{1 + 2e^{-2x}}$

131. ***Population*** The population P of Phoenix, Arizona (in thousands) from 1970 through 2000 can be modeled by $P = 590e^{0.027t}$, where t represents the year, with $t = 0$ corresponding to 1970. According to this model, when will the population reach 1.5 million? (Source: U.S. Census Bureau)

132. ***Radioactive Decay*** The half-life of radioactive uranium II (^{234}U) is 250,000 years. What percent of a present amount of radioactive uranium II will remain after 5000 years?

133. ***Compound Interest*** A deposit of $10,000 is made in a savings account for which the interest is compounded continuously. The balance will double in 5 years.

(a) What is the annual interest rate for this account?

(b) Find the balance after 1 year.

134. ***Bacteria Growth*** The number N of bacteria in a culture is given by the model $N = 200e^{kt}$, where t is the time in hours. If $N = 350$ when $t = 5$, estimate the time required for the population to triple in size.

In Exercises 135 and 136, find the exponential function $y = ae^{bx}$ that passes through the points.

135. $(0, 2), (4, 3)$

136. $\left(0, \frac{1}{2}\right), (5, 5)$

137. ***Test Scores*** The test scores for a biology test follow a normal distribution modeled by

$$y = 0.0499e^{-(x-71)^2/128}, \quad 40 \le x \le 100$$

where x is the test score.

(a) Use a graphing utility to graph the equation.

(b) From the graph, estimate the average test score.

138. ***Typing Speed*** In a typing class, the average number of words per minute typed after t weeks of lessons was found to be

$$N = \frac{157}{1 + 5.4e^{-0.12t}}.$$

Find the time necessary to type (a) 50 words per minute and (b) 75 words per minute.

139. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per square centimeter is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right).$$

Determine the intensity of a sound in watts per square centimeter if the decibel level is 125.

140. ***Geology*** On the Richter scale, the magnitude R of an earthquake of intensity I is

$$R = \log_{10}\frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity per unit of area for each value of R.

(a) $R = 8.4$ (b) $R = 6.85$ (c) $R = 9.1$

Synthesis

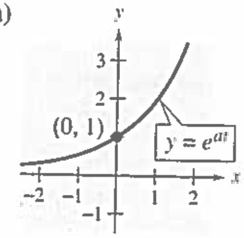
True or False? **In Exercises 141 and 142, determine whether the equation or statement is true or false. Justify your answer.**

141. $\log_b b^{2x} = 2x$

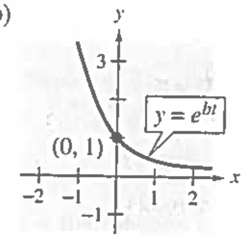
142. $\ln(x + y) = \ln x + \ln y$

143. The graphs of $y = e^{kt}$ are shown for $k = a, b, c,$ and d. Use the graphs to order $a, b, c,$ and d. Which of the four values are negative? Which are positive?

(a)



(b)



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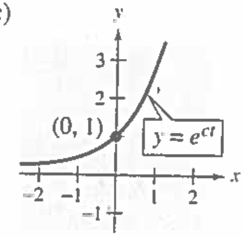

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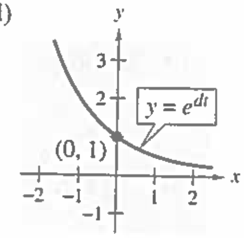


Chapter Test

The *Interactive* CD-ROM and *Internet* versions of this text offer Chapter Pre-Tests and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, evaluate the expression. Approximate your result to three decimal places.

1. $12.4^{2.79}$ **2.** $4^{3\pi/2}$ **3.** $e^{-7/10}$ **4.** $e^{3.1}$

In Exercises 5–7, construct a table of values. Then sketch the graph of the function.

5. $f(x) = 10^{-x}$ **6.** $f(x) = -6^{x-2}$ **7.** $f(x) = 1 - e^{2x}$

8. Evaluate (a) $\log_7 7^{-0.89}$ and (b) $4.6 \ln e^2$.

In Exercises 9–11, construct a table of values. Then sketch the graph of the function. Identify any asymptotes.

9. $f(x) = -\log_{10} x - 6$ **10.** $f(x) = \ln(x - 4)$ **11.** $f(x) = 1 + \ln(x + 6)$

In Exercises 12–14, evaluate the expression. Approximate your result to three decimal places.

12. $\log_7 44$ **13.** $\log_{2/5} 0.9$ **14.** $\log_{24} 68$

In Exercises 15 and 16, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms.

15. $\log_2 3a^4$ **16.** $\ln \dfrac{5\sqrt{x}}{6}$

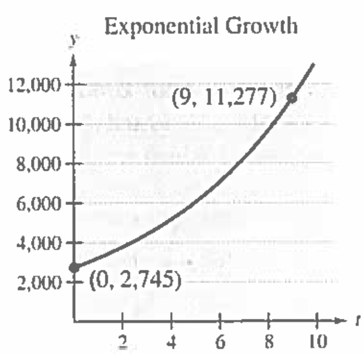


FIGURE FOR 21

In Exercises 17 and 18, condense the expression to the logarithm of a single quantity.

17. $\log_3 13 + \log_3 y$ **18.** $4 \ln x - 4 \ln y$

In Exercises 19 and 20, solve the equation algebraically. Approximate your result to three decimal places.

19. $\dfrac{1025}{8 + e^{4x}} = 5$ **20.** $\log_{10} x - \log_{10}(8 - 5x) = 2$

21. Find an exponential growth model for the graph shown in the figure.

22. The half-life of radioactive actinium (^{227}Ac) is 22 years. What percent of a present amount of radioactive actinium will remain after 19 years?

23. A model that can be used for predicting the height H (in centimeters) of a child based on his or her age is $H = 70.228 + 5.104x + 9.222 \ln x$, $\frac{1}{4} \le x \le 6$, where x is the age of the child in years. (Source: Snapshots of Applications in Mathematics)

(a) Construct a table of values. Then sketch the graph of the model.

(b) Use the graph from part (a) to estimate the height of a four-year-old child. Then calculate the actual height using the model.

