Lukas Fu - HW1 problem 1

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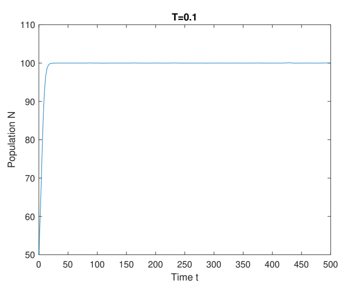
Problem 1

The dynamics analysed is the Allee effect for population growth, where a time delay parameter is included. The following is the growth model with a time delay with Allee effect:

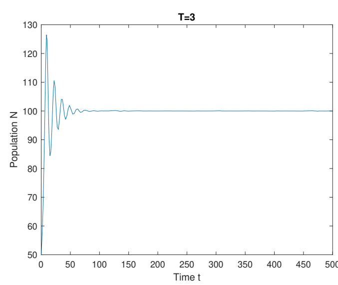
$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-T)}{k}\right)\left(\frac{N(t)}{A} - 1\right). \tag{1}$$

Here $N(t)$ is the population size as a function of time, r is the growth parameter, k is the carrying capacity and A is Allee effect parameter.

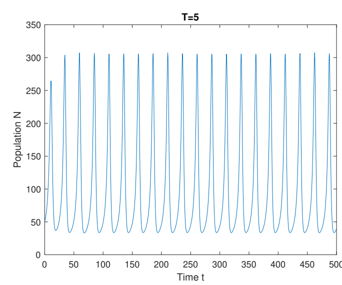
The following figures, figure 1a, 1b and 1c show the different dynamics that occur for different values of time delay. For low values the dynamics show no oscillation, for intermediate values the dynamics show damped oscillations, and for large values the dynamics show stable oscillations.



(a) No oscillation, $T = 0.1$



(b) Damped oscillation, $T = 3$



(c) Stable oscillation, $T = 5$

Figure 1: Graph for the three different cases of oscillations: no oscillations, damped oscillations and stable oscillations.

To estimate the starting point of the damped oscillation, the amplitude difference is taken of the largest and smallest oscillation of each value of time delay T. This difference is then plotted against the time delay to show when the difference starts to increase away from 0. In figure 2 the amplitude difference starts to increase around $T = 1.2$, and remains over 0 for the remainder of time delay values.

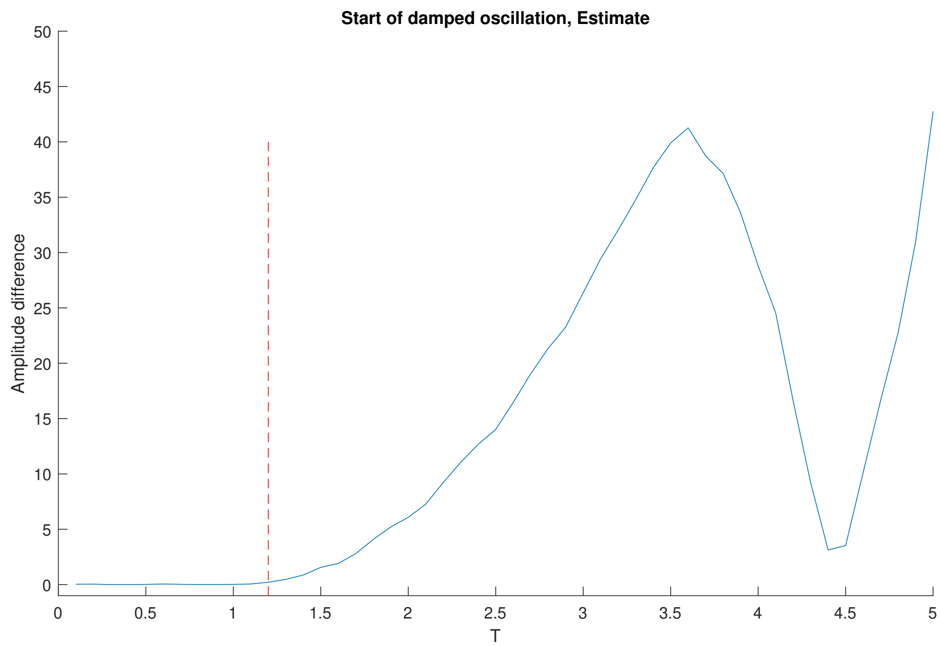
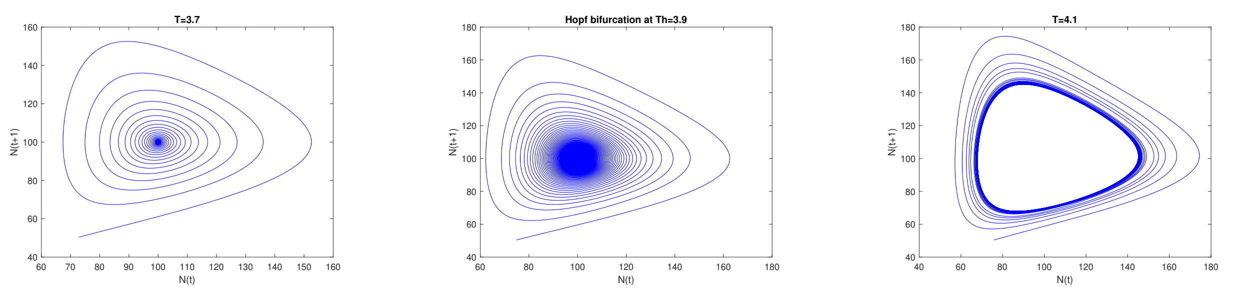


