Lecture 4: Priority Queues, Sorting, and Heaps

Pure Consciousness is a Field of Perfect Order

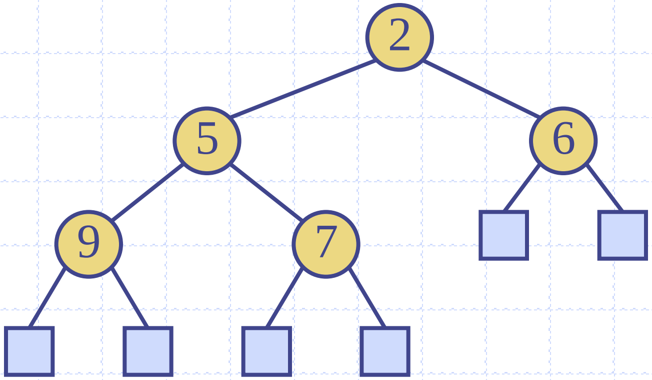
Wholeness Statement

The Priority Queue ADT stores any kind of object as a *key object* pair, but the *keys* must be objects that have a *total order relation* (or *linear ordering*). Each individual has access to the source of thought which is a field perfect order and balance.

Overview

- Priority Queue ADT
- Sorting with a Priority Queue
- Heap Data Structure

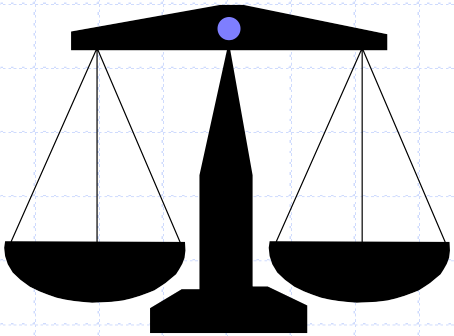
Priority Queues



Priority Queue ADT (§ 2.4.1)

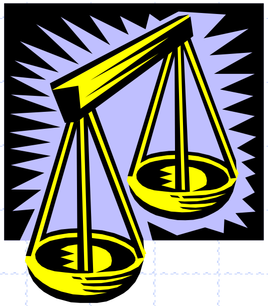
- A priority queue stores a collection of items
- An item is a pair (key, element)
- Main methods of the Priority Queue ADT
  - insertItem(k, e) inserts an item with key k and element e
  - removeMin() removes the item with smallest key and returns its element
- Additional methods
  - minKey() returns, but does not remove, the smallest key of an item
  - minElement() returns, but does not remove, the element of an item with smallest key
  - size(), isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Total Order Relation



- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct items in a priority queue can have the same key
- Mathematical concept of total order relation $\leq$
  - **Reflexive** property: $x \leq x$
  - **Antisymmetric** property: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - **Transitive** property: $x \leq y \wedge y \leq z \Rightarrow x \leq z$
  - **Totality** property: $x \leq y \vee y \leq x$

Comparator ADT (§ 2.4.1)



- A comparator encapsulates the action of comparing two objects according to a given total order relation
- A generic priority queue uses an auxiliary comparator
- The comparator is external to the keys being compared
- When the priority queue needs to compare two keys, it uses its comparator

- Methods of the Comparator ADT, all with Boolean return type
  - isLessThan(x, y)
  - isLessThanOrEqualTo(x,y)
  - isEqualTo(x,y)
  - isGreaterThan(x, y)
  - isGreaterThanOrEqualTo(x,y)
  - isComparable(x)

Sorting with a Priority Queue (§ 2.4.2)

- We can use a priority queue to sort a set of comparable elements
  - Insert the elements one by one with a series of insertItem(e, e) operations
  - Remove the elements in sorted order with a series of removeMin() operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm *PQ-Sort*(*S*, *C*)
  Input sequence *S*, comparator *C* for the elements of *S*
  Output sequence *S* sorted in increasing order according to *C*
  $P \leftarrow$ new priority queue using C
  while $\neg$*S.isEmpty* () **do**
    $e \leftarrow$ *S.remove* (*S. first* ())
    P.insertItem(*e*, *e*)
  while $\neg$*P.isEmpty*() **do**
    $e \leftarrow$ *P.removeMin*()
    S.insertLast(*e*)

Sequence-based Priority Queue

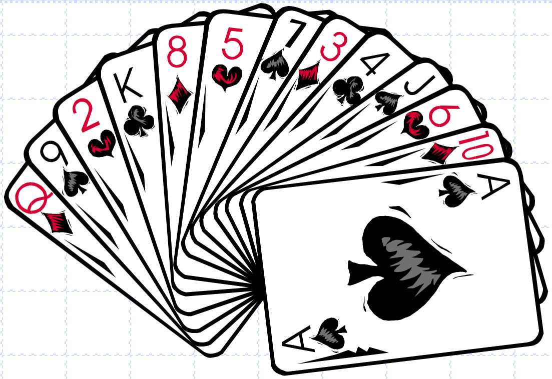
- Implementation with an unsorted list

- Performance:
  - insertItem takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin, minKey and minElement take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list

- Performance:
  - insertItem takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin, minKey and minElement take $O(1)$ time since the smallest key is at the beginning of the sequence

Selection-Sort



- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence

- Running time of Selection-sort:
  - Inserting the elements into the priority queue with n insertItem operations takes $O(n)$ time
  - Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$n + \ldots + 2 + 1$$

- Selection-sort runs in $O(n^2)$ time

Insertion-Sort

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence

- Running time of Insertion-sort:
  - Inserting the elements into the priority queue with n insertItem operations takes time proportional to
    $$1 + 2 + \ldots + n$$
  - Removing the elements in sorted order from the priority queue with a series of n removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

Main Point

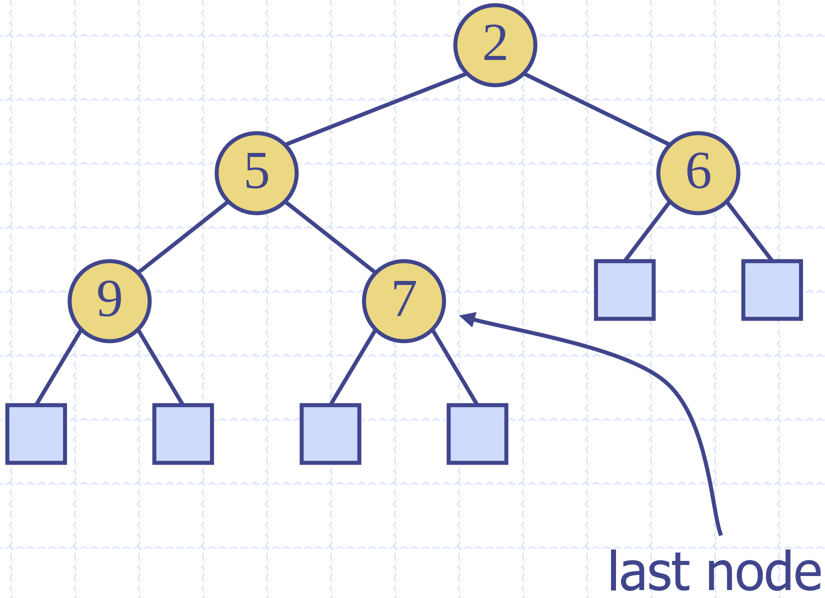
1. Insertion sort starts with an initial list with one element, then inserts each new element such that the resulting sequence is also in order. Selection sort selects the smallest element each iteration from an unsorted list and inserts it at the end of the target list. Neither of these algorithms is optimal. Pure intelligence always follows the optimal law of least action.

