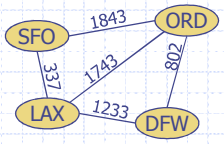
Lecture 12: Graphs and Graph Traversal

Principle of Transcending



Wholeness Statement

Graphs have many useful applications in different areas of computer science. However, to be useful we have to be able to traverse them. There are two primary ways that graphs are systematically explored, either using depth-first or breadth-first search. The TM technique provides a simple, effortless way to systematically explore the different levels of the conscious mind until the process of thinking is transcended and unbounded silence is experienced; this is the field of wholeness of individual and cosmic intelligence.

Graphs Outline and Reading

- Graphs (§6.1)
  - Definition
  - Applications
  - Terminology
  - Properties
  - ADT
- Data structures for graphs (§6.2)
  - Edge list structure
  - Adjacency list structure
  - Adjacency matrix structure

Depth-First Search Outline and Reading

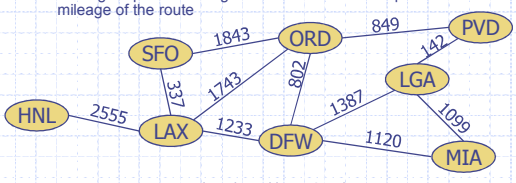
- Definitions (§6.1)
  - Subgraph
  - Connectivity
  - Spanning trees and forests
- Depth-first search (§6.3.1)
  - Algorithm
  - Example
  - Properties
  - Analysis
- Applications of DFS (§6.5)
  - Path finding
  - Cycle finding

Breadth-First Search Outline and Reading

- Breadth-first search (§6.3.3)
  - Algorithm
  - Example
  - Properties
  - Analysis
  - Applications
- DFS vs. BFS (§6.3.3)
  - Comparison of applications
  - Comparison of edge labels

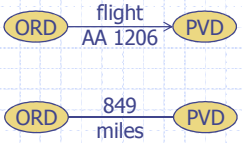
Graph

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



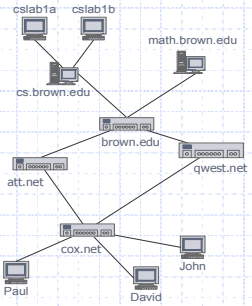
Edge Types

- *Directed edge*
  - ordered pair of vertices (u,v)
  - first vertex u is the *origin*
  - second vertex v is the *destination*
  - e.g., a flight
- *Undirected edge*
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- *Directed graph*
  - all the edges are directed
  - e.g., flight network
- *Undirected graph*
  - all the edges are undirected
  - e.g., route network



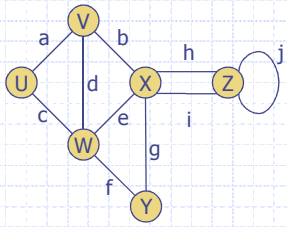
Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



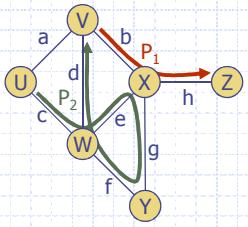
Terminology

- *End vertices* (or *endpoints*)
  - the 2 vertices joined by an edge
  - U and V are the endpoints of **a**
- *Vertices are adjacent*
  - if they are endpoints of the same edge
  - U and V are adjacent
- Edge is *incident* on a vertex
  - if the vertex is one the edge's endpoints,
  - **a**, **d**, and **b** are incident on V
- *Degree* of a vertex
  - number of incident edges
  - X has degree 5
- h and i are parallel edges
- *j is a self-loop*
- A ***Simple Graph*** has no parallel edges or self-loops
  - We will assume graphs are simple



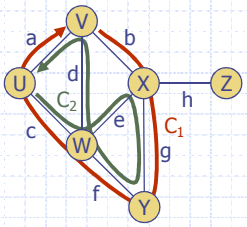
Terminology (cont.)

- *Path*
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- *Simple path*
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



Terminology (cont.)

