Runtime analysis of the SLOWSORT algorithm

Lucio Santi
lsanti at dc.uba.ar

1 Introduction

This document provides a runtime analysis of the SLOWSORT algorithm[1]. Despite being presented in a satirical article, its recursion pattern turns out to be rather complex, which in turn makes it harder to analyze. On the other hand, the authors do not provide a thorough derivation of the claimed running time in the aforementioned article. The asymptotic bounds explored here do not match tightly with those, as they differ roughly by a linear factor.

2 Slowsort algorithm

The following figure shows the pseudocode of SLOWSORT :

```
SLOWSORT (A[1...n]):
    If n = 1: return.
    Let m := (n+1)/2.
    SLOWSORT (A[1...m]).
    SLOWSORT (A[m+1...n]).
    If A[n] < A[m]: swap A[n] and A[m].
    SLOWSORT (A[1...n-1]).
```

3 Runtime analysis

3.1 Recurrence relation

Let $T(n)$ be the function that characterizes the worst-case running time of SLOWSORT when given an input array of size n. Being a recursive algorithm, we can define T recurrently like so:

$$T(n) = 2T(n/2) + T(n-1) + \Theta(1)$$

Note that we intentionally leave out base cases as we aim at finding asymptotic bounds for T. From this, we can see that $T(i) - T(i-1) = 2T(i/2) + \Theta(1)$ for a given i. Thus,

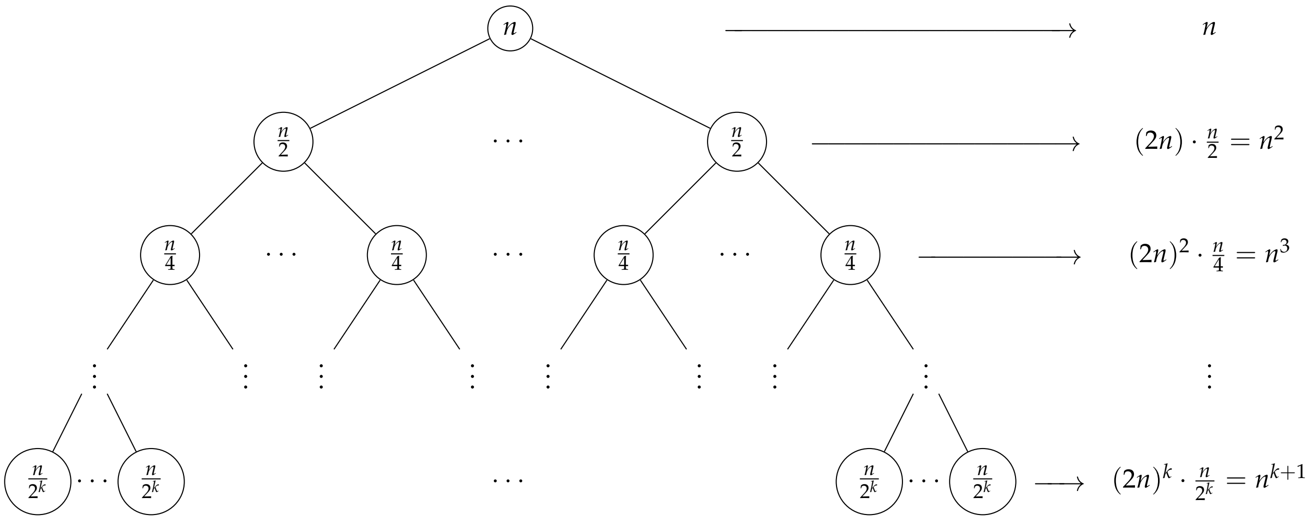
$$\begin{aligned}
\sum_{i=1}^{n} T(i) - T(i-1) &= \sum_{i=1}^{n} 2T(i/2) + \Theta(1) \\
\Rightarrow \quad T(n) - T(0) &= \sum_{i=1}^{n} 2T(i/2) + \sum_{i=1}^{n} \Theta(1) \\
\Rightarrow \quad T(n) - T(0) &= \sum_{i=1}^{n} 2T(i/2) + \Theta(n) \\
\Rightarrow \quad T(n) &= \sum_{i=1}^{n} 2T(i/2) + \Theta(n)
\end{aligned}$$

3.2 Asymptotic bounds

We shall look now for asymptotic bounds for T. Let us consider first the following recurrence U:

$$\begin{aligned} U(n) &= \sum_{i=1}^{n} 2U(n/2) + \Theta(n) \\ &= 2n\, U(n/2) + \Theta(n) \end{aligned}$$

Clearly, we have that $T(n) \leq U(n)$ for all n. In order to find an asymptotic solution for U, let's have a look at its recursion tree:



From this, it follows that

$$\begin{aligned} U(n) &= \sum_{k=0}^{\lfloor \log_2(n) \rfloor} n^{k+1} \\ &= n \sum_{k=0}^{\lfloor \log_2(n) \rfloor} n^{k} \\ &= \frac{n}{n-1} \left(n^{\lfloor \log_2(n) \rfloor + 1} - 1 \right) \\ &\in \Theta\left(n^{\log_2(n)+1} \right) \end{aligned}$$

