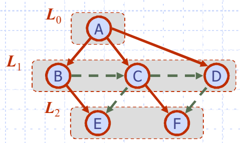
BFS Levels

When actually implemented, the levels are normally merged into a single sequence/queue

How could we keep track of the level of a vertex?



BFS

BFS Algorithm (revised)

- The BFS algorithm if we need to know the level

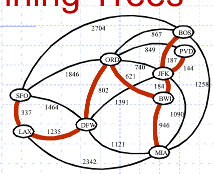
```
Algorithm BFS(G)
  Input graph G
  Output labeling of the edges
      and partition of the
      vertices of G
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      BFS(G, v)\
```

```
Algorithm BFS(G, s)
  L ← new empty sequence
  L.insertLast(s)
  setLevel(s, 0) {use only if level # is needed}
  setLabel(s, VISITED)
  while ¬L.isEmpty()
      v ← L.remove (L.first())
      for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            setLabel(w, VISITED)
            L.insertLast(w)
            setLevel(w, getLevel(v)+1)
          else
            setLabel(e, CROSS)
```

Shows how we could keep track of the level?

Lecture 14: Minimum Spanning Trees

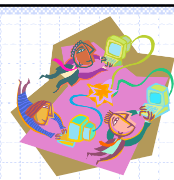
Infinite Correlation



Wholeness Statement

A minimum spanning tree is a spanning tree subgraph with minimum total edge weight. Efficient greedy algorithms have been developed to compute MST both with and without special data structures. Pure creative intelligence is the source of all creative algorithms. Regular practice of TM improves our ability to make use of our own innate creative potential.

Outline and Reading



- Minimum Spanning Trees (§7.3)
  - Definitions
  - Cycle Property
  - Partition Property
- The Prim-Jarnik Algorithm (1957, 1930) (§7.3.2)
- Kruskal's Algorithm (1956) (§7.3.1)
- Baruvka's Algorithm (1926) (§7.3.3)

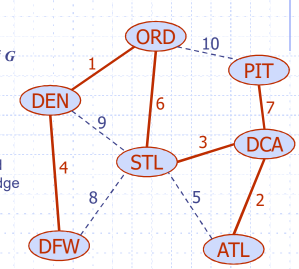
Minimum Spanning Tree

Spanning subgraph
- Subgraph of a graph ***G*** containing all the vertices of ***G***

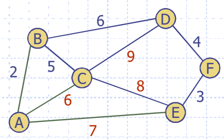
Spanning tree
- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)
- Spanning tree of a weighted graph with minimum total edge weight

- Applications
  - Communications networks
  - Transportation networks

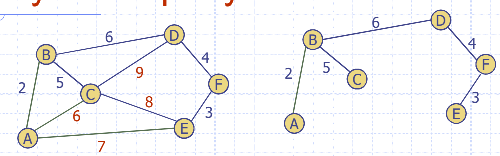


Cycle Property



If the weight of an edge ***e*** of a cycle C is larger than the weights of other edges of C, then this edge cannot belong to a MST.

Cycle Property



If the weight of an edge ***e*** of a cycle C is larger than the weights of other edges of C, then this edge cannot belong to a MST.

Cycle Property

- **Cycle Property**: For any cycle C in a graph, if the weight of an edge ***e*** of C is larger than the weights of other edges of C, then this edge cannot belong to a MST.
- **Proof**: Assume the contrary, i.e. that e belongs to an MST ***T1***; then deleting ***e*** will break ***T1*** into two subtrees with the two ends of ***e*** in different subtrees. The remainder of C reconnects the subtrees, hence there is an edge ***f*** of C with ends in different subtrees, i.e., it reconnects the subtrees into a tree ***T2*** with weight less than that of ***T1***, because the weight of ***f*** is less than the weight of ***e***; thus ***T1*** cannot be a MST.

