Stochastic Optimisation Algorithm
Home Problem 1

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Problem 1.1 - Penalty Method

Calculations and Results of steps 1-3

We have the function $f(x_1, x_2)$ that is subject to the constraint $g(x_1, x_2)$.

$$f(x_1, x_2) = (x_1 - 1)^2 + 2(x_2 - 2)^2 \tag{1}$$

$$g(x_1, x_2) = x_1^2 + x_2^2 - 1 \leq 0 \tag{2}$$

$$P(\mathbf{x}, \mu) = \mu(x_1^2 + x_2^2 - 1)^2 \tag{3}$$

This would then lead to the function where the constraint is fulfilled and unfulfilled respectively:

$$f_p(\mathbf{x}; \mu) = f(x) + P(\mathbf{x}, \mu) = (x_1 - 1)^2 + 2(x_2 - 2)^2 + \mu(x_1^2 + x_2^2 - 1)^2 \tag{4}$$

$$f_p = (x_1 - 1)^2 + 2(x_2 - 2)^2 \tag{5}$$

The unconstrained minimum is then the minimum of the function where the constraint is not fulfilled, where the gradient of function 5 is 0.

$$\frac{\partial f_p}{\partial x_1} = 2x_1 - 2 = 0 \Rightarrow x_1 = 1 \tag{6}$$

$$\frac{\partial f_p}{\partial x_2} = 4x_1 - 8 = 0 \Rightarrow x_1 = 2 \tag{7}$$

This gives then the starting point $x_0 = (1, 2)$.

The gradient of the constrained function f_p is derived similarly, $\nabla f_p = \frac{\partial f_p}{\partial x_1}\hat{x_1} + \frac{\partial f_p}{\partial x_2}\hat{x_2}$.

$$\frac{\partial f_p}{\partial x_1} = 2(x_1 - 1) + 4\mu x_1(x_1^2 + x_2^2 - 1) \tag{8}$$

$$\frac{\partial f_p}{\partial x_2} = 4(x_2 - 2) + 4\mu x_2(x_1^2 + x_2^2 - 1) \tag{9}$$

μ	x_1	x_2
1	0.4338	1.2102
10	0.3314	0.9955
50	0.3159	0.9602
100	0.3137	0.9553
500	0.3120	0.9512
1000	0.3118	0.9507
2000	0.3117	0.9505

Tabell 1: Values of x_1 and x_2 for increasing values of μ.

Judging by the values in table, x_1 converges towards 0.3117 and x_2 towards 0.9505, probably a little lower when we let μ increase more.

Problem 1.3 - Basic GA program

Subproblem 1.3a

The following set of parameters were used for the values in the table over 10 runs of RunSingle.m:

1. Tournament Size = 5
2. Tournament Probability = 0.6
3. Crossover Probability = 0.8
4. Mutation Probability = 0.05
5. Number of Generations = 2000

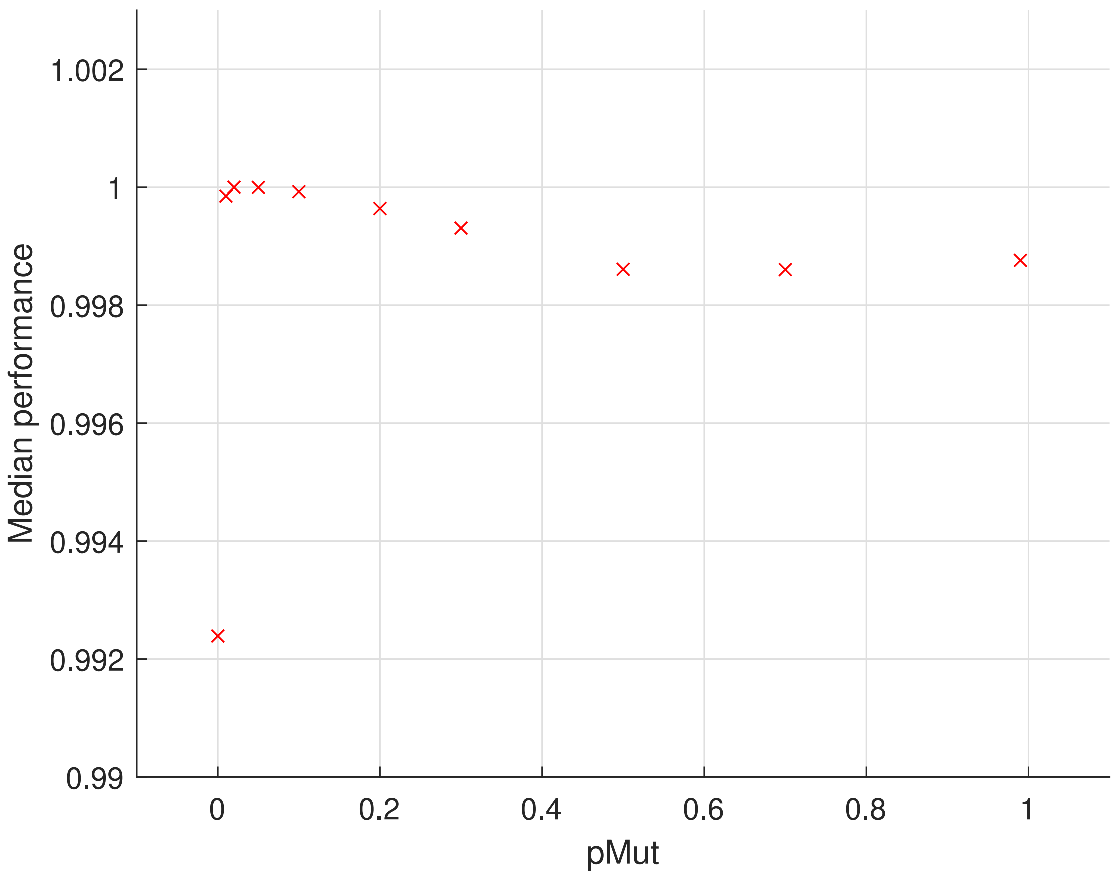
g	x_1	x_2
1.00000000220190	3.00011122227047	0.500030681491813
1.00000000002125	2.99998873472180	0.499997004866510
1.00000000000079	3.00000214576728	0.500000581145304
1.00000000015167	3.00003075599762	0.500007733702891
1.00000000014101	2.99997174739753	0.499992236494786
1.00000000000002	3.00000005960465	0.499999985098838
1.00000000015167	3.00003075599762	0.500007733702891
1.00000000014134	2.99997234344400	0.499992236494786
1.00000000000184	3.00000333786021	0.500000879168537
1.00000000001349	3.00000810623193	0.500002369284701

