

Initial check

2023-11-03

Log precision

Let

$$X_i \stackrel{iid}{\sim} \mathcal{N}(0, \tau^{-1}), \quad \tau \sim \chi^2(\nu)$$

The posterior is

$$\pi(\tau|\mathbf{x}) \propto \tau^{\frac{n}{2}} \exp \left\{ -\frac{\tau}{2} \sum_{i=1}^n x_i^2 \right\} \tau^{\frac{\nu}{2}-1} e^{-\tau/2} = \tau^{\frac{n+\nu}{2}-1} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n x_i^2 + 1 \right] \right\},$$

so

$$\tau|\mathbf{x} \sim \text{Gamma} \left(\frac{n+\nu}{2}, \frac{1}{2} \left[\sum_i x_i^2 + 1 \right] \right).$$

Now, consider the transformation $\theta = \log \tau$, then the posterior, in terms of θ is

$$\pi(\theta|\mathbf{X}) \propto \prod_{i=1}^n p(x_i|e^\theta) \pi(e^\theta) e^\theta = \Gamma \left(e^\theta; \frac{n+\nu}{2}, \frac{1}{2} \left[\sum_i x_i^2 + 1 \right] \right) e^\theta = e^{\theta \alpha_n} \exp \{ -e^\theta \beta_n \},$$

where $\alpha_n = (n+\nu)/2$ and $\beta_n = \sum_i x_i^2 + 1$. Then

$$\frac{\partial}{\partial \theta} \log \pi(\theta|\mathbf{x}) = \alpha_n - \beta_n e^\theta, \quad \frac{\partial^2}{\partial \theta^2} \log \pi(\theta|\mathbf{x}) = -\beta_n e^\theta.$$

The posterior mode is $\hat{\theta} = \log \frac{\alpha_n}{\beta_n}$, so

$$- \left(\frac{\partial^2}{\partial \theta^2} \log \pi(\theta|\mathbf{x}) \right) \Big|_{\theta=\hat{\theta}} = \alpha_n$$

The Laplace approximation to the posterior is $\mathcal{N}(\theta; \hat{\theta}, 1/\alpha_n)$.

Example using stan

```
library(rstan)
library(tidyverse)
```

Generate data

```
tau = 0.4
n = 50
set.seed(123)
x = rnorm(n, 0, 1/tau)
```

Fit the Stan model, comparison shown in Fig.1.

```
sp_d <- list(N = n, y = x)
sp_fit <- stan('wishart.stan', data = sp_d,
              iter = 5e3, chains = 1)
```

```

theta = log(sp_fit0sim$samples[[1]]$tau)
x_d = seq(-3,0,length.out = 500)
alpha_n = (n+1)/2
beta_n = (sum(x^2)+1)/2

ggplot() +
  geom_density(aes(x = theta), color = "blue") +
  geom_line(aes(x = x_d,
                y = dnorm(x_d, log(alpha_n/beta_n),
                          1/sqrt(alpha_n))), color = "red") +
  ylab("") + theme_bw()

```

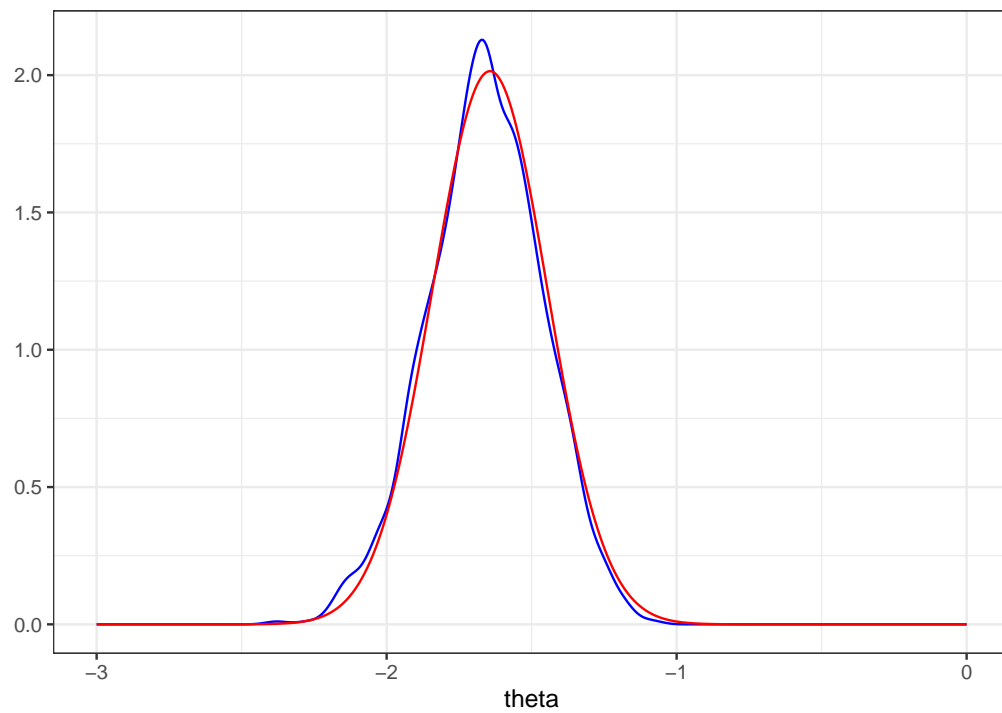


Figure 1: Blue -> stan, Red -> Laplace