Chapter 3

Section 3.1 (page 206)

1. 946.852 3. 0.006 5. 0.472

7. d 8. c 9. a 10. b

11. Shift the graph of four units to the right.

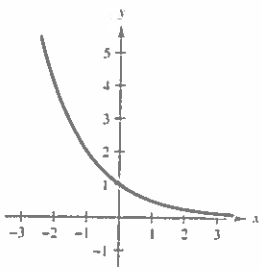
13. Shift the graph of five units upward.

15. Reflect f in the x-axis and shift four units to the left.

17. Reflect f in the x-axis and shift five units upward.

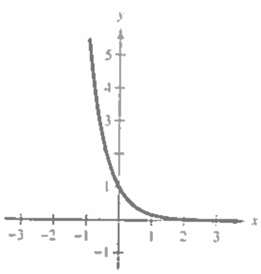
19.

x	-2	-1	0	1	2
$f(x)$	4	2	1	0.5	0.25



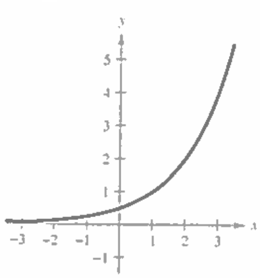
21.

x	-2	-1	0	1	2
$f(x)$	36	6	1	0.167	0.028



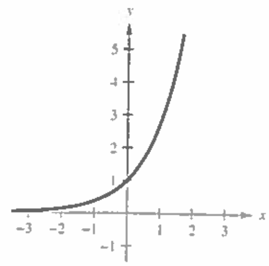
23.

x	-2	-1	0	1	2
$f(x)$	0.125	0.25	0.5	1	2



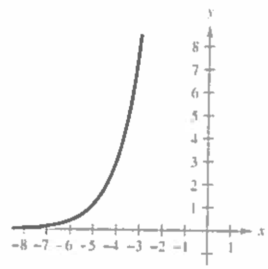
25.

x	-2	-1	0	1	2
$f(x)$	0.135	0.368	1	2.718	7.389



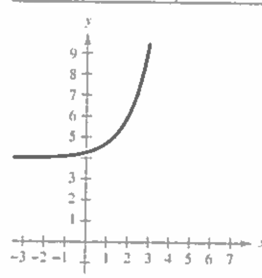
27.

x	-8	-7	-6	-5	-4
$f(x)$	0.055	0.149	0.406	1.104	3



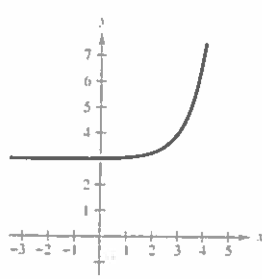
29.

x	-2	-1	0	1	2
$f(x)$	4.037	4.100	4.271	4.736	6



31.

x	-1	0	1	2	3
$f(x)$	3.004	3.016	3.063	3.25	4



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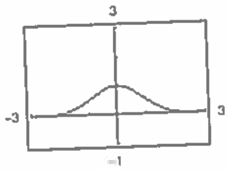

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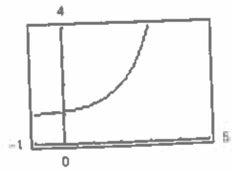

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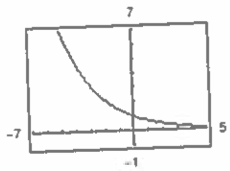
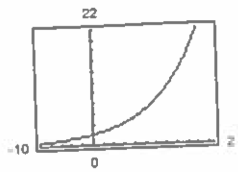
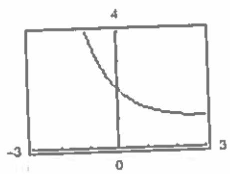


39.



41.



43.

n	1	2	4
A	\$5397.31	\$5477.81	\$5520.10

n	12	365	Continuous
A	\$5549.10	\$5563.36	\$5563.85

45.

n	1	2	4
A	\$11,652.39	\$12,002.55	\$12,188.60

n	12	365	Continuous
A	\$12,317.01	\$12,380.41	\$12,382.58

47.

t	10	20
A	\$26,706.49	\$59,436.39

t	30	40	50
A	\$132,278.12	\$294,390.36	\$655,177.80

49.

t	10	20
A	\$22,986.49	\$44,031.56

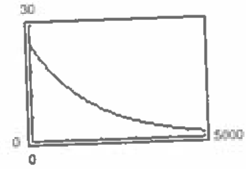
t	30	40	50
A	\$84,344.25	\$161,564.86	\$309,484.08

51. \$222,822.57

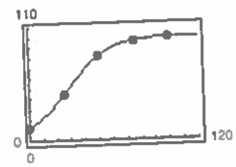
53. \$35.45 **55.** (a) 100 (b) 300 (c) 900

57. (a) 25 grams (b) 16.30 grams

(c)



59. (a)



(b)

x	0	25	50	75	100
y	13	45	82	96	99

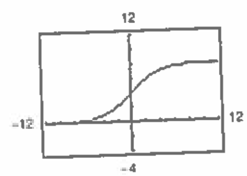
(c) 63.1% (d) 38.2

61. True. As $x \to -\infty$, $f(x) \to -2$ but never reaches -2.

63. $f(x) = h(x)$ **65.** $f(x) = g(x) = h(x)$

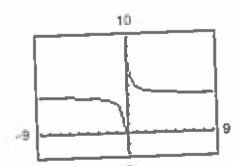
67. (a) $x < 0$ (b) $x > 0$

69. (a)



Horizontal asymptotes: $y = 0$, $y = 8$

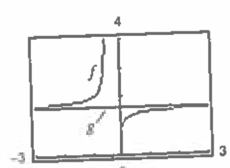
(b)



Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 0$

71.

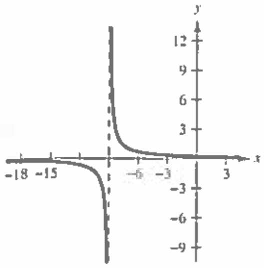


As $x \to \infty$, $f(x) \to g(x)$.

