Homework #5 MATH270

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Wednesday, February 27th

1. The theorem is correct and properly proved directly. Here is an alternate proof.

   Theorem 1. *For any sets A, B, and C, if $A \setminus B \subseteq C$ and $A \not\subseteq C$ then $A \cap B \neq \emptyset$.*

   Proof. Consider the contrapositive *if* $A \cap B = \emptyset$ *then* $A \setminus B \not\subseteq C$ *or* $A \subseteq C$. The consequence states $A \not\subseteq C$ or $A \subseteq C$, which is always true. Hence, the implication is true. $\square$

2. If $A_x = [-x, x]$, what is $\bigcap_{x\in\mathbb{R}^+} A_x$ and $\bigcup_{x\in\mathbb{R}^+} A_x$?

$$\bigcap_{x\in\mathbb{R}^+} A_x = \{0\}$$

   Proof. If $y \in \bigcap_{x\in\mathbb{R}^+} A_x$, then $y \in A_x$ for all $x \in \mathbb{R}^+$. Considering the interval $A_x = [-x, x]$, if we have some number $\epsilon \in \mathbb{R}^+$, we can always choose a smaller number $\frac{\epsilon}{2} \in \mathbb{R}^+$. Therefore, the only number to be in all intervals $[-x, x]$ is 0. And so $y \in \{0\}$, thus $\bigcap_{x\in\mathbb{R}^+} A_x \subseteq \{0\}$.

   Now let $y \in \{0\}$. For each $x \in \mathbb{R}^+$, we may write $-x \leq y \leq x$. And so $y \in A_x$ for all $x \in \mathbb{R}^+$. This shows $y \in \bigcap_{x\in\mathbb{R}^+} A_x$, hence $\{0\} \subseteq \bigcap_{x\in\mathbb{R}^+} A_x$.

   Combining the two implications yields the equality, $\bigcap_{x\in\mathbb{R}^+} A_x = \{0\}$. $\square$

$$\bigcup_{x\in\mathbb{R}^+} A_x = (-\infty, \infty)$$

   Proof. If $y \in \bigcup_{x\in\mathbb{R}^+} A_x$, then $y \in [-x, x]$ for some $x \in \mathbb{R}^+$. Since x can be arbitrarily large, $y \in (-\infty, \infty)$, therefore $\bigcup_{x\in\mathbb{R}^+} A_x \subseteq (-\infty, \infty)$.

   Now if $y \in (-\infty, \infty)$, then for some $x \in \mathbb{R}^+$ we may have $y \in [-x, x]$. So $y \in A_x$, hence $(-\infty, \infty) \subseteq \bigcup_{x\in\mathbb{R}^+} A_x$.

   These two arguments combined to form the equality $\bigcup_{x\in\mathbb{R}^+} A_x = (-\infty, \infty)$. $\square$

3. Prove the following set inclusion

$$\bigcup_{b\in\mathbb{R}^+}\{(x,y)\in\mathbb{R}^2 : x+y=b\}\subseteq\bigcap_{s\in\mathbb{R}^-}\{(x,y)\in\mathbb{R}^2 : x+y>s\}.$$

Proof. Let $z\in\bigcup_{b\in\mathbb{R}^+}\{(x,y)\in\mathbb{R}^2 : x+y=b\}$, then $z\in\{(x,y)\in\mathbb{R}^2 : x+y=b\}$ for some $b\in\mathbb{R}^+$. Now for all $s\in\mathbb{R}^-$ we have $z\in\{(x,y)\in\mathbb{R}^2 : x+y=b>s\}$, or $z\in\{(x,y)\in\mathbb{R}^2 : x+y>s\}$. Therefore $z\in\bigcap_{s\in\mathbb{R}^-}\{(x,y)\in\mathbb{R}^2 : x+y>s\}$ and the theorem is complete. □

4. (a) False, let $A=\emptyset$, $\{\emptyset\}\not\subseteq\emptyset$
(b) True, $\emptyset$ is a subset of any set. However, $\emptyset$ is never equal to $\mathscr{P}(A)$ because $\mathscr{P}(A)$ is a set that always contains at least one element (the element being $\emptyset$).
(c) False, $\{\emptyset\}\in\mathscr{P}(A)$, but $\{\emptyset\}\not\subseteq\{x,y\}$.
(d) False, $\mathscr{P}(\{\{x,y\}\})=\{\emptyset,\{\{x,y\}\}\}$
(e) False, let $A=\emptyset$, $\mathscr{P}(\emptyset)=\{\emptyset\},\{\emptyset\}\notin\{\emptyset\}$

5. Show that

$$\mathscr{P}(A)\cup\mathscr{P}(B)\subseteq\mathscr{P}(A\cup B), \tag{1}$$

but

$$\mathscr{P}(A)\cup\mathscr{P}(B)\neq\mathscr{P}(A\cup B). \tag{2}$$

Proof. Let $x\in\mathscr{P}(A)\cup\mathscr{P}(B)$, then $x\in\mathscr{P}(A)$ or $x\in\mathscr{P}(B)$. Assume that $x\in\mathscr{P}(A)$, then $x\subseteq A$. It follows that $x\subseteq A\cup B$, hence $x\in\mathscr{P}(A\cup B)$. Similarly, assume $x\in\mathscr{P}(B)$, then $x\subseteq B$. Now $x\subseteq A\cup B$, therefore $x\in\mathscr{P}(A\cup B)$. This proves (1).

Now we can show (2) by counter example. If $A=\{1\}$ and $B=\{2\}$, then $\mathscr{P}(A)\cup\mathscr{P}(B)=\{\emptyset,\{1\},\{2\}\}$ and $\mathscr{P}(A\cup B)=\{\emptyset,\{1\},\{2\},\{1,2\}\}$. This shows the two sets are not equivalent. □

6. Let $\{A_\alpha : \alpha\in I\}$ be a nonempty indexed collection of sets. Prove that

$$\mathscr{P}\left(\bigcap_{\alpha\in I}A_\alpha\right)=\bigcap_{\alpha\in I}\mathscr{P}(A_\alpha)$$

Proof. Let $x\in\mathscr{P}(\bigcap_{\alpha\in I}A_\alpha)$, then $x\subseteq\bigcap_{\alpha\in I}A_\alpha$. Let $z\in x$, then $z\in\bigcap_{\alpha\in I}A_\alpha$. By definition, $z\in A_\alpha$ for all $\alpha\in I$. Since $z\in x$ implied $z\in A_\alpha$, then $x\subseteq A_\alpha$. Therefore, $x\in\mathscr{P}(A_\alpha)$ for all $\alpha\in I$. Hence $x\in\bigcap_{\alpha\in I}\mathscr{P}(A_\alpha)$.

Now let $x\in\bigcap_{\alpha\in I}\mathscr{P}(A_\alpha)$. Therefore, $x\in\mathscr{P}(A_\alpha)$ for all $\alpha\in I$. Now, by definition, $x\subseteq A_\alpha$. If we let $z\in x$, then $z\in A_\alpha$, and therefore $z\in\bigcap_{\alpha\in I}A_\alpha$. Since $z\in x$ implied $z\in\bigcap_{\alpha\in I}A_\alpha$, then $x\subseteq\bigcap_{\alpha\in I}A_\alpha$. It follows $x\in\mathscr{P}(\bigcap_{\alpha\in I}A_\alpha)$.

Since containment was proved in both directions, the two sets are equivalent. □