The Heap Data Structure

What is a heap (§2.4.3)

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - Heap-Order: for every internal node v other than the root,
    $key(v) \geq key(parent(v))$
  - Complete Binary Tree: let h be the height of the heap
    - for $i = 0, \ldots, h - 1$, there are 2^i nodes of depth i
    - at depth $h - 1$, the internal nodes are to the left of the external nodes

- The last node of a heap is the rightmost internal node of depth $h - 1$



Heap-Order Property

- For all internal nodes v (except the root):

  $$key(v) \geq key(parent(v))$$

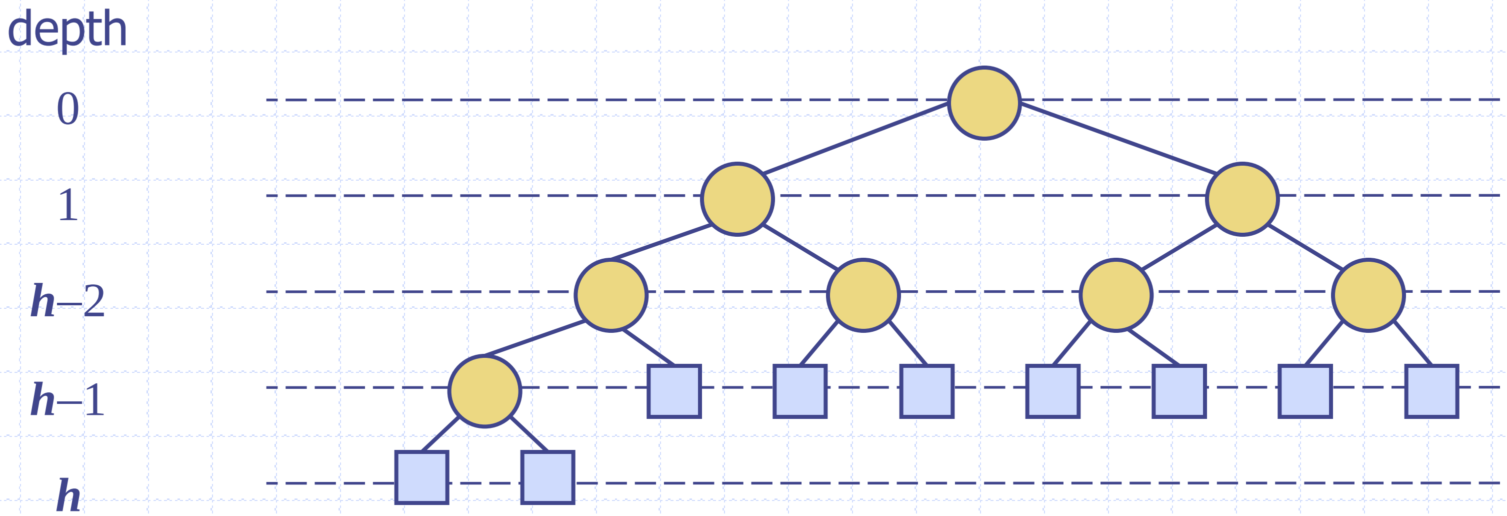
  - That is, the key of every child node is greater than or equal to the key of its parent node

Other Properties of a Heap

- A heap is a binary tree whose values are in ascending order on every path from root to leaf
- Values are stored in internal nodes only
- A heap is a binary tree whose root contains the minimum value and whose subtrees are heaps

Heap

- All leaves of the tree are on two adjacent levels
- The binary tree is complete on every level except the deepest level.

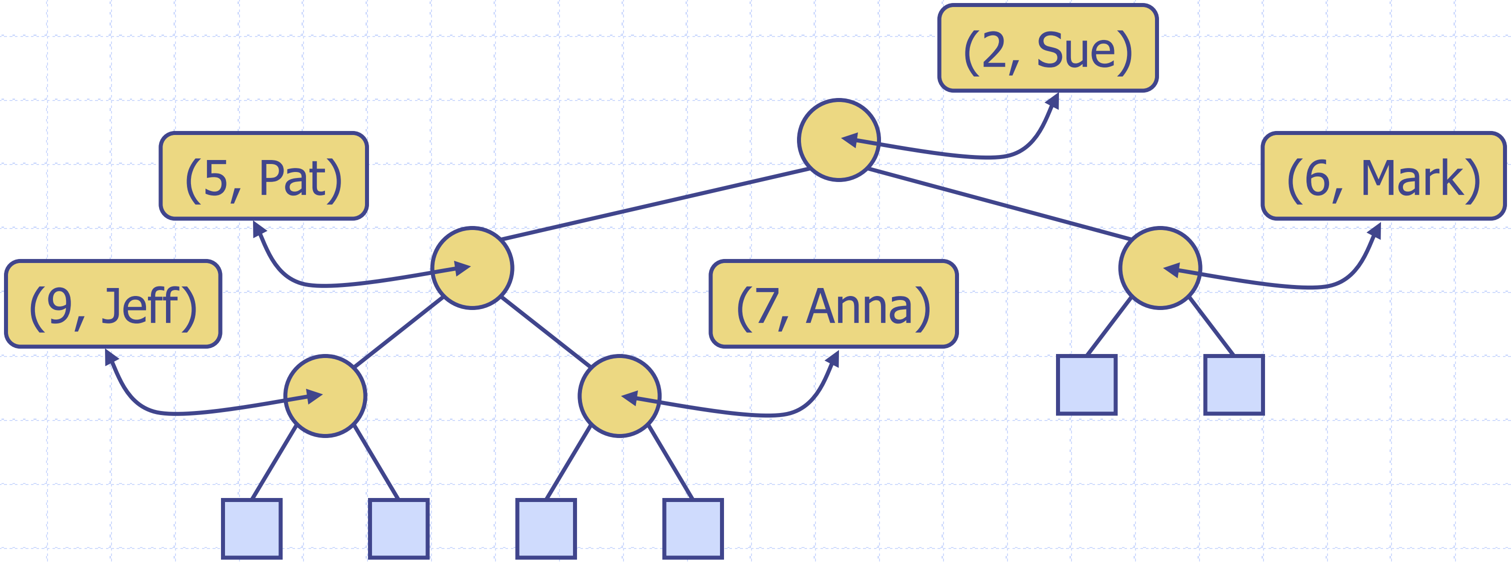


Adding Nodes to a Heap

- New nodes must be added left to right at the lowest level, i.e., the level containing internal and external nodes or containing all external nodes

