Homework #11 MATH270

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1. Suppose that $A \subseteq B \subseteq C$, $A \approx C$, and C is countable. Is $A \approx B$?

   Proof. Yes. If C is finite, then $|A| \leq |B| \leq |C|$. But since $A \approx C$, then $|A| = |C|$. This would force B to be equal cardinality as A and C, hence $|A| = |B|$, or $A \approx B$.

   If C were to be countable infinite then $C \approx \mathbb{N}$. We also know $A \approx C$, which lets us conclude $A \approx \mathbb{N}$. Since $A \subseteq B \subseteq C$, $A \approx C$, we know $|A| \leq |B| \leq |C|$. From the just previous conclusion we find $|\mathbb{N}| \leq |B| \leq |\mathbb{N}|$. This forces B to have the same cardinality as $\mathbb{N}$, hence $B \approx \mathbb{N}$. Therefore $A \approx B$. $\square$

2. Prove that set A is uncountable if there is an injective function $f : (0, 1) \to A$.

   Proof. Since there is an injective function $f : (0, 1) \to A$, we know the cardinalty of $(0, 1)$ is less than or equal to A. Hence, since $(0, 1)$ is uncountable (because it is an interval over $\mathbb{R}$), A must be uncountable. $\square$

3. Let X and Y be two nonempty finite sets. Let $F(X, Y)$ denote the set of all function from X to Y. Is this set finite, countably infinite, or uncountable?

   Proof. There are a finite number of functions. For every element in X, it can be sent to any element in Y. Hence, every element in X has $|Y|$ possibles. Therefore there would be $|Y|^{|X|}$ potential functions. This number would be finite as both X and Y are finite. $\square$

4. Prove that the set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

   Proof. Assume that $\mathbb{R} \setminus \mathbb{Q}$ is countable. We know $\mathbb{Q}$ is countable. This would imply $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$ is countable. This is false however, so $\mathbb{R} \setminus \mathbb{Q}$ must be uncountable. $\square$

5. Prove that if $A \approx B$ then $\mathscr{P}(A) \approx \mathscr{P}(B)$.

   Proof. Since $A \approx B$, then $|A| = |B|$. Therefore have an bijective function $f : A \to B$, where $f(a) = b$. We can construct another function $F : \mathscr{P}(A) \to \mathscr{P}(B)$, where $F(A_0) = \{f(a) : a \in A_0\}$. We can show function F is injective. If $F(A_1) = F(A_2)$, then $\{f(a_1) : a_1 \in A_1\} = \{f(a_2) : a_2 \in A_2\}$. Since f is a bijection, that would imply $\{a_1 : a_1 \in A_1\} = \{a_2 : a_2 \in A_2\}$, or $A_1 = A_2$. This shows that F is injective, since the same argument could be shown for $F : \mathscr{P}(B) \to \mathscr{P}(A)$ we have an injection both ways. Hence $\mathscr{P}(A) \approx \mathscr{P}(B)$. $\square$

   $\Phi = \vec{E} \cdot \vec{A}$