Homework #4 MATH270

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Wednesday, February 20th

1. Given

$$A = \{(x, y) \in \mathbb{R}^2 : xy > 0\},$$
$$B = \{(x, y) \in \mathbb{R}^2 : y > |x|\},$$
$$C = \{(x, y) \in \mathbb{R}^2 : 0 < x < y\}.$$

Prove that $A \cap B = C$.

Proof. If $p \in A \cap B$, then $p \in A$ and $p \in B$. Thus $p = (x, y)$ where $xy > 0$ and $y > |x|$. Since $y > 0$ by $y > |x|$, $x > 0$ too in order to satisfy the inequality $xy > 0$. Now notice that $-y < x < y$. Put these results together shows that $0 < x < y$. We can now conclude that $p \in C$, so $A \cap B \subseteq C$

To complete the proof we must show that $C \subseteq A \cap B$. So if $p \in C$, then $p = (x, y)$ where $0 < x < y$. Since both x and y are positive $xy > 0$. So $p \in A$. Also $-y < 0 < x < y$ which implies $y > |x|$. So $p \in B$. Thus $p \in A \cap B$ and $C \subseteq A \cap B$

Since containment was proved in both directions, we can conclude the two sets are equal. $\square$

2. Given

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\},$$
$$B = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}.$$

Is $A \subset B$, $B \subset A$, $A \subseteq B$, or $B \subseteq A$?

We can show $B \subseteq A$ and $B \subset A$.

Proof. If $p \in B$, then $p = (x, y)$ where $|x| + |y| \leq 1$. If we square both sides we find $(|x| + |y|)^2 \leq 1^2$, $x^2 + y^2 + 2|x||y| \leq 1$, $x^2 + y^2 \leq 1 - 2|x||y| \leq 1$. This shows $p \in A$, so $B \subseteq A$.

To show $B \subset A$ we see that $A \nsubseteq B$. The point $q = (\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}) \sim (0.7, 0.7)$ is in A, but not in B.

Therefore, $A \not\subset B$, $B \subset A$, $A \nsubseteq B$, and $B \subseteq A$. $\square$

3. Given the following definition, do the following.

Definition 1. *The symmetric difference of two sets A and B is the set $A \triangle B$ defined by*

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

(a) Draw a Venn diagram for the symmetric difference.

(b) Prove that

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Proof. Let $x \in A \triangle B$, then $x \in (A \setminus B) \cup (B \setminus A)$. So $x \in A \setminus B$ or $x \in B \setminus A$. Suppose $x \in A$ and $x \notin B$, then $x \in A \cup B$ and $x \notin A \cap B$. So $x \in (A \cup B) \setminus (A \cap B)$. Similarly, now suppose $x \in B$ and $x \notin A$, then $x \in A \cup B$ and $x \notin A \cap B$. So $x \in (A \cup B) \setminus (A \cap B)$. Thus we can conclude $A \triangle B \subseteq (A \setminus B) \cup (B \setminus A)$.

Now we need to show $(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$. If $x \in (A \cup B) \setminus (A \cap B)$, then $x \in A \cup B$ and $x \notin A \cap B$. Suppose $x \in A$ and $x \notin B$, then $x \in A \setminus B$ and therefore $x \in (A \setminus B) \cup (B \setminus A)$ which means $x \in A \triangle B$. Similarly, now suppose $x \in B$ and $x \notin A$, then $x \in B \setminus A$ and therefore $x \in (A \setminus B) \cup (B \setminus A)$ which means $x \in A \triangle B$. So $(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$.

Since containment in both directions was proved, then we may conclude that the two sets are equal. $\square$

(c #1) Prove that $A \triangle A = \emptyset$.

Proof. Suppose there is an $x \in A \triangle A$, then $x \in (A \setminus A) \cup (A \setminus A)$. This would require x to simultaneously be an element and not an element of A, which is impossible, so $x \in \emptyset$. Thus $A \triangle A = \emptyset$. $\square$

(c #2) Prove that $A \triangle \emptyset = A$.

Proof. Let $x \in A \triangle \emptyset$, then $x \in (A \setminus \emptyset) \cup (\emptyset \setminus A)$. Since $x \in A \setminus \emptyset$, $x \in A$, so $A \triangle \emptyset \subseteq A$. Now let $x \in A$. Therefore $x \in A \setminus \emptyset$, and $x \in (A \setminus \emptyset) \cup (\emptyset \setminus A)$. So $A \subseteq A \triangle \emptyset$. Together we have proven that $A \triangle \emptyset = A$. $\square$

(d) Prove that for sets A, B, we have $A \triangle B = A \setminus B$ if and only if $B \subseteq A$.

Proof. First we will prove that if $A \triangle B = A \setminus B$, then $B \subseteq A$. Let $x \in B$. Considering $(A \setminus B) \cup (B \setminus A) = A \setminus B$, the only way for this equality to work is if $B \setminus A = \emptyset$. If $B \setminus A = \emptyset$, that means all elements in B are a part of A. Hence $x \in A$ and $B \subseteq A$.

Now we'll prove that if $B \subseteq A$, then $A \triangle B = A \setminus B$. Let $x \in A \triangle B$, so $x \in (A \setminus B) \cup (B \setminus A)$. Since $B \subseteq A$, $B \setminus A = \emptyset$. Thus $x \in A \setminus B$, therefore $A \triangle B \subseteq A \setminus B$. Now let $x \in A \setminus B$, then $x \in (A \setminus B) \cup (B \setminus A)$. Hence $x \in A \triangle B$, and $A \setminus B \subseteq A \triangle B$.

Both directions of implication were proved, thus proving equivalence. $\square$

4. Prove that the union of two sets can always be written as a union of disjoint sets. (Show that the sets $A \setminus B$ and B are disjoint and that $A \cup B = (A \setminus B) \cup B$).

Proof. The sets $A \setminus B$ and B are disjoint. An element cannot be both in B and a set that excludes all elements of B.

We can show $A \cup B = (A \setminus B) \cup B$. To prove this first we'll prove $A \cup B \subseteq (A \setminus B) \cup B$, then that $(A \setminus B) \cup B \subseteq A \cup B$.

Let $x \in A \cup B$. If $x \in A$ and $x \notin B$, then $x \in A \setminus B$ and $x \in (A \setminus B) \cup B$. Otherwise, if $x \in B$ (regardless if $x \in A$), then $x \in (A \setminus B) \cup B$. So $x \in (A \setminus B) \cup B$.

Now let $x \in (A \setminus B) \cup B$. So $x \in A \setminus B$ or $x \in B$. If $x \in A \setminus B$, then $x \in A$. Therefore $x \in A \cup B$. Otherwise, if $x \in B$, then $x \in A \cup B$.

Since containment in both directions was proved, equality is. $\square$