Curious OCaml

Lukasz Stafiniak

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Chapter 1: Logic

From logic rules to programming constructs

1.1 In the Beginning there was Logos

What logical connectives do you know? Before we write any code, let us take a step back and think about logic itself. The connectives listed below form the foundation of reasoning, and as we will discover, they also form the foundation of programming.

$\top$	$\bot$	$\wedge$	$\vee$	$\to$
		$a \wedge b$	$a \vee b$	$a \to b$
truth	falsehood	conjunction	disjunction	implication
"trivial"	"impossible"	a and b	a or b	a gives b
	shouldn't get	got both	got at least one	given a, we get b

How can we define these connectives precisely? The key insight is to think in terms of *derivation trees*. A derivation tree shows how we arrive at conclusions from premises, building up knowledge step by step:

$$\frac{\dfrac{\overline{\text{a premise}}\quad\overline{\text{another premise}}}{\text{some fact}}\qquad\dfrac{\overline{\text{this we have by default}}}{\text{another fact}}}{\text{final conclusion}}$$

We define connectives by providing rules for using them. For example, a rule $\frac{a\ b}{c}$ matches parts of the tree that have two premises, represented by variables a and b, and have any conclusion, represented by variable c. These variables act as placeholders that can match any proposition.

Design principle: When defining a connective, we try to use only that connective in its definition. This keeps definitions self-contained and avoids circular dependencies between connectives.

1.2 Rules for Logical Connectives

Each logical connective comes with two kinds of rules:

Introduction rules tell us how to *produce* or *construct* a connective. If you want to prove "A and B", the introduction rule tells you what you need: proofs of both A and B.

Elimination rules tell us how to *use* or *consume* a connective. If you already have "A and B", the elimination rules tell you what you can get from it: either A or B (your choice).

In the table below, text in parentheses provides informal commentary. Letters like a, b, and c are variables that can stand for any proposition.

Connective	Introduction Rules	Elimination Rules
$\top$	$\frac{}{\top}$	doesn't have
$\bot$	doesn't have	$\frac{\bot}{a}$ (i.e., anything)
$\wedge$	$\frac{a \quad b}{a \wedge b}$	$\frac{a \wedge b}{a}$ (take first) $\frac{a \wedge b}{b}$ (take second)

Connective	Introduction Rules	Elimination Rules
$\vee$	$\frac{a}{a\vee b}$ (put first) $\frac{b}{a\vee b}$ (put second)	$\frac{a\vee b \quad \begin{matrix}[a]^x\\ \vdots\\ c\end{matrix} \quad \begin{matrix}[b]^y\\ \vdots\\ c\end{matrix}}{c}$ using x, y
$\to$	$\frac{\begin{matrix}[a]^x\\ \vdots\\ b\end{matrix}}{a\to b}$ using x	$\frac{a\to b \quad a}{b}$

Notation for Hypothetical Derivations The notation $\begin{matrix}[a]^x\\ \vdots\\ b\end{matrix}$ (sometimes written as a tree) matches any subtree that derives b and can use a as an assumption (marked with label x), even though a might not otherwise be warranted. The square brackets around a indicate that this is a *hypothetical* assumption, not something we have actually established. The superscript x is a label that helps us track which assumption gets "discharged" when we complete the derivation.

This is the key to proving implications: to prove "if A then B", we temporarily assume A and show we can derive B. For example, we can derive "sunny $\to$ happy" by showing that *assuming* it is sunny, we can derive happiness:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{\text{sunny}}\,x}{\text{go outdoor}}}{\text{playing}}}{\text{happy}}}{\text{sunny} \to \text{happy}}\ \text{using } x$$

Notice how the assumption "sunny" (marked with x) appears at the top of the derivation tree. We use this assumption to derive "go outdoor", then "playing", and finally "happy". Once we complete the derivation, the assumption is *discharged*: we no longer need to assume it is sunny because we have established the conditional "sunny $\to$ happy".

A crucial point: such assumptions can only be used within the matched subtree! However, they can be used *multiple times* within that subtree. For example, if someone's mood is more difficult to influence and requires multiple sunny conditions:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{\text{sunny}}\,x}{\text{go outdoor}}}{\text{playing}} \quad \dfrac{\dfrac{}{\text{sunny}}\,x \quad \dfrac{\dfrac{}{\text{sunny}}\,x}{\text{go outdoor}}}{\text{nice view}}}{\text{happy}}}{\text{sunny} \to \text{happy}}\ \text{using } x$$

In this more complex derivation, the assumption "sunny" (labeled x) is used three times: once to derive "go outdoor", and twice more in deriving "nice view". All three uses are valid because they occur within the same hypothetical subtree.

Reasoning by Cases The elimination rule for disjunction deserves special attention because it represents **reasoning by cases**, one of the most fundamental proof techniques.

Suppose we know "A or B" is true, but we do not know which one. How can we still derive a conclusion C? We must show that C follows *regardless* of which alternative holds. In other words, we need to prove: (1) assuming A, we can derive C, and (2) assuming B, we can derive C. Since one of A or B must be true, and both lead to C, we can conclude C.

Here is a concrete example: How can we use the fact that it is sunny ∨ cloudy (but not rainy)?

$$\dfrac{\dfrac{}{\text{sunny}\vee\text{cloudy}}\,\text{forecast} \qquad \dfrac{\dfrac{}{\text{sunny}}\,x}{\text{no-umbrella}} \qquad \dfrac{\dfrac{}{\text{cloudy}}\,y}{\text{no-umbrella}}}{\text{no-umbrella}}\ \text{using } x, y$$

We know that it will be sunny or cloudy (by watching the weather forecast). Now we reason by cases: *If* it will be sunny, we will not need an umbrella. *If* it will be cloudy, we will not need an umbrella. Since one of these must be the case, and both lead to the same conclusion, we can confidently say: we will not need an umbrella.

Reasoning by Induction We need one more kind of rule to do serious math: **reasoning by induction**. This rule is somewhat similar to reasoning by cases, but instead of considering a finite number of alternatives, it allows us to prove properties that hold for infinitely many cases, such as all natural numbers.

Here is the example rule for induction on natural numbers:

$$\dfrac{p(0) \qquad \begin{array}{c}[p(x)]^x \\ \vdots \\ p(x+1)\end{array}}{p(n)}\ \text{by induction, using } x$$

This rule says: we get property p for *any* natural number n, provided we can do two things:

1. **Base case:** Establish $p(0)$, that is, prove the property holds for zero.
2. **Inductive step:** Show that *assuming* $p(x)$ holds for some arbitrary x, we can derive $p(x+1)$. This assumption $p(x)$ is called the *induction hypothesis*.

Here x is a unique variable representing an arbitrary natural number. We cannot substitute a particular number for it because we write "using x" on the side, indicating that the derivation works for any choice of x.

The power of induction lies in this: once we have the base case and the inductive step, we have implicitly covered *all* natural numbers. Starting from $p(0)$, we can derive $p(1)$, then $p(2)$, then $p(3)$, and so on, reaching any natural number n we wish.

1.3 Logos was Programmed in OCaml

We now arrive at one of the most remarkable discoveries in the foundations of computer science: the **Curry-Howard correspondence**, also known as "propositions as types" or the "proofs-as-programs" interpretation. This deep correspondence reveals that logical proofs and computer programs are, in a precise sense, the same thing!

Under this correspondence:

- **Propositions** (logical statements) correspond to **types**
- **Proofs** (derivations showing a proposition is true) correspond to **programs** (expressions of a given type)
- **Introduction rules** correspond to **constructors** (ways to build values)
- **Elimination rules** correspond to **destructors** (ways to use values)

This is not merely an analogy. The formal rules for logic and the formal rules for type checking are *identical*. When you write a well-typed program, you are simultaneously constructing a proof!

The following table shows how each logical connective corresponds to a programming construct in OCaml:

Logic	Type	Expression	Intuition
$\top$	`unit`	`()`	The trivially true proposition; the type with exactly one value
$\perp$	`'a`	`raise`	Falsehood; a type with no values (exceptions escape normal typing)
$\land$	`*`	`(,)`	Conjunction corresponds to pairs: having both A and B
$\lor$	`|`	`match`	Disjunction corresponds to variants: having either A or B
$\to$	`->`	`fun`	Implication corresponds to functions: given A, produce B
induction	`-`	`rec`	Inductive proofs correspond to recursive functions

Let us now see the precise typing rules for each OCaml construct, presented in the same style as our logical rules:

Typing rules for OCaml constructs:

- **Unit (truth):** $\frac{}{\texttt{()}:\texttt{unit}}$

  The unit value `()` always has type `unit`. This is like $\top$ in logic: we can always produce it without any premises.

- **Exception (falsehood):** $\frac{\text{oops!}}{\texttt{raise exn}:c}$ can produce any type

  The `raise` expression can have *any* type c. This corresponds to the principle of "explosion" in logic: from falsehood, anything follows. In practice, `raise` never actually produces a value; it transfers control to an exception handler. The type system allows it to have any type because the expression will never complete normally.

- **Pair (conjunction):**
  - Introduction: $\frac{s:a \quad t:b}{(s,t):a*b}$
  - Elimination: $\frac{p:a*b}{\texttt{fst }p:a}$ and $\frac{p:a*b}{\texttt{snd }p:b}$

  To construct a pair, you need both components. To use a pair, you can extract either component. This mirrors conjunction perfectly: to prove "A and B", you need proofs of both; given "A and B", you can conclude either A or B.

- **Variant (disjunction):**
  - Introduction: $\frac{s:a}{\texttt{A}(s):\texttt{A of }a\texttt{ | B of }b}$
  - Elimination (match): given t of variant type and branches for each case, produce result c

  To construct a variant, you only need one of the alternatives. To use a variant, you must handle *all* possible cases (pattern matching). This mirrors disjunction: to prove "A or B", you only need one; to use "A or B", you must consider both possibilities.

- **Function (implication):**
  - Introduction: $\frac{\substack{[x:a] \\ e:b}}{\texttt{fun }x\texttt{->}e:a\rightarrow b}$
  - Elimination (application): $\frac{f:a\rightarrow b \quad t:a}{f\ t:b}$

  To construct a function, you assume you have an input of type a (the parameter x) and show how to produce a result of type b. To use a function, you apply it to an argument. This mirrors implication: to prove "A implies B", assume A and derive B; given "A implies B" and A, conclude B.

- **Recursion (induction):** $\frac{\substack{[x:a] \\ e:a}}{\texttt{rec }x=e:a}$

In recursion, the function being defined can refer to itself. This corresponds to induction: we can use the property we are trying to prove (the induction hypothesis) in the inductive step.

1.3.1 Definitions Writing out expressions and types repetitively quickly becomes tedious. More importantly, without definitions we cannot give names to our concepts, making code harder to understand and maintain. This is why we need definitions.

Type definitions are written: `type ty =` some type.

- Writing `A(s) : A of a | B of b` in the table above was a simplification. In practice, we usually have to define the type first and then use it. For example, using `int` for a and `string` for b:

  ```
  type int_string_choice = A of int | B of string
  ```

  This allows us to write `A(s) : int_string_choice`.

