Homework #8 MATH270

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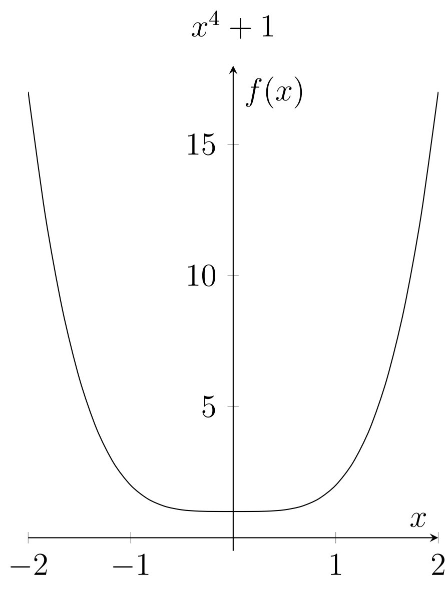
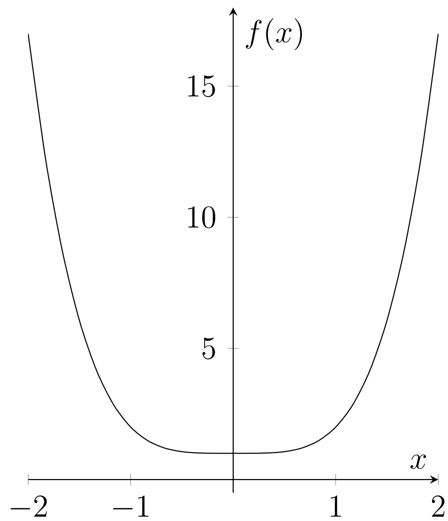
1. For function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^2$, find
   (a) $f((0,1)) = (0,2)$
   (b) $f((-1,3)) = (0,18)$
   (c) $f^{-1}((-2,1)) = (-\sqrt{1/2}, \sqrt{1/2})$
   (d) $f^{-1}((0,2)) = (-1,1)$
   (e)
   $$f^{-1}((a,b)) = \begin{cases} (-\sqrt{\frac{b}{2}}, -\sqrt{\frac{a}{2}}) \cup (\sqrt{\frac{a}{2}}, \sqrt{\frac{b}{2}}) & a, b > 0 \\ (-\sqrt{\frac{b}{2}}, \sqrt{\frac{b}{2}}) & a \leq 0, b > 0 \\ \emptyset & a, b < 0 \end{cases}$$
   (Here (a,b) is the set defined by $(a,b) = \{x \in \mathbb{R} : a < x < b\}$)
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4 + 1$.
   (a) Make a graph of f.



(b) Using your graph, show how you can guess $f([0,2])$.

$x^4 + 1$



$f([0,2]) = [1,17]$

(c) Prove that your guess for $f([0,2])$ is correct.

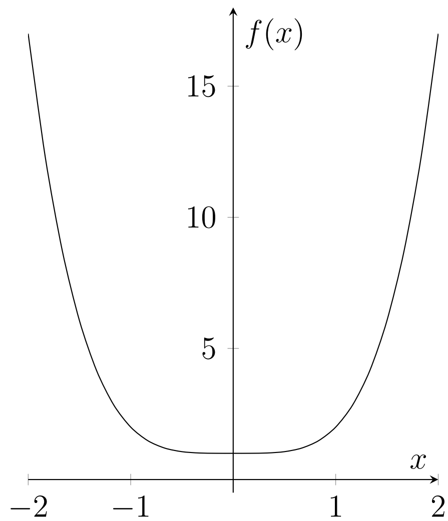
Proof. First we will prove that $f([0,2]) \subseteq [1,17]$. Let $y \in f([0,2])$, then $y \in \{f(x) : 0 \leq x \leq 2\}$. Hence $y \in \{x^4+1 : 0 \leq x \leq 2\}$, $y = x^4+1$, where $x \leq x \leq 2$. The minimum of y is when $x = 0$, and the maximum is when $x = 2$. Therefore the range is $y \in [f(0), f(2)]$, or $y \in [1,17]$.

Now we will show $[1,17] \subseteq f([0,2])$. Let $y \in [1,17]$. If we define $x = \sqrt[4]{y-1}$. The smallest x is when $y = 1$, and the largest is when $y = 2$. When $y = 1$, $x = 0$; when $y = 17$, $x = 2$. Hence the interval for x is $[0,2]$ and $y \in f([0,2])$.

Since containment was shown in both directions, equality is proven. $\square$

(d) Use your graph to find $f^{-1}([2, 17])$.

$x^4 + 1$



$f^{-1}([2, 17]) = [-2, -1] \cup [1, 2]$

(e) Prove that your guess for $f^{-1}([2, 17])$.

