

4.B Cartesian products

from *Random Walks on infinite Graphs and Groups* by W.Woess

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From Chapter 4.A I

Definition

\mathcal{N} (or (X, P)) satisfies an \mathcal{F} -isoperimetric inequality $IS_{\mathcal{F}}$, if there is a constant $\kappa > 0$ such that

$$\mathcal{F}(m(A)) \leq \kappa a(\partial A)$$

for every finite $A \subset X$.

If this holds for the SRW then we say that the graph X itself satisfies $IS_{\mathcal{F}}$.

If $\mathcal{F}(t) = t^{1-\frac{1}{d}} (1 \leq d \leq \infty)$ then we call it d -dimensional isoperimetric inequality, IS_d .

From Chapter 4.A II

For a function $f : X \rightarrow \mathbb{R}$, the *Sobolev norm* is

$S_P(f) = \frac{1}{2} \sum_{x,y \in X} |f(x) - f(y)| a(x,y)$ and its norm in $\ell^p(X, m)$ is

$$\|f\|_p = (\sum_{x \in X} |f(x)|^p m(x))^{1/p}$$

Proposition

(X, P) satisfies IS_d ($1 \leq d \leq \infty$) if and only if

$$\|f\|_{\frac{d}{d-1}} \leq \kappa S_P(f) \text{ for every } f \in \ell_0(X).$$

Direct and Cartesian product of graphs

- $X_1, X_2 \dots$ graphs
- **Direct product** $X_1 \otimes X_2$: the graph with vertex set $\{x_1x_2, x_i \in X_i\}$
- Neighbourhood given by:
$$x_1x_2 \sim y_1y_2 \iff x_1 \sim y_1 \text{ and } x_2 \sim y_2$$
- **Cartesian product** $X_1 \times X_2$: same vertex set as above
- Neighbourhood given by:
$$x_1x_2 \sim y_1y_2 \iff x_1 \sim y_1 \text{ and } x_2 = y_2 \text{ or } x_1 = y_1 \text{ and } x_2 \sim y_2$$
- If both factors are bipartite graphs (they have no closed paths with odd length), then the direct product is disconnected.
- Γ_1, Γ_2 finitely generated groups with Cayley graphs $X(\Gamma_i, S_i)$, then the direct product is not necessarily a Cayley graph of $\Gamma_1 \times \Gamma_2$ - it **is** true for the Cartesian product.
- The SRW on the Cartesian product of two regular graphs is a Cartesian product of the SRWs on the factors.

Direct and Cartesian product of networks

- $\mathcal{N}_1, \mathcal{N}_2 \dots$ networks with conductance functions a_1, a_2
- **Direct product** $\mathcal{N}_1 \otimes \mathcal{N}_2$: with conductance function
$$a(x_1x_2, y_1y_2) = a_1(x_1, y_1)a_2(x_2, y_2)$$
- Not necessarily connected.
- **Cartesian product** $\mathcal{N}_1 \times \mathcal{N}_2$: with conductance function
$$a(x_1x_2, y_1y_2) = a_1(x_1, y_1)\delta_{x_2}(y_2) + a_2(x_2, y_2)\delta_{x_1}(y_1)$$
- The reversible MC associated with the Cartesian product of two networks is not a Cartesian product of the reversible MC on the factors.

Direct and Cartesian product of transition matrices

- $P_1, P_2 \dots$ transition matrices over X_1, X_2
- **Direct (or tensor) product** $P_1 \otimes P_2$:
 $p(x_1 x_2, y_1 y_2) = p_1(x_1, y_1) p_2(x_2, y_2)$
- In general it does not preserve irreducibility.
- **Cartesian product** $P_1 \times P_2$:
 $P = cP_1 \otimes I_2 + (1 - c)I_1 \otimes P_2$ with $0 < c < 1$.

Theorem

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Let $\mathcal{N}_i, i = 1, 2$ be networks with associated invariant measures $m_i : \sup_{\mathbf{x}_i} m_i(\mathbf{x}) < \infty$. If \mathcal{N}_1 satisfies IS_{d_1} and \mathcal{N}_2 satisfies IS_{d_2} then $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ satisfies $IS_{d_1+d_2}$.

Proof.

See blackboard / handout.



Remark and Corollary

Remark

Let $\mathcal{N}_1, \mathcal{N}_2$ be networks which satisfy IS_{d_1} and IS_{d_2} and $P_i \geq c_i l_i$ holds elementwise ($c_i > 0$, l_i identity over X_i), then $\mathcal{N}_1 \otimes \mathcal{N}_2$ satisfies $IS_{d_1+d_2}$.

Corollary

*Let X_1, X_2 be infinite graphs with bounded geometry. If X_1 satisfies IS_{d_1} and X_2 satisfies IS_{d_2} then $X_1 \times X_2$ satisfies $IS_{d_1+d_2}$.
In every case $X_1 \times X_2$ satisfies IS_2 .*

The SRW on $X_1 \times X_2$ may be replaced with a Cartesian product of the SRWs on the factors.

References

- [1] Wolfgang Woess. *Random Walks on Infinite Graphs and Groups*. Cambridge: Cambridge University Press, Feb. 2000. DOI: [10.1017/CB09780511470967](https://doi.org/10.1017/CB09780511470967).