- Why do we need to define variant types? The reasons are: exhaustiveness checks, performnance of generated code, and ease of type inference. When OCaml sees `A(5)`, it needs to figure out (or "infer") the type. Without a type definition, how would OCaml know whether this is `A of int | B of string` or `A of int | B of float | C of bool`? The definition tells OCaml exactly what variants exist. When you match `| A i -> ...`, the compiler will warn you if you forgot to also cover `C b` in your match patterns.

- OCaml does provide an alternative: *polymorphic variants*, written with a backtick. We can write `` `A(s) : [`A of a | `B of b] ``. With `` ` `` variants, OCaml does infer what other variants might exist based on usage. These types are powerful and flexible, we will discuss them in chapter 11.

- Tuple elements do not need labels because we always know at which position a tuple element stands: the first element is first, the second is second, and so on. However, having labels makes code much clearer, especially when tuples have many components or components of the same type. For this reason, we can define a *record type*:

  ```
  type int_string_record = {a: int; b: string}
  ```

  and create its values: `{a = 7; b = "Mary"}`. OCaml 5.4 and newer also support labeled tuples, we will not discuss these.

- We access the *fields* of records using the dot notation: `{a=7; b="Mary"}.b = "Mary"`. Unlike tuples where you must remember "the second element is the name", with records you can write `.b` to get the field named `b`.

1.3.2 Expression Definitions The recursive expression `rec x = e` that appeared in our table was a simplification: `rec` (usually called `fix` in programming language theory) cannot appear alone in OCaml! It must always be part of a `let` definition.

This brings us to **expression definitions**, which let us give names to values. The typing rules for definitions are a bit more complex than what we have seen so far:

$$\frac{e_1 : a \quad \frac{[x{:}a]}{e_2{:}b}}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2 : b}$$

This rule says: if e_1 has type a, and assuming x has type a we can show that e_2 has type b, then the whole `let` expression has type b. Interestingly, this rule is equivalent to introducing a function and immediately applying it: `let x = e1 in e2` behaves the same as `(fun x -> e2) e1`. This equivalence reflects a deep connection in the Curry-Howard correspondence.

For recursive definitions, we need an additional rule:

$$\frac{\frac{[x{:}a]}{e_1{:}a} \quad \frac{[x{:}a]}{e_2{:}b}}{\texttt{let rec}\ x = e_1\ \texttt{in}\ e_2 : b}$$

Notice the crucial difference: in the recursive case, x can appear in e_1 itself! This is what allows functions to call themselves. The name x is visible both in its own definition (e_1) and in the body that uses the definition (e_2).

These rules are slightly simplified. The full rules involve a concept called **polymorphism**, which we will cover in a later chapter. Polymorphism explains how the same function can work with different types.

1.3.3 Scoping Rules Understanding *scope*—where names are visible—is essential for reading and writing OCaml programs.

- **Type definitions** we have seen above are *global*: they need to be at the top-level (not nested in expressions), and they extend from the point they occur till the end of the source file or interactive session. You cannot define a type inside a function.
- **`let-in` definitions** for expressions: `let x = e1 in e2` are *local*—the name x is only visible within e_2. Once you exit the `in` part, x no longer exists. This is useful for temporary values that should not pollute the global namespace.
- **`let` definitions** without `in` are global: placing `let x = e1` at the top-level makes x visible from after e_1 till the end of the source file or interactive session. This is how you define functions and values that the rest of your program can use.

- In the interactive session (toplevel/REPL), we mark the end of a top-level "sentence" with `;;`. This tells OCaml "I am done typing, please evaluate this." In source files compiled by the build system, `;;` is unnecessary because the end of each definition is clear from context.

1.3.4 Operators Operators like `+`, `*`, `<`, `=` are simply names of functions. In OCaml, there is nothing magical about operators; they are ordinary functions that happen to have special characters in their names and can be used in infix position (between their arguments).

Just like other names, you can define your own operators:

```
let (+:) a b = String.concat "" [a; b];;  (* Special way of defining *)
"Alpha" +: "Beta";;  (* but normal way of using operators *)
```

Notice the asymmetry here: when *defining* an operator, we wrap it in parentheses to tell OCaml "this is the name I am defining". When *using* the operator, we write it in the normal infix position between its arguments. This asymmetry exists because the definition syntax needs to distinguish between "the name `+:`" and "the expression `a +: b`".

An important feature of OCaml is that operators are **not overloaded**. This means that a single operator cannot work for multiple types. Each type needs its own set of operators: - `+`, `*`, `/` work for integers - `+.`, `*.`, `/.` work for floating point numbers

This design choice makes type inference simpler and more predictable. When you see `x + y`, OCaml knows immediately that `x` and `y` must be integers.

Exception: The comparison operators `<`, `=`, `<=`, `>=`, `<>` do work for all values other than functions. These are called *polymorphic comparisons*.

1.4 Exercises

The following exercises are adapted from *Think OCaml: How to Think Like a Computer Scientist* by Nicholas Monje and Allen Downey. They will help you get comfortable with OCaml's syntax and type system.

1. Assume that we execute the following assignment statements:

   ```
   let width = 17
   let height = 12.0
   let delimiter = '.'
   ```

   For each of the following expressions, write the value of the expression and the type (of the value of the expression), or the resulting type error.

   1. `width/2`
   2. `width/.2.0`
   3. `height/3`
   4. `1 + 2 * 5`

5. `delimiter * 5`

2. Practice using the OCaml interpreter as a calculator:

   1. The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$. What is the volume of a sphere with radius 5? (*Hint:* 392.6 is wrong!)
   2. Suppose the cover price of a book is \$24.95, but bookstores get a 40% discount. Shipping costs \$3 for the first copy and 75 cents for each additional copy. What is the total wholesale cost for 60 copies?
   3. If I leave my house at 6:52 am and run 1 mile at an easy pace (8:15 per mile), then 3 miles at tempo (7:12 per mile) and 1 mile at easy pace again, what time do I get home for breakfast?

3. You've probably heard of the Fibonacci numbers before, but in case you haven't, they're defined by the following recursive relationship:

$$\begin{cases} f(0) = 0 \\ f(1) = 1 \\ f(n+1) = f(n) + f(n-1) \quad \text{for } n = 2, 3, \ldots \end{cases}$$

   Write a recursive function to calculate these numbers.

4. A palindrome is a word that is spelled the same backward and forward, like "noon" and "redivider". Recursively, a word is a palindrome if the first and last letters are the same and the middle is a palindrome.

   The following are functions that take a string argument and return the first, last, and middle letters:

```
let first_char word = word.[0]
let last_char word =
  let len = String.length word - 1 in
  word.[len]
let middle word =
  let len = String.length word - 2 in
  String.sub word 1 len
```

   1. Enter these functions into the toplevel and test them out. What happens if you call `middle` with a string with two letters? One letter? What about the empty string `""`?
   2. Write a function called `is_palindrome` that takes a string argument and returns `true` if it is a palindrome and `false` otherwise.

5. The greatest common divisor (GCD) of a and b is the largest number that divides both of them with no remainder.

   One way to find the GCD of two numbers is Euclid's algorithm, which is based on the observation that if r is the remainder when a is divided by b, then $\gcd(a, b) = \gcd(b, r)$. As a base case, we can consider $\gcd(a, 0) = a$.

Write a function called `gcd` that takes parameters `a` and `b` and returns their greatest common divisor.

If you need help, see http://en.wikipedia.org/wiki/Euclidean_algorithm.

Chapter 2: Algebra

Algebraic Data Types and some curious analogies

In this chapter, we will deepen our understanding of OCaml's type system by working through type inference examples by hand. Then we will explore algebraic data types—a cornerstone of functional programming that allows us to define rich, structured data. Along the way, we will discover a surprising and beautiful connection between these types and ordinary polynomials from high-school algebra.

2.1 A Glimpse at Type Inference

For a refresher, let us apply the type inference rules introduced in Chapter 1 to some simple examples. We will start with the identity function `fun x -> x`—perhaps the simplest possible function, yet one that reveals important aspects of polymorphism. In the derivations below, [?] means "dunno yet" (type unknown).

We begin with an incomplete derivation:

$$\frac{[?]}{\texttt{fun x -> x} : [?]}$$

Using the $\rightarrow$ introduction rule, we need to derive the body `x` assuming `x` has some type a:

$$\frac{\frac{}{\texttt{x}:a}x}{\texttt{fun x -> x} : [?] \rightarrow [?]}$$

The premise $\frac{}{\texttt{x}:a}x$ matches the pattern for hypothetical derivations since $e = \texttt{x}$. Since the body `x` has type a (from our assumption), and the parameter `x` also has type a, we conclude:

$$\frac{\frac{}{\texttt{x}:a}x}{\texttt{fun x -> x} : a \rightarrow a}$$

Because a is arbitrary (we made no assumptions constraining it), OCaml introduces a *type variable* `'a` to represent it. This is how polymorphism emerges naturally from the inference process—the identity function can work with values of any type:

```
# fun x -> x;;
- : 'a -> 'a = <fun>
```

A More Complex Example Now let us try something that will constrain the types more: `fun x -> x+1`. This is the same as `fun x -> ((+) x) 1` (try it in OCaml to verify!). The addition operator forces specific types upon us.

We will use the notation $[?\alpha]$ to mean "type unknown yet, but the same as in other places marked $[?\alpha]$." This notation helps us track how constraints propagate through the derivation.

Starting the derivation and applying $\to$ introduction:

$$\dfrac{\dfrac{[?]}{\texttt{((+) x) 1}:[?\alpha]}}{\texttt{fun x -> ((+) x) 1} : [?] \to [?\alpha]}$$

Applying $\to$ elimination (function application) to `((+) x) 1`:

$$\dfrac{\dfrac{\dfrac{[?]}{\texttt{(+) x}:[?\beta]\to[?\alpha]} \quad \dfrac{[?]}{\texttt{1}:[?\beta]}}{\texttt{((+) x) 1}:[?\alpha]}}{\texttt{fun x -> ((+) x) 1} : [?] \to [?\alpha]}$$

We know that `1 : int`, so $[?\beta] = \texttt{int}$:

$$\dfrac{\dfrac{\dfrac{[?]}{\texttt{(+) x}:\texttt{int}\to[?\alpha]} \quad \dfrac{}{\texttt{1}:\texttt{int}}\,\text{(constant)}}{\texttt{((+) x) 1}:[?\alpha]}}{\texttt{fun x -> ((+) x) 1} : [?] \to [?\alpha]}$$

Applying function application again to `(+) x`:

$$\dfrac{\dfrac{\dfrac{\dfrac{[?]}{\texttt{(+)}:[?\gamma]\to\texttt{int}\to[?\alpha]} \quad \dfrac{[?]}{\texttt{x}:[?\gamma]}}{\texttt{(+) x}:\texttt{int}\to[?\alpha]} \quad \dfrac{}{\texttt{1}:\texttt{int}}\,\text{(constant)}}{\texttt{((+) x) 1}:[?\alpha]}}{\texttt{fun x -> ((+) x) 1} : [?\gamma] \to [?\alpha]}$$

Since `(+) : int -> int -> int`, we have $[?\gamma] = \texttt{int}$ and $[?\alpha] = \texttt{int}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{(+)}:\texttt{int}\to\texttt{int}\to\texttt{int}}\,\text{(constant)} \quad \dfrac{}{\texttt{x}:\texttt{int}}\,x}{\texttt{(+) x}:\texttt{int}\to\texttt{int}} \quad \dfrac{}{\texttt{1}:\texttt{int}}\,\text{(constant)}}{\texttt{((+) x) 1}:\texttt{int}}}{\texttt{fun x -> ((+) x) 1} : \texttt{int} \to \texttt{int}}$$

2.1.1 Curried Form When there are several arrows "on the same depth" in a function type, it means that the function returns a function. For example, `(+) : int -> int -> int` is just a shorthand for `(+) : int -> (int -> int)`. The arrow associates to the right, so we can omit the parentheses.

