Study Guide for Exam 2

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1. You are supposed to know how to compute the integration of the form

(1) $\displaystyle\int \sin^m x \cos^n x \; dx$

- Case: m odd $\to$ Use $u = \cos x$ substitution
- Case: n odd $\to$ Use $u = \sin x$ substitution
- Case: m & n even $\to$ Reduce the degree by double angle formula

(2) $\displaystyle\int \tan^m x \sec^n x \; dx$

- Case: $n > 0$ even $\to$ Use $u = \tan x$ substitution
- Case: $n > 0$ odd & m odd (or n even & m odd)
  $\to$ Use $u = \sec x$ substitution
- Case: $n > 0$ odd & m even $\to$ Integration by parts
- Case: $n = 0 \to$ Use $\tan^2 x = \sec^2 x - 1$ to reduce to the case $n > 0$ and to the lower degree case

NOTE: Look at the document "Strategy fir Trigonometric Integration" posted on the Brightspace for more details.

Example Problems

1.1. Compute the following integrals:

(i) $\displaystyle\int \sin^3 x \cos^2 x \; dx$

(ii) $\displaystyle\int \sin^5 x \cos^{\frac{3}{2}} x \; dx$

(iii) $\displaystyle\int \sin^2 x \cos^3 x \; dx$

(iv) $\displaystyle\int_0^{3\pi/2} \sin^5 x \cos^2 x \; dx$

(v) $\displaystyle\int \sin^3 x \cos^3 x \; dx$

(vi) $\displaystyle\int \sin^{\frac{7}{2}} x \cos^3 x \; dx$

(vii) $\displaystyle\int \sin^2 x \cos^2 x \; dx$

(viii) $\displaystyle\int \sin^4 x \cos^2 x \; dx$

(ix) $\displaystyle\int_0^{\pi/2} \cos^2(3x) \; dx$

(x) $\displaystyle\int_0^{\pi/4} \sin^2 x \; dx$

1.2. Compute the following integrals:

$$
\begin{array}{ll}
\text{(i)} & \displaystyle\int \tan^2 x \sec^4 x \; dx \\
\text{(ii)} & \displaystyle\int_0^{\pi/3} \tan^2 x \sec^4 x \; dx \\
\text{(iii)} & \displaystyle\int \tan^3 x \sec^4 x \; dx \\
\text{(iv)} & \displaystyle\int \tan x \;\; \sec^3 x \; dx \\
\text{(v)} & \displaystyle\int \tan^3 x \;\; \sec^5 x \; dx \\
\text{(vi)} & \displaystyle\int_0^{\pi/4} \tan x \;\; \sec^5 x \; dx \\
\text{(vii)} & \displaystyle\int \sec^3 x \; dx. \\
\text{(viii)} & \displaystyle\int \tan^2 x \sec x \; dx. \\
\text{(ix)} & \displaystyle\int \tan^5 x \; dx. \\
\text{(x)} & \displaystyle\int_0^{\pi/3} \tan^3 x \; dx.
\end{array}
$$

1.3. (i) Compute $\displaystyle\int \sec x \; dx = \int \frac{1}{\cos^2 x} \cos x \; dx = \int \frac{1}{1 - \sin^2 x} \cos x \; dx$ using the substitution $u = \sin x$ and then using the partial fraction.

(ii) Check that the result obtained in (i) coincides with the well-known formula $\int \sec x \; dx = \ln|\sec x + \tan x| + C$.

1.4. We would like to compute

$$\int \tan x \; dx = \int \frac{\sin x}{\cos x} \; dx$$

in the following two ways.

(i) Use substitution $u = \sin x$ to get

$$
\begin{aligned}
\int \tan x \; dx &= \int \frac{\sin x}{\cos x} \; dx = \int \frac{\sin x}{\cos^2 x} \; \cos x dx \\
&= \int \frac{\sin x}{1 - \sin^2 x} \; \cos x dx = \int \frac{u}{1 - u^2} \; du
\end{aligned}
$$

and then use the partial fractions.

(ii) Use the same substitution as above and then use another substitution $v = 1 - u^2$ to get

$$\int \tan x \; dx = \int \frac{u}{1-u^2} \; du = \int -\frac{1}{2}\frac{dv}{v}$$

and compute.

(iii) Check that the results obtained in (i) and (ii) coincide with the well known formula $\int \tan x \; dx = \ln|\sec x| + C$.

2. You are supposed to know how to use the 3 types of trigonometric substitution, and carry out the integration accordingly.