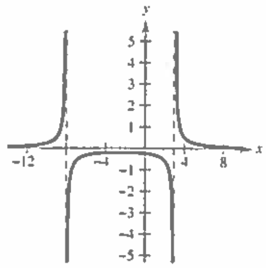
As $x \to -\infty$, $f(x) \to g(x)$.

73. c, d **75.** $y = \frac{1}{7}(2x + 14)$ **77.** $y = \pm\sqrt{25 - x^2}$

79.



81.



Section 3.2 *(page 216)*

1. $4^3 = 64$ **3.** $7^{-2} = \frac{1}{49}$ **5.** $32^{2/5} = 4$

7. $e^{-0.693\ldots} = \frac{1}{2}$ **9.** $\log_5 125 = 3$ **11.** $\log_{81} 3 = \frac{1}{4}$

13. $\log_6 \frac{1}{36} = -2$ **15.** $\ln 20.0855\ldots = 3$

17. $\ln 4 = x$ **19.** 4 **21.** 0 **23.** 3 **25.** 2

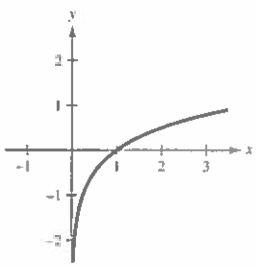
27. -0.097 **29.** 2.913 **31.** -0.575 **33.** c

34. f **35.** d **36.** e **37.** b **38.** a

39. Domain: $(0, \infty)$

x-intercept: $(1, 0)$

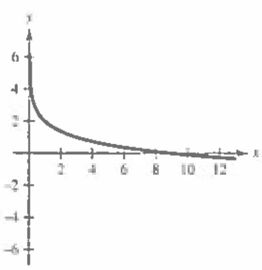
Vertical asymptote: $x = 0$



41. Domain: $(0, \infty)$

x-intercept: $(9, 0)$

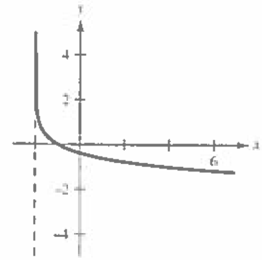
Vertical asymptote: $x = 0$



43. Domain: $(-2, \infty)$

x-intercept: $(-1, 0)$

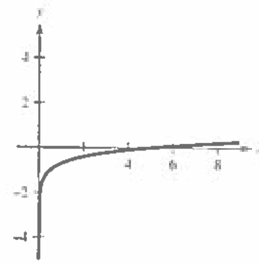
Vertical asymptote: $x = -2$



45. Domain: $(0, \infty)$

x-intercept: $(5, 0)$

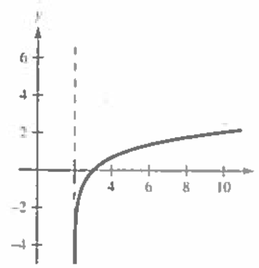
Vertical asymptote: $x = 0$



47. Domain: $(2, \infty)$

x-intercept: $(3, 0)$

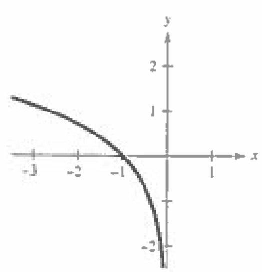
Vertical asymptote: $x = 2$



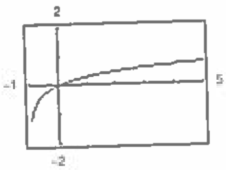
49. Domain: $(-\infty, 0)$

x-intercept: $(-1, 0)$

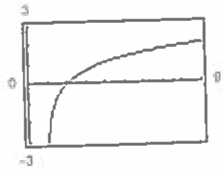
Vertical asymptote: $x = 0$



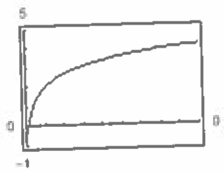
51.



53.



55.

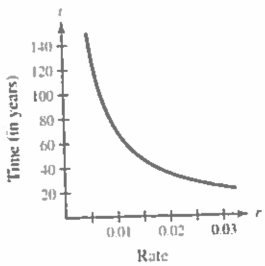


57. (a) 30 years; 20 years (b) \$396,234; \$301,123.20
(c) \$246,234; \$151,123.20
(d) $x = 1000$; The monthly payment must be greater than \$1000.

59. (a)

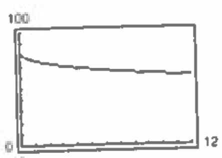
r	0.005	0.01	0.015	0.02	0.025	0.03
t	138.6	69.3	46.2	34.7	27.7	23.1

(b)



(c) Answers will vary.

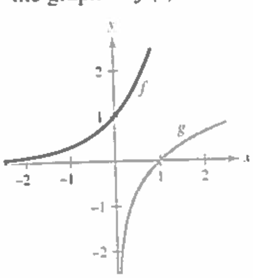
61. (a)



(b) 80 (c) 68.1 (d) 62.3

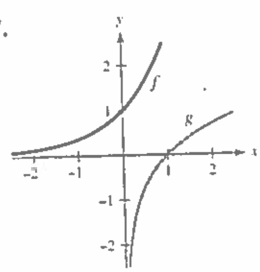
63. False. Reflecting $g(x)$ about the line $y = x$ will determine the graph of $f(x)$.

65.



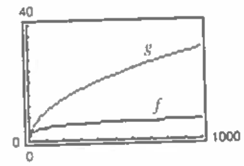
$g = f^{-1}$

67.



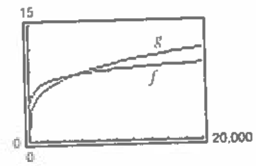
$g = f^{-1}$

69. (a)



$g(x)$: The natural log function grows at a slower rate than the square root function.

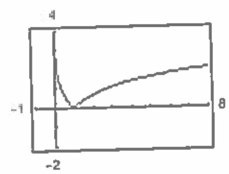
(b)



$g(x)$: The natural log function grows at a slower rate than the fourth root function.