This is very different from:

$$\texttt{fun f -> (f 1) + 1} : (\texttt{int} \to \texttt{int}) \to \texttt{int}$$

In the first case, `(+)` is a function that takes an integer and returns a function from integers to integers. In the second case, we have a function that takes a function as an argument—a *higher-order function*. The parentheses around `int -> int` are essential here; without them, the meaning would be completely different.

This style of defining multi-argument functions, where each function takes one argument and returns another function expecting the remaining arguments, is called *curried form* (named after logician Haskell Curry). It enables a powerful technique called *partial application*.

For example, instead of writing `(fun x -> x+1)`, we can simply write `((+) 1)`. Here we apply `(+)` to just one argument, getting back a function that adds 1 to its input. What expanded form does `((+) 1)` correspond to exactly (computationally)?

Think about it before reading on…

It corresponds to `fun y -> 1 + y`. We have "baked in" the first argument, and the resulting function waits for the second.

We will become more familiar with functions returning functions when we study the *lambda calculus* in a later chapter.

2.2 Algebraic Data Types

In Chapter 1, we learned about the `unit` type and variant types like:

```
type int_string_choice = A of int | B of string
```

We also covered tuple types, record types, and type definitions. Now let us explore these concepts more deeply, building up to the powerful notion of *algebraic data types*.

Variants Without Arguments Variants do not have to carry arguments. Instead of writing `A of unit`, we can simply use `A`. This is more convenient and idiomatic:

```
type color = Red | Green | Blue
```

This defines a type with exactly three possible values—no more, no less. The compiler knows this, which enables exhaustive pattern matching checks.

A subtle point about OCaml: In OCaml, variants take multiple arguments rather than taking tuples as arguments. This means `A of int * string` is different from `A of (int * string)`. The first takes two separate arguments, while the second takes a single tuple argument. This distinction is usually not important—until you get bitten by it in some corner case! For most purposes, you can ignore it.

Recursive Type Definitions Here is where things get really interesting: type definitions can be recursive! This allows us to define data structures of arbitrary size using a finite definition:

```
type int_list = Empty | Cons of int * int_list
```

Let us see what values inhabit `int_list`. The definition tells us there are two ways to build an `int_list`: - `Empty` represents the empty list—a list with no elements - `Cons (5, Empty)` is a list containing just 5 - `Cons (5, Cons (7, Cons (13, Empty)))` is a list containing 5, 7, and 13

Notice how `Cons` takes an integer and another `int_list`, allowing us to chain together as many elements as we like. This recursive structure is the essence of how functional languages represent unbounded data.

The built-in type `bool` can be viewed as if it were defined as `type bool = true | false`—just a variant with two constructors. Similarly, `int` can be thought of as a very large variant: `type int = 0 | -1 | 1 | -2 | 2 | ...` (though of course the compiler implements it more efficiently!)

Parametric Type Definitions Our `int_list` type only works with integers. But what if we want a list of strings? Or a list of booleans? We would have to define separate types for each, duplicating the same structure.

Type definitions can be *parametric* with respect to the types of their components. This allows us to define generic data structures that work with any element type. For example, a list of elements of arbitrary type:

```
type 'elem list = Empty | Cons of 'elem * 'elem list
```

The `'elem` is a *type parameter*—a placeholder that gets filled in when we use the type. We can have a `string list`, an `int list`, or even an `int list list` (a list of lists of integers).

Several conventions and syntax rules apply to parametric types:

- Type variables must start with `'`, but since OCaml will not remember the names we give, it is customary to use the names OCaml uses: `'a`, `'b`, `'c`, `'d`, etc.
- The OCaml syntax places the type parameter before the type name, mimicking English word order. A silly example that reads almost like English:

  ```
  type 'white_color dog = Dog of 'white_color
  ```

  This defines a "white-color dog" type—the syntax reads naturally!
- With multiple parameters, OCaml uses parentheses:

```
type ('a, 'b) choice = Left of 'a | Right of 'b
```

Compare this to F# syntax: `type choice<'a,'b> = Left of 'a | Right of 'b`

And Haskell syntax: `data Choice a b = Left a | Right b`

Different languages have different conventions, but the underlying concept is the same.

2.3 Syntactic Bread and Sugar

OCaml provides various syntactic conveniences—sometimes called *syntactic sugar*—that make code more pleasant to write and read. Let us survey the most important ones.

Constructor Naming Names of variants, called *constructors*, must start with a capital letter. If we wanted to define our own booleans, we would write:

```
type my_bool = True | False
```

Only constructors and module names can start with capital letters in OCaml. Everything else (values, functions, type names) must start with a lowercase letter. This convention makes it easy to distinguish constructors at a glance.

Modules are organizational units (like "shelves") containing related values. For example, the `List` module provides operations on lists, including `List.map` and `List.filter`. We will learn more about modules in later chapters.

Accessing Record Fields Did we mention that we can use dot notation to access record fields? The syntax `record.field` extracts a field value. For example, if we have `let person = {name="Alice"; age=30}`, we can write `person.name` to get `"Alice"`.

Function Definition Shortcuts Several syntactic shortcuts make function definitions more concise. These are worth memorizing, as you will see them constantly in OCaml code:

- `fun x y -> e` stands for `fun x -> fun y -> e`. Note that `fun x -> fun y -> e` parses as `fun x -> (fun y -> e)`. This shorthand aligns with curried form—we can write multi-argument functions without nesting `fun` expressions.
- `function A x -> e1 | B y -> e2` stands for `fun p -> match p with A x -> e1 | B y -> e2`. The general form is: `function PATTERN-MATCHING` stands for `fun v -> match v with PATTERN-MATCHING`. This is handy when you want to immediately pattern-match on a function's argument.

- `let f ARGS = e` is a shorthand for `let f = fun ARGS -> e`. This is probably the most common way to define functions in practice.

2.4 Pattern Matching

Pattern matching is one of the most powerful features of OCaml and similar languages. It lets us examine the structure of data and extract components in a single, elegant construct.

Recall that we introduced `fst` and `snd` as means to access elements of a pair. But what about larger tuples? There is no built-in `thd` for the third element. The fundamental way to access any tuple—or any algebraic data type—uses the `match` construct. In fact, `fst` and `snd` can easily be defined using pattern matching:

```
let fst = fun p -> match p with (a, b) -> a
let snd = fun p -> match p with (a, b) -> b
```

The pattern `(a, b)` *destructures* the pair, binding its first component to `a` and its second to `b`. We then return whichever component we want.

Matching on Records Pattern matching also works with records, letting us extract multiple fields at once:

```
type person = {name: string; surname: string; age: int}

let greet_person () =
  match {name="Walker"; surname="Johnnie"; age=207}
  with {name=n; surname=sn; age=a} -> "Hi " ^ sn ^ "!"
```

Here we match against a record pattern, binding each field to a variable. Note that we bind `name` to `n`, `surname` to `sn`, and `age` to `a`—then use `sn` in the greeting.

Understanding Patterns The left-hand sides of `->` in `match` expressions are called **patterns**. Patterns describe the structure of values we want to match against. They can include: - Constants (like `1`, `"hello"`, or `true`) - Variables (which bind to the matched value) - Constructors (like `None`, `Some x`, or `Cons (h, t)`) - Tuples and records - Nested combinations of all the above

Patterns can be nested to arbitrary depth, allowing us to match complex structures in one go:

```
match Some (5, 7) with
| None -> "sum: nothing"
| Some (x, y) -> "sum: " ^ string_of_int (x+y)
```

Here `Some (x, y)` is a nested pattern: we match `Some` of *something*, and that something must be a pair, whose components we bind to `x` and `y`.

Simple Patterns and Wildcards A pattern can simply bind the entire value without destructuring. Writing `match f x with v -> ...` is the same as `let v = f x in ...`. This is occasionally useful when you want the syntax of `match` but do not need to take the value apart.

When we do not need a value in a pattern, it is good practice to use the underscore `_`, which is a *wildcard*. The wildcard matches anything but does not bind it to a name. This signals to the reader (and the compiler) that we are intentionally ignoring that part:

```
let fst (a, _) = a
let snd (_, b) = b
```

Using `_` instead of an unused variable name avoids compiler warnings about unused bindings.

Pattern Linearity A variable can only appear once in a pattern. This property is called *linearity*. You might think this is a limitation—what if we want to check that two parts of a structure are equal? We cannot write `(x, x)` to match pairs with equal components.

However, we can add conditions to patterns using `when`, so linearity is not really a limitation in practice:

```
let describe_point p =
  match p with
  | (x, y) when x = y -> "diag"
  | _ -> "off-diag"
```

The `when` clause acts as a guard: the pattern matches only if both the structure matches *and* the condition is true.

Here is a more elaborate example showing how to implement a comparison function:

```
let compare a b = match a, b with
  | (x, y) when x < y -> -1
  | (x, y) when x = y -> 0
  | _ -> 1
```

Notice how we match against the tuple `(a, b)` in different ways, using guards to distinguish the cases.

Partial Record Patterns We can skip unused fields of a record in a pattern. Only the fields we care about need to be mentioned. This keeps patterns concise and means we do not have to update every pattern when we add a new field to a record type.

Or-Patterns We can compress patterns by using `|` inside a single pattern to match multiple alternatives. This is different from having multiple pattern clauses—it lets us share a single right-hand side for several patterns:

```
type month =
  | Jan | Feb | Mar | Apr | May | Jun
  | Jul | Aug | Sep | Oct | Nov | Dec

type weekday = Mon | Tue | Wed | Thu | Fri | Sat | Sun

type date =
  {year: int; month: month; day: int; weekday: weekday}

let day =
  {year = 2012; month = Feb; day = 14; weekday = Wed};;

match day with
  | {weekday = Sat | Sun; _} -> "Weekend!"
  | _ -> "Work day"
```

The pattern `Sat | Sun` matches either `Sat` or `Sun`. This is much cleaner than writing two separate clauses with the same right-hand side.

Named Patterns with `as` Sometimes we want to both destructure a value *and* keep a reference to the whole thing (or some intermediate part). We use `(pattern as v)` to name a nested pattern, binding the matched value to `v`:

```
match day with
  | {weekday = (Mon | Tue | Wed | Thu | Fri as wday); _}
      when not (day.month = Dec && day.day = 24) ->
    Some (work (get_plan wday))
  | _ -> None
```

This example demonstrates several features working together:

- An or-pattern matches any weekday from Monday to Friday
- The `as wday` clause binds the matched weekday to the variable `wday`
- A `when` guard checks that it is not Christmas Eve
- The bound variable `wday` is then used in the expression `get_plan wday`

This combination of features makes OCaml's pattern matching remarkably expressive.

