Lecture 13: Template Methods and Shortest Paths in a Weighted Graph

Path of Least Action

Wholeness Statement

In a weighted graph, the shortest path algorithm finds the path between a given pair of vertices such that the sum of the weights of that path's edges is the minimum. Natural law always chooses the path of least action, the shortest path to the goal with no wasted effort.

Template Method Pattern

Depth-first search is to graphs what the Euler tour is to binary trees

Recall Our Earlier Example of the Template Method Pattern

- Generic algorithm that can be specialized by redefining certain steps
- Implemented by means of an abstract Java class
- Visit methods that can be redefined by subclasses
- Template method eulerTour
  - Recursively called on the left and right children
  - A Result object with fields leftResult, rightResult and finalResult keeps track of the output of the recursive calls to eulerTour

```
public abstract class EulerTour {
    protected BinaryTree tree;
    protected void visitExternal(Position p, Result r) { }
    protected void visitPreOrder(Position p, Result r) { }
    protected void visitInOrder(Position p, Result r) { }
    protected void visitPostOrder(Position p, Result r) {}
    protected Object eulerTour(Position p) {
        Result r = new Result();
        if tree.isExternal(p) { visitExternal(p, r); }
        else {
            visitPreOrder(p, r);
            r.leftResult = eulerTour(tree.leftChild(p));
            visitInOrder(p, r);
            r.rightResult = eulerTour(tree.rightChild(p));
            visitPostOrder(p, r);
            return r.finalResult;
        } ...
```


Specializations of EulerTour

- We show how to specialize class EulerTour to evaluate an arithmetic expression
- Assumptions
  - External nodes store Integer objects
  - Internal nodes store Operator objects supporting method operation (Integer, Integer)

```
public class EvaluateExpression
                    extends EulerTour {
    protected void visitExternal(Position p, Result r) {
        r.finalResult = (Integer) p.element();
    }
    protected void visitPostOrder(Position p, Result r) {
        Operator op = (Operator) p.element();
        r.finalResult = op.operation(
                            (Integer) r.leftResult,
                            (Integer) r.rightResult
                            );
    }
    ...
}
```


DFS Template Example

Depth-First Search (review)

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G

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Template Version of DFS

```
Algorithm DFS(G)
  Input graph G
  Output the edges of G are
         labeled as discovery edges
         and back edges

  initResult( G )
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
    postInitVertex(u)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
    postInitEdge(e)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      preComponentVisit(G, v)
      DFS(G, v)
      postComponentVisit(G, v)

  return result( G )
```

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  startVertexVisit(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        preDiscoveryTraversal(G, v, e, w)
        DFS(G, w)
        postDiscoveryTraversal(G, v, e, w)
      else
        setLabel(e, BACK)
        backTraversal(G, v, e, w)
  finishVertexVisit(G, v)
```

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Path Finding
Override hook operations

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  startVertexVisit(G, v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        preDiscoveryTraversal(G, v, e, w)
        DFS(G, w)
        postDiscoveryTraversal(G, v, e, w)
      else
        setLabel(e, BACK)
        backEdgeVisit(G, v, e, w)
  finishVertexVisit(G, v)
```

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z then
    path ← S.elements()
  for all e ∈ G.incidentEdges(v) do
    if getLabel(e) = UNEXPLORED then
      w ← opposite(v, e)
      if getLabel(w) = UNEXPLORED then
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop()          { e gets popped }
      else
        setLabel(e, BACK)

  S.pop()              { v gets popped }
```

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Overriding hook methods in a subclass FindSimplePath

```
Algorithm findPath(G, u, v)
    S ← new empty stack   {S is a subclass field}
    z ← v                 {z is a subclass field & is the target vertex}
    for all u ∈ G.vertices()
      setLabel(u, UNEXPLORED)
    for all e ∈ G.edges()
      setLabel(e, UNEXPLORED)
    DFS(G, u)
    return(path)