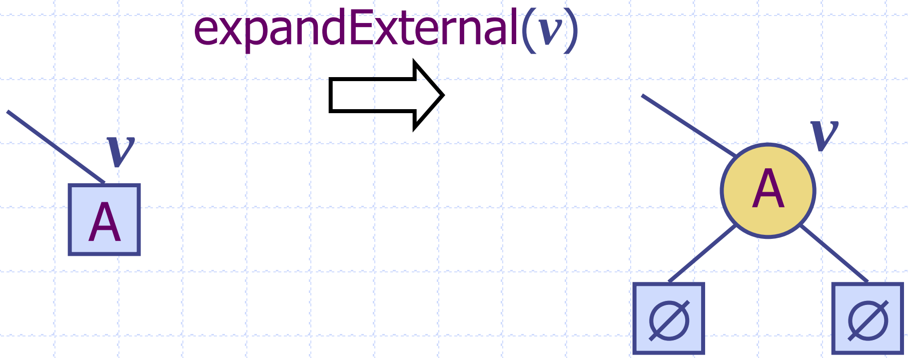
Heaps and Priority Queues

- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
- For simplicity, we show only the keys in diagrams

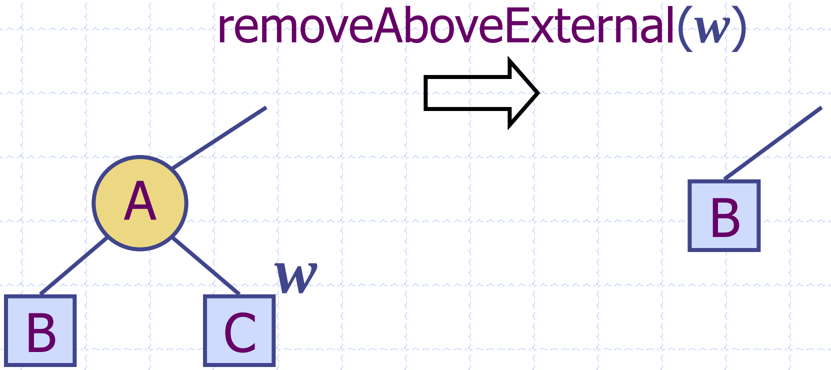


Additional Tree Update Methods

- expandExternal(v)

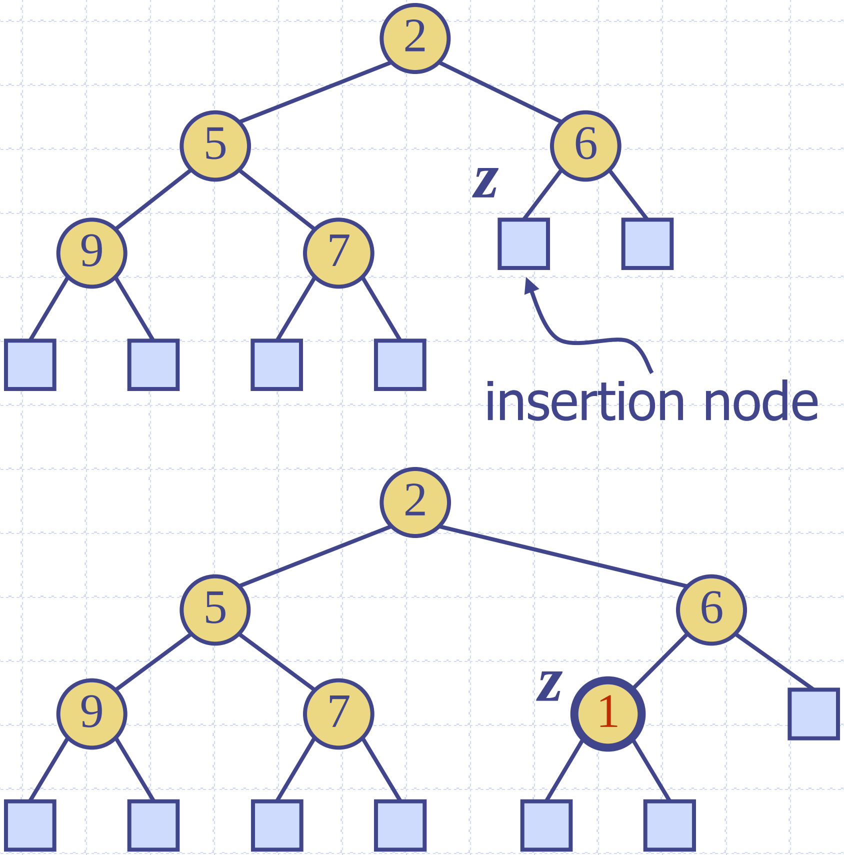


- removeAboveExternal(w)



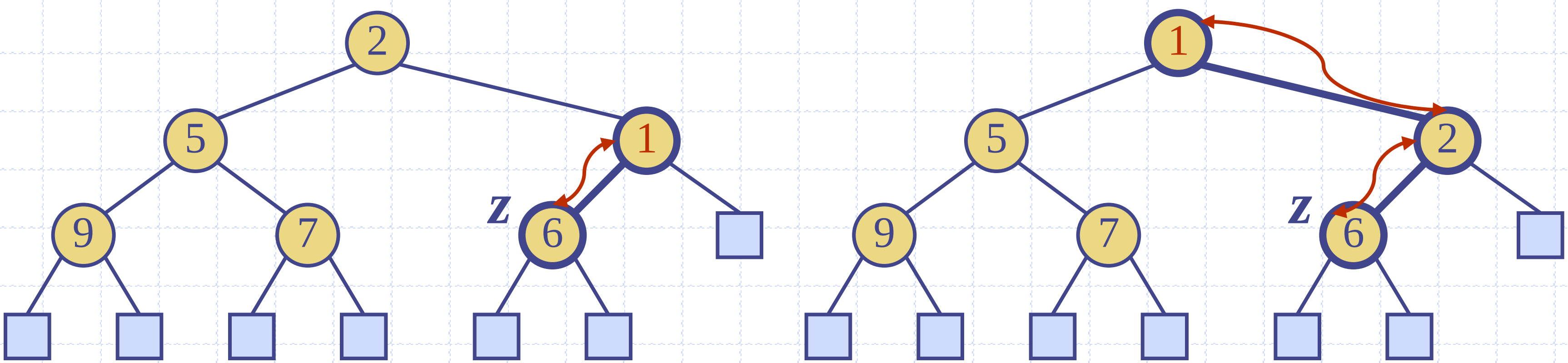
Insertion into a Heap (§2.4.3)

- Method insertItem of the priority queue ADT corresponds to the insertion of a key ***k*** into the heap
- The insertion algorithm consists of three steps
  1. Find the insertion node ***z*** (the new last node)
  2. Store ***k*** at ***z*** and expand ***z*** into an internal node
  3. Restore the heap-order property