2.5 Interpreting Algebraic Data Types as Polynomials

Now we come to one of the most delightful aspects of algebraic data types: they really are *algebraic* in a precise mathematical sense. Let us explore a curious analogy between types and polynomials that turns out to be surprisingly deep.

The translation from types to mathematical expressions works as follows:

- Replace `|` (variant choice) with $+$ (addition)
- Replace `*` (tuple product) with $\times$ (multiplication)
- Treat record types as tuple types (erasing field names and translating `;` as $\times$)

We also need translations for some special types:

- The **void type** (a type with no constructors, hence no values):

  ```
  type void
  ```

  (Yes, this is its complete definition, with no `= something` part.) Since no values can be constructed, it represents emptiness—translate it as 0.

- The **unit type** has exactly one value, so translate it as 1. Since variants without arguments behave like variants `of unit`, translate them as 1 as well.

- The **bool type** has exactly two values (`true` and `false`), so translate it as 2.

- Types like `int`, `string`, `float`, and type parameters are treated as variables. We do not care about their exact number of values; we just give them symbolic names like x, y, etc.

- Defined types translate according to their definitions (substituting variables as necessary).

Give a name to the type being defined (representing a function of the introduced variables). Now interpret the result as an ordinary numeric polynomial! (Or a "rational function" if recursively defined.)

This might seem like a mere curiosity, but it leads to real insights. Let us have some fun with it!

Example: Date Type

```
type date = {year: int; month: int; day: int}
```

A date is a record with three `int` fields. Translating to a polynomial (using x for `int`):

$$D = x \times x \times x = x^3$$

The cube makes sense: a date is essentially a triple of integers.

Example: Option Type The built-in option type is defined as:

```
type 'a option = None | Some of 'a
```

Translating (using x for the type parameter `'a`):

$$O = 1 + x$$

This reads as: an option is either nothing (1) or something of type x. The polynomial $1 + x$ is beautifully simple!

Example: List Type

```
type 'a my_list = Empty | Cons of 'a * 'a my_list
```

Translating (where L represents the list type itself, and x represents the element type):

$$L = 1 + x \cdot L$$

This is a recursive equation! A list is either empty (1) or an element times another list ($x \cdot L$). If you solve this equation algebraically, you get $L = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$, which corresponds to: a list is either empty, or has one element, or has two elements, etc.

Example: Binary Tree Type

```
type btree = Tip | Node of int * btree * btree
```

Translating:

$$T = 1 + x \cdot T \cdot T = 1 + x \cdot T^2$$

A binary tree is either a tip (1) or a node containing a value and two subtrees ($x \cdot T^2$).

Type Isomorphisms Here is the remarkable payoff: when translations of two types are equal according to the laws of high-school algebra, the types are *isomorphic*. This means there exist bijective (one-to-one and onto) functions between them—you can convert from one type to the other and back without losing any information.

Let us play with the binary tree polynomial and see where algebra takes us:

$$\begin{aligned}
T &= 1 + x \cdot T^2 \\
&= 1 + x \cdot T + x^2 \cdot T^3 \\
&= 1 + x + x^2 \cdot T^2 + x^2 \cdot T^3 \\
&= 1 + x + x^2 \cdot T^2 \cdot (1 + T) \\
&= 1 + x \cdot (1 + x \cdot T^2 \cdot (1 + T))
\end{aligned}$$

Each step uses standard algebraic manipulations: substituting $T = 1 + xT^2$, expanding, factoring, and rearranging. The result is a different but algebraically equivalent expression.

Now let us translate this resulting expression back to a type:

```
type repr =
  (int * (int * btree * btree * btree option) option) option
```

Reading the polynomial $1 + x \cdot (1 + x \cdot T^2 \cdot (1 + T))$ from outside in: we have an option (the outermost $1 + \ldots$), whose `Some` case contains an `int` times another option, and so on.

The challenge is to find isomorphism functions with signatures:

```
val iso1 : btree -> repr
val iso2 : repr -> btree
```

These functions should satisfy: for all trees `t`, `iso2 (iso1 t) = t`, and for all representations `r`, `iso1 (iso2 r) = r`. Can you write them?

My First (Failed) Attempt Here is my first attempt, trying to guess the pattern directly:

```
# let iso1 (t : btree) : repr =
  match t with
    | Tip -> None
    | Node (x, Tip, Tip) -> Some (x, None)
    | Node (x, Node (y, t1, t2), Tip) ->
      Some (x, Some (y, t1, t2, None))
    | Node (x, Node (y, t1, t2), t3) ->
      Some (x, Some (y, t1, t2, Some t3));;

Warning 8: this pattern-matching is not exhaustive.
Here is an example of a value that is not matched:
Node (_, Tip, Node (_, _, _))
```

I forgot about one case! The case `Node (_, Tip, Node (_, _, _))`—a node with an empty left subtree and non-empty right subtree—was not covered. It seems difficult to guess the solution directly when trying to map the complex final form all at once.

Have you found it on your first try? If so, congratulations! Most people do not. This illustrates an important principle: complex transformations are easier to get right when broken into smaller steps.

Breaking Down the Problem Let us divide the task into smaller steps corresponding to intermediate points in the polynomial transformation. Instead of jumping from $T = 1 + xT^2$ directly to the final form, we will introduce intermediate types for each algebraic step:

```
type ('a, 'b) choice = Left of 'a | Right of 'b

type interm1 =
  ((int * btree, int * int * btree * btree * btree) choice)
  option

type interm2 =
  ((int, int * int * btree * btree * btree option) choice)
  option
```

Now we can define each step:

```
let step1r (t : btree) : interm1 =
  match t with
    | Tip -> None
    | Node (x, t1, Tip) -> Some (Left (x, t1))
    | Node (x, t1, Node (y, t2, t3)) ->
      Some (Right (x, y, t1, t2, t3))

let step2r (r : interm1) : interm2 =
  match r with
    | None -> None
    | Some (Left (x, Tip)) -> Some (Left x)
    | Some (Left (x, Node (y, t1, t2))) ->
      Some (Right (x, y, t1, t2, None))
    | Some (Right (x, y, t1, t2, t3)) ->
      Some (Right (x, y, t1, t2, Some t3))

let step3r (r : interm2) : repr =
  match r with
    | None -> None
    | Some (Left x) -> Some (x, None)
    | Some (Right (x, y, t1, t2, t3opt)) ->
      Some (x, Some (y, t1, t2, t3opt))

let iso1 (t : btree) : repr =
  step3r (step2r (step1r t))
```

Each step function handles one small transformation, and the compiler verifies that our pattern matching is exhaustive. No more missed cases!

Exercise: Define `step1l`, `step2l`, `step3l`, and `iso2`.

Hint: Now it is straightforward—each step is simply the inverse of its corresponding forward step. The left-going functions undo what the right-going functions do.

Take-Home Lessons This exploration of type isomorphisms teaches us two valuable principles:

1. **Design for validity:** Try to define data structures so that only meaningful information can be represented—as long as it does not overcomplicate the data structures. Avoid catch-all clauses when defining functions. The compiler will then tell you if you have forgotten about a case. The exhaustiveness checker is your friend.
2. **Divide and conquer:** Break solutions into small steps so that each step can be easily understood and verified. When I tried to write `iso1` directly, I made a mistake. When I broke it into three simple steps, each step was obviously correct, and composing them gave the right answer.

2.6 Differentiating Algebraic Data Types

Of course, you might object that the pompous title is wrong—we will differentiate the translated polynomials, not the types themselves. Fair enough! But what sense does differentiating a type's polynomial make?

It turns out that taking the partial derivative of a polynomial (translated from a data type), when translated back, gives a type representing a "one-hole context"—a data structure with one piece missing. This missing piece corresponds to the variable with respect to which we differentiated. The derivative tells us: "Here are all the ways to point at one element of this type."

Example: Differentiating the Date Type Let us start with our familiar date type:

```
type date = {year: int; month: int; day: int}
```

The translation and its derivative:

$$\begin{aligned} D &= x \cdot x \cdot x = x^3 \\ \frac{\partial D}{\partial x} &= 3x^2 = x \cdot x + x \cdot x + x \cdot x \end{aligned}$$

We could have left it as $3 \cdot x \cdot x$, but expanding it as a sum shows the structure more clearly. The derivative $3x^2$ says: there are three ways to "point at" an `int` in a date, and each way leaves two other `int`s behind.

Translating the expanded form back to a type:

```
type date_deriv =
  Year of int * int | Month of int * int | Day of int * int
```

Each variant represents a "hole" at a different position: - `Year (m, d)` means the year field is the hole (and we have the month `m` and day `d`) - `Month (y, d)` means the month field is the hole (and we have year `y` and day `d`) - `Day (y, m)` means the day field is the hole

Now we can define functions to introduce and eliminate this derivative type:

```
let date_deriv {year=y; month=m; day=d} =
  [Year (m, d); Month (y, d); Day (y, m)]

let date_integr n = function
  | Year (m, d) -> {year=n; month=m; day=d}
  | Month (y, d) -> {year=y; month=n; day=d}
  | Day (y, m) -> {year=y; month=m; day=n}
;;

List.map (date_integr 7)
  (date_deriv {year=2012; month=2; day=14})
```

The `date_deriv` function produces all contexts (one for each field)—it "differentiates" a date into a list of one-hole contexts. The `date_integr` function fills in a hole with a new value—it "integrates" by putting a value back into the context. Notice how the naming follows the calculus analogy!

The example above takes the date February 14, 2012, produces three contexts (one for each field), and then fills each hole with the number 7, producing three modified dates.

Example: Differentiating Binary Trees Now let us tackle the more challenging case of binary trees:

```
type btree = Tip | Node of int * btree * btree
```

The translation and differentiation:

$$
\begin{aligned}
T &= 1 + x \cdot T^2 \\
\frac{\partial T}{\partial x} &= 0 + T^2 + 2 \cdot x \cdot T \cdot \frac{\partial T}{\partial x} = T \cdot T + 2 \cdot x \cdot T \cdot \frac{\partial T}{\partial x}
\end{aligned}
$$

Something interesting happened: the derivative is recursive! It refers to itself via $\frac{\partial T}{\partial x}$. This makes perfect sense when you think about it:

- $T \cdot T$ represents pointing at the root: the hole is at the current node, and we have the two subtrees.
- $2 \cdot x \cdot T \cdot \frac{\partial T}{\partial x}$ represents pointing deeper in the tree: we choose left or right (the factor of 2), remember the current node's value (x), keep the other subtree (T), and then have a context in the chosen subtree ($\frac{\partial T}{\partial x}$).

Instead of translating 2 as `bool`, we introduce a more descriptive type to make the code clearer:

```
type btree_dir = LeftBranch | RightBranch

type btree_deriv =
```

```
  | Here of btree * btree
  | Below of btree_dir * int * btree * btree_deriv
```

The `Here` constructor means the hole is at the current position, and we have the left and right subtrees. The `Below` constructor means we go down one level, remembering which direction we went, the value at the node we passed, and the subtree we did not enter.

(You might someday hear about *zippers*—they are "inverted" relative to our type. In a zipper, the hole comes first, and the context trails behind. Both representations are useful in different situations.)