(1) $\sqrt{a^2 - x^2}, x = a\sin\theta, dx = a\cos\theta, \sqrt{a^2 - x^2} = a\cos\theta,$
(2) $\sqrt{a^2 + x^2}, x = a\tan\theta, dx = a\sec^2\theta, \sqrt{a^2 + x^2} = a\sec\theta,$
(3) $\sqrt{x^2 - a^2}, x = a\sec\theta, dx = a\tan\theta\sec\theta, \sqrt{x^2 - a^2} = a\tan\theta.$

Example Problems

2.1. Compute the following integrals:

(i) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

(ii) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}} \; dx$

(iii) $\displaystyle\int_0^{7/\sqrt{2}} \frac{x^2}{\sqrt{49-x^2}} \; dx$

(iv) $\displaystyle\int \sqrt{5-4x^2} \; dx$

(v) $\displaystyle\int \frac{\sqrt{x^2-9}}{x} \; dx \quad (x > 3)$

(vi) $\displaystyle\int_1^{\sqrt{2}} x^3\sqrt{x^2-1} \; dx$

(vii) $\displaystyle\int \sqrt{x^2+1} \; dx$

(viii) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+1}}$

$$\text{(ix)} \quad \int \sqrt{7+4x+x^2}\ dx$$

$$\text{(x)} \quad \int \sqrt{7+12x+4x^2}\ dx$$

$$\text{(xi)} \quad \int \frac{x}{\sqrt{3-2x-x^2}} dx$$

$$\text{(xii)} \quad \int \frac{x^2-2x+2}{\sqrt{x^2-2x+10}} dx$$

$$\text{(xiii)} \quad \int \frac{x}{\sqrt{3+2x-x^2}} dx$$

$$\text{(xiv)} \quad \int_0^{1/2} \frac{x^3\ dx}{(1-x^2)^2}$$

$$\text{(xv)} \quad \int_{-3}^{3} \frac{dx}{\sqrt{x^2+9}}$$

2.2. Verify that the area of a circle of radius r is πr^2.

2.3 Compute the length of the curve

$$y = f(x) = \frac{x^2}{2} \text{ over the interval } [1, \sqrt{3}].$$

3. You are supposed to know
∘ the proper form of the partial fractions,
∘ how to determine the approppariate constants appearing in the partial fraction,
∘ how to compute the integral accordingly.

Example Problems

3.1. Determine the proper form of th partial fractions for the following. (You do not have to calculate the constants.)

$$\text{(i)} \quad \frac{1}{(x+2)(x^2-4)(x^2+x+1)^2}$$

$$\text{(ii)} \quad \frac{x^3}{(x-1)(x^3-1)(x^2+4x+5)}$$

$$\text{(iii)} \quad \frac{x^7}{(x-2)^2(x^2-4)(x^2+4)}$$

3.2. Compute the following integrals:

$$
\begin{array}{ll}
\text{(i)} & \displaystyle\int \frac{3x+5}{(x-1)(x+3)}\,dx \\
\text{(ii)} & \displaystyle\int \frac{x^2}{(x-1)^2}\,dx \\
\text{(iii)} & \displaystyle\int \frac{x}{(x^2-1)(x-2)}\,dx \\
\text{(iv)} & \displaystyle\int \frac{2x^3+3x^2-2x-1}{x^2-1}\,dx \\
\text{(v)} & \displaystyle\int \frac{x+2}{x^2+2x+2}\,dx \\
\text{(vi)} & \displaystyle\int \frac{x^2}{(x-1)^2(x^2+1)}\,dx \\
\text{(vii)} & \displaystyle\int \frac{x^2+x+1}{x^3+x}\,dx \\
\text{(viii)} & \displaystyle\int \frac{x^2+x+2}{x^2+4x+5}\,dx \\
\text{(ix)} & \displaystyle\int \frac{7x^2+23x+25}{x(x^2+4x+5)}\,dx \\
\text{(x)} & \displaystyle\int \frac{5}{(x-1)(x^2+2)}\,dx
\end{array}
$$

4. You are supposed to know why a given improper integral is improper, and accordingly to be able to determine if the given improper integral is convergent/divergent. In case it is convergent, you should be able to compute its value.

Example Problems

4.1. Evaluate the following improper integrals

$$
\begin{array}{ll}
\text{(i)} & \displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx \\
\text{(ii)} & \displaystyle\int_{e^5}^\infty \frac{1}{x\{\ln(x)\}^2}\,dx \\
\text{(iii)} & \displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx \\
\text{(iv)} & \displaystyle\int_0^\infty \frac{e^{2x}}{e^{2x}+1}\,dx
\end{array}
$$

(v) $\displaystyle\int_0^9 \frac{1}{x-1}\,dx$

(vi) $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

(vii) $\displaystyle\int_{-\infty}^{\infty} x\,dx$

(viii) $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

(ix) $\displaystyle\int_{-\infty}^{0} xe^{-x^2}\,dx$

(x) $\displaystyle\int_e^{\infty} \frac{1}{x\ln x}\,dx$

5. You are supposed to know the definition of a sequence $\{a_n\}$ and its limit. You should know the clear difference between a sequence $\{a_n\}$ and series $\sum a_n$. You are supposed to be able to compute the value of a telescopic series.

NOTE: The main body of the discussion of a sequenece and series will be the subject of Exam 3. Here in Exam 2, you are only expected to know the very basics.

Example Problems

5.1. Compute the following telescopic series.

(i) $\sum_{n=1}^{\infty} \dfrac{3}{n^2+5n+6}$

(ii) $\sum_{n=1}^{\infty} \dfrac{2}{n^2+2n}$