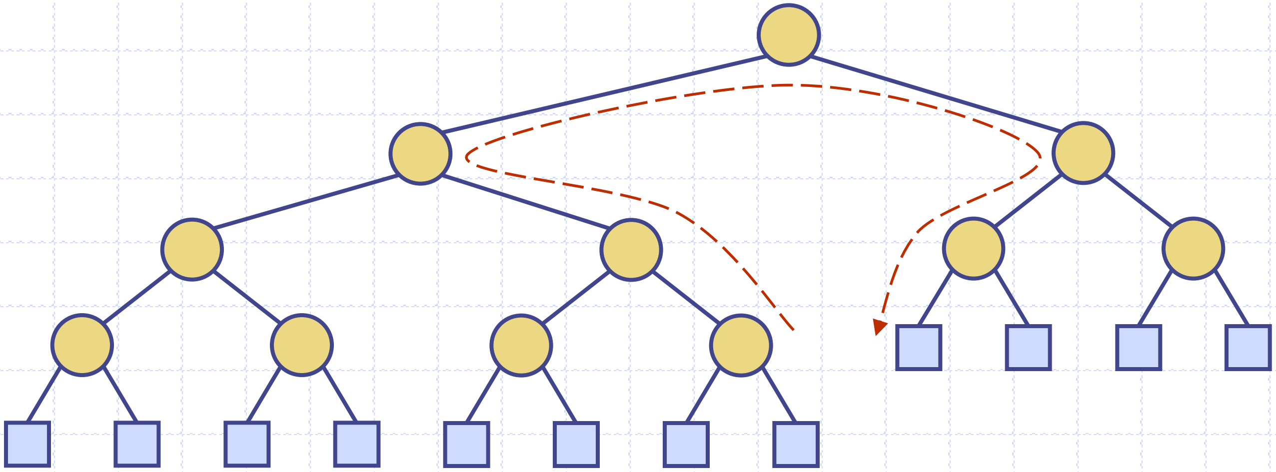
Upheap

- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



Finding the Insertion Node and Updating the Last Node

- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - While the current node is a right child, go to the parent node
  - If the current node is a left child, go to the right child
  - While the current node is internal, go to the left child
- Similar algorithm for updating the last node after a removal



Exercise

- Insert the following keys into a heap represented as a Tree:

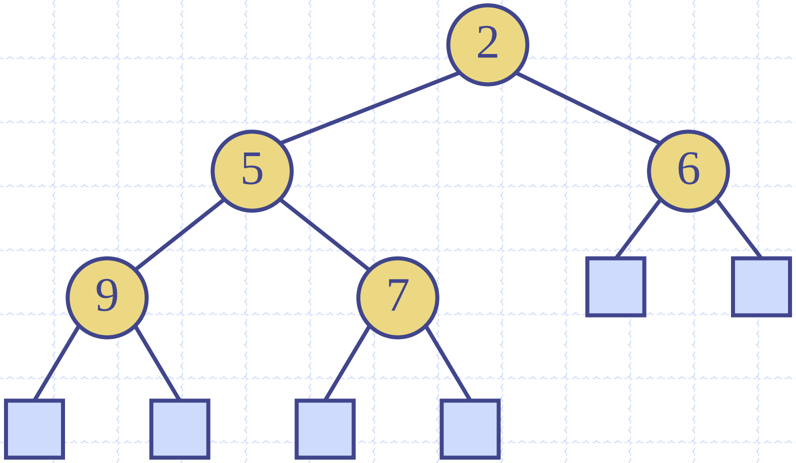
  9, 6, 5, 14, 4, 12, 15, 3, 2

Efficient Representation of A Heap

Use an Array, Vector, or Sequence with random access

Vector (or Array) based Heap Implementation (§2.4.3)

- We can represent a heap with n keys by means of a vector of length $n + 1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i + 1$
- Links between nodes are not explicitly stored
- The leaves are not represented
- The cell at rank 0 is not used
- Operation insertItem corresponds to inserting at rank $n + 1$
- Operation removeMin corresponds to removing at rank n
- Yields in-place heap-sort



	2	5	6	9	7
0	1	2	3	4	5

Exercise

- Insert the following keys into a heap represented as a Vector/Array:

  9, 6, 5, 14, 4, 12, 15, 3, 2

Implementation of upHeap

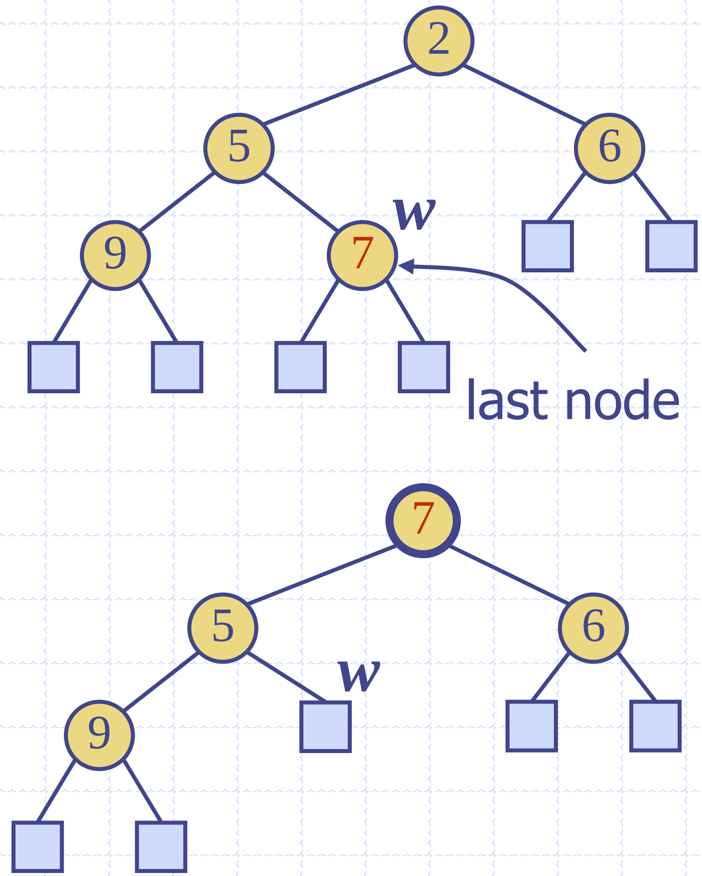
Algorithm ***upHeap***(***H***, ***i***)

Input Array H representing a heap and index i of an element in the heap
Output H with the heap property restored

parent ← **i / 2**
if 1 ≤ **parent** ∧ ***H*[parent]** > ***H*[i]** **then**
temp ← ***H*[parent]**
***H*[parent]** ← ***H*[i]** **{swap elements}**
H*[i]** ← ***temp
upHeap(***H***, ***parent***)