Algorithm startVertexVisit(G, v)
    S.push(v)
    if v=z  then      {z is a subclass field & is the target}
      path ← S.elements()  {path is a subclass field & is the result}

Algorithm preDiscoveryTraversal(G, v , e, w)
    S.push(e)

Algorithm postDiscoveryTraversal(G, v, e, w)
    S.pop()           {pop e off the stack}

Algorithm finishVertexVisit(G, v)
    S.pop()           {pop v off the stack}
```

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Template Version of DFS (v2)

```
Algorithm DFS(G)
  Input graph G
  Output the edges of G are
         labeled as discovery edges
         and back edges

  initResult( G )
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
    postInitVertex(u)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
    postInitEdge(e)
  for all v ∈ G.vertices()
    if isNextComponent(G, v)
      preComponentVisit(G, v)
      DFS(G, v)
      postComponentVisit(G, v)

  return result( G )

Algorithm isNextComponent(G, v)
  return getLabel(v) = UNEXPLORED
```

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  startVertexVisit(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        preDiscoveryTraversal(G, v, e, w)
        DFS(G, w)
        postDiscoveryTraversal(G, v, e, w)
      else
        setLabel(e, BACK)
        backTraversal(G, v, e, w)
  finishVertexVisit(G, v)
```

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Overriding hook methods in a subclass FindSimplePath (v2)

```
Algorithm findPath(G, u, v)
    S ← new empty stack   {S is a subclass field}
    start ← u             {start is a subclass field & is the starting vertex}
    dest ← v              {dest is a subclass field & is the destination vertex}
    path ← Ø              {path is a subclass field & is the path from u to v}
    return DFS(G)

Algorithm result(G)
    return(path)

Algorithm isNextComponent(G, v)
    return v=start        {start the component traversal at vertex start}

Algorithm startVertexVisit(G, v)
    S.push(v)
    if v=dest  then       {dest is a subclass field & is the destination vertex}
      path ← S.elements()  {path is a subclass field & is the result}

Algorithm preDiscoveryTraversal(G, v , e, w)
    S.push(e)

Algorithm postDiscoveryTraversal(G, v, e, w)
    S.pop()           {pop e off the stack}

Algorithm finishVertexVisit(G, v)
    S.pop()           {pop v off the stack}
```

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Exercise: Cycle Finding
Override hook operations

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  startVertexVisit(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        preDiscoveryTraversal(G, v, e, w)
        DFS(G, w)
        postDiscoveryTraversal(G, v, e, w)
      else
        setLabel(e, BACK)
        backEdgeVisit(G, v, e, w)
  finishVertexVisit(G, v)
```

```
Algorithm cycleDFS(G, v)
  setLabel(v, VISITED)
  if cycle = null then return
  S.push(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      S.push(e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        cycleDFS(G, w)
        S.pop()
      else
        setLabel(e, BACK)
        cycle ← new empty sequence
        o ← w
        repeat
          cycle.insertLast(o)
          o ← S.pop()
        until o = w
  S.pop()
```

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Exercise: Cycle Finding
Override hook operations (v2)

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  startVertexVisit(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      preEdgeTraversal(G, v, e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        preDiscoveryTraversal(G, v, e, w)
        DFS(G, w)
        postDiscoveryTraversal(G, v, e, w)
      else
        setLabel(e, BACK)
        backEdgeVisit(G, v, e, w)
  finishVertexVisit(G, v)
```

```
Algorithm cycleDFS(G, v)
  setLabel(v, VISITED)
  if cycle = null then return
  S.push(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      S.push(e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        cycleDFS(G, w)
        S.pop()
      else
        setLabel(e, BACK)
        cycle ← new empty sequence
        o ← w
        repeat
          cycle.insertLast(o)
          o ← S.pop()
        until o = w
  S.pop()
```

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Overriding template methods in subclass FindCycles

```
Algorithm startVertexVisit(G, v)
  if ¬ cycleFound then S.push(v)

Algorithm finishVertexVisit(G, v)
  if ¬ cycleFound then S.pop()

Algorithm preEdgeTraversal(G, v, e)
  if ¬ cycleFound then S.push(e)

Algorithm postDiscoveryTraversal(G, v, e, w)
  if ¬ cycleFound then S.pop()

Algorithm backEdgeVisit (G, v, e, w)
  if ¬ cycleFound then
      cycle ← new empty sequence
      o ← w
      repeat
          cycle.insertLast(o)
          o ← S.pop()
      until o = w
      cycleFound ← true  {cycleFound is a subclass field, initially false}
```

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- What additional method(s) do we need create or need to override?