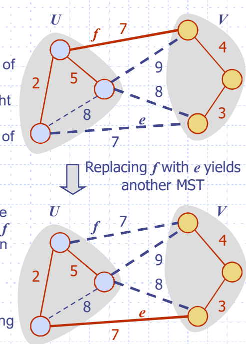
Partition Property- A Crucial Fact

Partition Property:

- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property, $weight(f) \le weight(e)$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e



Generic MST Algorithm

Algorithm ***GenericMST***(***G***)
    T ← a tree with all vertices in G, but no edges
    while ***T does not form a spanning tree*** **do**
        (***u, v***) ← a safe edge of ***G***
        T ← (***u, v***) U ***T***
    return ***T***

A *safe edge* is one that when added to T forms a subgraph of a MST

Main Point

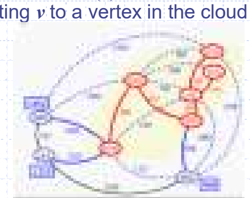
1. A minimum spanning tree algorithm gradually grows a (sub-solution) tree by adding a “safe edge” that connects a vertex in the tree to a vertex not yet in the tree.
   The nature of life is to grow and progress to the state of enlightenment, fulfillment.

Prim(1957)-Jarnik(1930) Algorithm

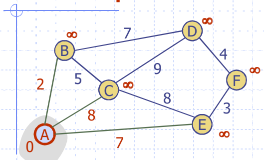
AKA Dijkstra-Prim Algorithm

Prim-Jarnik's Algorithm

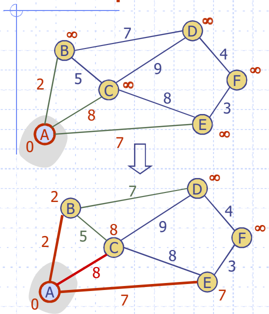
- Similar to Dijkstra's shortest path algorithm (for a connected graph)
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v a label $d(v)$ = the smallest weight of an edge connecting v to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u



Example



Example



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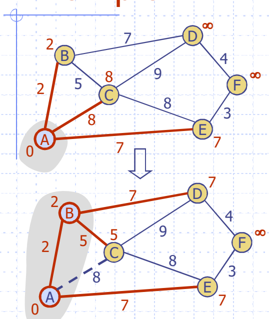
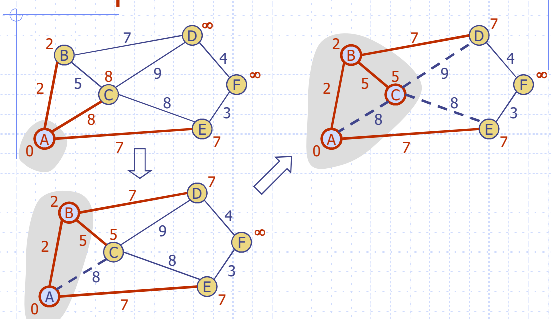
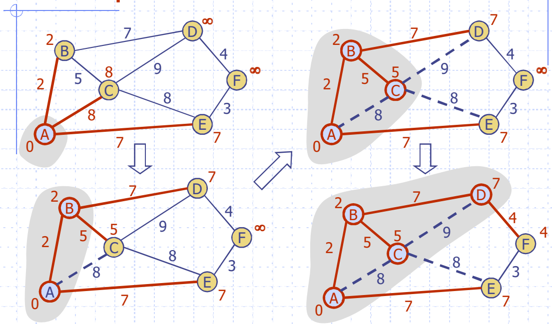


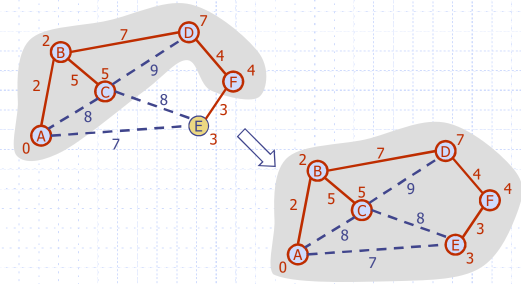
Example



Example



Example (contd.)



Prim-Jarnik's Algorithm (cont.)

- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- Locator-based methods
  - *insert*(k, e) returns a locator
  - *replaceKey*(l, k) changes the key of an item
- We store three labels with each vertex:
  - Distance
  - Parent edge in MST
  - Locator in priority queue
- Correction in red inspired by Bereket Chalew (May 2014)

```
Algorithm PrimJarnikMST(G)
  Q ← new heap-based priority queue
  s ← G.aVertex ()
  for all  v ∈ G.vertices() do
    if  v = s  then
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    setParent(v, ∅)
    l ← Q.insert(getDistance(v), v)
    setLocator(v, l)
  while  ¬Q.isEmpty() do
    u ← Q.removeMin()
    setLocator(u, ∅)  {u is now in MST}
    for all  e ∈ G.incidentEdges(u) do
      z ← G.opposite(u, e)
      r ← weight(e)
      l ← getLocator (z)
      if  l ¬= ∅   {z not yet in MST}
          ∧ r < getDistance(z) then
        setDistance(z, r)
        setParent(z, e)
        Q.replaceKey(l, r)
```


Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
  - Recall that $\Sigma_v \deg(v) = 2m$
- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
- The running time is $O(m \log n)$ since the graph is connected

Main Point

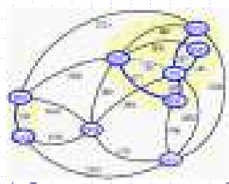
2. A defining feature of the Minimum Spanning Tree (and shortest path) greedy algorithms is that once a vertex becomes in-tree (or "inside the cloud"), the resulting subtree is optimal and nothing can change this state.
   A defining feature of enlightenment is that once this state is reached, one's consciousness is optimal and nothing can change this state.

Kruskal's Algorithm

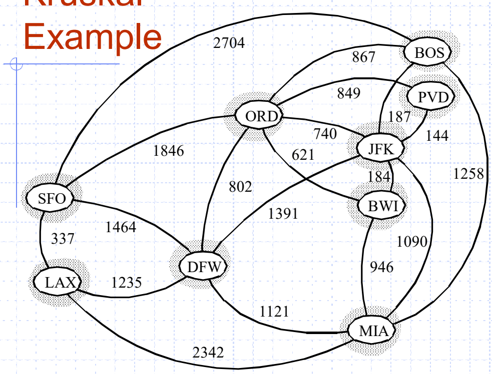
Based on the Partition Property

Basic Idea of the Kruskal Algorithm

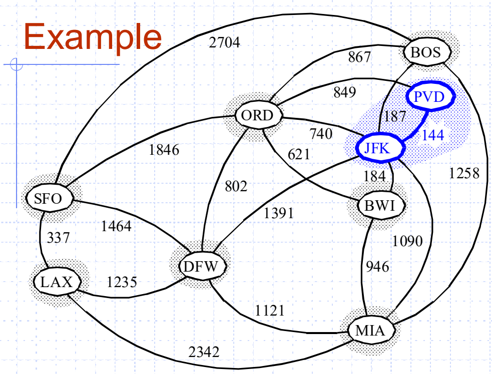
- The algorithm maintains a forest of trees
- Select the edge with the smallest weight
  - The edge is accepted if it connects distinct trees
- Keep adding edges until the tree has n-1 edges



Kruskal Example



Example



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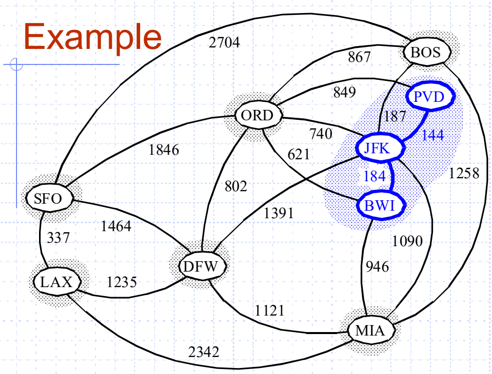