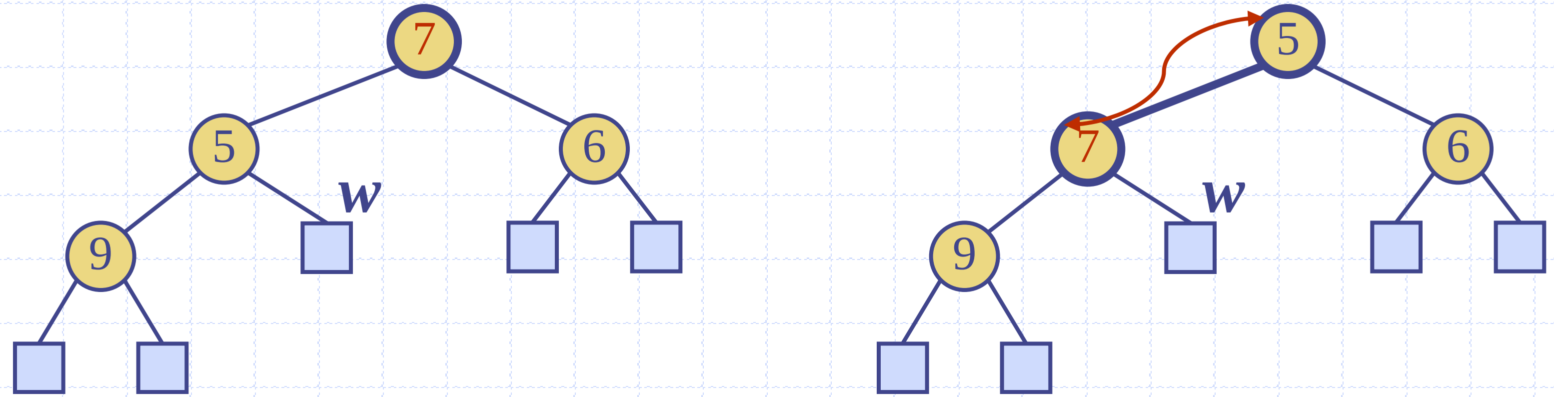
Removal from a Heap (§2.4.3)

- Method removeMin of the priority queue ADT corresponds to the removal of the root key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node w
  - Compress w and its children into a leaf
  - Restore the heap-order property



Downheap

- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root
- Downheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time



Exercise:

- Write the pseudocode for downHeap
  - You can have any interface you wish
  - You will need an extra argument, i.e., the size of the heap (Why?)

  - The interface for a recursive version of upHeap was upHeap(H, i)
    - So the interface of a recursive version would be downHeap(H, i, size)
    - An iterative version would have interface downHeap(H, size)

Recursive Version

Algorithm ***downHeap***(***H, r, size***)

Input Array H representing a heap, rank ***r*** of an element in H, and the size of the heap H

Output H with the heap property restored

smallest ← rankOfMin(H, ***r***, *size*) **{min of r and its children}**

if ***smallest*** **¬ = r then**

  temp ← ***H*****[smallest]**

  H**[smallest]** ← ***H*****[r]** **{swap elements}**

  H**[r]** ← ***temp***

  downHeap(***H, smallest, size***)

Helper for downHeap Algorithm

Algorithm ***rankOfMin***(***A, r, size***)

Input array *A*, a rank ***r*** (containing an element of A), and ***size*** of the heap stored in *A*

Output the rank of element in *A* containing the smallest value

smallest ← r
left ← **2*r**
right ← **2*r + 1**
if left ≤ size ∧ *A*[left] < *A*[smallest] then
  smallest ← left
if right ≤ size ∧ *A*[right] < *A*[smallest] then
  smallest ← right
return ***smallest***

Iterative Version

Algorithm ***downHeap***(***H, size***)

Input Array H representing a heap and the size of H (size $\geq$ 1)

Output H with the heap property restored

```
property ← false
i ← 1
while ¬ property  do
    smallest ← rankOfMin(H, i, size)
    if smallest  != i  then
        temp ← H[smallest]
        H[smallest] ← H[i]                {swap elements}
        H[i] ← temp
        i ← smallest
    else
        property ← true

return H
```

Analysis of Heap Operations

- Upheap()
- Downheap()

Analysis of Heap-based Priority Queue

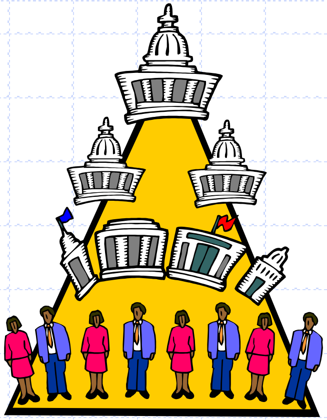
- insertItem(k, e)
- removeMin()
- minKey()
- minElement()
- size()
- isEmpty()

Analysis of Sorting with a Heap-based Priority Queue

- What is the running time of this sorting method if the priority queue is implemented as a Heap?

Algorithm *PQ-Sort*(*S*, *C*)
    Input sequence ***S***, comparator ***C*** for the elements of ***S***
    Output sequence ***S*** sorted in increasing order according to ***C***
    $P \leftarrow$ priority queue with comparator ***C***
    while $\neg$***S.isEmpty*** () **do**
        e $\leftarrow$ ***S.remove*** (***S. first*** ())
        P.insertItem(***e***, ***e***)
    while $\neg$***P.isEmpty***() **do**
        e $\leftarrow$ ***P.removeMin***()
        S.insertLast(***e***)

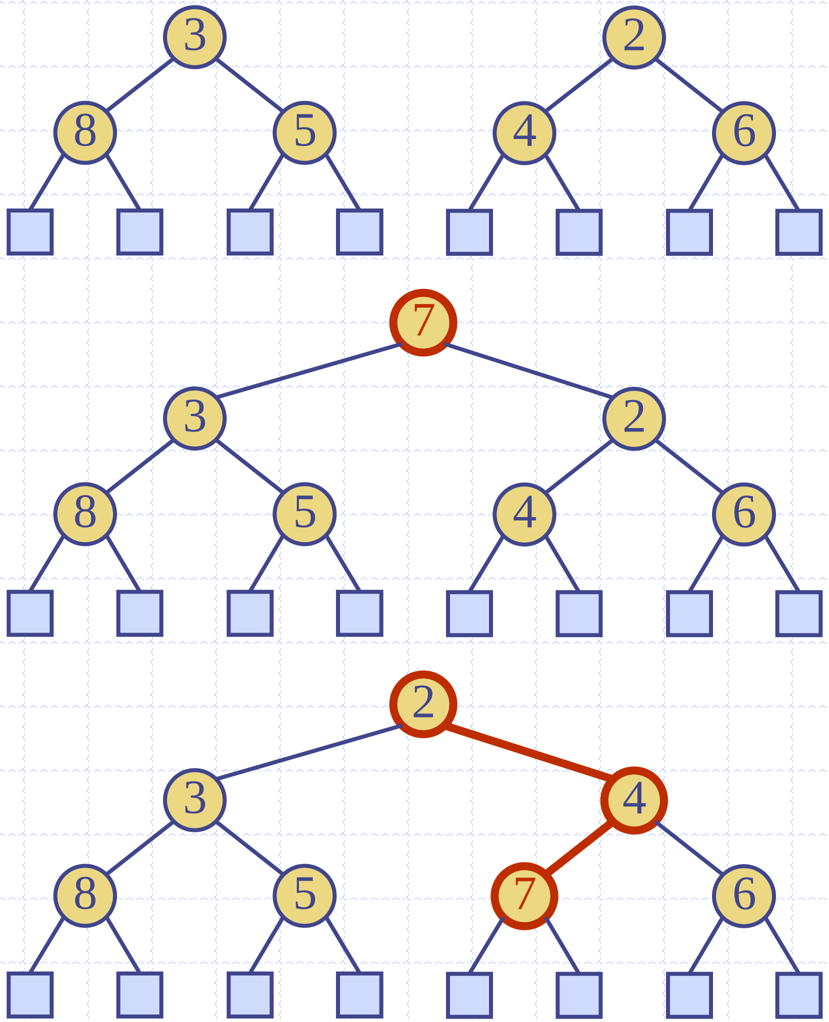
Analysis of Heap-Based Priority Queue (§2.4.4)



- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insertItem and removeMin take $O(\log n)$ time
  - methods size, isEmpty, minKey, and minElement take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

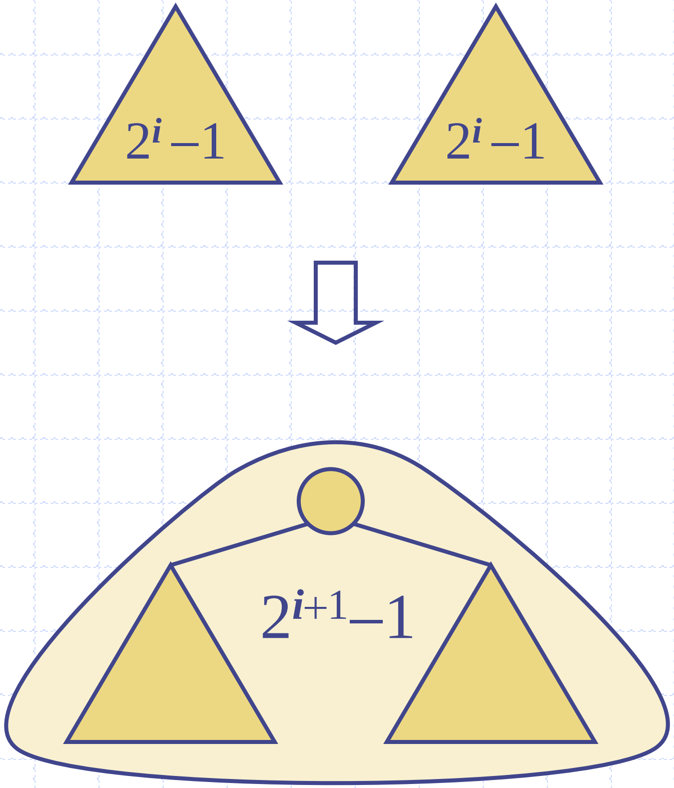
Merging Two Heaps

- We are given two heaps and a key ***k***
- We create a new heap with the root node storing ***k*** and with the two heaps as subtrees
- We call downHeap to restore the heap-order property

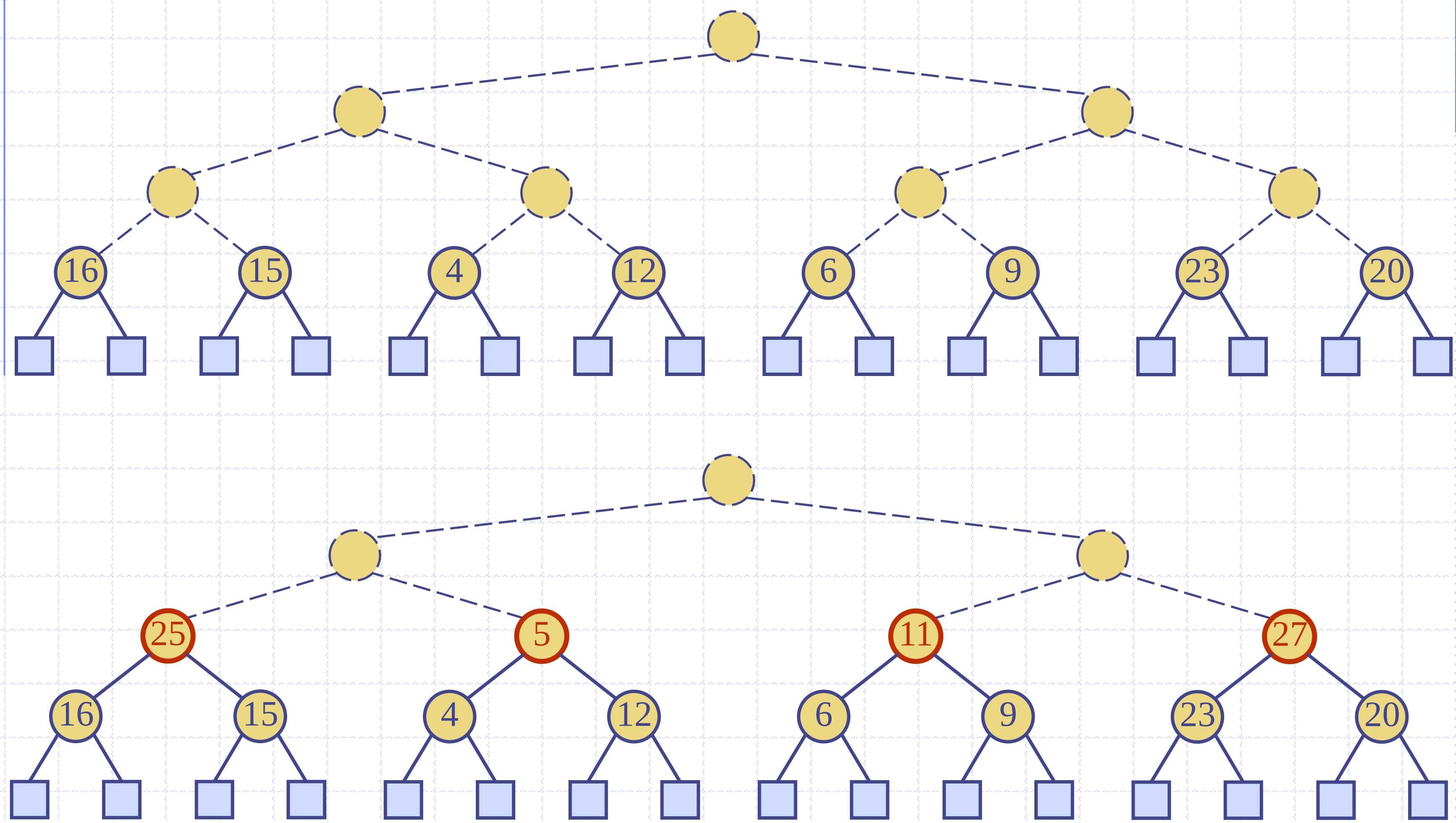


Bottom-up Heap Construction (§2.4.4)

