Homework #7 MATH270

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Wednesday, March 27th

1. Are the following functions?

   (a) $f : \mathbb{R} \to \mathbb{R} = \{(x, y) : x^2 + y^2 = 4\}$

   No, if $x > 2$, then there is no corresponding y.

   (b) $f : \mathbb{R} \to \mathbb{R}, f(x) = 1/(x + 1)$

   No, f is not defined at $-1 \in \mathbb{R}$.

   (c) $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x + y$

   Yes.

   (i) for all $(x, y) \in \mathbb{R}^2$ there is an element $x + y \in \mathbb{R}$.

   (ii) for all $(x, y) \in \mathbb{R}^2$, if $f((x, y)) = a$ and $f((x, y)) = b$, then $x + y = a$ and $x + y = b$, hence $a = b$.

   (d) $f : \mathbb{Q} \to \mathbb{R}$,

$$f(x) = \begin{cases} x + 1, & x \in 2\mathbb{Z} \\ x - 1, & x \in 3\mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

   No, $(6, 7) \in f$ and $(6, 5) \in f$. Therefore failing the vertical line test.

   (e) The domain of f is the set of all circles in the plane $\mathbb{R}^2$, the codomain is $\mathbb{R}$, and if c is a circle in the domain, $f(c)$ is the circumference of c.

   Yes.

   (i) every circle in the domain has a corresponding radius $f(c)$.

   (ii) for every circle, if $f(a) = b$ and $f(a) = c$, then $b = c$ as the circumference of the same circle is the same.

2. For $x \in \mathbb{R}$ we define the greatest integer of x by $\lfloor x \rfloor = n$, where $n \in \mathbb{Z}$ and $n \leq x < n + 1$. The floor function $f : \mathbb{R} \to \mathbb{Z}$ is defined by $f(x) = \lfloor x \rfloor$.

   (a) Prove that f is a well defined function.

Proof. We must prove two conditions.

(i) for all $x \in \mathbb{R}$, there exists a $m \in \mathbb{Z}$ such that $m \leq x < m + 1$. There is always two integers surrounding a real number.

(ii) for all $a \in \mathbb{R}$, if $f(a) = n$ and $f(a) = m$, then $n = \lfloor a \rfloor$ and $m = \lfloor a \rfloor$, so $n = m$. Since the two conditions of a a function were met, f is a function. □

(b) Determine the range of f. The range is $\mathbb{Z}$.

Proof. First we will show $\operatorname{ran}(f) \subseteq \mathbb{Z}$. If $n \in \operatorname{ran}(f)$, then $n \in \mathbb{Z}$.

Now we will show $\mathbb{Z} \subseteq \operatorname{ran}(f)$. Let $n \in \mathbb{Z}$. Let $n \leq x < n+1$ where $x \in \mathbb{R}$. Hence $x \in \operatorname{dom}(f)$. Now evaluate $f(x)$ to find m where $m \leq x < nm+1$. Now $m = n$ because in order for $x \in [n, n+1)$ and $x \in [m, m+1)$ they have to be the same integer. Hence $f(x) = n$. □

(c) Define the function $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = \lfloor x \rfloor + \lfloor -x \rfloor$. Find its range and prove that your answer is correct. The range is $\{-1, 0\}$.

Proof. First we will show $\operatorname{ran}(f) \subseteq \{-1, 0\}$. Let $y \in \operatorname{ran}(f)$. Then $y \in \mathbb{Z}$. So $\operatorname{ran}(f) \subseteq \mathbb{Z}$. To show that y is only -1 or 0 we will consider what will happen if x in $f(x)$ is an integer. Then there is a n such that $n \leq x < n+1$ and an m such that $m \leq -x < m+1$. Since x is an integer, $n = x$ and $m = -x$, so adding these together is 0. Now consider if x is a non-integer real number. Then there is an $n \leq x < n+1$ and $m \leq -x < m+1$. If x is positive, then n will be the first integer less than x, and m will be the first integer more negative than x. Hence, added together $n+m$ will be -1. So $y \in \{-1, 0\}$, and $\operatorname{ran}(f) \subseteq \{-1, 0\}$.

Now we will show $\{-1, 0\} \subseteq \operatorname{ran}(f)$. We know $-1 \in \operatorname{ran}(f)$ because $f(1.5) = -1$. We also know $0 \in \operatorname{ran}(f)$ because $f(0) = 0$. Hence $\{-1, 0\} \subseteq \operatorname{ran}(f)$.

