Initial check

2023-11-03

Log precision

Let

$$X_i \overset{iid}{\sim} \mathcal{N}(0, \tau^{-1}), \quad \tau \sim \chi^2(\nu)$$

The posterior is

$$\pi(\tau|\mathbf{x}) \propto \tau^{\frac{n}{2}} \exp\left\{-\frac{\tau}{2}\sum_{i=1}^n x_i^2\right\} \tau^{\frac{\nu}{2}-1} e^{-\tau/2} = \tau^{\frac{n+\nu}{2}-1} \exp\left\{-\frac{\tau}{2}\left[\sum_{i=1}^n x_i^2 + 1\right]\right\},$$

so

$$\tau|\mathbf{x} \sim \text{Gamma}\left(\frac{n+\nu}{2},\ \frac{1}{2}\Big[\sum_i x_i^2 + 1\Big]\right).$$

Now, consider the transformation $\theta = \log \tau$, then the posterior, in terms of θ is

$$\pi(\theta|\mathbf{X}) \propto \prod_{i=1}^n p(x_i|e^\theta)\pi(e^\theta)e^\theta = \Gamma\left(e^\theta;\ \frac{n+\nu}{2},\ \frac{1}{2}\Big[\sum_i x_i^2 + 1\Big]\right) e^\theta = e^{\theta\alpha_n} \exp\left\{-e^\theta \beta_n\right\},$$

where $\alpha_n = (n+\nu)/2$ and $2\beta_n = \sum_i x_i^2 + 1$. Then

$$\frac{\partial}{\partial\theta} \log \pi(\theta|\mathbf{x}) = \alpha_n - \beta_n e^\theta, \quad \frac{\partial^2}{\partial\theta^2} \log \pi(\theta|\mathbf{x}) = -\beta_n e^\theta.$$

The posterior mode is $\hat{\theta} = \log \frac{\alpha_n}{\beta_n}$, so

$$-\left(\frac{\partial^2}{\partial\theta^2} \log \pi(\theta|\mathbf{x})\right)\Big|_{\theta=\hat{\theta}} = \alpha_n$$

The Laplace approximation to the posterior is $\mathcal{N}(\theta; \hat{\theta}, 1/\alpha_n)$.

Example using stan

```r
library(rstan)
library(tidyverse)
```

Generate data

```r
tau = 0.4
n = 50
set.seed(123)
x = rnorm(n, 0, 1/tau)
```

Fit the Stan model, comparison shown in Fig.1.

```r
sp_d <- list(N = n, y = x)
sp_fit <- stan('wishart.stan', data = sp_d,
               iter = 5e3, chains = 1)
```

```r
theta = log(sp_fit@sim$samples[[1]]$tau)
x_d = seq(-3,0,length.out = 500)
alpha_n = (n+1)/2
beta_n = (sum(x^2)+1)/2

ggplot() +
  geom_density(aes(x = theta), color = "blue") +
  geom_line(aes(x = x_d,
                y = dnorm(x_d, log(alpha_n/beta_n),
                          1/sqrt(alpha_n))), color = "red") +
  ylab("") + theme_bw()
```

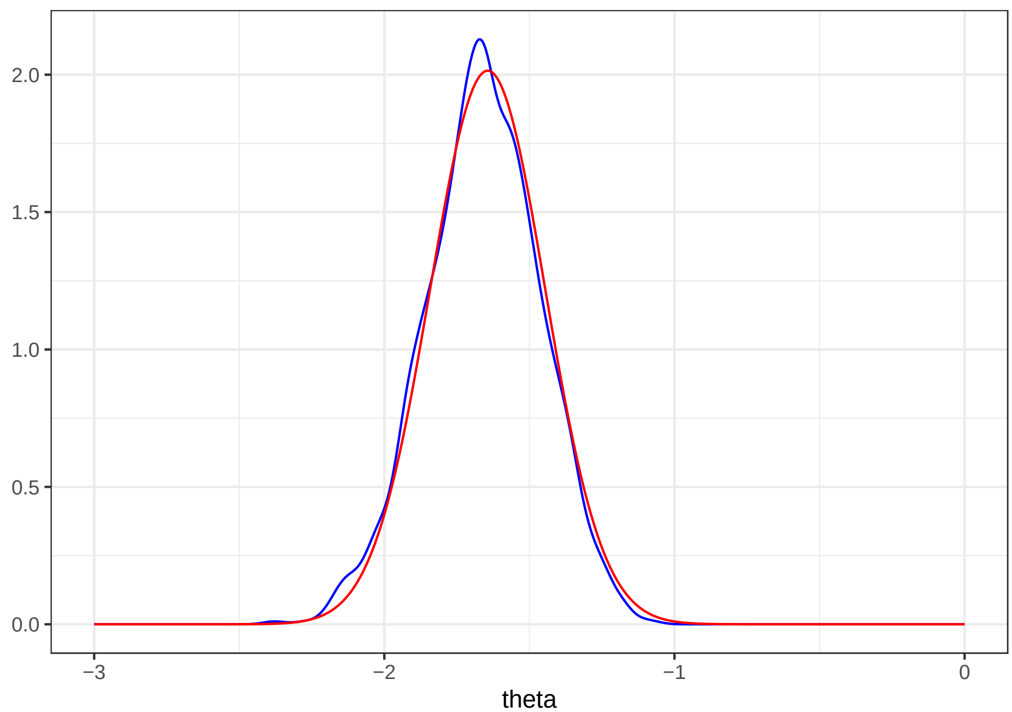

Figure 1: Blue -> stan, Red -> Laplace

Precision

Consider now the χ^2 approximation to the posterior of the precision rather than the log transformed case.

$$\frac{\partial}{\partial \tau} \log \pi(\tau|\mathbf{x}) = \frac{\alpha_n - 1}{\tau} - \beta_n, \quad \frac{\partial^2}{\partial \tau^2} \log \pi(\tau|\mathbf{x}) = -\frac{\alpha_n - 1}{\tau^2}.$$

The posterior mode is $\hat{\tau} = \frac{\alpha_n - 1}{\beta_n}$ so

$$-H(\hat{\tau}) = \frac{\beta_n^2}{\alpha_n - 1}.$$

Thus a χ^2 approximation to the posterior of τ is based on $\mathcal{N}(\tau; \hat{\tau}, (\alpha_n - 1)/\beta_n$ using that, for $X \sim \mathcal{N}(\mu, \sigma)$ we have that $(X/\sigma)^2 \sim \chi_1(\mu^2/\sigma^2)$. So the Laplace approximation to τ is given by

$$\chi_1^2\left(\hat{\tau}^2/H(\hat{\tau})\right), \quad \hat{\tau}^2/H(\hat{\tau}) = \alpha_n - 1 = n + \nu - 1$$