Homework #6 MATH270

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1. (a) Prove $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

   Proof. Let $x \in A \times (B \cup C)$, then $x = (y, z)$, where $y \in A$ and $z \in B \cup C$. In other words, $y \in A$ and $z \in B$, or the case $y \in A$ and $z \in C$. Thus $x \in A \times B$ or $x \in A \times C$. Therefore $x \in (A \times B) \cup (A \times C)$.

   To prove containment in the other direction, let $x \in (A \times B) \cup (A \times C)$. Therefore $x \in A \times B$ or $x \in B \times C$. Then $x = (y, z)$, where $y \in A$ and $z \in B$ or the case $y \in A$ and $z \in B$. Hence $y \in A$ and $z \in B \cup C$. Therefore $x \in A \times (B \cup C)$.

   Since containment was shown in both directions, equality is proved. □

   (b) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   Proof. Let $x \in A \times (B \cap C)$. Then $x = (y, z)$ where $y \in A$ and $z \in B \cap C$. Hence $z \in B$ and $z \in C$. So it's the case $y \in A$ and $z \in B$, and the case $y \in A$ and $z \in C$. Therefore $x \in (A \times B) \cap (A \times C)$.

   Now let $x \in (A \times B) \cap (A \times C)$. So $x = (y, z)$ where the case $y \in A$ and $z \in B$ and the case $y \in A$ and $z \in C$. Hence, $z \in B \cap C$. Since $y \in A$ and $z \in B \cap C$, $x \in A \times (B \cap C)$.

   Containment was proved in both directions showing equality. □

2. On the paper stapled.

3. Let $X = \{1, 2, 3, 4, 5\}$. (a) define an equivalence relation. (b) define a reflexive, but not symmetric nor transitive relation. (c) define a symmetric, but not reflexive nor transitive relation. (d) define a transative, but not reflexive nor symmetric relation.
   (a) $x \sim y \iff T$
   (b) $x \sim y \iff x - y$ is odd and $x - y \geq 0$
   (c) $x \sim y \iff x \neq y$
   (d) $x \sim y \iff x \mid y$ and $x \neq y$

4. Define a relation $\mathbb{R}$ as follows: $x \sim y$ if and only if $x^2 - y^2 \in \mathbb{Z}$. Prove its and equivalence relation and give five different real numbers in the equivalence class $E_{\sqrt{2}}$.

Proof. To prove that it is an equivalence relation we will show it is reflexive, symmetric, and transitive.

Let $x \in \mathbb{R}$. Then $x^2 - x^2 = 0 \in \mathbb{Z}$. Hence $x \sim x$.

If $x \sim y$, then $x^2 - y^2 \in \mathbb{Z}$. Negating yields another integer $y^2 - x^2$. Since $y^2 - x^2 \in \mathbb{Z}$, $y \sim x$.

If $x \sim y$ and $y \sim z$, then $x^2 - y^2 \in \mathbb{Z}$ and $y^2 - z^2 \in \mathbb{Z}$. Adding $x^2 - y^2$ and $y^2 - z^2$ yields another integer $x^2 - z^2$. So $x^2 - z^2 \in \mathbb{Z}$, therefore $x \sim z$.

Hence, the relation is indeed equivalent. $\square$

Five real numbers in $E_{\sqrt{2}}$ include 0, 1, 2, 3, and 4.