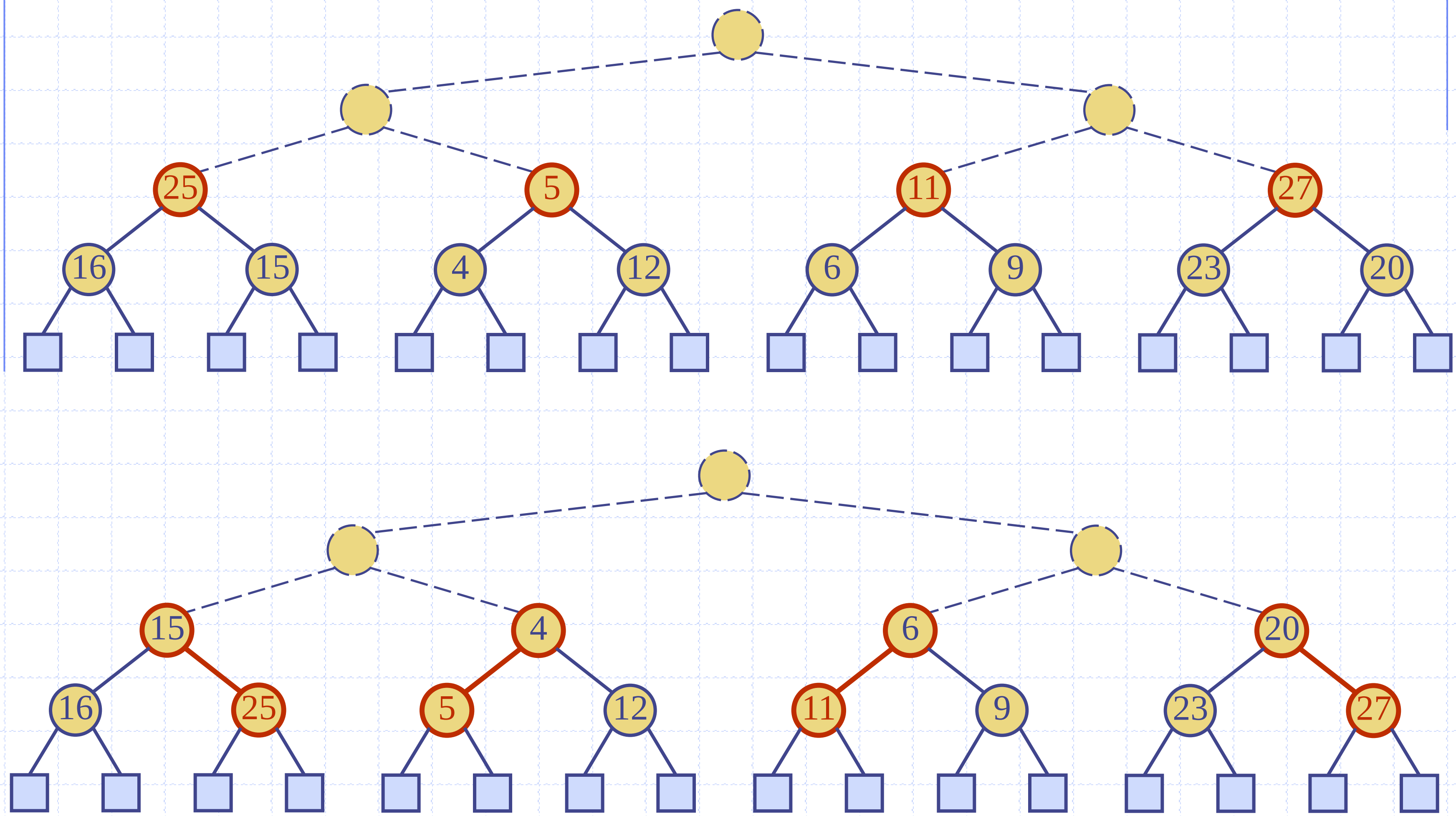
- We can construct a heap storing n given keys using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1} - 1$ keys



Example



Example (contd.)



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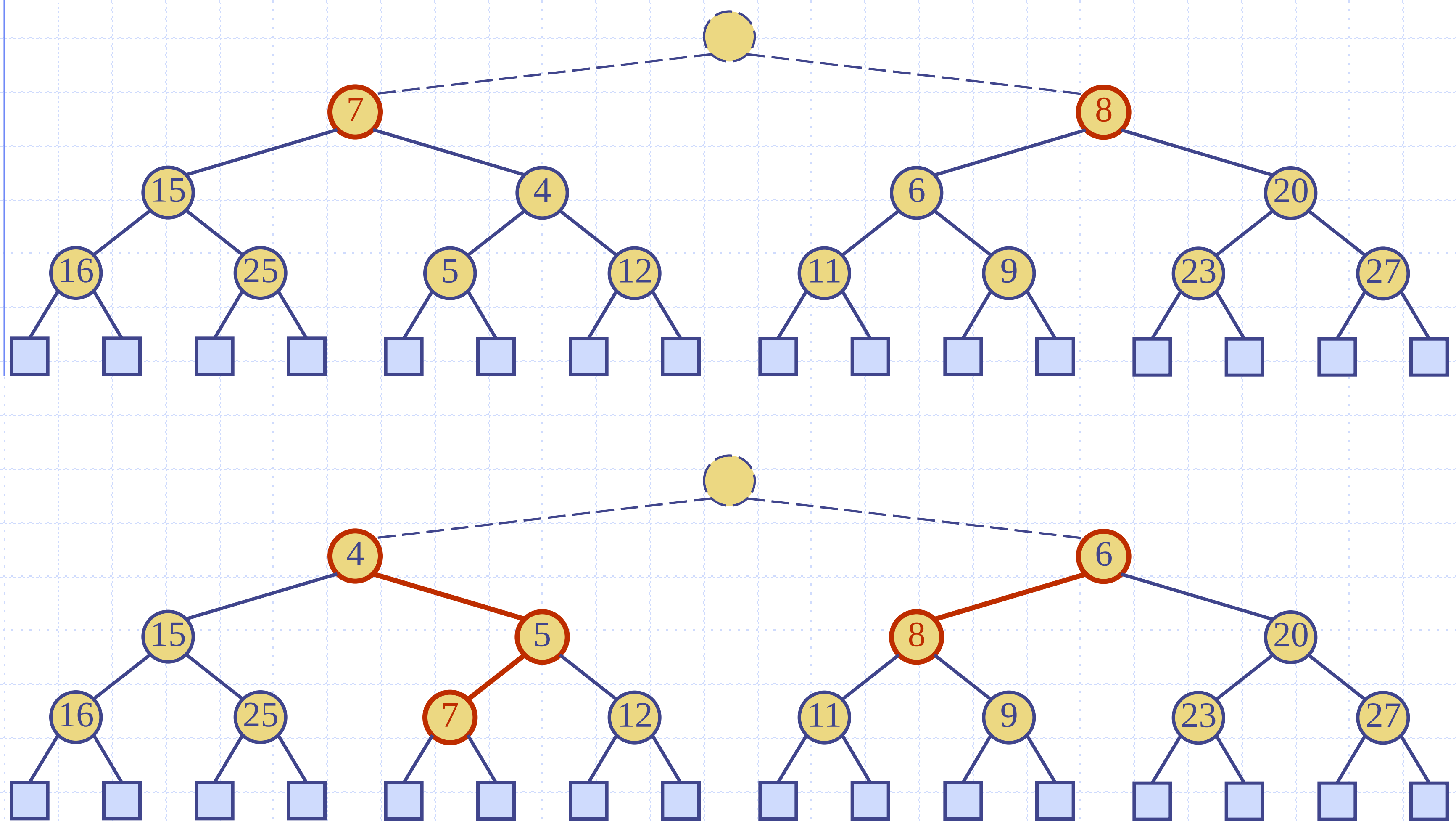
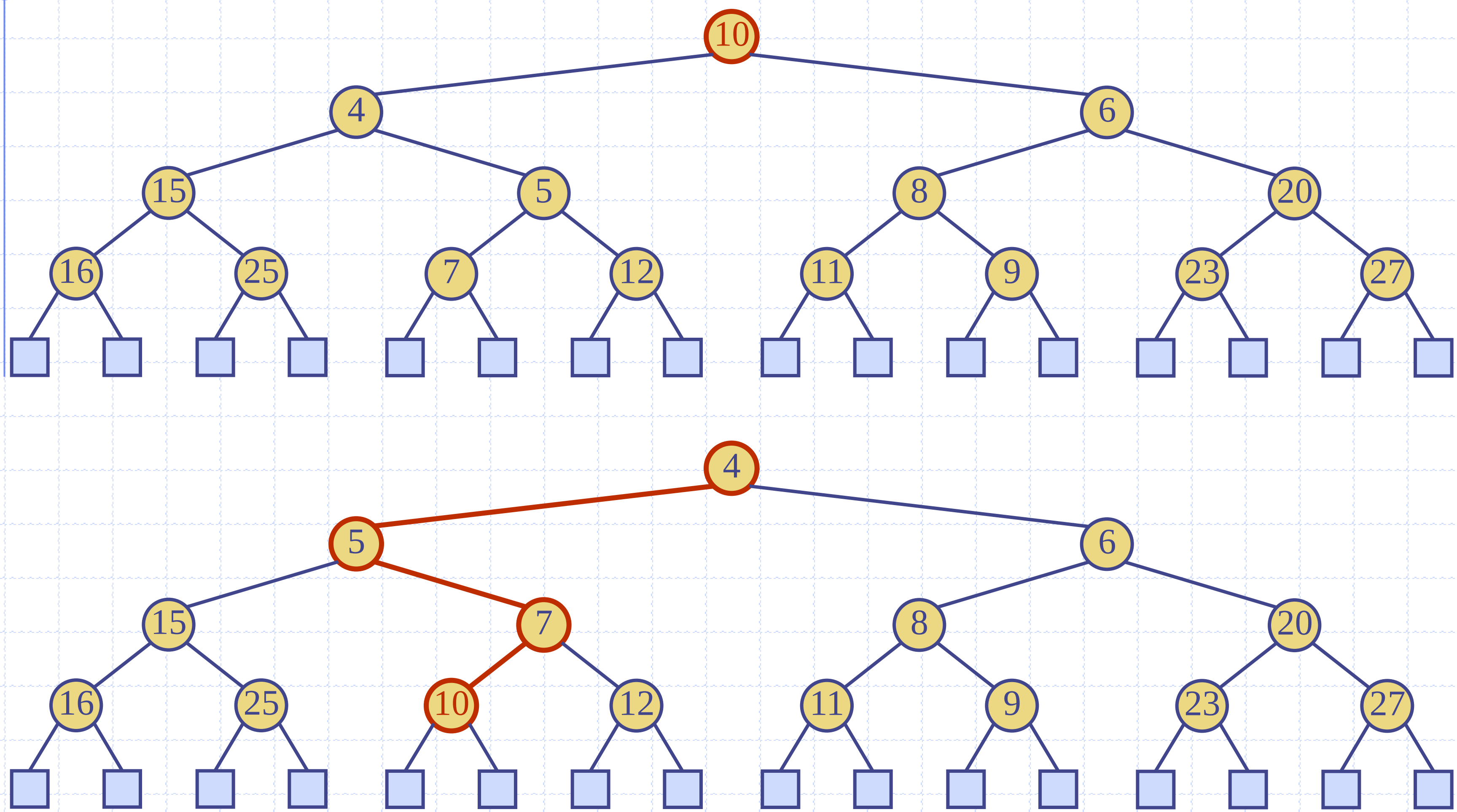
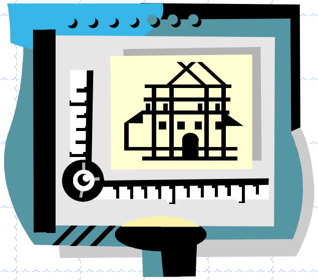