Exercise: Write a function that takes a number and a `btree_deriv`, and builds a `btree` by putting the number into the "hole" in `btree_deriv`.

Solution

The integration function fills the hole with a value. It must be recursive because the derivative type is recursive—we may need to descend through multiple `Below` constructors before reaching the `Here` where the hole actually is:

```
let rec btree_integr n = function
  | Here (ltree, rtree) -> Node (n, ltree, rtree)
  | Below (LeftBranch, m, rtree, deriv) ->
    Node (m, btree_integr n deriv, rtree)
  | Below (RightBranch, m, ltree, deriv) ->
    Node (m, ltree, btree_integr n deriv)
```

When we reach `Here`, we create a node with the new value `n` and the two subtrees. When we see `Below`, we reconstruct the node we passed through and recursively integrate into the appropriate subtree.

2.7 Exercises

Exercise 1: Designing Valid Data Structures *Due to Yaron Minsky.*

This exercise practices the principle of "making invalid states unrepresentable." Consider a datatype to store internet connection information. The time `when_initiated` marks the start of connecting and is not needed after the connection is established (it is only used to decide whether to give up trying to connect). The ping information is available for established connections but not straight away.

```
type connectionstate = Connecting | Connected | Disconnected

type connectioninfo = {
  state : connectionstate;
  server : Inetaddr.t;
  lastpingtime : Time.t option;
  lastpingid : int option;
```

```
  sessionid : string option;
  wheninitiated : Time.t option;
  whendisconnected : Time.t option;
}
```

(The types `Time.t` and `Inetaddr.t` come from the *Core* library. You can replace them with `float` and `Unix.inet_addr`. Load the Unix library in the interactive toplevel with `#load "unix.cma";;`.)

The problem with this design is that it allows many nonsensical combinations: a `Connecting` state with ping information, a `Disconnected` state with a session ID, etc. The optional fields (all those `option` types) make it unclear which fields are valid in which states.

Rewrite the type definitions so that the datatype will contain only reasonable combinations of information. Use separate record types for each connection state, with only the fields that make sense for that state.

Exercise 2: Labeled and Optional Arguments In OCaml, functions can have labeled arguments and optional arguments (parameters with default values that can be omitted). This exercise explores these features.

Labels can differ from the names of argument values:

```
let f ~meaningfulname:n = n + 1
let _ = f ~meaningfulname:5  (* We do not need the result so we ignore it. *)
```

When the label and value names are the same, the syntax is shorter:

```
let g ~pos ~len =
  StringLabels.sub "0123456789abcdefghijklmnopqrstuvwxyz" ~pos ~len

let () =  (* A nicer way to mark computations that return unit. *)
  let pos = Random.int 26 in
  let len = Random.int 10 in
  print_string (g ~pos ~len)
```

When some function arguments are optional, the function must take non-optional arguments after the last optional argument. Optional parameters with default values:

```
let h ?(len=1) pos = g ~pos ~len
let () = print_string (h 10)
```

Optional arguments are implemented as parameters of an option type. This allows checking whether the argument was provided:

```
let foo ?bar n =
  match bar with
    | None -> "Argument = " ^ string_of_int n
    | Some m -> "Sum = " ^ string_of_int (m + n)
```

We can use it in various ways:

```
let _ = foo 5
let _ = foo ~bar:5 7
```

We can also provide the option value directly:

```
let test_foo () =
  let bar = if Random.int 10 < 5 then None else Some 7 in
  foo ?bar 7
```

1. Observe the types that functions with labeled and optional arguments have. Come up with coding style guidelines for when to use labeled arguments. When might they improve readability? When might they be overkill?

2. Write a rectangle-drawing procedure that takes three optional arguments: left-upper corner, right-lower corner, and a width-height pair. It should draw a correct rectangle whenever two of the three arguments are given (since any two determine the third), and raise an exception otherwise. Load the graphics library with `#load "graphics.cma";;`. Use `invalid_arg`, `Graphics.open_graph`, and `Graphics.draw_rect`.

3. Write a function that takes an optional argument of arbitrary type and a function argument, and passes the optional argument to the function without inspecting it. This tests your understanding of how optional arguments work at the type level.

Exercise 3: Type Inference Practice *From a past exam.*

These exercises help you internalize how type inference works. Try to work them out by hand before checking with the OCaml toplevel.

1. Give the (most general) types of the following expressions, either by guessing or by inferring by hand:
   1. `let double f y = f (f y) in fun g x -> double (g x)`
   2. `let rec tails l = match l with [] -> [] | x::xs -> xs::tails xs in fun l -> List.combine l (tails l)`
2. Give example expressions that have the following types (without using type constraints). There are many possible answers for each:
   1. `(int -> int) -> bool`
   2. `'a option -> 'a list`

Exercise 4: Types as Exponents We have seen that algebraic data types can be related to analytic functions (the subset definable from polynomials via recursion)—by literally interpreting sum types (variant types) as sums and product types (tuple and record types) as products. We can extend this interpretation to function types by interpreting $a \rightarrow b$ as b^a (i.e., b to the power of a). Note that the b^a notation is actually used to denote functions in set theory.

This interpretation makes sense: a function from a set with a elements to a set with b elements is choosing, for each of the a inputs, one of b outputs—giving b^a possible functions.

1. Translate a^{b+cd} and $a^b \cdot (a^c)^d$ into OCaml types, using any distinct types for a, b, c, d, and using `type ('a,'b) choice = Left of 'a | Right of 'b` for $+$. Write the bijection functions in both directions. Verify algebraically that $a^{b+cd} = a^b \cdot (a^c)^d$ using the laws of exponents.
2. Come up with a type `'t exp` that shares with the exponential function the following property: $\frac{\partial \exp(t)}{\partial t} = \exp(t)$, where we translate a derivative of a type as a context (i.e., the type with a "hole"), as in this chapter. In other words, the derivative of the type should be isomorphic to the type itself! Explain why your answer is correct. *Hint:* in computer science, our logarithms are mostly base 2.

Further reading: Algebraic Type Systems - Combinatorial Species

Exercise 5 (Homework): Finding Contexts Write a function `btree_deriv_at` that takes a predicate over integers (i.e., a function `f: int -> bool`) and a `btree`, and builds a `btree_deriv` whose "hole" is in the first position for which the predicate returns true. It should return a `btree_deriv option`, with `None` if the predicate does not hold for any node.

This function lets you "search" a tree and get back a context pointing to the found element. Think about what order you want to search in (pre-order, in-order, or post-order) and what "first" means in that context.

Chapter 3: Computation

Reduction semantics and operational reasoning

References:

- "Using, Understanding and Unraveling the OCaml Language" by Didier Remy, Chapter 1
- "The OCaml system" manual, the tutorial part, Chapter 1

In this chapter, we explore how functional programs actually execute. We will learn how to reason about computation step by step using *reduction semantics*, and discover important optimization techniques like *tail call optimization* that make functional programming practical. Along the way, we will encounter our first taste of *continuation passing style*, a powerful programming technique that will reappear throughout this book.

3.1 Function Composition

Function composition is one of the most fundamental operations in functional programming. It allows us to build complex transformations by combining simpler functions. The usual way function composition is defined in mathematics is "backward"—the notation follows the convention of mathematical function application:

$$(f \circ g)(x) = f(g(x))$$

This means that when we write $f \circ g$, we first apply g and then apply f to the result. The function written on the left is applied last—hence the term "backward" composition. Here is how this is expressed in different functional programming languages:

Language	Definition
Math	$(f \circ g)(x) = f(g(x))$
OCaml	`let (-|) f g x = f (g x)`
F#	`let (<<) f g x = f (g x)`
Haskell	`(.) f g = \x -> f (g x)`

This backward composition looks like function application but needs fewer parentheses. Do you recall the functions `iso1` and `iso2` from the previous chapter on type isomorphisms? Using backward composition, we could write:

```
let iso2 = step1l -| step2l -| step3l
```

While backward composition matches traditional mathematical notation, many programmers find a "forward" composition more intuitive. Forward composition follows the order in which computation actually proceeds—data flows from left to right, matching how we typically read code in most programming languages:

Language	Definition
OCaml	`let (\\|-) f g x = g (f x)`
F#	`let (>>) f g x = g (f x)`

With forward composition, you can read a pipeline of transformations in the natural order:

```
let iso1 = step1r |- step2r |- step3r
```

Here, the data first passes through `step1r`, then the result goes to `step2r`, and finally to `step3r`. This "pipeline" style of programming is particularly popular in languages like F# and has influenced the design of many modern programming languages.

Partial Application Both composition examples above rely on **partial application**, a technique we introduced in the previous chapter. Recall that `((+) 1)` is a function that adds 1 to its argument—we have provided only one of the two arguments that `(+)` requires. Partial application occurs whenever we supply fewer arguments than a function expects; the result is a new function that waits for the remaining arguments.

Consider the composition `step1r |- step2r |- step3r`. How exactly does partial application come into play here? The composition operator `(|-)` is defined as `let (|-) f g x = g (f x)`, which means it takes *three* arguments: two functions `f` and `g`, and a value `x`. When we write `step1r |- step2r`, we are partially applying `(|-)` with just two arguments. The result is a function that still needs the final argument `x`.

Exercise: Think about the types involved. If `step1r` has type `'a -> 'b` and `step2r` has type `'b -> 'c`, what is the type of `step1r |- step2r`?

Power Function Now we define iterated function composition—applying a function to itself repeatedly. This is written mathematically as:

$$f^n(x) := \underbrace{(f \circ \cdots \circ f)}_{n \text{ times}}(x)$$

In other words, f^0 is the identity function, $f^1 = f$, $f^2 = f \circ f$, and so on. In OCaml, we first define the backward composition operator, then use it to implement `power`:

```
let (-|) f g x = f (g x)

let rec power f n =
  if n <= 0 then (fun x -> x) else f -| power f (n-1)
```

When `n <= 0`, we return the identity function `fun x -> x`. Otherwise, we compose `f` with `power f (n-1)`, which gives us one more application of `f`. Notice how elegantly this definition expresses the mathematical concept—we are literally composing `f` with itself `n` times.

This `power` function is surprisingly versatile. For example, we can use it to define addition in terms of the successor function:

```
let add n = power ((+) 1) n
```

Here `add 5 7` would compute $7 + 1 + 1 + 1 + 1 + 1 = 12$. We could even define multiplication:

```
let mult k n = power ((+) k) n 0
```

This computes $0 + k + k + \ldots + k$ (adding k a total of n times), giving us $k \times n$. While not the most efficient implementation, these examples show how higher-order functions like `power` can express fundamental mathematical operations.

Numerical Derivative A beautiful application of `power` is computing higher-order derivatives. First, let us define a numerical approximation of the derivative using the standard finite difference formula:

```
let derivative dx f = fun x -> (f(x +. dx) -. f(x)) /. dx
```

This definition computes $\frac{f(x+dx)-f(x)}{dx}$, which approximates $f'(x)$ when `dx` is small. Notice the explicit `fun x -> ...` syntax, which emphasizes that `derivative dx f` is itself a function—we are transforming a function `f` into its derivative function.

We can write the same definition more concisely using OCaml's curried function syntax:

```
let derivative dx f x = (f(x +. dx) -. f(x)) /. dx
```

Both definitions are equivalent, but the first makes the "function returning a function" structure more explicit, while the second is more compact.

