4.B Cartesian products

from Random Walks on infinite Graphs and Groups by W.Woess

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From Chapter 4.A I

Definition

$\mathcal{N}$ (or (X, P)) satisfies an $\mathcal{F}$-isoperimetric inequality $IS_{\mathcal{F}}$, if there is a constant $\kappa > 0$ such that

$$\mathcal{F}(m(A)) \leq \kappa a(\partial A)$$

for every finite $A \subset X$.
If this holds for the SRW then we say that the graph X itself satisfies $IS_{\mathcal{F}}$.

- If $\mathcal{F}(t) = t^{1-\frac{1}{d}}(1 \leq d \leq \infty)$ then we call it d-dimensional isoperimetric inequality, IS_d.
- For a function $f : X \to \mathbb{R}$, the *Sobolev norm* is $S_P(f) = \frac{1}{2}\sum_{x,y\in X} |f(x) - f(y)|a(x, y)$ and its norm in $\ell^p(X, m)$ is $\|f\|_p = (\sum_{x\in X} |f(x)|^p m(x))^{1/p}$

From Chapter 4.A II

Proposition

(X, P) satisfies $IS_d (1 \leq d \leq \infty)$ if and only if

$$\|f\|_{\frac{d}{d-1}} \leq \kappa S_P(f) \text{ for every } f \in \ell_0(X).$$

Graphs

- X_1, X_2 ... graphs
- **Direct product** $X_1 \otimes X_2$: the graph with vertex set $\{x_1 x_2, x_i \in X_i\}$
- Neighbourhood given by:
  $x_1 x_2 \sim y_1 y_2 \iff x_1 \sim y_1$ and $x_2 \sim y_2$
- **Cartesian product** $X_1 \times X_2$: same vertex set as above
- Neighbourhood given by:
  $x_1 x_2 \sim y_1 y_2 \iff x_1 \sim y_1$ and $x_2 = y_2$ or $x_1 = y_1$ and $x_2 \sim y_2$

Networks

Analoguous for networks

- $\mathcal{N}_1, \mathcal{N}_2$ ... networks with conductance functions a_1, a_2
- **Direct product** $\mathcal{N}_1 \otimes \mathcal{N}_2$: with conductance function $a(x_1x_2, y_1y_2) = a_1(x_1, y_1)a_2(x_2, y_2)$
- **Cartesian product** $\mathcal{N}_1 \times \mathcal{N}_2$: with conductance function $a(x_1x_2, y_1y_2) = a_1(x_1, y_1)\delta_{x_2}(y_2) + a_2(x_2, y_2)\delta_{x_1}(y_1)$

Transition matrices

Transition matrices

- P_1, P_2 ... transition matrices over X_1, X_2
- **Direct product** (tensor product) $P_1 \otimes P_2$:
  $p(x_1x_2, y_1y_2) = p_1(x_1, y_1)p_2(x_2, y_2)$
- **Cartesian product** $P_1 \times P_2$:
  $P = cP_1 \otimes I_2 + (1-c)I_1 \otimes P_2$ with $0 < c < 1$.

Theorem

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Let $\mathcal{N}_i, i = 1, 2$ be networks with associated invariant measures $m_i : \sup_{X_i} m_i(x) < \infty$. If $\mathcal{N}_1$ satisfies IS_{d_1} and $\mathcal{N}_2$ satisfies IS_{d_2} then $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ satisfies $IS_{d_1+d_2}$.

Remarks

- Let $\mathcal{N}_1, \mathcal{N}_2$ be networks which satisfy IS_{d_1} and IS_{d_2} and $P_i \geq c_i I_i$ holds elementwise ($c_i > 0, I_i$ identity over X_i), then $\mathcal{N}_1 \otimes \mathcal{N}_2$ satisfies $IS_{d_1+d_2}$.

Corollary

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Let X_1, X_2 *be infinite graphs with bounded geometry. If* X_1 *satisfies* IS_{d_1} *and* X_2 *satisfies* IS_{d_2} *then* $X_1 \times X_2$ *satisfies* $IS_{d_1+d_2}$*.*
In every case $X_1 \times X_2$ *satisfies* IS_2*.*