Homework #8 MATH270

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1. Functions $f : A \to B$ are given below. For each of them find the range of f. Further, if possible, find $f^{-1} : B \to A$. Rigorous proofs are not required, but provide explanations.

   (a) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = 1/x$

   The range is $\mathbb{R} \setminus \{0\}$, one can always find an x for any arbitrary $f(x)$ using the equation $x = 1/f(x)$. The exception is you can't find an x for when $f(x) = 0$, hence the range is $\mathbb{R} \setminus \{0\}$.

   The inverse of f is $f^{-1}(y) = 1/y$. However, it is not a perfect inverse because its domain is $\mathbb{R} \setminus \{0\}$, not $\mathbb{R}$. An inverse whose domain is $\mathbb{R}$ would be impossible.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x + y$

   The range is $\mathbb{R}$ as you can write any real number as a sum of 0 and the same real number.

   There is no inverse because there are infinity many ways to write a real number as a sum of two real numbers.

   (c) $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (y, x)$

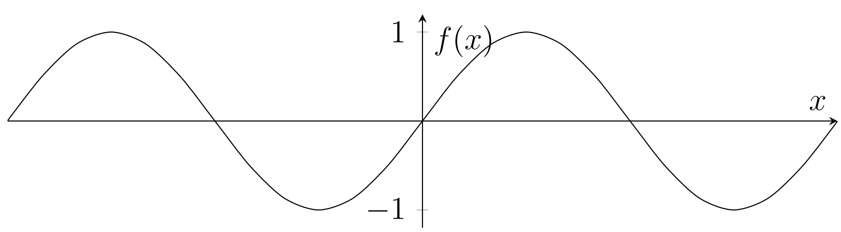
   The range is $\mathbb{R}^2$ because any point (a, b) in the Cartesian plane can be written as $f(b, a)$.

   The inverse of f would be $f^{-1}(y, x) = (x, y)$. It is the same as f.

   (d) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$

   The range of f is $[-1, 1]$. Looking at a plot of the function:

$\sin x$



   We see that $\sin x$ varies between -1 and 1.

   The inverse $\sin^{-1} x$ would be whatever function that makes $\sin^{-1}(\sin x) = x$. The domain of this inverse could only be valid over $[-1, 1]$.

(e) $f : \{x \in \mathbb{R} : -\pi/2 < x < \pi/2\} \to \mathbb{R}, f(x) = \tan x$

The range of f is $\mathbb{R}$. This is because as x approaches $-\pi/2$ from the right $f(x)$ approaches $-\infty$. And as x approaches $\pi/2$ from the left $f(x)$ approaches ∞.

There is an inverse of $f(x)$, it is called $\tan^{-1} x$, or more sanely notated, $\arctan x$.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, x + y)$. Show that f has an inverse and the inverse function.

First we will prove that f is a bijection.

Proof. We will first prove f is one-to-one. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. If $f(x_1) = f(x_2)$, then $(x_1, x_1 + x_2) = (y_1, y_1 + y_2)$. Hence $x_1 = y_1$, and $x_1 + x_2 = y_1 + y_2$. Subtracting x_1 and y_1 from the last equation yields $x_2 = y_2$. Hence f is one-to-one.

Now we will show f is onto. Let $(y_1, y_2) \in \mathbb{R}^2$ and let $(x_1, x_2) = (y_1, y_1 - y_2)$. Then $(x_1, x_2) \in \mathbb{R}^2$, and $f((x_1, x_2)) = (y_1, y_1 - y_2 + y_1) = (y_1, y_2)$. This shows f is onto. □

The inverse of f is $f^{-1}(x, y) = (x, x - y)$. It follows $f^{-1}(f((x, y))) = f^{-1}((x, x + y)) = (x, y)$ and $f(f^{-1}((x, y))) = f(x, x - y) = (x, y)$.

3. Prove the following theorem.

Theorem 1. *If $f : A \to B$, then the following are equivalent.*

(a) f is a bijection.

(b) f has an inverse.

(c) There is $h : B \to A$ such that $h \circ f = i_A$ and $f \circ h = i_B$.

Proof. Theorem 16.4 tells us that $(a) \implies (b)$ and $(a) \implies (c)$. Theorem 16.8 tells us $(c) \implies (b)$. Now if we prove $(b) \implies (a)$ equivalence is shown.

Suppose f has an inverse $f^{-1} : B \to A$ such that $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

If $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$. Hence $i_A(x_1) = i_A(x_2)$. Therefore $x_1 = x_2$. Showing f is one-to-one.

Now we will show f is onto. Let $y \in B$ and set $x = f^{-1}(y)$. Then $x \in A$, and $f(x) = f(f^{-1}(y)) = y$. Lemma 15.1 implies that f is onto.

Since f was shown to be one-to-one and onto, it a bijection.

All the implications that needed to shown were. □

4. Give an example of sets A and B, and functions $f : A \to B$ and $g : B \to A$ such that $f \circ g = i_B$, but $g \circ f \neq i_A$ (thus the existence of a function g such that $f \circ g = i_b$ is not enough to conclude that f has an inverse). Give another example, such that $g \circ f = i_A$ but $f \circ g \neq i_B$.

5. Prove that $(f^{-1})^{-1} = f$.

Proof. By definition of the inverse of a function, $(f^{-1})^{-1}(x) = y$ if and only if $f^{-1}(y) = x$. Also, by the same definition, $f^{-1}(y) = x$ if and only if $f(x) = y$. Since for all x $(f^{-1})^{-1}(x) = f(x) = y$, then $(f^{-1})^{-1} = f$. $\square$