71. $(0, \infty)$ 73. $3 < x < 4$

75. (a)



(b) Increasing: $(1, \infty)$
Decreasing: $(0, 1)$
(c) Relative minimum: $(1, 0)$

77. $83.95 + 37.50t$

Section 3.3 *(page 223)*

1. (a) $\dfrac{\log_{10} x}{\log_{10} 5}$ (b) $\dfrac{\ln x}{\ln 5}$ 3. (a) $\dfrac{\log_{10} x}{\log_{10} \frac{1}{5}}$ (b) $\dfrac{\ln x}{\ln \frac{1}{5}}$

5. (a) $\dfrac{\log_{10} \frac{3}{10}}{\log_{10} x}$ (b) $\dfrac{\ln \frac{3}{10}}{\ln x}$ 7. (a) $\dfrac{\log_{10} x}{\log_{10} 2.6}$ (b) $\dfrac{\ln x}{\ln 2.6}$

9. 1.771 11. −2.000 13. −0.417 15. 2.633

17. $\log_4 5 + \log_4 x$ 19. $4 \log_8 x$ 21. $1 - \log_5 x$

23. $\frac{1}{2} \ln z$ 25. $\ln x + \ln y + 2 \ln z$

27. $\ln z + 2 \ln(z - 1)$ 29. $\frac{1}{2} \log_2(a - 1) - \log_2 9$

31. $\frac{1}{3} \ln x - \frac{1}{3} \ln y$ 33. $4 \ln x + \frac{1}{2} \ln y - 5 \ln z$

35. $2 \log_5 x - 2 \log_5 y - 3 \log_5 z$

37. $\frac{3}{4} \ln x + \frac{1}{4} \ln(x^2 + 3)$ 39. $\ln 3x$ 41. $\log_4 \dfrac{z}{y}$

43. $\log_2(x + 4)^2$ 45. $\log_3 \sqrt[4]{5x}$

47. $\ln \dfrac{x}{(x + 1)^3}$ 49. $\log_{10} \dfrac{xz^3}{y^2}$ 51. $\ln \dfrac{x}{(x^2 - 4)^4}$

53. $\ln \sqrt[3]{\dfrac{x(x + 3)^2}{x^2 - 1}}$ 55. $\log_8 \dfrac{\sqrt[3]{y(y + 4)^2}}{y - 1}$

57. $\log_2 \frac{32}{4} = \log_2 32 - \log_2 4$; Property 2 **59.** 2

61. $\frac{3}{4}$ **63.** 2.4 **65.** -9 is not in the domain of $\log_3 x$.

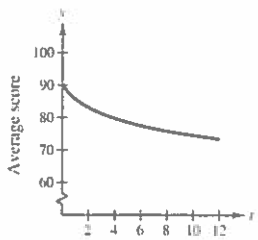
67. 4.5 **69.** $-\frac{1}{2}$ **71.** 7 **73.** 2

75. $\frac{3}{2}$ **77.** $-3 - \log_5 2$ **79.** $6 + \ln 5$

81. (a) 90 (b) 77 (c) 73 (d) 9 months

(e) $90 - \log_{10}(t + 1)^{15}$

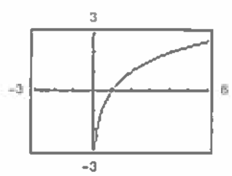
(f)

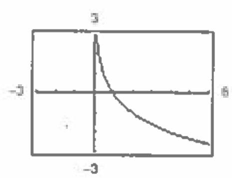


83. False. $\ln 1 = 0$ **85.** False. $\ln(x - 2) \neq \ln x - \ln 2$

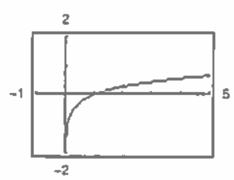
87. False. $u = v^2$ **89.** Answers will vary.

91. $f(x) = \dfrac{\log_{10} x}{\log_{10} 2} = \dfrac{\ln x}{\ln 2}$ **93.** $f(x) = \dfrac{\log_{10} x}{\log_{10} \frac{1}{2}} = \dfrac{\ln x}{\ln \frac{1}{2}}$

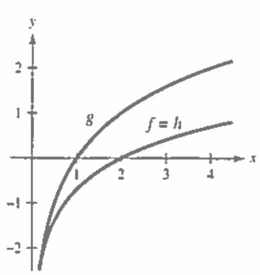




95. $f(x) = \dfrac{\log_{10} x}{\log_{10} 11.8} = \dfrac{\ln x}{\ln 11.8}$



97. $f(x) = h(x)$; Property 2



99. $\dfrac{3x^4}{2y^3}$, $x \neq 0$ **101.** 1, $x \neq 0$, $y \neq 0$

103. $-1, \dfrac{1}{3}$ **105.** $\dfrac{-1 \pm \sqrt{97}}{6}$

Section 3.4 *(page 232)*

1. (a) Yes (b) No

3. (a) No (b) Yes (c) Yes, approximate

5. (a) Yes, approximate (b) No (c) Yes **7.** 2

9. 4 **11.** -2 **13.** -5 **15.** 3 **17.** 2

19. $\ln 2 \approx 0.693$ **21.** $e^{-1} \approx 0.368$ **23.** 64

25. $\frac{1}{10}$ **27.** (3, 8) **29.** (9, 2)

31. $\dfrac{\ln 5}{\ln 3} \approx 1.465$ **33.** $\ln 5 \approx 1.609$

35. $\ln 28 \approx 3.332$ **37.** $\dfrac{\ln 80}{2 \ln 3} \approx 1.994$ **39.** 2

41. 4 **43.** $3 - \dfrac{\ln 565}{\ln 2} \approx -6.142$

45. $\dfrac{1}{3}\log_{10}\left(\dfrac{3}{2}\right) \approx 0.059$ **47.** $1 + \dfrac{\ln 7}{\ln 5} \approx 2.209$

49. $\dfrac{\ln 12}{3} \approx 0.828$ **51.** $-\ln\dfrac{3}{5} \approx 0.511$ **53.** 0

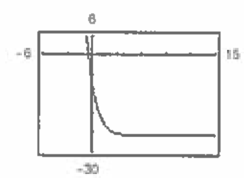
55. $\dfrac{\ln \frac{8}{3}}{3 \ln 2} + \dfrac{1}{3} \approx 0.805$ **57.** $\ln 5 \approx 1.609$

59. $\ln 4 \approx 1.386$ **61.** $2 \ln 75 \approx 8.635$

63. $\frac{1}{2} \ln 1498 \approx 3.656$ **65.** $\dfrac{\ln 4}{365 \ln\left(1 + \frac{0.065}{365}\right)} \approx 21.330$

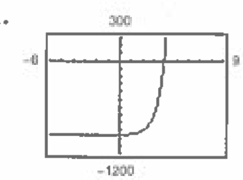
67. $\dfrac{\ln 2}{12 \ln\left(1 + \frac{0.10}{12}\right)} \approx 6.960$

69.



