1 Introduction

This document provides a runtime analysis of the SLOWSORT algorithm[1]. Despite being presented in a satirical article, its recursion pattern turns out to be rather complex, which in turn makes it harder to analyze. On the other hand, the authors do not provide a thorough derivation of the claimed running time in the aforementioned article. The asymptotic bounds explored here do not match tightly with those, as they differ roughly by a linear factor.

2 Slowsort algorithm

The following figure shows the pseudocode of SLOWSORT :

```
SLOWSORT (A[1...n]):
    If n = 1: return.
    Let m := (n+1)/2.
    SLOWSORT (A[1...m]).
    SLOWSORT (A[m+1...n]).
    If A[n] < A[m]: swap A[n] and A[m].
    SLOWSORT (A[1...n-1]).
```

3 Runtime analysis

3.1 Recurrence relation

Let $T(n)$ be the function that characterizes the worst-case running time of SLOWSORT when given an input array of size n. Being a recursive algorithm, we can define T recurrently like so:

$$T(n) = 2T(n/2) + T(n-1) + \Theta(1)$$

Note that we intentionally leave out base cases as we aim at finding asymptotic bounds for T. From this, we can see that $T(i) - T(i-1) = 2T(i/2) + \Theta(1)$ for a given i. Thus,

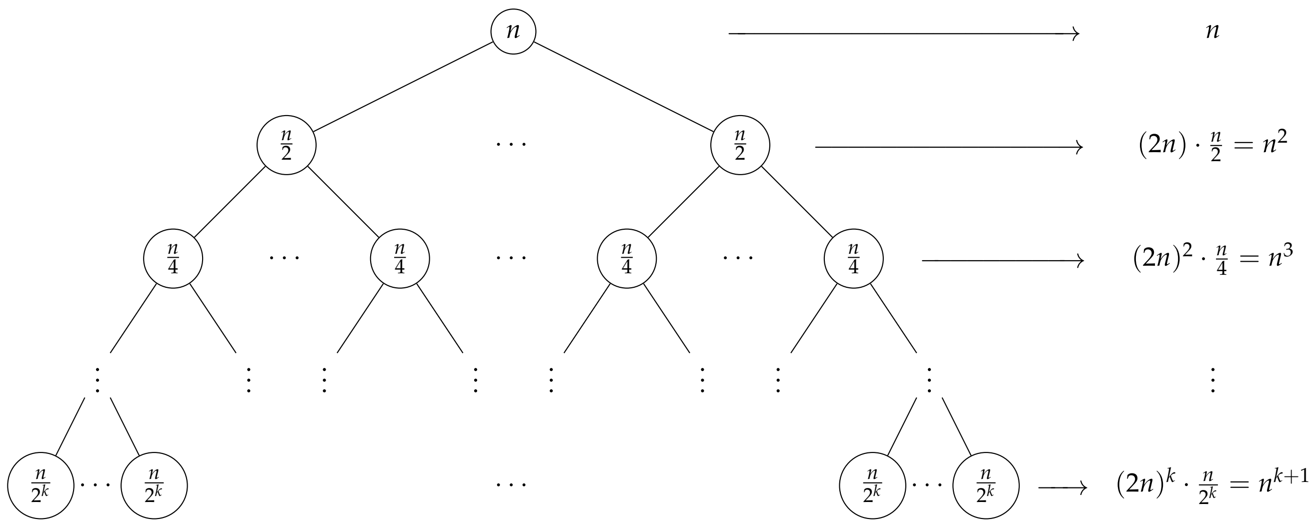
$$\begin{aligned}
\sum_{i=1}^{n} T(i) - T(i-1) &= \sum_{i=1}^{n} 2T(i/2) + \Theta(1) \\
\Rightarrow \quad T(n) - T(0) &= \sum_{i=1}^{n} 2T(i/2) + \sum_{i=1}^{n} \Theta(1) \\
\Rightarrow \quad T(n) - T(0) &= \sum_{i=1}^{n} 2T(i/2) + \Theta(n) \\
\Rightarrow \quad T(n) &= \sum_{i=1}^{n} 2T(i/2) + \Theta(n)
\end{aligned}$$

3.2 Asymptotic bounds

We shall look now for asymptotic bounds for T. Let us consider first the following recurrence U:

$$\begin{aligned}
U(n) &= \sum_{i=1}^{n} 2U(n/2) + \Theta(n) \\
&= 2n\,U(n/2) + \Theta(n)
\end{aligned}$$

Clearly, we have that $T(n) \leq U(n)$ for all n. In order to find an asymptotic solution for U, let's have a look at its recursion tree:



From this, it follows that

$$\begin{aligned}
U(n) &= \sum_{k=0}^{\lfloor \log_2(n) \rfloor} n^{k+1} \\
&= n \sum_{k=0}^{\lfloor \log_2(n) \rfloor} n^k \\
&= \frac{n}{n-1} \left(n^{\lfloor \log_2(n) \rfloor + 1} - 1 \right) \\
&\in \Theta \left(n^{\log_2(n)+1} \right)
\end{aligned}$$

This yields an asymptotic upper bound for T:

$$T(n) \in \mathrm{O} \left(n^{\log_2(n)+1} \right) \tag{1}$$

In order to find an asymptotic lower bound, let's first rewrite T in the following convenient way:

$$T(n) = \sum_{i=0}^{n-1} 2T(n/2 - i/2) + \Theta(n)$$

Now, define the recurrence $L(n)$ as follows:

$$L(n) = \sum_{i=0}^{\lfloor \sqrt{n} \rfloor - 1} 2L(n/2 - i/2) + \Theta(n)$$

Such an L nearly satisfies every hypotheses required by the Akra-Bazzi theorem[2]. The only apparent exception is that the summation in the right-hand side goes up to a function of n, and the theorem states that this value should be a constant $k \in \mathbb{N}$. However, the proof outlined in [2] does not seem to rely strongly on this fact[1]. In what follows, then, we will use the Akra-Bazzi theorem to obtain an asymptotic solution for L.

Before proceeding, we observe that the perturbation functions $h_i(n)$ are properly bounded:

[1] If someone sufficiently qualified to confirm or disprove this assertion ever reads this, please let me know. Thanks in advance!

$$\begin{aligned}
|h_i(n)| &= |-\frac{i}{2}| \\
&= \frac{i}{2} \\
&\leq \frac{\lfloor\sqrt{n}\rfloor - 1}{2} \\
&\leq \sqrt{n} \\
&\in \mathrm{O}\left(\frac{n}{\log^2 n}\right) \qquad (2)
\end{aligned}$$

(2) holds since $\frac{\sqrt{n}}{n/\log^2 n}$ goes to 0 as $n \to \infty$.

References

[1] A. Broder and J. Stolfi, "Pessimal algorithms and simplexity analysis," *ACM SIGACT News*, vol. 16, pp. 49–53, 1984.

[2] T. Leighton, "Notes on better master theorems for divide-and-conquer recurrences," in *Lecture notes, MIT*, 1996.