- *Cycle*
  - circular sequence of alternating vertices and edges
  - i.e., a path with the same start and end vertices
- *Simple cycle*
  - cycle such that all its vertices and edges are distinct
  - i.e., the path is simple
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,V) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,U) is a cycle that is not simple



List of Terms

- Graph
- Vertex, vertices
- End vertices
- Adjacent vertices
- Degree of a vertex
- Edges
- Incident edges
- Directed edge, undirected edge
- Directed graph, undirected graph, mixed graph
- Path, simple path
- Cycle, simple cycle

Main Point

1. A path in a graph is a sequence of alternating vertices and edges, starting with a vertex and ending with a vertex. A path is simple if all its vertices and edges are distinct.
   The path to enlightenment is simple: regular practice of the TM technique and a balanced daily routine to stabilize the gains during meditation.

Properties

Property 1

$\Sigma_v \deg(v) = 2m$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no parallel edges

$m \leq n(n-1)/2$

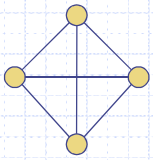
Proof: each vertex has degree at most $(n-1)$

What is the bound for a directed graph?

$m \leq n(n-1)$

Notation

n	number of vertices
m	number of edges
$\deg(v)$	degree of vertex v



Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$

Main Methods of the Graph ADT

- Vertices and edges
  - are Positions
  - store elements
- Accessor methods
  - aVertex()
  - incidentEdges(v)
  - endVertices(e)
  - isDirected(e)
  - origin(e)
  - destination(e)
  - opposite(v, e)
  - areAdjacent(v, w)
- Update methods
  - insertVertex(o)
  - insertEdge(v, w, o)
  - insertDirectedEdge(v, w, o)
  - removeVertex(v)
  - removeEdge(e)
- Generic methods
  - numVertices()
  - numEdges()
  - vertices()
  - edges()
  - degree(v)

Graph Data Structures

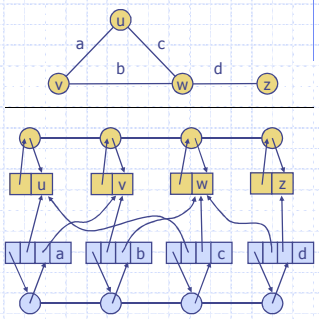
Edge list

Adjacency list

Adjacency matrix

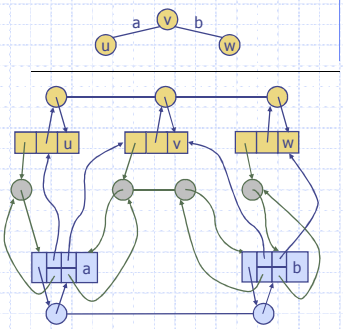
Edge List Structure

- Vertex object
  - element
  - reference to position in vertex sequence
- Edge object
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence
- Vertex sequence
  - sequence of vertex objects
- Edge sequence
  - sequence of edge objects



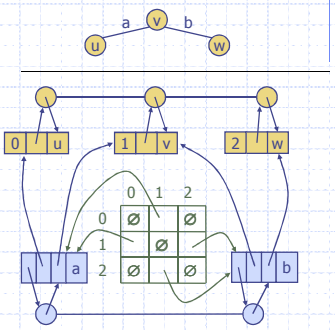
Adjacency List Structure

- Edge list structure
- Incidence sequence for each vertex
  - sequence of references to edge objects of incident edges
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices



Adjacency Matrix Structure

- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D adjacency array
  - Reference to edge object for adjacent vertices
  - Null for nonadjacent vertices
- The "old fashioned" version just has 0 for no edge and 1 for edge



Asymptotic Performance

n vertices, m edges no parallel edges no self-loops Bounds are "big-Oh"	Edge List	Adjacency List	Adjacency Matrix
Space			
incidentEdges(v)			
areAdjacent(v, w)			
insertVertex(o)			
insertEdge(v, w, o)			
removeVertex(v)			
removeEdge(e)			

Asymptotic Performance

n vertices, m edges no parallel edges no self-loops Bounds are "big-Oh"	Edge List	Adjacency List	Adjacency Matrix
Space	$n + m$		
incidentEdges(v)	m		
areAdjacent(v, w)	m		
insertVertex(o)	1		
insertEdge(v, w, o)	1		
removeVertex(v)	m		
removeEdge(e)	1		