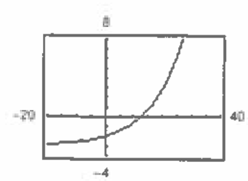
-0.427

71.



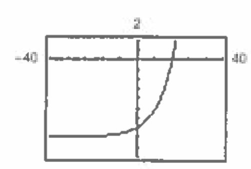
3.847

73.



12.207

75.



16.636

77. $e^{-3} \approx 0.050$ **79.** $\dfrac{e^{2.4}}{2} \approx 5.512$ **81.** 1,000,000

83. $2(3^{11/6}) \approx 14.988$ **85.** $\dfrac{e^{10/3}}{5} \approx 5.606$

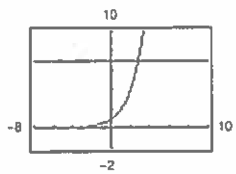
87. $e^2 - 2 \approx 5.389$ **89.** $e^{-2/3} \approx 0.513$

91. No solution **93.** $1 + \sqrt{1+e} \approx 2.928$

95. No solution **97.** 7 **99.** $\dfrac{-1+\sqrt{17}}{2} \approx 1.562$

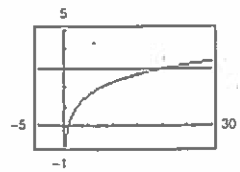
101. 2 **103.** $\dfrac{725 + 125\sqrt{33}}{8} \approx 180.384$

105.



2.807

107.

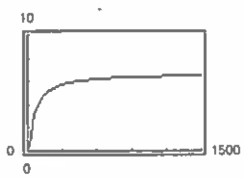


20.086

109. 8.2 years **111.** 12.9 years

113. (a) 1426 units (b) 1498 units

115. (a)



(b) $V = 6.7$; the yield will approach 6.7 million cubic feet per acre.

(c) 29.3 years

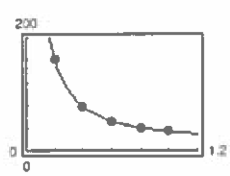
117. (a) $y = 100$ and $y = 0$; The range falls between 0% and 100%.

(b) Males: 69.71 inches Females: 64.51 inches

119. (a)

x	0.2	0.4	0.6	0.8	1.0
y	162.6	78.5	52.5	40.5	33.9

(b)



The model appears to fit the data well.

(c) 1.2 meters

(d) No. According to the model, when the number of g's is less than 23, x is between 2.276 meters and 4.404 meters, which isn't realistic in most vehicles.

121. $\log_b uv = \log_b u + \log_b v$

True by Property 1 in Section 3.3.

123. $\log_b(u - v) = \log_b u - \log_b v$

False.

$1.95 \approx \log_{10}(100 - 10) \neq \log_{10} 100 - \log_{10} 10 = 1$

125. Yes. See Exercise 95.

127. Yes. Time to double: $t = \dfrac{\ln 2}{r}$;

Time to quadruple: $t = \dfrac{\ln 4}{r} = 2\left(\dfrac{\ln 2}{r}\right)$

129. $4|x|y^2\sqrt{3y}$ **131.** $5\sqrt[3]{3}$ **133.** $M = kp^3$

135. $d = kab$ **137.** 1.226 **139.** -5.595

Section 3.5 *(page 243)*

1. c **2.** e **3.** b

4. a **5.** d **6.** f

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 years*
7.	\$1000	12%	5.78 yr	\$3320.12
9.	\$750	8.9438%	7.75 yr	\$1834.37
11.	\$500	11.0%	6.3 yr	\$1505.00
13.	\$6376.28	4.5%	15.4 yr	\$10,000.00

15. \$112,087.09

17. (a) 6.642 years (b) 6.330 years

(c) 6.302 years (d) 6.301 years

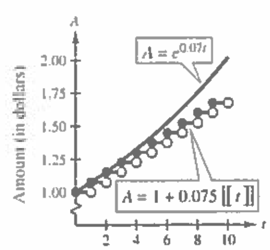
19.

r	2%	4%	6%	8%	10%	12%
t	54.93	27.47	18.31	13.73	10.99	9.16

21.

r	2%	4%	6%	8%	10%	12%
t	55.48	28.01	18.85	14.27	11.53	9.69

23.



Continuous compounding

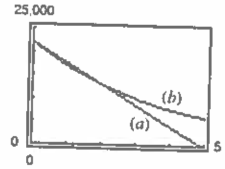
	Isotope	*Half-life (years)*	*Initial Quantity*	*Amount After 1000 Years*
25.	^{226}Ra	1620	10 g	6.52 g
27.	^{14}C	5730	2.26 g	2 g
29.	^{239}Pu	24,360	2.16 g	2.1 g

31. $y = e^{0.7675x}$ **33.** $y = 5e^{-0.4024x}$ **35.** 2003

37. $k = 0.0274$; 720,738 **39.** 3.15 hours **41.** 95.8%

43. (a) $V = -4500t + 22{,}000$ (b) $V = 22{,}000e^{-0.263t}$

(c)



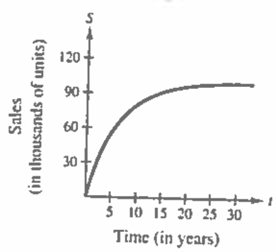
Exponential

(d) 1 year: Straight-line, $17,500;
Exponential, $16,912
3 years: Straight-line, $8500;
Exponential, $9995

(e) The value decreases $4500 per year.