Tabell 2: Table over the values of g and x for different a set of parameters, over the course of 10 runs of the GA.

Subproblem 1.3b

Mutation Rate	Median performance
0.00	0.992390544147112
0.01	0.999847804698992
0.02	0.999999990045126
0.05	0.999996708943629
0.10	0.999924042691684
0.20	0.999637934406341
0.30	0.999306222449832
0.50	0.998607592370938
0.70	0.998602063866222
0.99	0.998759415970438

Tabell 3: Tabular over the mutation rates and the median performance (fitness value).



Figur 1: The median performance using 10 different mutation rates. Data points shown in the tabular 3.

Overall the values of the median performance are maximized when the mutation rate is kept low and non-zero. The performance is slightly lower and seems to converge to a somewhat stable value when mutation rate approaches. In fact the initial rate given, $p_mut = 0.2 = 1/m$, and mutation rates slightly higher and lower relative to that also performs well.

Subproblem 1.3c

Judging by the results from subproblem 1.3a, I would estimate the values to $g(x_1, x_2) = 1$, $x_1 = 3$, $x_2 = 0.5$. To check whether this really is the minimum we can check the value of $f_{xx}f_{yy} - f_{xy}^2$, as well as the values of f_{xx} and f_{yy}. If all three of these values are larger than zero for the given point, the point is a minimum point.

$$g(x_1, x_2) = (1.5 - x_1 + x_1x_2)^2 + (2.25 - x_1 + x_1x_2^2)^2 + (2.65 - x_1 + x_1x_2^3)^2 \tag{10}$$

$$g_{x_1} = 2(1.5 - x_1 + x_1x_2)(x_2 - 1) + 2(2.25 - x_1 + x_1x_2^2)(x_2^2 - 1) + 2(2.65 - x_1 + x_1x_2^3)(x_2^3 - 1) \tag{11}$$

$$g_{x_2} = 2x_1(1.5 - x_1 + x_1x_2) + 4x_1x_2(1.5 - x_1 + x_1x_2^2) + 6x_1x_2^2(2.65 - x_1 + x_1x_2^3) \tag{12}$$

$$g_{x_1x_1} = 2(x_2 - 1)^2 + 2(x_2^2 - 1)^2 + 2(x_2^3 - 1)^2 \tag{13}$$

$$g_{x_2x_2} = 2x_1x_2 + 4x_1x_2(2x_1x_2) + 4x_1(1.5 - x_1 + x_1x_2^2) + 6x_1x_2^2(3x_1x_2^2) + 12x_1x_2(2.65 - x_1 + x_1x_2^3) \tag{14}$$

$$g_{x_1x_2} = 2(2.5 - x_1 + x_1x_2) + 2x_1(x_1 - 1) + 4x_2(1.5 - x_1 + x_1x_2^2) + 4x_1x_2(x_2^2 - 1) + 6x_2^2(2.65 - x_1 + x_1x_2^3) + 6x_1x_2^2(x_2^3 - 1) \tag{15}$$

Just to clarify:

$$g_{x_1x_2} = \frac{\partial}{\partial x_1}(\frac{\partial g}{\partial x_2}) \tag{16}$$

Using the estimated values of the minimum point we calculate the following:

$$g_{x_1x_1}(3, 0.5)g_{x_2x_2}(3, 0.5) - g_{x_1x_2}^2(3, 0.5) = 54.4423 \tag{17}$$

$$g_{x_1x_1}(3, 0.5) = 3.1563 \tag{18}$$

$$g_{x_2x_2}(3, 0.5) = 22.5750 \tag{19}$$

As all of these values are larger than zero, the point is therefore confirmed to be a stationary point as well as the minimum. Numerical calculations were done through Matlab.