Asymptotic Performance

n vertices, m edges no parallel edges no self-loops Bounds are "big-Oh"	Edge List	Adjacency List	Adjacency Matrix
Space	$n + m$	$n + m$	
incidentEdges(v)	m	$\deg(v)$	
areAdjacent(v, w)	m	$\min(\deg(v), \deg(w))$	
insertVertex(o)	1	1	
insertEdge(v, w, o)	1	1	
removeVertex(v)	m	$\deg(v)$	
removeEdge(e)	1	1	

Asymptotic Performance

n vertices, m edges no parallel edges no self-loops Bounds are "big-Oh"	Edge List	Adjacency List	Adjacency Matrix
Space	$n + m$	$n + m$	n^2
incidentEdges(v)	m	$\deg(v)$	n
areAdjacent(v, w)	m	$\min(\deg(v), \deg(w))$	1
insertVertex(o)	1	1	n^2
insertEdge(v, w, o)	1	1	1
removeVertex(v)	m	$\deg(v)$	n^2
removeEdge(e)	1	1	1

Asymptotic Performance

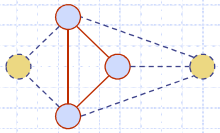
n vertices, m edges no parallel edges no self-loops Bounds are "big-Oh"	Edge List	Adjacency List	Adjacency Matrix
aVertex()			
edges()			
vertices()			
endVertices(e)			
opposite(v, e)			
degree(v)			
numEdges()			

Asymptotic Performance

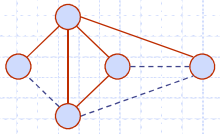
n vertices, m edges no parallel edges no self-loops Bounds are "big-Oh"	Edge List	Adjacency List	Adjacency Matrix
aVertex()	1	1	1
edges()	m	m	m
vertices()	n	n	n
endVertices(e)	1	1	1
opposite(v, e)	1	1	1
degree(v)	m	1	n
numEdges()	1	1	1

Subgraphs

- A subgraph S of a graph G is a graph such that
  - vertices(S) $\subseteq$ vertices(G)
  - edges(S) $\subseteq$ edges(G)
- A spanning subgraph of G is a subgraph that contains all the vertices of G, i.e., vertices(S) = vertices(G)



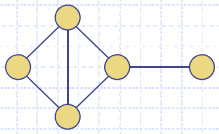
Subgraph



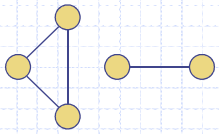
Spanning subgraph

Connectivity

- Two vertices are *connected* if there is a path between them
- A graph is *connected* if there is a path between every pair of vertices
- A *connected component* of a graph G is a maximal connected subgraph of G



Connected graph

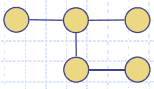


Non connected graph with two connected components

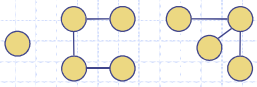
Trees and Forests

- A (free) *tree* is an **undirected** graph T such that
  - T is **connected**
  - T has no **cycles**

  This definition is different from the definition of a rooted tree
- A *forest* is an undirected graph without cycles
- The connected components of a forest are trees



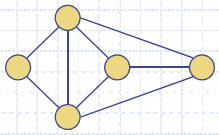
Tree



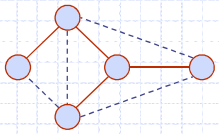
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph

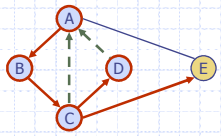


Spanning tree

More Terms

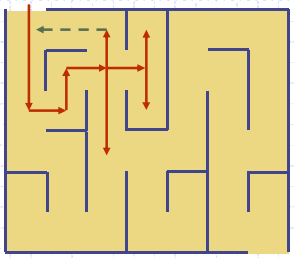
- Subgraph
- Connectivity
  - Connected Vertices (path between them)
  - Connected Graph (all vertices are connected)
  - Connected Component (maximal connected subgraph)
- Tree (connected, no cycles)
- Forest (one or more trees)
- Spanning Tree and Spanning Forest