45. (a) $S(t) = 100(1 - e^{-0.1625t})$

(b)



(c) 55,625

47. (a) $S = 10(1 - e^{-0.0575x})$ (b) 3314 units

49. (a) $N = 30(1 - e^{-0.050t})$ (b) 36 days

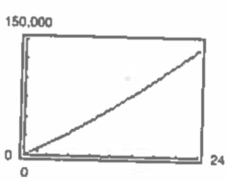
51. (a) 7.91 (b) 7.68 (c) 5.40

53. (a) 20 decibels (b) 70 decibels
(c) 95 decibels (d) 120 decibels

55. 95% 57. 4.64 59. 1.58×10^{-6} moles per liter

61. 10^7 63. 3:00 A.M.

65. (a)



(b) ≈ 21 years; Yes

67. False. The domain can be the set of real numbers for a logistic growth function.

69. False. The graph of $f(x)$ is the graph of $g(x)$ shifted upward five units.

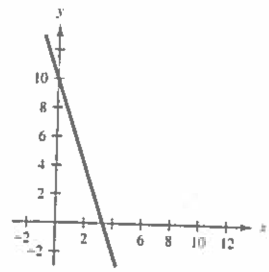
71. (a) Logarithmic (b) Logistic (c) Exponential
(d) Linear (e) None of the above (f) Exponential

73. Rises to the right.
Falls to the left.

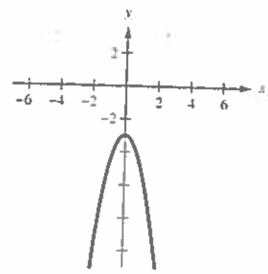
75. Rises to the left.
Falls to the right.

77. $4x^2 - 12x + 9$ 79. $2x^2 + 3 + \dfrac{3}{x - 4}$

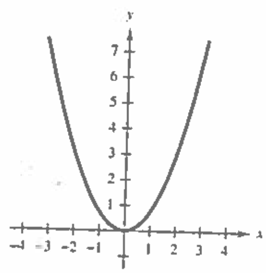
81.



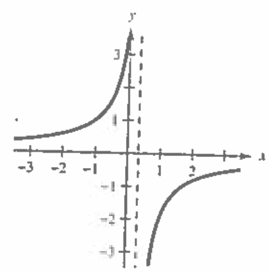
83.



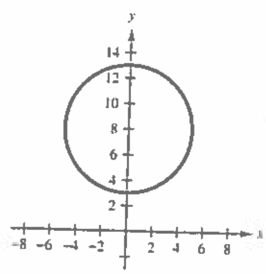
85.



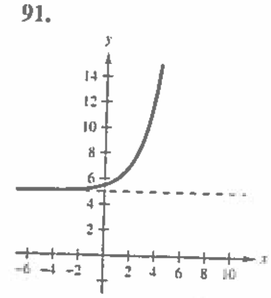
87.



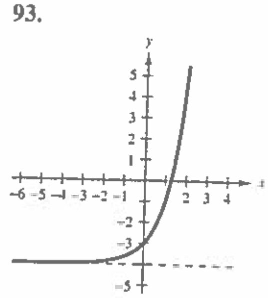
89.



91.



93.



Review Exercises *(page 250)*

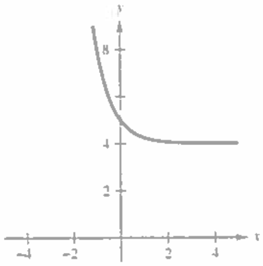
1. 76.699 3. 0.337 5. 1201.845 7. c
8. d 9. a 10. b

11. Shift the graph of one unit to the right.

13. Reflect f in the x-axis and shift two units to the left.

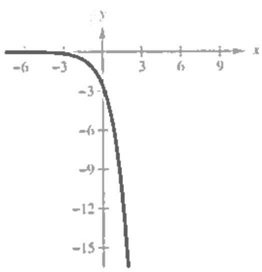
15.

x	-1	0	1	2	3
$f(x)$	8	5	4.25	4.063	4.016



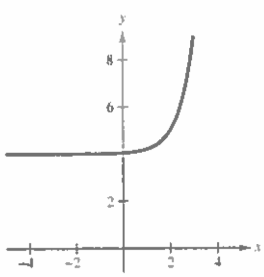
17.

x	-2	-1	0	1	2
$f(x)$	-0.377	-1	-2.65	-7.023	-18.61



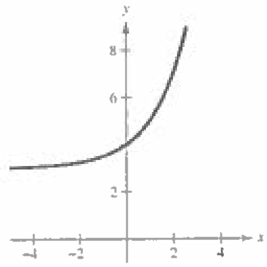
19.

x	-1	0	1	2	3
$f(x)$	4.008	4.04	4.2	5	9



21.

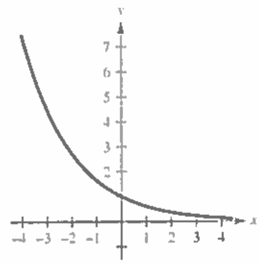
x	-2	-1	0	1	2
$f(x)$	3.25	3.5	4	5	7



23. 2980.958 **25.** 0.183

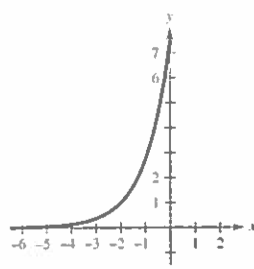
27.

x	-2	-1	0	1	2
$h(x)$	2.72	1.65	1	0.61	0.37



29.

x	-3	-2	-1	0	1
$f(x)$	0.37	1	2.72	7.39	20.09



31.