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Template Version of DFS

```
Algorithm DFS(G)
  Input graph G
  Output the edges of G are
         labeled as discovery edges
         and back edges

  initResult( G )
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
    postInitVertex(u)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
    postInitEdge(e)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      preComponentVisit(G, v)
      DFS(G, v)
      postComponentVisit(G, v)

  return result( G )
```

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  startVertexVisit(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      preEdgeTraversal(G, v, e, w)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        preDiscoveryTraversal(G, v, e, w)
        DFS(G, w)
        postDiscoveryTraversal(G, v, e, w)
      else
        setLabel(e, BACK)
        backTraversal(G, v, e, w)
  finishVertexVisit(G, v)
```

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BFS Template Method

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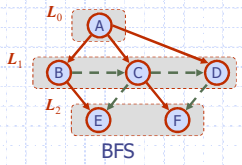
Applications of BFS (review)

- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

BFS Levels

When actually implemented, the levels are normally merged into a single sequence/queue



BFS

BFS Algorithm (revised)

- The BFS algorithm using a single list/sequence/Queue L

```
Algorithm BFS(G)
  Input graph G
  Output labeling of the edges
      and partition of the
      vertices of G
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      BFS(G, v)\
```

```
Algorithm BFS(G, s)
  L ← new empty List
  L.insertLast(s)
  setLabel(s, VISITED)
  while ¬L.isEmpty()
    v ← L.remove (L.first())
    for all e ∈ G.incidentEdges(v)
      if getLabel(e) = UNEXPLORED then
        w ← G.opposite(v,e)
        if getLabel(w) = UNEXPLORED
        then
          setLabel(e, DISCOVERY)
          setLabel(w, VISITED)
          L.insertLast(w)
        else
          setLabel(e, CROSS)
```

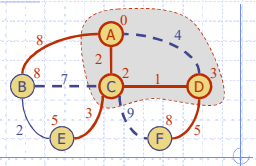

Main Point

1. The Template Method Pattern implements the changing and non-changing parts of an algorithm in the superclass; it then allows subclasses to override certain (changeable) steps of an algorithm without modifying the basic structure of the original algorithm.

   The changing and non-changing aspects of creation are unified in the field pure intelligence that we experience every day during our TM program.

Shortest Paths

Path of Least Action

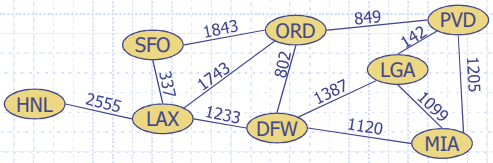


Outline and Reading

- Weighted graphs (§7.1)
  - Shortest path problem
  - Shortest path properties
- Dijkstra's algorithm (§7.1.1)
  - Algorithm
  - Edge relaxation

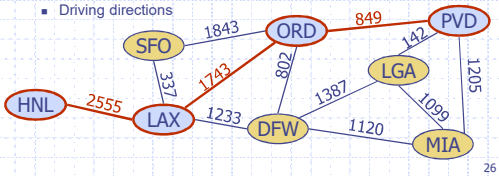
Weighted Graphs

- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