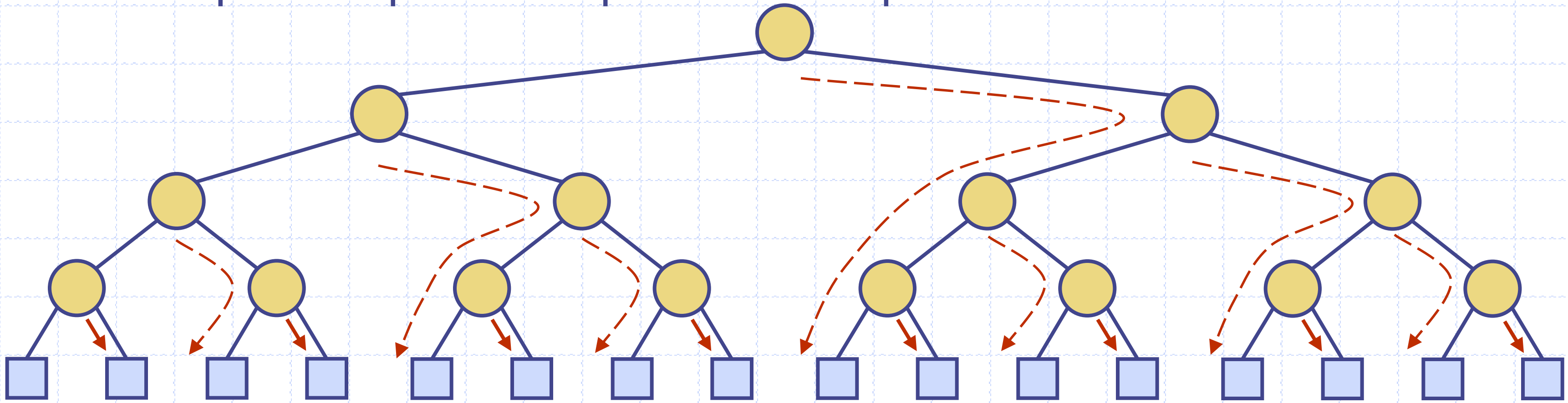
Example (end)



Analysis of Bottom-up Heap Construction



- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



Main Point

2. A heap is a binary tree that stores *key object* pairs at each internal node and maintains *heap-order* and is *complete*. Heap-order means that for every node *v* (except the root), $key(v) \geq key(parent(v))$.

   Pure consciousness is the field of wholeness, perfectly orderly, and complete.

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◈ slow ◈ in-place ◈ for small data sets (< 1K)
insertion-sort	$O(n^2)$	◈ slow ◈ in-place ◈ for small data sets (< 1K)
heap-sort	$O(n \log n)$	◈ fast ◈ in-place ◈ for large data sets (1K — 1M)

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. Sorting with a Priority Queue is a simple process of inserting the elements in the queue and removing them using the *removeMin* operation.

2. How the Priority Queue is implemented determines its efficiency when used in a sort, i.e., if implemented as a Heap, then the sorting algorithm is optimal, $O(n \log n)$.

3. <u>Transcendental Consciousness</u> is the unbounded field of pure order and efficiency.
4. **<u>Impulses within Transcendental Consciousness</u>**: The laws of nature are non-changing and universal which provide a reliable basis for the integrity of the universe.
5. **<u>Wholeness moving within itself</u>** : In Unity Consciousness, life is spontaneously lived in accord with natural law for maximum achievement with minimum effort.