Homework #3 MATH270

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1. Prove that if n is an integer, then $4n^2 + 4n + 8$ is an even integer. What method of proof did you use?

   Proof. We will prove this directly. By definition, if an integer n is even, then $n = 2m, m \in \mathbb{Z}$. Since $4n^2 + 4n + 8 = 2(2n^2 + 2n + 4)$, then by definition $4n^2 + 4n + 8$ is an even integer. ■

2. Complete the proof for the theorem:

   Theorem 1. *Let x and y be real numbers. If $xy > 1/2$ then $x^2 + y^2 > 1$.*

   Proof. The proof will proceed by considering the contrapositive. So suppose $x^2 + y^2 \leq 1$. Now we know that $(x^2 - y^2) \geq 0$ (this is always true).

$$\begin{aligned}
1 &\leq (x-y)^2 + 1 \\
x^2 + y^2 &\leq (x-y)^2 + 1 \\
x^2 + y^2 &\leq x^2 - 2xy + y^2 + 1 \\
0 &\leq -2xy + 1 \\
2xy &\leq 1 \\
xy &\leq \frac{1}{2}
\end{aligned}$$

   This shows the contrapositive is true. So then the theorem is true. ■

3. Prove that $\sqrt{3}$ is not rational. Before we prove that, we will prove that if n^2 is divisible by 3, then n is divisible by 3.

Proof. We will prove that if $3 \mid n^2$, then $3 \mid n$ by proving the contrapositive. If $3 \nmid n$, then by definition there is not integer m such that $n = 3m$. Then there should also be no m such that $n^2 = 3(3m^2)$, which means $3 \nmid n^2$. ■

Proof. We will prove this by contradiction. Assume $\sqrt{3}$ is rational. Then $\frac{p}{q} = \sqrt{3}$, $p, q \in \mathbb{Z}$ where p and q are coprime. Squaring both sides yields $\frac{p^2}{q^2} = 3$. So $p^2 = 3q^2$. By the previous proof, since $3 \mid p^2$, then $3 \mid p$. So $p = 3m$, then $p^2 = 9m^2 = 3q^2$, or $q^2 = 3m^2$. Since $3 \mid q^2$, then $3 \mid q$. Now we have shown that 3 divides both p and q, showing they have a common factor of 3, contradicting the assumption that they were coprime.
This contradiction proves that $\sqrt{3}$ must not be rational. ■

4. Let x be a real number.
(a) Prove $-|x| \leq x \leq |x|$*

Proof. We will prove this by looking at each possible case. The definition of the absolute value function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

there are two cases, $x \geq 0$ and $x < 0$.
Case: If $x \geq 0$, then $|x| = x$. Substituting for $|x|$ into (*) yields $-x \leq x \leq x$, which is true.
Case: If $x < 0$, then $|x| = -x$. Substituting for $|x|$ into (*) yields $x \leq x \leq -x$. This is true. The largest term is positive, where as the two left ones are negative. ■

(b) Let $a \geq 0$. Prove $|x| \leq a$ if and only if $-a \leq x \leq a$.

Proof. First we will prove that if $|x| \leq a$, then $-a \leq x \leq a$ by cases.
Case: If $x \geq 0$, then $x \leq a$, which means $-a \leq 0 \leq x \leq a$. And that is true.
Case: If $x < 0$, then $-x \leq a \implies x \geq -a$, then $-a \leq x < 0 < a$
Next we will prove that if $-a \leq x \leq a$, then $|x| \leq a$, again by cases.
Case: $-a < 0 \leq x \leq a$. In this case x is positive, so $|x| = x$, therefore $|x| \leq a$.
Case: $-a \leq x < 0 < a \implies -a \leq x \implies -x \leq a$. Since $x < 0$, then $|x| = -x$, therefore $|x| \leq a$. ■

(c) Prove the following.

Theorem 2. *Let x and y be real numbers. Then $|x+y| \leq |x| + |y|$*

Proof.

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|$$

Summing these two formulas yields

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

Implying

$$|x+y| \leq ||x| + |y||$$

$|x| + |y|$ is always positive, so $||x| + |y|| = |x| + |y|$, therefore proving

$$|x+y| \leq |x| + |y|$$

■

(d) Prove the following corollary to Theorem 2.

Corollary 2.1. *For any $x, y \in \mathbb{R}$, $||x| - |y|| \leq |x - y|$.*

Proof.

$$\begin{aligned} |x| &= |x - y + y| \\ |x - y + y| &\leq |x - y + y| \\ |x - y + y| &\leq |x - y| + |y| \\ |x - y + y| - |y| &\leq |x - y| \\ |x| - |y| &\leq |x - y| \qquad (1) \end{aligned}$$

Proving a second result

$$\begin{aligned} x &\leq |x| \\ y &\leq |y| \\ x - y &\leq |x| - |y| \\ -|x - y| &\leq x - y \qquad \text{by 4a} \\ -|x - y| &\leq |x| - |y| \qquad (2) \end{aligned}$$

Combining (1) and (2) gives

$$-|x-y| \leq |x| - |y| \leq |x - y|$$

Using the result from 4b

$$||x| - |y|| \leq |x - y|$$

■