A note on OCaml's numeric operators: OCaml uses different operators for floating-point arithmetic than for integers. The type of `(+)` is `int -> int -> int`, so we cannot use `+` with `float` values. Instead, operators followed by a dot work on `float` numbers: `+.`, `-.`, `*.`, and `/.`. This might seem inconvenient at first, but it catches type errors at compile time and avoids the implicit conversions that cause subtle bugs in other languages.

Computing Higher-Order Derivatives Now comes the payoff. With `power` and `derivative`, we can elegantly compute higher-order derivatives:

```
let pi = 4.0 *. atan 1.0
let sin''' = (power (derivative 1e-5) 3) sin;;
sin''' pi
```

Here `sin'''` is the third derivative of sine. The expression `(power (derivative 1e-5) 3)` creates a function that applies the derivative operation three times—exactly what we need for the third derivative.

Mathematically, the third derivative of $\sin(x)$ is $-\cos(x)$, so `sin''' pi` should give us $-\cos(\pi) = 1$. The actual result will be close to 1, with some numerical error due to the finite difference approximation (the error compounds with each derivative we take).

This example demonstrates the power of treating functions as first-class values. We have built a general-purpose derivative operator and combined it with our `power` function to create an nth-derivative calculator—all in just a few lines of code.

3.2 Evaluation Rules (Reduction Semantics)

So far, we have written OCaml programs and observed their results, but we have not precisely described *how* those results are computed. To understand how OCaml programs execute, we need to formalize the evaluation process. This section presents **reduction semantics** (also called *operational semantics*), which describes computation as a series of rewriting steps that transform expressions until we reach a final value.

Understanding reduction semantics is valuable for several reasons. It helps us predict what our programs will do, reason about their efficiency, and understand subtle behaviors like infinite loops and non-termination. The ideas here also form the foundation for understanding more advanced topics like type systems and program verification.

Expressions Programs consist of **expressions**. Here is the grammar of expressions for a simplified version of OCaml (we omit some features for clarity):

$$
\begin{array}{rcll}
a & := & x & \text{variables} \\
 & | & \texttt{fun } x \texttt{ -> } a & \text{(defined) functions} \\
 & | & a\ a & \text{applications} \\
 & | & C^0 & \text{value constructors of arity 0} \\
 & | & C^n(a, \ldots, a) & \text{value constructors of arity } n \\
 & | & f^n & \text{built-in values (primitives) of arity } n \\
 & | & \texttt{let } x = a \texttt{ in } a & \text{name bindings (local definitions)} \\
 & | & \texttt{match } a \texttt{ with} & \\
 & & \quad p \texttt{ -> } a \texttt{ | } \cdots \texttt{ | } p \texttt{ -> } a & \text{pattern matching} \\
 & & & \\
p & := & x & \text{pattern variables} \\
 & | & (p, \ldots, p) & \text{tuple patterns} \\
 & | & C^0 & \text{variant patterns of arity 0} \\
 & | & C^n(p, \ldots, p) & \text{variant patterns of arity } n
\end{array}
$$

Arity means how many arguments something requires. For constructors, arity tells us how many components the constructor holds; for functions (primitives), it tells us how many arguments they need before they can compute a result. For tuple patterns, arity is simply the length of the tuple.

The `fix` Primitive Our grammar above includes functions defined with `fun`, but what about recursive functions defined with `let rec`? To keep our semantics simple, we introduce a primitive `fix` that captures the essence of recursion:

$$\texttt{let rec } f\ x = e_1 \texttt{ in } e_2 \equiv \texttt{let } f = \texttt{fix (fun } f\ x \texttt{ -> } e_1\texttt{) in } e_2$$

The `fix` primitive is a *fixpoint combinator*. It takes a function that expects to receive “itself” as its first argument and produces a function that, when called, behaves as if it has access to itself for recursive calls. This might seem mysterious now, but we will see exactly how it works when we examine its reduction rule below.

Values Expressions evaluate (i.e., compute) to **values**. Values are expressions that cannot be reduced further—they are the "final answers" of computation:

$$
\begin{array}{rcll}
v & := & \texttt{fun } x \texttt{ -> } a & \text{(defined) functions} \\
 & \mid & C^n(v_1, \ldots, v_n) & \text{constructed values} \\
 & \mid & f^n\ v_1\ \cdots\ v_k & k < n \text{ (partially applied primitives)}
\end{array}
$$

Note that functions are values: `fun x -> x + 1` is already fully evaluated—there is nothing more to compute until the function is applied to an argument. Similarly, constructed values like `Some 42` or `(1, 2, 3)` are values when all their components are values.

Partially applied primitives like `(+) 3` are also values. The expression `(+) 3` has received one argument but needs another before it can compute a sum. Until that second argument arrives, there is nothing more to do, so `(+) 3` is a value.

Substitution The heart of evaluation is **substitution**. To substitute a value v for a variable x in expression a, we write $a[x := v]$. This notation means that every occurrence of x in a is replaced by v.

For example, if a is the expression `x + x * y` and we substitute 3 for `x`, we get `3 + 3 * y`. In our notation: `(x + x * y)[x := 3] = 3 + 3 * y`.

Implementation note: Although we describe substitution as "replacing" variables with values, the actual implementation in OCaml does not duplicate the value v in memory each time it appears. Instead, OCaml uses closures and sharing to ensure that values are stored once and referenced wherever needed. This is both more efficient and essential for handling recursive data structures.

Reduction Rules (Redexes) Now we can describe how computation actually proceeds. Reduction works by finding reducible expressions called **redexes** (short for "reducible expressions") and applying reduction rules that rewrite them into simpler forms. We write $e_1 \rightsquigarrow e_2$ to mean "expression e_1 reduces to expression e_2 in one step."

Here are the fundamental reduction rules:

Function application (beta reduction):

$$(\texttt{fun } x \texttt{ -> } a)\ v \rightsquigarrow a[x := v]$$

This is the most important rule. When we apply a function `fun x -> a` to a value v, we substitute v for the parameter x throughout the function body

a. This rule is traditionally called "beta reduction" in the lambda calculus literature.

For example: `(fun x -> x + 1) 5` $\rightsquigarrow$ `5 + 1` $\rightsquigarrow$ `6`.

Let binding:

$$\texttt{let } x = v \texttt{ in } a \rightsquigarrow a[x := v]$$

A let binding works similarly: once the bound expression has been evaluated to a value v, we substitute it into the body. Notice that `let x = e in a` is essentially equivalent to `(fun x -> a) e`—both bind x to the result of evaluating e within the expression a.

Primitive application:

$$f^n\ v_1\ \cdots\ v_n \rightsquigarrow f(v_1, \ldots, v_n)$$

When a primitive (like `+` or `*`) receives all the arguments it needs (determined by its arity n), it computes the result. Here $f(v_1, \ldots, v_n)$ denotes the actual result of the primitive operation—for example, `(+) 2 3` $\rightsquigarrow$ `5`.

Pattern matching with a variable pattern:

$$\texttt{match } v \texttt{ with } x \texttt{ -> } a \texttt{ | } \cdots \rightsquigarrow a[x := v]$$

A variable pattern always matches, binding the entire value to the variable.

Pattern matching with a non-matching constructor:

$$\frac{C_1 \neq C_2}{\texttt{match } C_1^n(v_1, \ldots, v_n) \texttt{ with } C_2^k(p_1, \ldots, p_k) \texttt{ -> } a \texttt{ | } pm \rightsquigarrow \texttt{match } C_1^n(v_1, \ldots, v_n) \texttt{ with } pm}$$

If the constructor in the value (C_1) does not match the constructor in the pattern (C_2), we skip this branch and try the remaining patterns (pm). This is how OCaml searches through pattern match cases from top to bottom.

Pattern matching with a matching constructor:

$$\texttt{match } C_1^n(v_1, \ldots, v_n) \texttt{ with } C_1^n(x_1, \ldots, x_n) \texttt{ -> } a \texttt{ | } \cdots \rightsquigarrow a[x_1 := v_1; \ldots; x_n := v_n]$$

If the constructor matches, we substitute all the values from inside the constructor for the corresponding pattern variables. For example, `match Some 42 with Some x -> x + 1 | None -> 0` reduces to `42 + 1` because `Some` matches `Some` and we substitute 42 for `x`.

If $n = 0$, then $C_1^n(v_1, \ldots, v_n)$ stands for simply C_1^0, a constructor with no arguments (like `None` or `[]`). We omit the more complex cases of nested pattern matching for brevity.

Rule Variables In these rules, we use *metavariables*—placeholders that can be replaced with actual expressions. Understanding them is key to applying the rules:

- x matches any variable name (like `foo`, `n`, or `result`)
- $a, a_1, \ldots, a_n$ match any expression (not necessarily a value)
- $v, v_1, \ldots, v_n$ match any *value* (expressions that are fully evaluated)

To apply a rule, find substitutions for these metavariables that make the left-hand side of the rule match your expression. Then the right-hand side (with the same substitutions applied) gives you the reduced expression.

For example, to apply the beta reduction rule to `(fun n -> n * 2) 5`: 1. Match `fun x -> a` with `fun n -> n * 2`, giving us $x =$ `n` and $a =$ `n * 2` 2. Match v with `5` 3. The right-hand side $a[x := v]$ becomes `(n * 2)[n := 5]` which equals `5 * 2`

Evaluation Context Rules The reduction rules above only apply when the arguments are already values. But what if we have `(fun x -> x + 1) (2 + 3)`? The argument `2 + 3` is not a value, so we cannot directly apply beta reduction. We need rules that tell us evaluation can proceed inside subexpressions.

If $a_i \rightsquigarrow a_i'$ (meaning a_i can take a reduction step), then:

$$
\begin{array}{lcl}
a_1\ a_2 & \rightsquigarrow & a_1'\ a_2 \\
a_1\ a_2 & \rightsquigarrow & a_1\ a_2' \\
C^n(a_1, \ldots, a_i, \ldots, a_n) & \rightsquigarrow & C^n(a_1, \ldots, a_i', \ldots, a_n) \\
\texttt{let}\ x = a_1\ \texttt{in}\ a_2 & \rightsquigarrow & \texttt{let}\ x = a_1'\ \texttt{in}\ a_2 \\
\texttt{match}\ a_1\ \texttt{with}\ pm & \rightsquigarrow & \texttt{match}\ a_1'\ \texttt{with}\ pm
\end{array}
$$

These rules describe *where* reduction can happen: - In a function application $a_1\ a_2$, either the function (a_1) or the argument (a_2) can be evaluated. The two rules allow evaluation in arbitrary order—this gives the implementation flexibility in how it schedules computation. - In a constructor application, any argument can be evaluated. - In a let binding `let x = a1 in a2`, the bound expression a_1 must be evaluated to a value before we can proceed. Notice there is no rule for evaluating a_2 directly—the body is only evaluated after the substitution happens. - In a match expression, the scrutinee (the expression being matched) must be evaluated before pattern matching can proceed.