Shortest Path Problem

- Given a weighted graph and two vertices u and v, we want to find a path of minimum total weight between u and v.
  - Length of a path is the sum of the weights of its edges.
- Example:
  - Shortest path between Providence and Honolulu
- Applications
  - Internet packet routing
  - Flight reservations
  - Driving directions

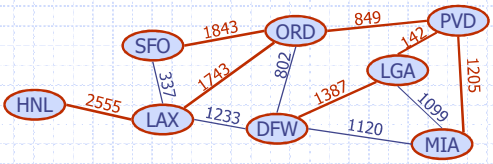


Shortest Path Properties

Property 1:
A subpath of a shortest path is itself a shortest path

Property 2:
There is a tree of shortest paths from a start vertex to all the other vertices

Example:
Tree of shortest paths from Providence



Dijkstra's Algorithm

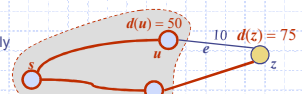
- The distance of a vertex v from a vertex s is the length of a shortest path between s and v
- Dijkstra's algorithm computes the shortest distances of all the vertices from a given start vertex s
- Assumptions:
  - the graph is connected
  - the edges are undirected
  - the edge weights are **nonnegative**

Dijkstra's Algorithm (Informal)

- We grow a "**cloud**" of vertices, beginning with s and eventually covering all the vertices
- We store with each vertex v a label $d(v)$
  - represents the distance of v from s in the subgraph consisting of the cloud and its adjacent vertices
- At each step
  - We add to the cloud a vertex u
    - outside the cloud
    - with the smallest distance label, $d(u)$
  - Then we update the labels of the vertices adjacent to u

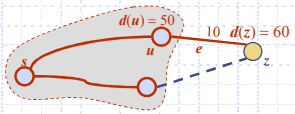
Edge Relaxation

- Consider an edge $e = (u,z)$ such that
  - u is the vertex most recently added to the cloud
  - z is not in the cloud

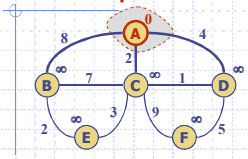


- The relaxation of edge e updates distance $d(z)$ as follows:

$d(z) \leftarrow \min\{d(z), d(u)+weight(e)\}$



Example



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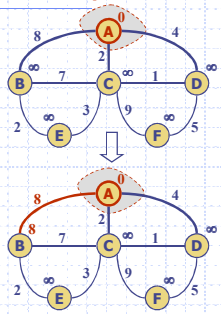
Example



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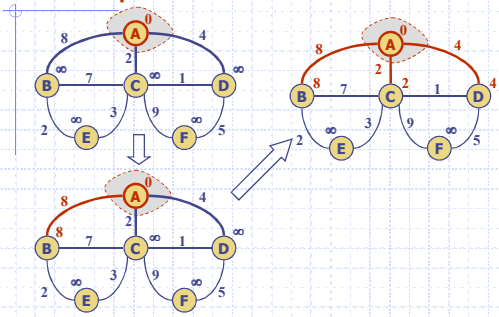
Example



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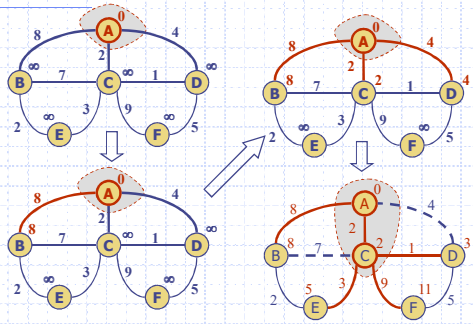
Example



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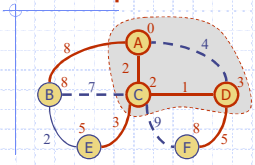
Example



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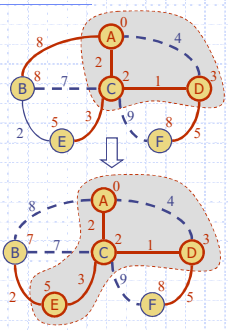
Example



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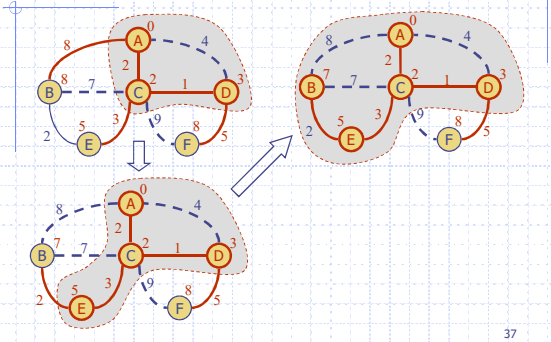
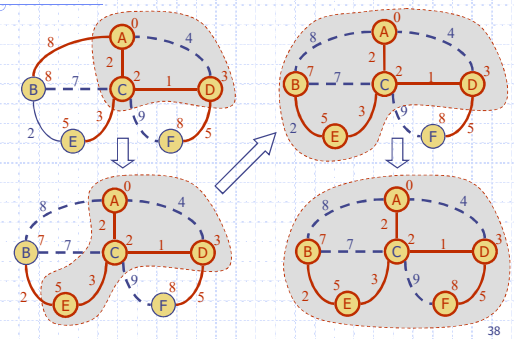
Example



Example

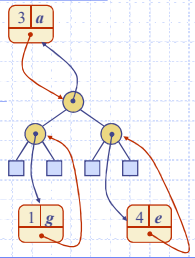


Dijkstra's Algorithm (version 1)

- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- We store the distance with each vertex:
  - Distance (d(v) label)
- In the Relax step we need to change the location of the vertex z in the priority queue
  - How do we do this efficiently?