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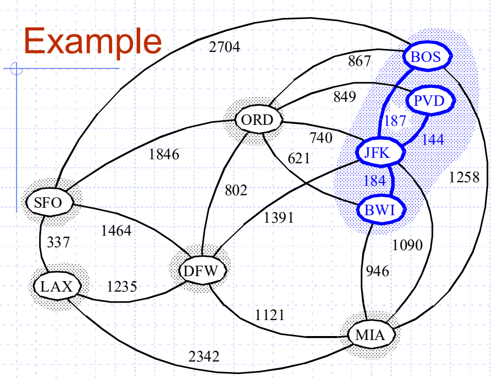

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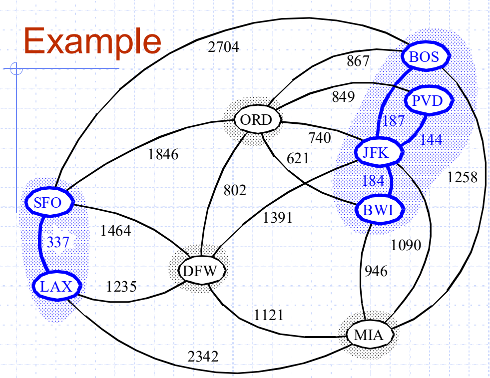

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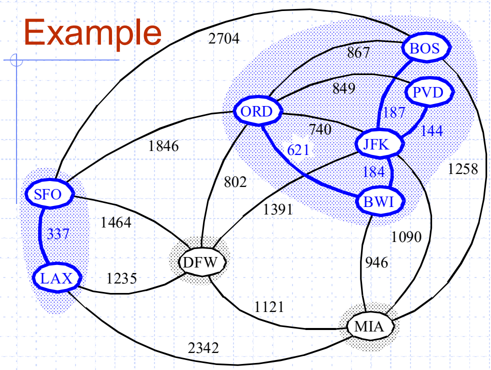
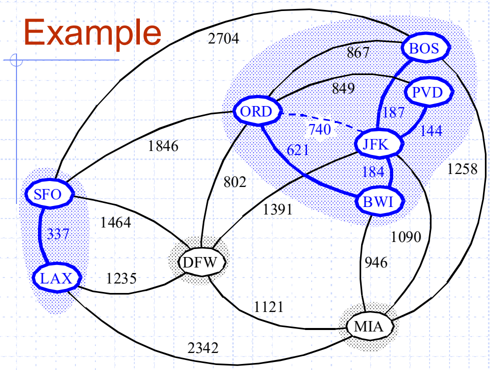
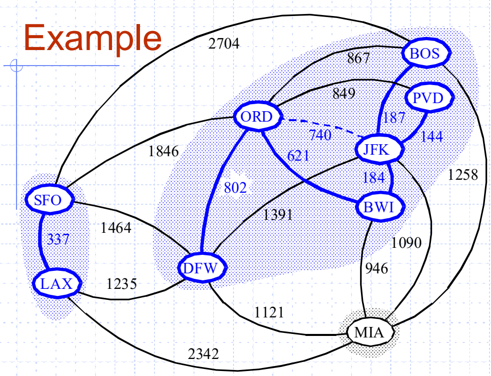


Example



Example



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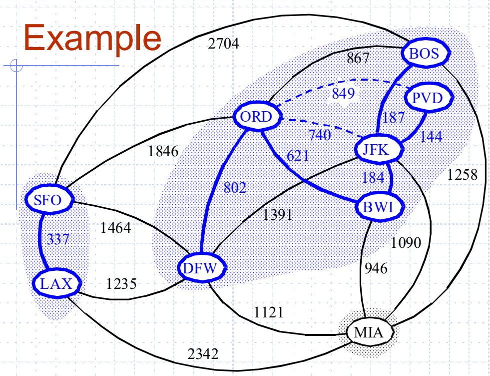
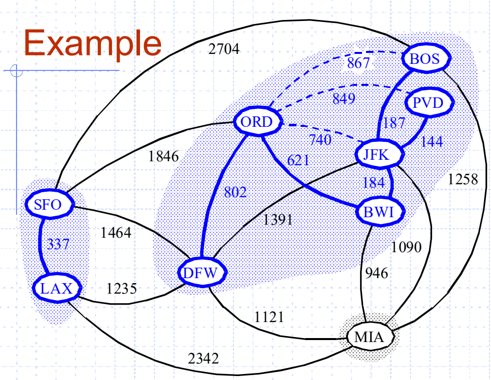