The `fix` Rule Finally, the rule for the `fix` primitive, which enables recursion:

$$\texttt{fix}^2\ v_1\ v_2 \rightsquigarrow v_1\ (\texttt{fix}^2\ v_1)\ v_2$$

This rule is subtle but powerful. Let us unpack it:

1. `fix` is a binary primitive (arity 2), meaning it needs two arguments before it computes.
2. When we apply `fix` to two values v_1 and v_2, it "unrolls" one level of recursion by calling v_1 with two arguments: `(fix v1)` (which represents "the recursive function itself") and v_2 (the actual argument to the recursive call).
3. Because `fix` has arity 2, the expression `(fix v1)` is a *partially applied primitive*—and partially applied primitives are values! This is crucial: it means `(fix v1)` will not be evaluated further until it is applied to another argument inside v_1.

This delayed evaluation is what prevents infinite loops. If `(fix v1)` were evaluated immediately, we would get an infinite chain of expansions. Instead, evaluation only continues when the recursive function actually makes a recursive call.

Practice The best way to understand reduction semantics is to work through examples by hand. Trace the evaluation of these expressions step by step:

Exercise 1: Evaluate `let double x = x + x in double 3`

Exercise 2: Evaluate `(fun f -> fun x -> f (f x)) (fun y -> y + 1) 0`

Exercise 3: Define the factorial function using `fix` and trace the evaluation of `factorial 3`

3.3 Symbolic Derivation Example

Let us see the reduction rules in action with a more substantial example. We will build a small computer algebra system that can represent mathematical expressions symbolically, evaluate them, and even compute their derivatives symbolically.

Consider the symbolic expression type from `Lec3.ml`:

```
type expression =
  | Const of float
  | Var of string
  | Sum of expression * expression    (* e1 + e2 *)
  | Diff of expression * expression   (* e1 - e2 *)
  | Prod of expression * expression   (* e1 * e2 *)
  | Quot of expression * expression   (* e1 / e2 *)

exception Unbound_variable of string

let rec eval env exp =
  match exp with
  | Const c -> c
  | Var v ->
```

```
    (try List.assoc v env with Not_found -> raise (Unbound_variable v))
  | Sum(f, g) -> eval env f +. eval env g
  | Diff(f, g) -> eval env f -. eval env g
  | Prod(f, g) -> eval env f *. eval env g
  | Quot(f, g) -> eval env f /. eval env g
```

The `expression` type represents mathematical expressions as a tree structure. Each constructor corresponds to a different kind of expression: constants, variables, and the four basic arithmetic operations. The `eval` function takes an environment `env` (a list of variable-value pairs) and recursively evaluates an expression to a floating-point number.

We can also define *symbolic differentiation*—computing the derivative of an expression without evaluating it numerically:

```
let rec deriv exp dv =
  match exp with
  | Const c -> Const 0.0
  | Var v -> if v = dv then Const 1.0 else Const 0.0
  | Sum(f, g) -> Sum(deriv f dv, deriv g dv)
  | Diff(f, g) -> Diff(deriv f dv, deriv g dv)
  | Prod(f, g) -> Sum(Prod(f, deriv g dv), Prod(deriv f dv, g))
  | Quot(f, g) -> Quot(Diff(Prod(deriv f dv, g), Prod(f, deriv g dv)),
                       Prod(g, g))
```

The `deriv` function implements the standard rules of calculus: - The derivative of a constant is 0. - The derivative of the variable we are differentiating with respect to is 1; any other variable is treated as a constant (derivative 0). - The sum and difference rules: $(f+g)' = f' + g'$ and $(f-g)' = f' - g'$. - The product rule: $(f \cdot g)' = f \cdot g' + f' \cdot g$. - The quotient rule: $(f/g)' = (f' \cdot g - f \cdot g')/g^2$.

For convenience, let us define some operators and variables so we can write expressions more naturally:

```
let x = Var "x"
let y = Var "y"
let (+:) f g = Sum (f, g)
let (-:) f g = Diff (f, g)
let ( *: ) f g = Prod (f, g)
let (/:) f g = Quot (f, g)
let (!:) i = Const i
```

These custom operators (ending in `:`) let us write symbolic expressions that look almost like regular mathematical notation.

Now let us evaluate the expression $3x + 2y + x^2y$ at $x = 1, y = 2$:

```
let example = !:3.0 *: x +: !:2.0 *: y +: x *: x *: y
let env = ["x", 1.0; "y", 2.0]
```

When we trace the evaluation using OCaml's `#trace` directive, we can see the recursive structure of the computation unfold:

```
eval_1_2 <-- 3.00 * x + 2.00 * y + x * x * y
  eval_1_2 <-- x * x * y
    eval_1_2 <-- y
    eval_1_2 --> 2.
    eval_1_2 <-- x * x
      eval_1_2 <-- x
      eval_1_2 --> 1.
      eval_1_2 <-- x
      eval_1_2 --> 1.
    eval_1_2 --> 1.
  eval_1_2 --> 2.
  eval_1_2 <-- 3.00 * x + 2.00 * y
    eval_1_2 <-- 2.00 * y
      eval_1_2 <-- y
      eval_1_2 --> 2.
      eval_1_2 <-- 2.00
      eval_1_2 --> 2.
    eval_1_2 --> 4.
    eval_1_2 <-- 3.00 * x
      eval_1_2 <-- x
      eval_1_2 --> 1.
      eval_1_2 <-- 3.00
      eval_1_2 --> 3.
    eval_1_2 --> 3.
  eval_1_2 --> 7.
eval_1_2 --> 9.
- : float = 9.
```

The arrows `<--` and `-->` show function calls and returns, respectively. Each level of indentation represents a nested function call. These indentation levels correspond to **stack frames**—the runtime structures that store the state of each function call. Each time `eval_1_2` is called recursively, a new stack frame is created to remember where to return and what computation remains.

The final result is $3 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 \cdot 2 = 3 + 4 + 2 = 9$, as expected.

This trace visualization brings us to an important question: what happens when we have very deep recursion? This leads us to our next topic.

3.4 Tail Calls and Tail Recursion

The call stack is finite, and each recursive call typically adds a new frame to it. This means that deeply recursive functions can exhaust the stack and crash—a notorious problem known as "stack overflow." Fortunately, functional language implementations have a trick to avoid this problem in many cases.

Excuse me for not formally defining what a *function call* is... Computers normally evaluate programs by creating **stack frames** on the call stack for each function call. A stack frame stores the local variables, the return address (where to continue after the function returns), and other bookkeeping information. The trace in the previous section illustrates this: each level of indentation represents a new stack frame.

What is a Tail Call? The key insight is that not all function calls require a new stack frame. A **tail call** is a function call that is performed as the very last action when computing a function—there is nothing more to do after the call returns except to return that value. For example:

```
let f x = g (x + 1)
```

The call to `g` is a tail call. Once `g` returns some value, `f` simply returns that same value—no further computation is needed.

In contrast:

```
let f x = 1 + g x
```

The call to `g` is *not* a tail call. After `g` returns, we still need to add 1 to the result before `f` can return. This means we need to remember to do the addition, which requires keeping the stack frame around.

Tail Call Optimization Functional language compilers (including OCaml's) recognize tail calls and optimize them by performing **tail call optimization** (TCO). Instead of creating a new stack frame, the compiler generates code that reuses the current frame by performing a "jump" to the called function. This means tail calls use constant stack space, no matter how deep the call chain goes.

This optimization is not just a nice-to-have; it is *essential* for functional programming. Without TCO, many natural recursive algorithms would be impractical because they would overflow the stack on moderately large inputs.

Tail Recursive Functions A function is **tail recursive** if all of its recursive calls (including calls to mutually recursive functions it depends on) are tail calls.

Writing tail recursive functions requires a shift in thinking. Instead of building up the result as recursive calls return, we build it up as we *make* the calls. This typically requires an extra **accumulator** argument that carries the partial result through the recursion.

The key insight is that with an accumulator, results are computed in "reverse order"—we do the work while climbing *into* the recursion (making calls) rather than while climbing *out* (returning from calls).

Example: Counting Let us see this in action with a simple counting function. Compare these two versions:

```
let rec count n =
  if n <= 0 then 0 else 1 + (count (n-1))
```

This version is *not* tail recursive. Look at the recursive case: after `count (n-1)` returns, we still need to add 1 to the result. Each recursive call must remember to do this addition, consuming a stack frame.

Now compare with the tail recursive version:

```
let rec count_tcall acc n =
  if n <= 0 then acc else count_tcall (acc+1) (n-1)
```

Here, the recursive call `count_tcall (acc+1) (n-1)` is the very last thing the function does—its result becomes our result directly. The accumulator `acc` carries the running count: we add 1 to it *before* the recursive call rather than *after* it returns. To count to 1000000, we call `count_tcall 0 1000000`.

Example: Building Lists The counting example does not really show the practical impact because the numbers are so small. Let us see a more dramatic example with lists:

```
let rec unfold n = if n <= 0 then [] else n :: unfold (n-1)
```

This function builds a list counting down from `n` to 1. It is not tail recursive because after the recursive call `unfold (n-1)` returns, we must cons `n` onto the front of the result.

```
# unfold 100000;;
- : int list = [100000; 99999; 99998; 99997; ...]

# unfold 1000000;;
Stack overflow during evaluation (looping recursion?).
```

With 100,000 elements, it works. But with a million elements, we run out of stack space and the program crashes! This is a serious problem for practical programming.

Now consider the tail-recursive version:

```
let rec unfold_tcall acc n =
  if n <= 0 then acc else unfold_tcall (n::acc) (n-1)
```

The accumulator `acc` collects the list as we go. We cons each element onto the accumulator *before* the recursive call. However, there is a catch: because we are building the list as we descend into the recursion (rather than as we return), the list comes out in reverse order:

```
# unfold_tcall [] 100000;;
- : int list = [1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; ...]
```

```
# unfold_tcall [] 1000000;;
- : int list = [1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; ...]
```

The tail-recursive version handles a million elements effortlessly. The trade-off is that we get `[1; 2; 3; ...]` instead of `[1000000; 999999; ...]`. If we need the original order, we could reverse the result at the end (which is an O(n) operation but uses only constant stack space).

A Challenge: Tree Depth Not all recursive functions can be easily converted to tail recursive form. Consider this problem: can we find the depth of a binary tree using a tail-recursive function?

```
type btree = Tip | Node of int * btree * btree
```

Here is the natural recursive approach:

```
let rec depth tree = match tree with
  | Tip -> 0
  | Node(_, left, right) -> 1 + max (depth left) (depth right)
```

This is not tail recursive: after both recursive calls return, we still need to compute `1 + max ...`. The fundamental challenge is that we have *two* recursive calls that we need to make. A simple accumulator will not work—we cannot proceed with one subtree until we know the result of the other.

This seems like an impossible situation. How can we make a function tail recursive when it inherently needs to explore two branches? The answer involves a technique called *continuation passing style*, which we explore in the next section.

Note on Lazy Languages The issue of tail recursion is more nuanced for **lazy** programming languages like Haskell. In a lazy language, expressions are only evaluated when their values are actually needed. The cons operation `(:)` does not immediately evaluate its arguments—it just builds a "promise" to compute them later.

This means that building a list with `n : unfold (n-1)` does not consume stack space in the same way as in OCaml. The `unfold (n-1)` is not evaluated immediately; it is just stored as an unevaluated expression (called a "thunk"). Stack space is only consumed later, when you actually traverse the list. This gives lazy languages different performance characteristics and trade-offs.