```
Algorithm DijkstraDistances(G, s)
  Q ← new heap-based priority queue
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    Q.insertItem(getDistance(v), v)
  while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
      { relax edge e }
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        Q.replaceKey(z,r)  {new method}
```


Locators



Outline and Reading

- Locators (§2.4.4)
- Locator-based methods (§2.4.4)
- Implementation
- Positions vs. Locators

Locator Design Pattern

- A mechanism for maintaining the association between an item and its current position in a container
- When an item is inserted into a container, we get back a locator for that item
  - this locator can be used later to refer to that same item in the container even if its position has changed
  - an item's position may change when items are inserted or an item's key is changed

Locators

- identifies and tracks a (key, element) item within a data structure
- sticks with a specific item, even if that element changes its position in the data structure
- Intuitive notion:
  - claim check
  - reservation number
- Methods of the Locator ADT:
  - key(): returns the key of the item associated with the locator
  - element(): returns the element of the item associated with the locator
- Application example:
  - Orders to purchase and sell a given stock are stored in two priority queues (sell orders and buy orders)
    - the key of an order is the price
    - the element is the number of shares
  - When an order is placed, a Locator to it is returned
  - Given a Locator, an order can be canceled or modified

Positions vs. Locators

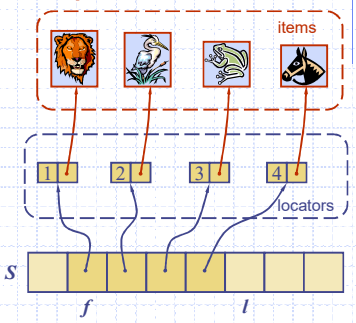
- Position
  - represents a "place" in a data structure
  - related to other positions in the data structure (e.g., previous/next or parent/child)
  - implemented as a node or an array cell
- Position-based ADTs are fundamental data storage schemes
- (e.g., list, sequence, and tree)
- Locator
  - identifies and tracks a (key, element) item
  - unrelated to other locators in the data structure
  - implemented as an object storing the item and its position in the underlying structure
- Key-based ADTs can be augmented with locator-based methods
  - (e.g., priority queue and dictionary)

Locator-based Methods added to the Priority Queue

- Locator-based methods:
  - insert(k, o): inserts the item (k, o) and returns a locator for it
  - min(): returns the locator of an item with smallest key
  - remove(l): remove the item with locator l
  - replaceKey(l, k): replaces the key of the item with locator l
  - replaceElement(l, o): replaces with o the element of the item with locator l
  - locators(): returns an iterator over the locators of the items in the priority queue (or dictionary)

Array-based Implementation

- We use an array storing locators
- A position object stores:
  - Item (key,element)
  - Rank/Index
- Indices f and l keep track of first and last locators



Sequences

Dijkstra's Algorithm with Locators

- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- Locator-based methods
  - *insert*(*k*,*e*) returns a locator
  - *replaceKey*(*l*,*k*) changes the key of an item
- We store two labels with each vertex:
  - Distance (d(v) label)
  - locator in priority queue