Figure 2: The estimated start of damped oscillation shown to be around $T = 1.2$.

The Hopf bifurcation occurs whenever the system transitions from damped oscillations to stable oscillations and a limit cycle appears. This can be visualized by plotting $N(t)$ against $N(t+1)$, from which it can be observed that the limit cycle starts appearing around $T_H = 3.9$.



(a) Damped oscillation showing a spiral, $T = 3.7$

(b) The last point before a limit cycle appears, $T_H = 3.9$

(c) Stable oscillation where we have a limit cycle, $T = 4.1$

Figure 3: Three figures to show the before, transition to and after the limit cycle.

Using linear stability analysis around the steady state N^* by linearising the original equation 1 and using a perturbation we get:

$$\lambda = (r - \frac{kr}{A})e^{-\lambda T}, \tag{2}$$

from which we can get the solution $\lambda = \lambda_{Re} + i\lambda_{Im} = \lambda' + i\lambda''$. In figure 4 the real and imaginary part of the solution are plotted against the time delay T. Here it can be observed that the real part changes signs at $T = 3.9$, where the Hopf bifurcation has occurred.

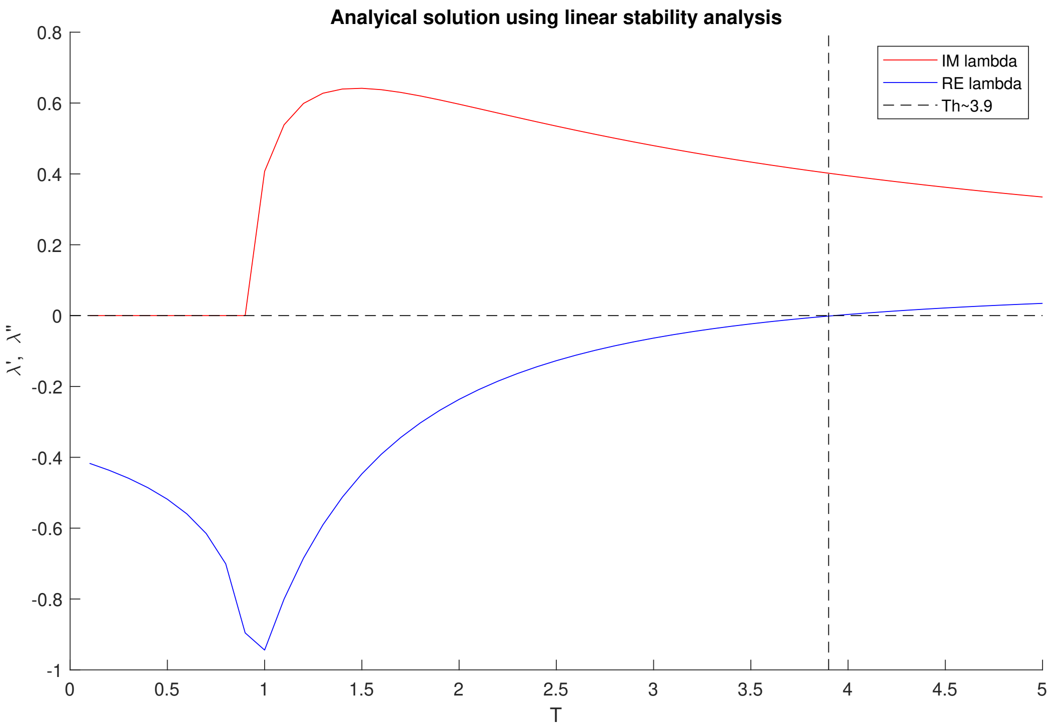


Figure 4: Comparing the real and imaginary parts of the analytical solution over T. When the real part of the solution λ switches signs from negative to positive, a Hopf bifurcation occurs.

While the result of the numerical analytical results were the same, the analytical one shows the changes in dynamics more clearly, as the numerical one requires some testing by hand. The analytical approach shows the curve crossing 0, while the numerical approach is judged by appearance.

Code

Problem 1

```
%% problem 1
%a

clear;clc;clf;

T_all = linspace(0.1,5,50);
for T=T_all
    plotN(T,500)
    pause(0.1)
end
tmax=500;T=[0.1,3,5];
hold off;
for i=1:3
    figure
    plotN(T(i),tmax)
    title('T=' + string(T(i)) )
end

%% b
clf
t=(0.1:0.1:5)';