Since containment was proved in both directions equality is shown. □

3. For each of the functions below determine whether or not the function is one-to-one and whether or not the function is onto. If the function is not one-to-one, give an explicit example to show what goes wrong. If it is not onto, determine the range.
   (a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 1/(x^2 + 1)$
   The function is not one-to-one because $f(1) = f(-1) = 1/2$, but $1 \neq -1$.
   The function is not onto, the range is $(0, 1]$.
   (b) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$
   The function is not one-to-one because $f(0) = f(2\pi) = 0$, but $0 \neq 2\pi$.
   The function is not onto, the range is $[-1, 1]$.
   (c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, f(n, m) = nm$
   The function is not one-to-one because $f(1, 2) = f(2, 1) = 2$, but $(1, 2) \neq (2, 1)$.
   The function is onto. If $n = 1$, then $f(1, m) = m$, where m can be any integer.
   (d) $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, f((x, y), (u, v)) = xu + yv$
   The function is not one-to-one because $f((1, 1), (0, 0)) = f((0, 0), (1, 1)) = 0$, but $((1, 1), (0, 0)) \neq ((0, 0), (1, 1))$.
   The function is onto, $f((x, 0), (1, 0)) = x$ where x can be any real number.
   (e) $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, f((x, y), (u, v)) = \sqrt{(x-u)^2 + (y-v)^2}$
   The function is not one-to-one because $f((1, 0), (0, 0)) = f((0, 1), (0, 0)) = 1$, but $((1, 0), (0, 0)) \neq ((0, 1), (0, 0))$.
   The function is not onto because f is never negative. The range is $[0, \infty)$.

(f) Let X be a nonempty set. Define $f : \mathscr{P}(X) \to \mathscr{P}(X)$ by $f(A) = X \setminus A$.

The function is not one-to-one because if $A, B \not\subseteq X$, then $f(A) = f(B) = X$, but A isn't necessarily equivalent to B.

The function is onto because any subset (hence any element of the power set) of X can be generated.

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 - 4x + 7, & x \leq 1, \\ 5 - x^2, & x > 1. \end{cases}$$

This function is well defined, prove it is a bijection.

Proof. First we will show that the function is one-to-one. Let $f(x_1) = f(x_2)$, if $x_1, x_2 \leq 1$, then $x_1^2 - 4x_1 + 7 = x_2^2 - 4x_2 + 7$, so $x_1 = x_2$. If $x_1, x_2 > 1$, then $5 - x_1^2 = 5 - x_2^2$. Hence $x_1 = x_2$. The last case we will show the contrapositive. If $x_1 \leq 1$ and $x_2 > 1$. Then $x_1 \neq x_2$, then $f(x_1) = x_1^2 - 4x_1 + 7$ and $f(x_2) = 5 - x_2^2$, so $f(x_1) \neq f(x_2)$.

Next we will show the function is onto. Then $\operatorname{ran}(f) = \mathbb{R}$. If $y \in \operatorname{ran}(f)$. Then $y \in \mathbb{R}$. Now let $y \in \mathbb{R}$. If $y \geq 4$, then we can have $x = 2 + \sqrt{y-3}$. If $y < 4$ then we can have $x = \sqrt{|5-y|}$. So $x \in \mathbb{R} = \operatorname{dom}(f)$ and $f(x) = y$ when $y \geq 4$ and $f(x) = y$ when $y < 4$. □

5. Define $f : \mathbb{R} \to (-1, 1)$ by

$$f(x) = \frac{x}{1 + |x|}$$

Prove f is bijective.

Proof. First we need to prove that f is one-to-one. If $f(a_1) = f(a_2)$, then

$$\frac{a_1}{1 + |a_1|} = \frac{a_2}{1 + |a_2|}$$

so $a_1 = a_2$. Hence f is one-to-one.

Now we need to prove f is onto. So we must prove $\operatorname{ran}(f) = (-1, 1)$. First we will start with $y \in \operatorname{ran}(f)$. Now let

$$x = \frac{y}{1 + y}$$

x can be any real number as y varies from $(-1, 1)$ and $f(x) = y$.

Since f was shown to be one-to-one and onto it is bijective. □