Depth-First Search

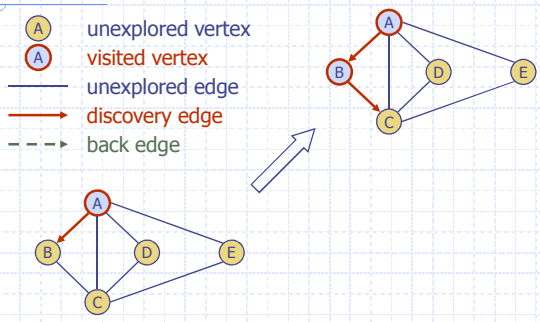


DFS and Maze Traversal

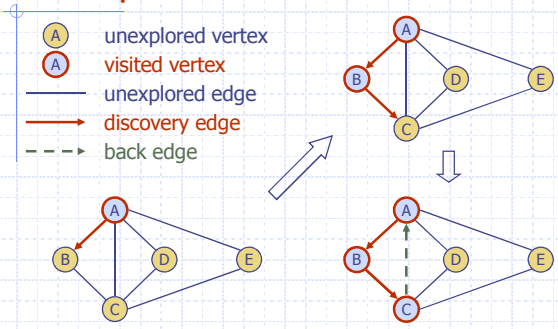
- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



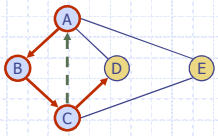
Depth-First Search Example



Example



Example (cont.)



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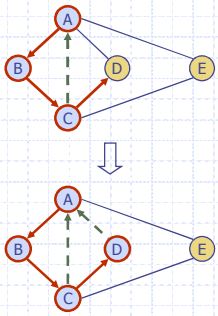

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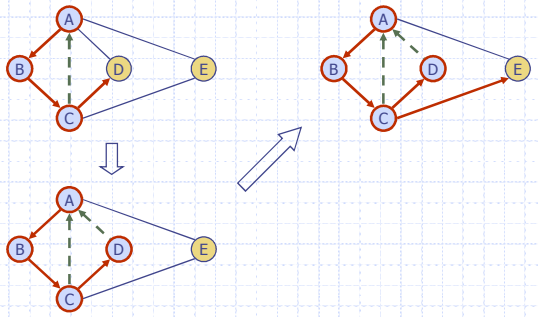

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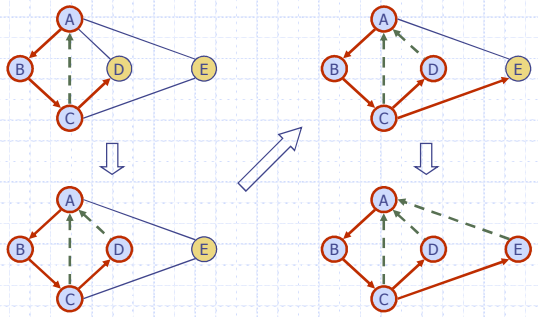
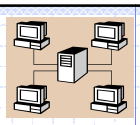


DFS Algorithm



- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm DFS(G)
  Input graph G
  Output labeling of the edges of G
      as discovery edges and
      back edges
  for all u ∈ G.vertices() do
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges() do
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices() do
    if getLabel(v) = UNEXPLORED
      DFS(G, v)
```

```
Algorithm DFS(G, v)
  Input graph G and a start vertex v of G
  Output labeling of the edges of G
      in the connected component of v
      as discovery edges and back edges
  setLabel(v, VISITED)
  for all e ∈ G.incidentEdges(v) do
    if getLabel(e) = UNEXPLORED
      w ← G.opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        DFS(G, w)
      else
        setLabel(e, BACK)
```

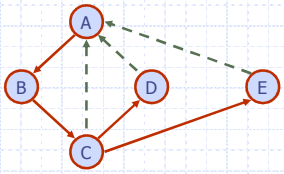

Properties of DFS

Property 1

$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v



Analysis of DFS

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G

Depth-First Search

- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what the Euler tour is to binary trees

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    path ← S.elements()
  for all e ∈ G.incidentEdges(v) do
    if getLabel(e) = UNEXPLORED
      w ← opposite(v, e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop()          { e gets popped }
      else
        setLabel(e, BACK)
  S.pop()                { v gets popped }
```


Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v)
  setLabel(v, VISITED)
  S.push(v)
  if cycle ¬= null then return
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      S.push(e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        cycleDFS(G, w)
        S.pop()
      else
        cycle ← new empty sequence
        o ← w
        repeat
          cycle.insertLast(o)
          o ← S.pop()
        until o = w
        setLabel(e, BACK)
  S.pop()
```


Recursive Programs

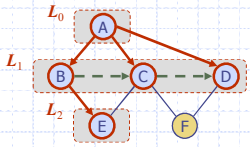
- The call structure can be described as a depth-first search of a rooted tree
  - Each non-root vertex corresponds to a recursive call
  - A tree is a logical construct, not an explicit data structure

Main Point

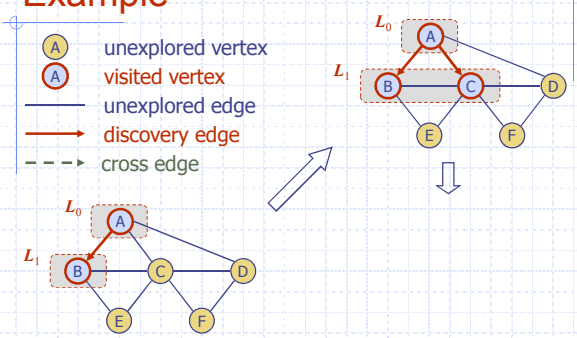
2. During dept-first search of a graph, each path is followed until the end is reached, then it backs up to branch out and explore new edges; all adjacent vertices are visited before backtracking.

   The mind is naturally seeking fields of greater happiness. The TM technique uses the nature of the mind to immediately and effortlessly take the mind to the deepest levels where true happiness and fulfillment can be gained.

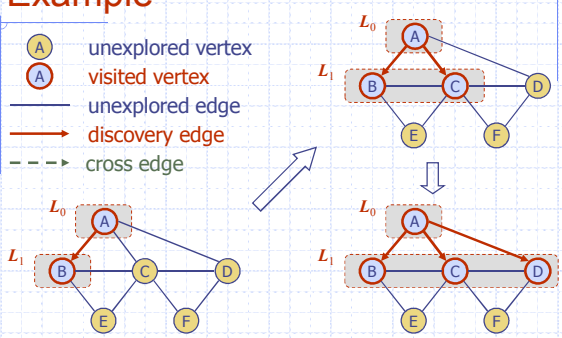
Breadth-First Search



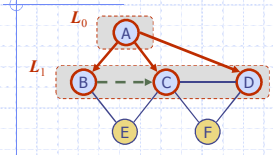
Example



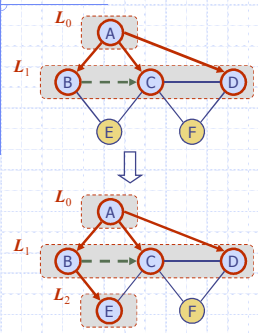
Example



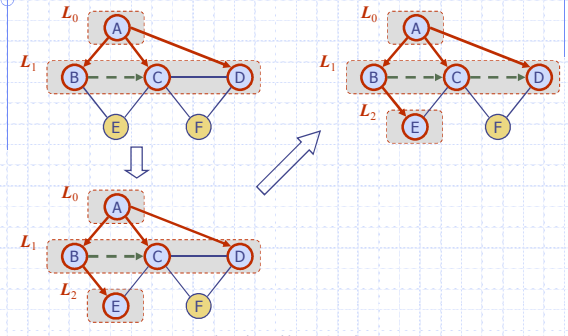
Example (cont.)



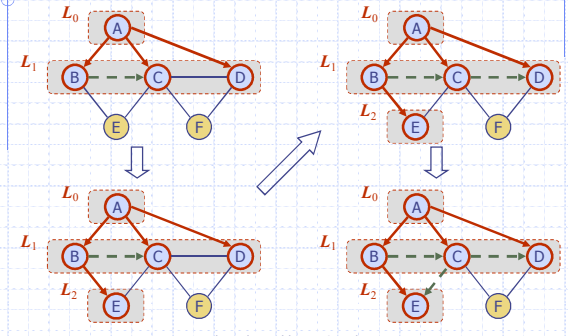
Example (cont.)



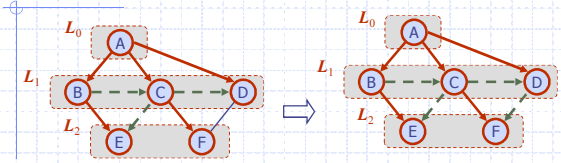
Example (cont.)



Example (cont.)



Example (cont.)



BFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm BFS(G)
    Input graph G
    Output labeling of the edges
        and partition of the
        vertices of G
    for all u ∈ G.vertices()
        setLabel(u, UNEXPLORED)
    for all e ∈ G.edges()
        setLabel(e, UNEXPLORED)
    for all v ∈ G.vertices()
        if getLabel(v) = UNEXPLORED
            BFS(G, v)
```

```
Algorithm BFS(G, s)
    L0 ← new empty sequence
    L0.insertLast(s)
    setLabel(s, VISITED)
    i ← 0
    while ¬Li.isEmpty()
        Li+1 ← new empty sequence
        for all v ∈ Li.elements()
            for all e ∈ G.incidentEdges(v)
                if getLabel(e) = UNEXPLORED
                    w ← opposite(v,e)
                    if getLabel(w) = UNEXPLORED
                        setLabel(e, DISCOVERY)
                        setLabel(w, VISITED)
                        Li+1.insertLast(w)
                    else
                        setLabel(e, CROSS)
        i ← i +1
```