This yields an asymptotic upper bound for T:

$$T(n) \in \mathrm{O}\left(n^{\log_2(n)+1} \right) \tag{1}$$

In order to find an asymptotic lower bound, let's first rewrite T in the following convenient way:

$$T(n) = \sum_{i=0}^{n-1} 2T(n/2 - i/2) + \Theta(n)$$

Now, let us choose an arbitrary $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$ and define the recurrence $L_\alpha(n)$ as follows:

$$L_\alpha(n) = \sum_{i=0}^{\lfloor n^\alpha \rfloor - 1} 2L_\alpha(n/2 - i/2) + \Theta(n)$$

Such an L_α nearly satisfies every hypotheses required by the Akra-Bazzi theorem[2]. The only apparent exception is that the summation in the right-hand side goes up to a function of n, and the theorem states that this value should be a constant $k \in \mathbb{N}$. However, the proof outlined in [2] does not seem to rely strongly on this fact[1]. In what follows, then, we will use the Akra-Bazzi theorem to obtain an asymptotic solution for L.

Before proceeding, we observe that the index perturbation functions $h_i(n)$ are properly bounded:

$$\begin{aligned}
|h_i(n)| &= |-\frac{i}{2}| \\
&= \frac{i}{2} \\
&\leq \frac{n^\alpha - 1}{2} \\
&\leq n^\alpha \\
&\in \mathrm{O}\left(\frac{n}{\log^2 n}\right) \qquad (2)
\end{aligned}$$

(2) holds since $\frac{n^\alpha}{n/\log^2 n}$ goes to 0 as $n \to \infty$.

Now, we need to find the value of p to plug into the theorem and find a tight asymptotic bound for L_α:

$$\begin{aligned}
\sum_{i=0}^{\lfloor n^\alpha \rfloor - 1} a_i b_i^p &= 1 \\
\Leftrightarrow \sum_{i=0}^{\lfloor n^\alpha \rfloor - 1} 2\,\frac{1}{2^p} &= 1 \\
\Leftrightarrow \frac{\lfloor n^\alpha \rfloor}{2^{p-1}} &= 1 \\
\Leftrightarrow \log_2\left(\lfloor n^\alpha \rfloor\right) &= p - 1 \\
\Leftrightarrow p &= \log_2\left(\lfloor n^\alpha \rfloor\right) + 1
\end{aligned}$$

With this in hand, we can calculate the following integral:

$$\begin{aligned}
\int_1^n \frac{g(x)}{x^{p+1}}\,dx &= \int_1^n \frac{x}{x^{\log_2(\lfloor n^\alpha \rfloor)+2}}\,dx \\
&= \int_1^n \frac{dx}{x^{\log_2(\lfloor n^\alpha \rfloor)+1}} \\
&= \left.\frac{-1}{\log_2\left(\lfloor n^\alpha \rfloor\right)\, x^{\log_2(\lfloor n^\alpha \rfloor)}}\right|_1^n \\
&= \frac{1}{\log_2\left(\lfloor n^\alpha \rfloor\right)} - \frac{1}{\log_2\left(\lfloor n^\alpha \rfloor\right)\, n^{\log_2(\lfloor n^\alpha \rfloor)}}
\end{aligned}$$

Finally,

$$\begin{aligned}
n^p\left(1 + \int_1^n \frac{g(x)}{x^{p+1}}\,dx\right) &= n^{\log_2(\lfloor n^\alpha \rfloor)+1}\left(1 + \frac{1}{\log_2\left(\lfloor n^\alpha \rfloor\right)} - \frac{1}{\log_2\left(\lfloor n^\alpha \rfloor\right)\, n^{\log_2(\lfloor n^\alpha \rfloor)}}\right) \\
&\in \Theta\left(n^{\log_2(\lfloor n^\alpha \rfloor)+1}\right) \\
&= \Theta\left(n^{\alpha \log_2(n)+1}\right)
\end{aligned}$$

Therefore,

$$L_\alpha(n) \in \Theta\left(n^{\alpha \log_2(n)+1}\right)$$

[1] If someone sufficiently qualified to confirm or disprove this assertion ever reads this, please let me know. Thanks in advance!

which, since $L_\alpha(n) \leq T(n)$ for all n, implies that

$$T(n) \in \Omega\left(n^{\alpha \log_2(n)+1}\right) \tag{3}$$

Putting everything together, it follows from (1) and (3) that, for sufficiently large n, there exist some constants $c_1, c_2 > 0$ such that

$$c_1\, n^{\alpha \log_2(n)+1} \leq T(n) \leq c_2\, n^{\log_2(n)+1}$$

for any $\alpha \in (0, 1)$.

3.3 Remarks about α

The lower bound we explored above is only valid when $0 < \alpha < 1$. We might be tempted to choose $\alpha = 1$ in order to obtain a tight asymptotic bound for T, but unfortunately doing so would violate the index perturbation hypothesis of the Akra-Bazzi theorem. Indeed, in such cases, $h_i(n) \leq \frac{n-1}{2}$, and thus some of these functions will not be bounded the way the theorem expects.

References

[1] A. Broder and J. Stolfi, "Pessimal algorithms and simplexity analysis," *ACM SIGACT News*, vol. 16, pp. 49–53, 1984.

[2] T. Leighton, "Notes on better master theorems for divide-and-conquer recurrences," in *Lecture notes, MIT*, 1996.