4.B Cartesian products

from *Random Walks on infinite Graphs and Groups* by W.Woess

VO Random Processes 2018/2019

Lukas Richter

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From Chapter 4.A I

Definition

$\mathcal{N}$ (or (X, P)) satisfies an $\mathcal{F}$-isoperimetric inequality $IS_{\mathcal{F}}$, if there is a constant $\kappa > 0$ such that

$$\mathcal{F}(m(A)) \leq \kappa a(\partial A)$$

for every finite $A \subset X$.
If this holds for the SRW then we say that the graph X itself satisfies $IS_{\mathcal{F}}$.

If $\mathcal{F}(t) = t^{1-\frac{1}{d}}(1 \leq d \leq \infty)$ then we call it d-dimensional isoperimetric inequality, IS_d.

From Chapter 4.A II

For a function $f : X \to \mathbb{R}$, the *Sobolev norm* is $S_P(f) = \frac{1}{2}\sum_{x,y\in X}|f(x) - f(y)|a(x,y)$ and its norm in $\ell^p(X,m)$ is $\|f\|_p = (\sum_{x\in X}|f(x)|^p m(x))^{1/p}$

Proposition

(X,P) satisfies $IS_d(1 \leq d \leq \infty)$ if and only if

$$\|f\|_{\frac{d}{d-1}} \leq \kappa S_P(f) \text{ for every } f \in \ell_0(X).$$

Direct and Cartesian product of graphs

- X_1, X_2 ... graphs
- **Direct product** $X_1 \otimes X_2$: the graph with vertex set $\{x_1 x_2, x_i \in X_i\}$
- Neighbourhood given by:
  $x_1 x_2 \sim y_1 y_2 \iff x_1 \sim y_1$ and $x_2 \sim y_2$
- **Cartesian product** $X_1 \times X_2$: same vertex set as above
- Neighbourhood given by:
  $x_1 x_2 \sim y_1 y_2 \iff x_1 \sim y_1$ and $x_2 = y_2$ or $x_1 = y_1$ and $x_2 \sim y_2$
- If both factors are bipartite graphs (they have no closed paths with odd length), then the direct product is disconnected.
- Γ_1, Γ_2 finitely generated groups with Cayley graphs $X(\Gamma_i, S_i)$, then the direct product is not necessarily a Cayley graph of $\Gamma_1 \times \Gamma_2$ - it **is** true for the Cartesian product.
- The SRW on the Cartesian product of two regular graphs is a Cartesian product of the SRWs on the factors.

Direct and Cartesian product of networks

- $\mathcal{N}_1, \mathcal{N}_2$ ... networks with conductance functions a_1, a_2
- **Direct product** $\mathcal{N}_1 \otimes \mathcal{N}_2$: with conductance function $a(x_1x_2, y_1y_2) = a_1(x_1, y_1)a_2(x_2, y_2)$
- Not necessarily connected.
- **Cartesian product** $\mathcal{N}_1 \times \mathcal{N}_2$: with conductance function $a(x_1x_2, y_1y_2) = a_1(x_1, y_1)\delta_{x_2}(y_2) + a_2(x_2, y_2)\delta_{x_1}(y_1)$
- The reversible MC associated with the Cartesian product of two networks is not a Cartesian product of the reversible MC on the factors.

Direct and Cartesian product of transition matrices

- P_1, P_2 ... transition matrices over X_1, X_2
- **Direct (or tensor) product** $P_1 \otimes P_2$:
  $p(x_1x_2, y_1y_2) = p_1(x_1, y_1)p_2(x_2, y_2)$
- In general it does not preserve irreducibility.
- **Cartesian product** $P_1 \times P_2$:
  $P = cP_1 \otimes I_2 + (1-c)I_1 \otimes P_2$ with $0 < c < 1$.

Theorem

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Let $\mathcal{N}_i, i = 1, 2$ *be networks with associated invariant measures* $m_i : \sup_{X_i} m_i(x) < \infty$. *If* $\mathcal{N}_1$ *satisfies* IS_{d_1} *and* $\mathcal{N}_2$ *satisfies* IS_{d_2} *then* $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ *satisfies* $IS_{d_1+d_2}$.

Proof.

See Woess [1]. □

Remark and Corollary

Remark

Let $\mathcal{N}_1, \mathcal{N}_2$ be networks which satisfy IS_{d_1} and IS_{d_2} and $P_i \geq c_i I_i$ holds elementwise ($c_i > 0$, I_i identity over X_i), then $\mathcal{N}_1 \otimes \mathcal{N}_2$ satisfies $IS_{d_1+d_2}$.

Corollary

Let X_1, X_2 be infinite graphs with bounded geometry. If X_1 satisfies IS_{d_1} and X_2 satisfies IS_{d_2} then $X_1 \times X_2$ satisfies $IS_{d_1+d_2}$.
In every case $X_1 \times X_2$ satisfies IS_2.

The SRW on $X_1 \times X_2$ may be replaced with a Cartesian product of the SRWs on the factors.

References

[1] Wolfgang Woess. *Random Walks on Infinite Graphs and Groups*. Cambridge: Cambridge University Press, Feb. 2000. DOI: 10.1017/CBO9780511470967.