Properties

Notation

G_s: connected component of s

Property 1

BFS(G, s) visits all the vertices and edges of G_s

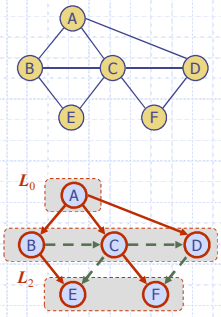
Property 2

The discovery edges labeled by **BFS**(G, s) form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges



Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G

Breadth-First Search

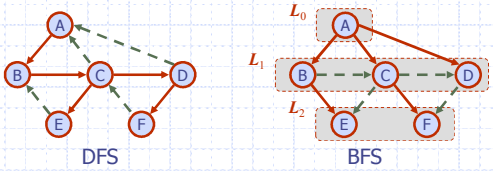
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

Applications

- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS

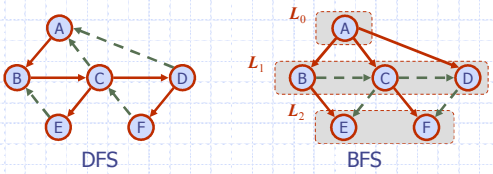
Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS vs. BFS (cont.)

Back edge (v,w)
- w is an ancestor of v in the tree of discovery edges

Cross edge (v,w)
- w is in the same level as v or in the next level in the tree of discovery edges



Main Point

3. During breadth-first search of a graph, the search repeatedly takes one step in all directions until all vertices and edges are visited. This is a bit like searching for fulfillment in waking state, i.e., floating on the surface of the mind or through one's daily activity.
   In contrast, Transcendental Meditation takes the mind immediately and effortlessly to the deepest levels where true fulfillment can be gained.

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. Almost any algorithm for solving a problem on a graph or digraph requires examining or processing each vertex or edge.

2. Depth-first and breadth-first search are two particularly useful and efficient search strategies requiring linear time.

3. **Transcendental Consciousness** is the goal of all searches, the field of complete fulfillment.
4. **Impulses within Transcendental Consciousness**: The dynamic natural laws within this unbounded field govern all activities and evolution of the universe.
5. **Wholeness moving within itself:** In Unity Consciousness, one experiences that the self-referral activity of the unified field gives rise to the whole breadth and depth of the universe.