Example



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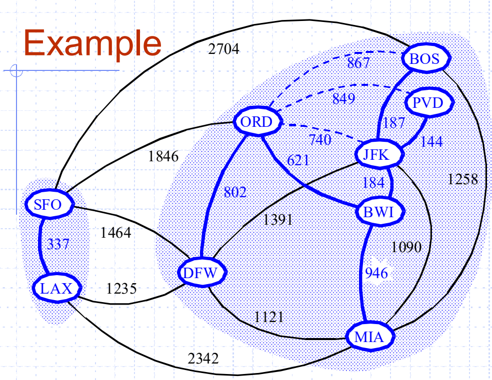

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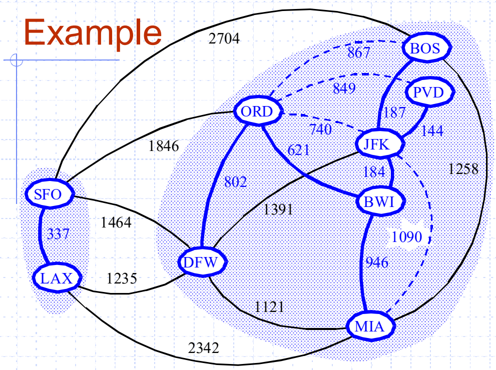
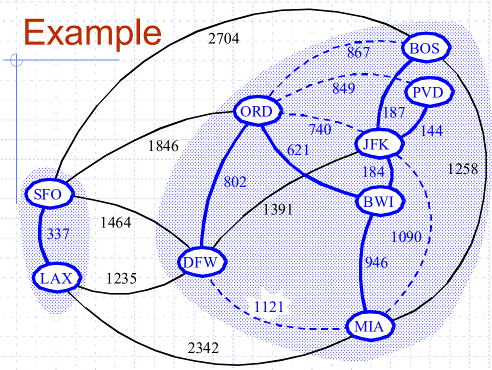
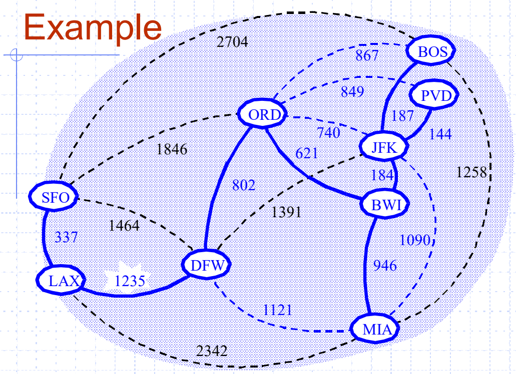


Example



Example



Kruskal's Algorithm (1956) (High Level)

- A priority queue stores the edges outside the cloud
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - We are left with one cloud that encompasses the MST
  - A tree *T* which is our MST

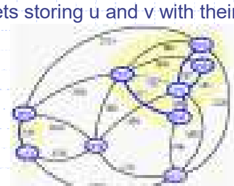
```
Algorithm KruskalMST(G)
    for each vertex v in G do
        define a Cloud(v) ← {v}
    Q ← new heap-based priority queue.
    for all  e ∈ G.edges()
        Q.insert(weight(e), e)
    T ← ∅
    while T has fewer than n-1 edges  do
        e ← Q.removeMin()
        (u, v) ← G.endVertices(e)
        if Cloud(v) ≠ Cloud(u) then
            Add edge e to T
            Merge Cloud(v) and Cloud(u)
    return T
```


Data Structure for Kruskal Algorithm

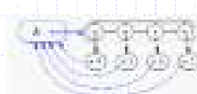
- The algorithm maintains a forest of trees
- An edge is accepted if it connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:

  -**find**(u): return the set storing u

  -**union**(u,v): replace the sets storing u and v with their union



Representation of a Partition



- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is $\min(n_u,n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- For each edge we do two finds
- We insert n-1 edges into the MST, so we do at most O(n) unions
- Each element is processed (copied into a different cloud) at most log n times (Why?)
  - Because whenever a vertex is processed in a union, it goes into a set of size at least double (Why?)
- O(log m) is O(log n) (Why?)
  - Because $m \le n(n-1)/2 < n^2$

Partition-Based Implementation

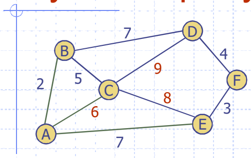
Performs cloud merges as unions and tests as finds.

Algorithm Kruskal(G):
Input: A weighted, simple, connected graph G.
Output: An MST T for G.

```
T ← new empty tree
Q ← new heap-based priority queue
for all v ∈ G.vertices() do
    insert vertex v into T
    define a Cloud(v) ← {v}
for all e ∈ G.edges() do
    Q.insert(weight(e), e)            {O(m log m) which is O(m log n)}
while T.numEdges() < n-1 do
    e ← Q.removeMin ()
    (u,v) ← G.endVertices(e)
    if P.find(u) != P.find(v) then
        insert edge e into T
        P.union(u,v)        {O(n log n) since a vertex is merged O(log n) times}
return T
```

Running time:
O((n+m)log n)

Cycle Property



By the Cycle Property, AC, CD, and CE cannot be in a MST. What about AE and BD?

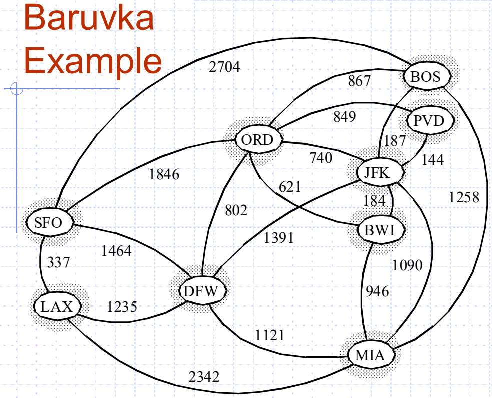
Let's run the algorithms to verify this.

Main Point

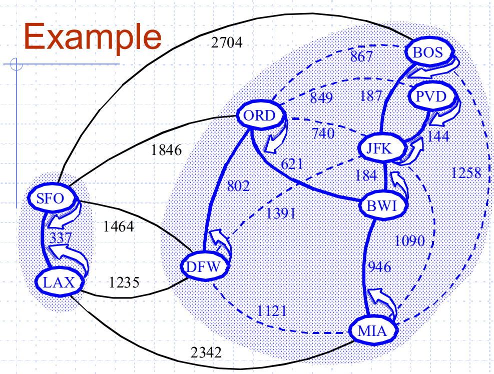
3. Kruskal's minimum spanning tree algorithm first finds the shortest edge, then tests it for feasibility; if it passes, it becomes part of the proposed solution.
   The fruits of an action cannot be determined precisely on the level of waking consciousness; it can only be determined from the level of infinite correlation where everything is interconnected. Thus only by establishing our awareness in pure awareness is right action possible.

Baruvka's Algorithm (1926)

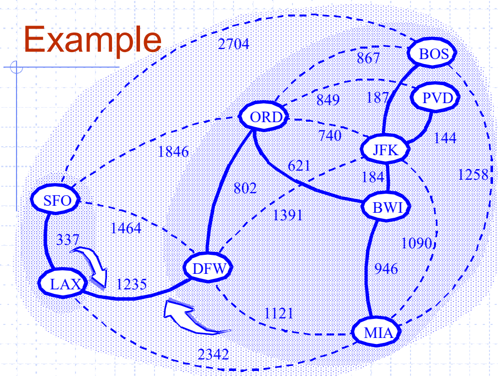
Baruvka Example



Example



Example



Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

```
Algorithm BaruvkaMST(G)
  T ← V {just the vertices of G, no edges, n connected components}
  while T has fewer than n-1 edges do {T is not yet an MST}
    for each connected component C in T do
      Find edge e with smallest-weight edge from C to
        another component in T.
      if e is not already in T then
        Add edge e to T
  return T
```

- Each iteration of the while-loop halves the number of connected components in T.