Proof. First we will prove that $f^{-1}([2, 17]) \subseteq [-2, -1] \cup [1, 2]$. Let $y \in f^{-1}([2, 17])$. Hence $y \in \{x \in \mathbb{R} : f(x) \in [2, 17]\}$. Hence $y \in \{x : x^4 + 1 \in [2, 17]\}$. There are two valid intervals, one less than zero, one greater than zero. For the interval less than zero, the smallest x is -2 and the largest is -1. For the interval greater than zero, the smallest x is 1 and the largest is 2. Hence $y \in [-2, -1] \cup [1, 2]$.

Next we will prove that $[-2, -1] \cup [1, 2] \subseteq f^{-1}([2, 17])$. Let $x \in [-2, -1] \cup [1, 2]$. Let $y = x^4 + 1$, then the range of y is $[2, 17]$. Hence $x \in f^{-1}([2, 17])$.

Since containment was proved in both directions, equality is shown. □

3. Let $f : X \to Y$. Show that in general

$$f(X \setminus A) \neq Y \setminus f(A), \qquad A \subseteq X$$

.

(a) If f is one-to-one, then $f(X \setminus A) \subseteq Y \setminus f(A)$.

Proof. Let $y \in f(X \setminus A)$. Then $y = f(a)$, where $a \in X \setminus A$. Hence $y \in Y$. Since f is one-to-one and $a \notin A$, then $f(a) \notin Y$. Thus $y \notin Y$, and $y \in Y \setminus f(A)$. □

(b) If f is onto, then $Y \setminus f(A) \subseteq f(X \setminus A)$.

Proof. Let $y \in Y \setminus f(A)$, then $y \in Y$ and $y \notin f(A)$. Hence $y \neq f(a)$ where $a \in A$. Since f is onto, then $\mathrm{ran}(f) = Y$, hence $y \in \mathrm{ran}(f)$. Thus $y = f(x)$, where $x \in X \setminus A$. So $y \in f(X \setminus A)$. □

(c) If f is a bijection, then $f(X \setminus A) = Y \setminus f(A)$.

Proof. By a and b, f is a bijection, then containment in both directions is proved. □

4. Show that for $f : \mathbb{R} \to \mathbb{R}$, $A, B \subseteq Y$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

Proof. First we will prove that $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$, then $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

Let $x \in f^{-1}(A \cap B)$. Then $x \in \{a \in \mathbb{R} : f(a) \in A \cap B\}$. Hence, $x = a$ where $f(a) \in A \cap B$. In other words, $f(a) \in A$ and $f(a) \in B$. Thus $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$, hence $x \in f^{-1}(A) \cap f^{-1}(B)$.

Now let $x \in f^{-1}(A) \cap f^{-1}(B)$. Thus $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Then $x \in \{a \in \mathbb{R} : f(a) \in A\}$ and $x \in \{b \in \mathbb{R} : f(b) \in B\}$. Therefore $x \in \{a \in \mathbb{R} : f(a) \in A \cap B\}$. Hence $x \in f^{-1}(A \cap B)$.

Since containment was proved in both directions, hence equivalence is proven. $\square$

5. Prove using mathematical induction that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Proof. First we check the base step: $P(1)$ is the statement that $1 = 1(1+1)/2$.

Now we check the induction step. Let $n \in \mathbb{Z}^+$ and suppose $P(n)$. Thus we suppose that for an $n \in \mathbb{Z}^+$ we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

.

We wish to show that $P(n+1)$ holds

$$1 + 2 + \cdots + (n+1) = \frac{(n+1)((n+1)+1)}{2}$$

Grouping the left side of $P(n+1)$ and then

$$\begin{aligned}
&1 + 2 + \cdots + n + (n+1) \\
=&(1 + 2 + \cdots + n) + (n+1) \\
=&\frac{n(n+1)}{2} + (n+1) \\
=&\frac{n(n+1)}{2} + (n+1) \\
=&\frac{n(n+1) + 2(n+1)}{2} \\
=&\frac{n^2 + n + 2n + 2}{2} \\
=&\frac{n^2 + 3n + 2}{2} \\
=&\frac{(n+1)(n+2)}{2} \\
=&\frac{(n+1)((n+1)+1)}{2}
\end{aligned}$$

By mathematical induction we conclude that assertion holds for all positive integers. $\square$

6. Prove using mathematical induction that $2^n \leq n!$ for all integers with $n \geq 5$.

Proof. Using the definition of factional and exponentiation.

$$2 * 2 * 2 * 2 * 2 * \cdots \leq 1 * 2 * 3 * 4 * 5 * \cdots$$

Checking the base step: $2^5 \leq 5!$, implies $32 \leq 120$, which is true.

Now we verify the induction stop when $n \geq 5$. Suppose the following is true

$$2 * 2 * \cdots * 2 \leq 1 * 2 * \cdots * n$$

Now we will see if the following holds

$$2 * 2 * \cdots * 2 * 2 \text{ (the nth two)} \leq 1 * 2 * \cdots * n * (n+1)$$

Grouping together

$$(2 * 2 * \cdots * 2) * 2 \text{ (the nth two)} \leq 1 * 2 * \cdots * n) * (n+1)$$

We know the left group is less than the right group, we also know $2 \leq (n+1)$ when $n \geq 5$. Hence the left product is less than the right product and the assertion holds. □