tmax = 100;
amplitudeDiff=zeros(length(t),1);
for iT=1:50
    T=iT/10;
    [x,y]=getxy(T,tmax);
    TF = islocalmax(y);
    amplitudeDiff(iT) = max(y(TF))-min(y(TF));
end
hold on
plot(t,amplitudeDiff) % amp diff starts increasing around T=1.2
plot([1.2,1.2],[-1,40],'r--')
xlim([0 5])
ylim([-1 50])
xlabel('T')
ylabel('Amplitude difference')
title('Start of damped oscillation, Estimate')

%% c
clf
%search for limitcycle --> grows/shrinks in beginning and periodic always
T=3.9; %also run for 3.7 and 4.1 to show difference before and after TH
TT=6000;
[x,y]=getxy(T,TT);
dt=0.1; delay=floor(T/dt);
x2 = 0.1:dt:TT;
y2 = interp1(x,y,x2,'spline');
nty1 = y2(delay+1:end);
nty2 = y2(1:end-delay);
plot(nty1,nty2,'b')
```

```
xlabel('N(t)')
ylabel('N(t+1)')
title('Hopf bifurcation at Th=3.9')
%title('T=3.7')
%title('T=4.1')

%% d
syms a
k=-2/5; %-2/5 for our problem
N=50;
lambda = zeros(N,1);
delay = zeros(N,1);

% solving each eq for eigenvalues
for iT=1:N
    iT
    T=iT*5/N;
    sol = solve(a == k*exp(-a*T));
    delay(iT)=T;
    lambda(iT)=sol;
end
clf
hold on;
plot(delay,imag(lambda),'r') %imaginary part
plot(delay,real(lambda),'b') %real part
plot([3.9,3.9],[-1,0.8],'k--') %hopf bifurcation point
plot([0,5],[0,0],'k--')
xlabel('T')
ylabel("\lambda',  \lambda''")
legend('IM lambda','RE lambda','Th~3.9')
title('Analyical solution using linear stability analysis')

%% functions

function plotN(T,tmax)
    tspan = [0 tmax];
    lags = T;
    sol = dde23(@ddefunc, lags, @history, tspan);
    plot(sol.x,sol.y,'')
    title(T)
    xlabel('Time t');
    ylabel('Population N');
end
function [x,y] = getxy(T,tmax)
    tspan = [0 tmax];
    lags = T;
    sol = dde23(@ddefunc, lags, @history, tspan);
    x = sol.x; y = sol.y;
end
function dydt = ddefunc(t,N,Z)
    Nlag = Z(:,1);
    A = 20;
    K = 100;
    r = 0.1;
```

```
      dydt = r*N * (1 - Nlag(1)/K) * (N/A - 1);
end
function s = history(t)
  s = 50;
end
```