Baruvka's Algorithm (more details)

```
Algorithm BaruvkaMST(G)
  for each e ∈G.edges() do {label edges NOT_IN_MST}
    setMSTLabel(e, NOT_IN_MST)  {no edges in MST}
  numEdges ← 0
  while numEdges < n-1 do
    labelVerticesOfEachComponent(G) {BFS}
    insertSmallest-WeightEdgeOutOfComponents(G)
  return G
```


Required functionality

- Does not use a priority queue or locators!
- Does not use union-find data structures!
- Maintains a forest T subject to edge insertion
  - Can be supported in O(1) time using labels on edges in MST
- Step 1: Mark vertices with number of the component to which they belong
  - Traverse forest T to identify connected components
    - O(1) time to label each vertex
    - Requires extra instance variable for each vertex
  - Takes O(n) time using a DFS or a BFS each time through the while-loop
- Step 2: Find a smallest-weight edge in E incident on each cluster/component C (insert into MST)
  - Scan adjacency lists of each vertex in each C to find minimum
  - Takes O(m) each time through the for-loop

Analysis of Baruvka's Algorithm

- While-loop: each iteration (at worst) halves the number of connected components in T
  - Thus executed log n times
- Identifying connected components (in for-loop)
  - Vertices are labelled with component name
  - DFS or BFS of T runs in O(n) time
- Find smallest edge incident on each component C
  - Scan adjacency lists of vertices in G
  - O(m) time
- The running time is O(m log n).

After labeling each vertex with its component number (Hw13)

```
Algorithm DFS(G)
  Input graph G
  Output the edges of G are labeled as
         discovery edges and back edges

    initResult( )
    for all  u ∈ G.vertices()
      setLabel(u, UNEXPLORED)
      postVertexInit(u)
    for all  e ∈ G.edges()
      setLabel(e, UNEXPLORED)
      postEdgeInit(e)
    for all  v ∈ G.vertices()
      if  getLabel(v) = UNEXPLORED
          preComponentVisit(v)
          DFS(G, v)
          postComponentVisit(v)

    result( )
```

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  startVertexVisit(v)
  for all  e ∈ G.incidentEdges(v)
    if  getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      edgeVisit(v, e, w)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        preDiscoveryTraversal(v, e, w)
        DFS(G, w)
        postDiscoveryTraversal(v, e, w)
      else
        setLabel(e, BACK)
        backTraversal(v, e, w)
  finishVertexVisit(v)
```

HW14: Insert into T the smallest-weight edge going out from each component

Lower Bound on MST Computation

- There are **randomized algorithms** that compute MST's in **expected linear time**
- Linear time seems to be the lower bound
- Unknown whether there is a deterministic algorithm that runs in linear time (open question)

Main Point

4. The "greedy" algorithms used by MST and shortest path only work for problems where localized attention can produce a globally optimal solution.
   An enlightened person maintains unbounded awareness along with localized awareness. The behavior of such a person is globally optimal for any problem.

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. Finding the minimum spanning tree can be done by an exhaustive search of all possible spanning trees, then choosing the one with minimum weight.

2. To devise a greedy strategy, we identify a set of candidate choices, determine a selection procedure, and consider whether there is a feasibility problem. Then we have to prove that the strategy works.

3. **Transcendental Consciousness** is the home of all the laws of nature, the source of all algorithms.
4. **Impulses within Transcendental Consciousness**: The natural laws within this unbounded field are the algorithms of nature governing all the activities of the universe.
5. **Wholeness moving within itself**: In Unity Consciousness, we perceive the spanning tree of natural law and appreciate the unity of all creation.