n	1	2	4	12
A	\$6569.98	\$6635.43	\$6669.46	\$6692.64

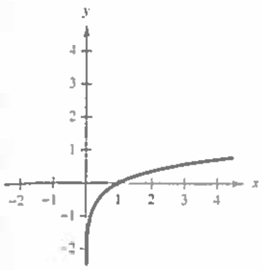
n	365	Continuous
A	\$6704.00	\$6704.39

33. (a) 0.154 (b) 0.487 (c) 0.811

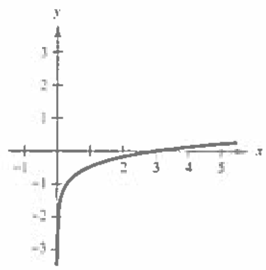
35. (a) \$1,069,047.14 (b) 7.9 years

37. $\log_4 64 = 3$ **39.** 3 **41.** -3

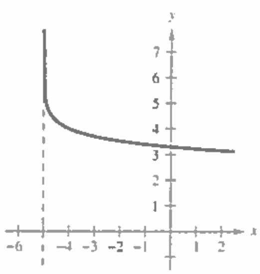
43. Domain: $(0, \infty)$
x-intercept: $(1, 0)$
Vertical asymptote: $x = 0$



45. Domain: $(0, \infty)$
x-intercept: $(3, 0)$
Vertical asymptote: $x = 0$

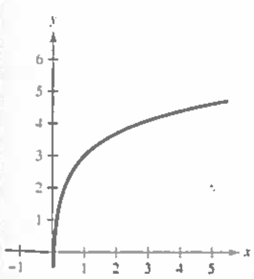


47. Domain: $(-5, \infty)$
x-intercept: $(9995, 0)$
Vertical asymptote: $x = -5$

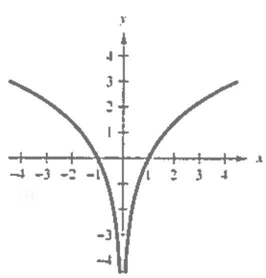


49. 3.118 **51.** -12 **53.** 2.034

55. Domain: $(0, \infty)$
x-intercept: $(e^{-3}, 0)$
Vertical asymptote: $x = 0$



57. Domain: $(-\infty, 0), (0, \infty)$
x-intercept: $(\pm 1, 0)$
Vertical asymptote: $x = 0$



59. 53.4 inches **61.** 1.585 **63.** -2.322

65. $1 + 2\log_5 |x|$ **67.** $1 + \log_3 2 - \frac{1}{3}\log_3 x$

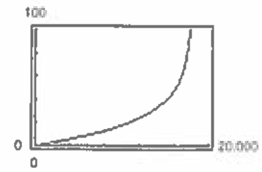
69. $2\ln x + 2\ln y + \ln z$ **71.** $\ln(x + 3) - \ln x - \ln y$

73. $\log_2 5x$ **75.** $\ln \dfrac{x}{\sqrt[4]{y}}$ **77.** $\log_8 y^7 \sqrt[3]{x + 4}$

79. $\ln \dfrac{\sqrt{|2x - 1|}}{(x + 1)^2}$

81. (a) $0 \le h < 18{,}000$

(b)



Vertical asymptote: $h = 18{,}000$

(c) The plane is climbing at a slower rate, so the time required increases.

(d) 5.46 minutes

83. 3 **85.** -3 **87.** $\ln 3 \approx 1.099$ **89.** 16

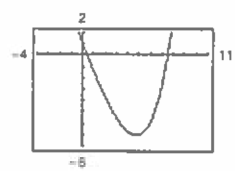
91. $e^4 \approx 54.598$

93. $\ln 12 \approx 2.485$ **95.** $-\dfrac{\ln 44}{5} \approx -0.757$

97. $\dfrac{\ln 22}{\ln 2} \approx 4.459$ **99.** $\dfrac{\ln 17}{\ln 5} \approx 1.760$

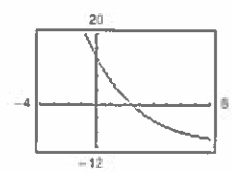
101. $\ln 2 \approx 0.693$, $\ln 5 \approx 1.609$

103.



0.39, 7.48

105.



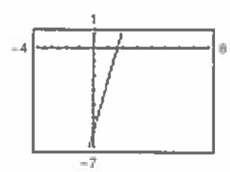
2.45

107. $\frac{1}{3}e^{8.2} \approx 1213.650$ **109.** $\frac{1}{4}e^{7.5} \approx 452.011$

111. $3e^2 \approx 22.167$ **113.** $e^4 - 1 \approx 53.598$

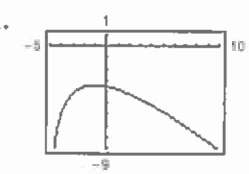
115. No solution **117.** 0.900

119.



1.64

121.



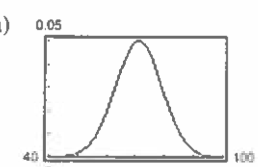
No solution

123. 15.2 years **125.** e **126.** b **127.** f

128. d **129.** a **130.** c **131.** 2004

133. (a) 13.8629% (b) \$11,486.98 **135.** $y = 2e^{0.1014x}$

137. (a)



(b) 71

139. $10^{-3.5}$ watt per square centimeter

141. True by the inverse properties

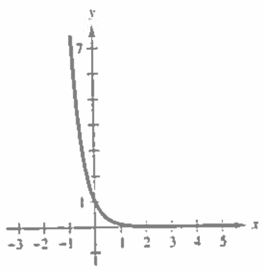
143. $b < d < a < c$
b and d are negative.
a and c are positive.

Chapter Test *(page 254)*

1. 1123.690 **2.** 687.291 **3.** 0.497 **4.** 22.198

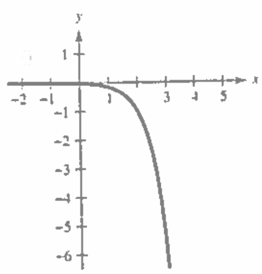
5.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$	10	3.162	1	0.316	0.1



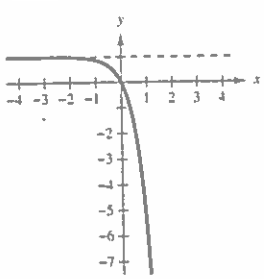
6.

x	-1	0	1	2	3
$f(x)$	-0.005	-0.028	-0.167	-1	-6



7.

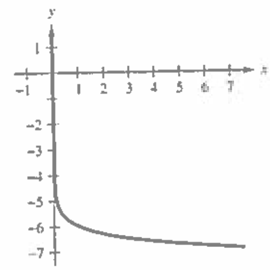
x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$	0.865	0.632	0	-1.718	-6.389



8. (a) -0.89 (b) 9.2

9.