```
Algorithm DijkstraDistances(G, s)
  Q ← new heap-based priority queue
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
  while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
      { relax edge e }
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        Q.replaceKey(getLocator(z),r)
```


Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
  - Recall that $\sum_v \deg(v) = 2m$
- Label operations of vertices
  - We set/get the distance and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Dijkstra's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
- The running time can also be expressed as $O(m \log n)$ since the graph is connected. Why?

- If the graph is connected, then m ≥ n-1
- Therefore, O(m + n) is …

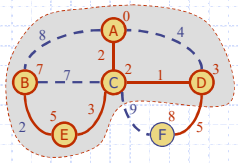
Template Pattern Extension

- Using the template method pattern, we can extend Dijkstra's algorithm to return a tree of shortest paths from the start vertex to all other vertices
- We store with each vertex a third label:
  - parent edge in the shortest path tree
- In the edge relaxation step, we update the parent label

```
Algorithm DijkstraShortestPathsTree(G, s)
    …
    for all v ∈ G.vertices()
      …
      setParent(v, ∅)
      …
    for all e ∈ G.incidentEdges(u)
      { relax edge e }
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setParent(z,e)
        Q.replaceKey(getLocator(z),r)
```

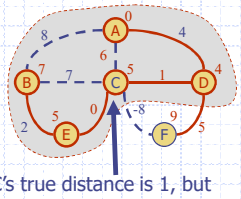

Why Dijkstra's Algorithm Works

- Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.
  - Suppose it didn't find all shortest distances. Let F be the first wrong vertex the algorithm processed.
  - When the previous node, D, on the true shortest path was considered, its distance was correct.
  - But the edge (D,F) was **relaxed** at that time!
  - Thus, as long as d(F)≥d(D), F's distance cannot be wrong. That is, there is no wrong vertex distance.



Why It Doesn't Work for Negative-Weight Edges

- Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.
  - If a node with a negative incident edge were to be added late to the cloud, it could mess up distances for vertices already in the cloud.



C's true distance is 1, but it is already in the cloud with d(C)=5!

Bellman-Ford Algorithm (later)

Works even with negative-weight edges

Must assume directed edges

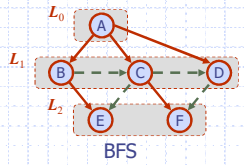
Main Point

2. By using the adjacency list data structure to represent the graph and a priority queue enhanced with locators to store the vertices not yet in the tree, the shortest path algorithm achieves a running time $O(m \log n)$.
   The algorithms of nature are always most efficient for maximum growth and progress.

BFS Levels

When actually implemented, the levels are normally merged into a single sequence/queue

How could we keep track of the level of a vertex? (HW)



BFS

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Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. Finding the shortest path to some desired goal is a common application problem in systems represented by weighted graphs, such as airline or highway routes.
2. By systematically extending short paths using data structures **especially suited** to this process, the shortest path algorithm operates in time $O(m \log n)$.

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3. **Transcendental Consciousness** is the silent field of infinite correlation where everything is eternally connected by the shortest path.
4. **Impulses within Transcendental Consciousness**: Because the natural laws within this unbounded field are infinitely correlated (no distance), they can govern all the activities of the universe simultaneously.
5. **Wholeness moving within itself:** In Unity Consciousness, the individual experiences the shortest path between one's Self and everything in the universe, a path of zero length.

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