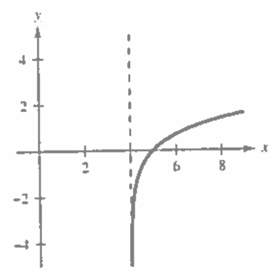
x	$\frac{1}{2}$	1	$\frac{3}{2}$	2	4
$f(x)$	-5.699	-6	-6.176	-6.301	-6.602



Vertical asymptote: $x = 0$

10.

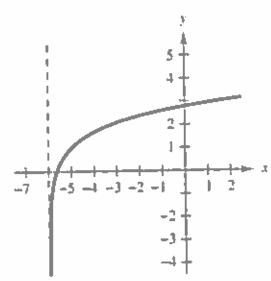
x	5	7	9	11	13
$f(x)$	0	1.099	1.609	1.946	2.197



Vertical asymptote: $x = 4$

11.

x	-5	-3	-1	0	1
$f(x)$	1	2.099	2.609	2.792	2.946



Vertical asymptote: $x = -6$

12. 1.945 **13.** 0.115 **14.** 1.328

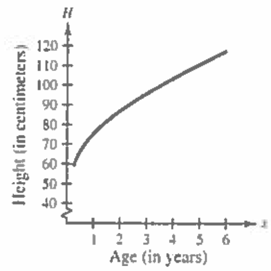
15. $\log_2 3 + 4\log_2|a|$ **16.** $\ln 5 + \frac{1}{2}\ln x - \ln 6$

17. $\log_3 13y$ **18.** $\ln \frac{x^4}{y^4}$ **19.** $\frac{\ln 197}{4} \approx 1.321$

20. $\frac{800}{501} \approx 1.597$ **21.** $y = 2745e^{0.1570t}$ **22.** 55%

23. (a)

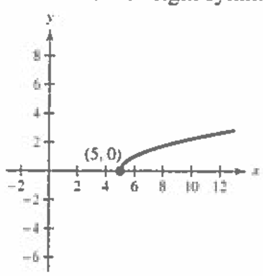
x	$\frac{1}{4}$	1	2	4	5	6
H	58.720	75.332	86.828	103.43	110.59	117.38



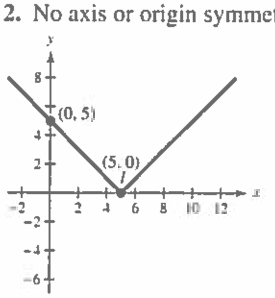
(b) 103 centimeters; 103.43 centimeters

Cumulative Test for Chapters 1–3
(page 255)

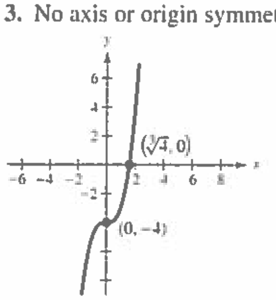
1. No axis or origin symmetry



2. No axis or origin symmetry



3. No axis or origin symmetry



4. $2x - y + 2 = 0$

5. For some values of x there correspond two values of y.

6. (a) $\dfrac{3}{2}$ (b) Division by 0 is undefined. (c) $\dfrac{s+2}{s}$

7. (a) Vertical shrink by $\frac{1}{2}$
   (b) Vertical shift of two units upward
   (c) Horizontal shift of two units to the left

8. (a) $5x - 2$ (b) $-3x - 4$ (c) $4x^2 - 11x - 3$
   (d) $\dfrac{x-3}{4x+1}$; Domain: all real numbers $x \neq -\dfrac{1}{4}$

9. (a) $\sqrt{x-1} + x^2 + 1$ (b) $\sqrt{x-1} - x^2 - 1$
   (c) $x^2\sqrt{x-1} + \sqrt{x-1}$ (d) $\dfrac{\sqrt{x-1}}{x^2+1}$; Domain: $x \geq 1$

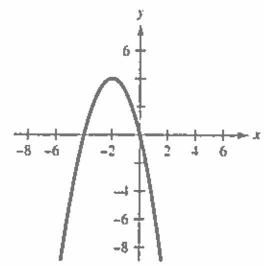
10. (a) $2x + 12$ (b) $\sqrt{2x^2 + 6}$

11. (a) $|x| - 2$ (b) $|x - 2|$

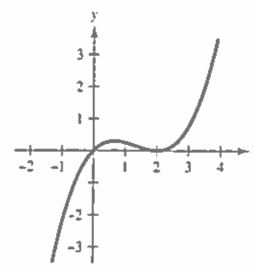
12. $h^{-1}(x) = \frac{1}{5}(x^2 + 3), x \geq 0$ 13. 2438.65 kilowatts

14. $y = -\frac{3}{4}(x + 8)^2 + 5$

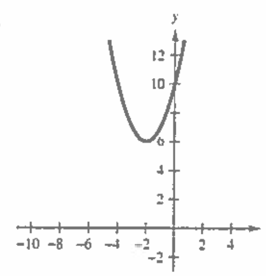
15.



16.



17.



18. Zeros: $-2, \pm 2i$; $f(x) = (x + 2)(x - 2i)(x + 2i)$

19. Zeros: $5, \frac{7}{2}, -\frac{4}{3}$; $f(x) = (x - 5)(2x - 7)(3x + 4)$

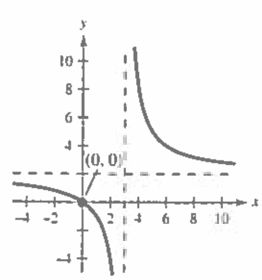
20. Zeros: $4, -\frac{1}{2}, 1 \pm 3i$;
    $f(x) = (x - 4)(2x + 1)(x - 1 - 3i)(x - 1 + 3i)$

21. $3x - 2 - \dfrac{3x - 2}{2x^2 + 1}$

22. $2x^3 - x^2 + 2x - 10 + \dfrac{25}{x+2}$

23. $[1, 2] \approx 1.20$ 24. $x^4 + 3x^3 - 11x^2 + 9x + 70$

25.



Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 2$