3.5 First Encounter of Continuation Passing Style

We can solve the tree depth problem using **Continuation Passing Style (CPS)**. This is a powerful technique that transforms programs in a surprising way: instead of returning values, functions receive an extra argument—a *continuation*—that tells them what to do with their result.

The key idea is to postpone doing actual work until the very last moment by passing around a continuation—a function that represents "what to do next with this result."

```
let rec depth_cps tree k = match tree with
  | Tip -> k 0
  | Node(_, left, right) ->
    depth_cps left (fun dleft ->
      depth_cps right (fun dright ->
        k (1 + (max dleft dright))))

let depth tree = depth_cps tree (fun d -> d)
```

Let us understand how this works step by step:

1. **The continuation parameter:** The function takes an extra parameter `k`, called the **continuation**. Instead of returning a value directly, `depth_cps` will call `k` with its result. You can think of `k` as meaning "and then do this with the answer."

2. **The base case (`Tip`):** When we reach a leaf, the depth is 0. Instead of returning 0, we call `k 0`—"give 0 to whoever is waiting for our answer."

3. **The recursive case (`Node`):** This is where CPS shines. We need to compute depths of both subtrees and combine them. Here is how we do it:

   - First, recursively compute the depth of the left subtree. But instead of waiting for the result, we pass a continuation: `fun dleft -> ...`
   - This continuation says "when you have the left depth (call it `dleft`), then..."
   - ...compute the depth of the right subtree, passing another continuation: `fun dright -> ...`
   - This inner continuation says "when you have the right depth (call it `dright`), then..."
   - ...finally call the original continuation `k` with the combined result `1 + max dleft dright`

4. **The wrapper function:** To use `depth_cps`, we need to provide an initial continuation. We pass the identity function `fun d -> d`, which just returns whatever it receives. This is the "final consumer" of the result.

The magic is that *every recursive call is now a tail call*! Look carefully: `depth_cps left (...)` is the last thing the function does in that branch—everything else is inside the continuation, which will be called later.

Where does the "pending work" go? Instead of being stored on the call stack, it is captured in the continuation closures. These closures are allocated on the heap. We have traded stack space for heap space.

Important caveat: This does not completely solve the stack overflow problem—we are just moving the problem from the stack to the heap. For very deep trees, the continuation closures can grow very large, potentially exhausting memory. True solutions for extreme cases involve techniques like *trampolining* (returning control to a loop) or using explicit data structures to represent the pending work. Nevertheless, CPS is often more space-efficient than direct recursion, and it is a fundamental technique that appears throughout functional programming.

We will encounter CPS again when studying monads and advanced control flow, where it provides the foundation for powerful abstractions.

3.6 Exercises

These exercises will help you practice the concepts from this chapter: function composition, reduction semantics, tail recursion, and continuation passing style.

Exercise 1: Tree Traversals

By "traverse a tree" below we mean: write a function that takes a tree and returns a list of values in the nodes of the tree. Use the `btree` type defined earlier.

1. Write a function (of type `btree -> int list`) that traverses a binary tree in **prefix order** (also called *preorder*)—first the value stored in a node, then values in all nodes to the left, then values in all nodes to the right.
2. Write a traversal in **infix order** (also called *inorder*)—first values in all nodes to the left, then the value stored in the node, then values in all nodes to the right. For a binary search tree, this would give you the elements in sorted order.
3. Write a traversal in **breadth-first order** (also called *level order*)—visit all nodes at depth 0, then all nodes at depth 1, and so on. Hint: you will need an auxiliary data structure (a queue) to keep track of nodes to visit.

Exercise 2: CPS Transformation

Turn the function from Exercise 1 (prefix or infix traversal) into continuation passing style. Compare the structure of your CPS version to the original. What are the trade-offs?

Exercise 3: Tree Derivatives Revisited

Do the homework from the end of Chapter 2: write `btree_deriv_at` that takes a predicate over integers and a `btree`, and builds a `btree_deriv` whose "hole" is in the first position (using your chosen traversal order) for which the predicate returns true.

Exercise 4: Expression Simplification

Write a function `simplify: expression -> expression` that simplifies symbolic expressions, so that for example the result of `simplify (deriv exp dv)` looks more like what a human would get computing the derivative of `exp` with respect to `dv`.

Some simplifications to consider: - $0 + x = x$ and $x + 0 = x$ - $0 \cdot x = 0$ and $x \cdot 0 = 0$ - $1 \cdot x = x$ and $x \cdot 1 = x$ - $x - 0 = x$ - $x/1 = x$

Approach this in two steps: 1. Write a `simplify_once` function that performs a single "pass" of simplification over the expression tree. 2. Wrap it using a general `fixpoint` function that performs an operation until a **fixed point** is reached: given f and x, it computes $f^n(x)$ such that $f^n(x) = f^{n+1}(x)$ (i.e., applying f one more time does not change the result).

Why do we need iteration to a fixed point rather than a single pass?

Exercise 5: Sorting Algorithms

Write two sorting algorithms working on lists: merge sort and quicksort.

1. **Merge sort** splits the list roughly in half, sorts the parts recursively, and merges the sorted parts into the sorted result. You will need a helper function to merge two sorted lists.
2. **Quicksort** splits the list into elements smaller than and greater-than-or-equal-to the first element (the "pivot"), sorts the parts recursively, and concatenates them.

Which of these algorithms can be implemented in a tail-recursive manner? What about the helper functions (merge, partition)?

Chapter 4: Functions

Programming in untyped lambda-calculus

This chapter explores the theoretical foundations of functional programming through the untyped lambda-calculus. We embark on a fascinating journey that reveals a surprising truth: every computation can be expressed using nothing but functions. No numbers, no booleans, no data structures—just functions all the way down.

We begin with a review of computation by hand using our reduction semantics, then introduce the lambda-calculus notation and show how to encode fundamental data types—booleans, pairs, and natural numbers—using only functions. The chapter concludes with an examination of recursion through fixpoint combinators and practical considerations for avoiding infinite loops in eager evaluation.

References:

- "Introduction to Lambda Calculus" by Henk Barendregt and Erik Barendsen
- "Lecture Notes on the Lambda Calculus" by Peter Selinger

4.1 Review: Computation by Hand

Before diving into the lambda-calculus, let us work through a complete example of evaluation using the reduction rules from Chapter 3. Computing a larger, recursive program by hand will solidify our understanding of how computation proceeds step by step and prepare us for the more abstract setting of lambda-calculus.

Recall that we use `fix` instead of `let rec` to simplify our rules for recursion. Also remember our syntactic conventions: `fun x y -> e` stands for `fun x -> (fun y -> e)`, and so forth.

Consider the following recursive `length` function applied to a two-element list:

```
let rec fix f x = f (fix f) x

type int_list = Nil | Cons of int * int_list

let length =
  fix (fun f l ->
    match l with
      | Nil -> 0
      | Cons (x, xs) -> 1 + f xs)

length (Cons (1, (Cons (2, Nil))))
```

Let us trace through this computation step by step. First, we eliminate the `let` binding:

$$\texttt{let}\ x = v\ \texttt{in}\ a \Downarrow a[x := v]$$

This gives us:

```
fix (fun f l ->
    match l with
      | Nil -> 0
      | Cons (x, xs) -> 1 + f xs) (Cons (1, (Cons (2, Nil))))
```

Next, we apply the `fix` rule:

$$\texttt{fix}^2\ v_1\ v_2 \Downarrow v_1\ (\texttt{fix}^2\ v_1)\ v_2$$

This unfolds to:

```
(fun f l ->
    match l with
      | Nil -> 0
      | Cons (x, xs) -> 1 + f xs)
    (fix (fun f l ->
```

```
      match l with
        | Nil -> 0
        | Cons (x, xs) -> 1 + f xs))
    (Cons (1, (Cons (2, Nil))))
```

Function application reduces according to:

$$(\texttt{fun}\ x\ \texttt{->}\ a)\ v \rightsquigarrow a[x := v]$$

After substituting both `f` and `l`, we get:

```
(match Cons (1, (Cons (2, Nil))) with
    | Nil -> 0
    | Cons (x, xs) -> 1 + (fix (fun f l ->
      match l with
        | Nil -> 0
        | Cons (x, xs) -> 1 + f xs)) xs)
```

Pattern matching against a non-matching constructor moves to the next branch:

$$\begin{array}{l}\texttt{match}\ C_1^n(v_1, \ldots, v_n)\ \texttt{with} \\ C_2^n(p_1, \ldots, p_k)\ \texttt{->}\ a\ \texttt{|}\ pm \Downarrow \texttt{match}\ C_1^n(v_1, \ldots, v_n)\ \texttt{with}\ pm\end{array}$$

Pattern matching against a matching constructor performs substitution:

$$\begin{array}{l}\texttt{match}\ C_1^n(v_1, \ldots, v_n)\ \texttt{with} \\ C_1^n(x_1, \ldots, x_n)\ \texttt{->}\ a\ \texttt{|}\ \ldots \Downarrow a[x_1 := v_1; \ldots ; x_n := v_n]\end{array}$$

After matching and substitution:

```
1 + (fix (fun f l ->
      match l with
        | Nil -> 0
        | Cons (x, xs) -> 1 + f xs)) (Cons (2, Nil))
```

Continuing the evaluation, we apply `fix` again and work through the pattern match for `Cons (2, Nil)`, eventually reaching:

```
1 + (1 + (fix (fun f l ->
             match l with
               | Nil -> 0
               | Cons (x, xs) -> 1 + f xs)) Nil)
```

One more unfolding and pattern match against `Nil` gives:

```
1 + (1 + 0)
```

Finally, applying the built-in addition:

$$f^n\ v_1\ \dots\ v_n \Downarrow f(v_1, \dots, v_n)$$

We obtain the result: `2`.

4.2 Language and Rules of the Untyped Lambda-Calculus

The lambda-calculus, introduced by Alonzo Church in the 1930s, is a minimal formal system for expressing computation. It may seem surprising that such a stripped-down language can be computationally complete, but that is precisely what we will demonstrate in this chapter. To work with lambda-calculus, we first simplify our language in several ways:

1. **Forget about types.** In pure lambda-calculus, there is no type system constraining which terms can be combined. Any function can be applied to any argument—including itself!
2. **Introduce notation.** We write $\lambda x.a$ for `fun x -> a`, and $\lambda xy.a$ for `fun x y -> a`, and so forth. This notation is more compact and traditional in the literature.
3. **Reduce to essentials.** We keep only functions (lambda abstractions) and variables—no constructors, no built-in primitives. Everything else will be *encoded* using functions.

The core reduction rule of lambda-calculus is called β**-reduction**:

$$(\texttt{fun}\ x\ \texttt{->}\ a_1)\ a_2 \rightsquigarrow a_1[x := a_2]$$

Note that this rule is more general than the one we use for OCaml evaluation. In our OCaml semantics, we require the argument to be a value: $(\texttt{fun}\ x\ \texttt{->}\ a)\ v \rightsquigarrow a[x := v]$. The general β-reduction rule allows substituting any expression, not just values.

Lambda-calculus also uses α**-conversion** (bound variable renaming), or equivalent techniques, to avoid **variable capture**—the unintended binding of free variables during substitution. We will explore the implications of β-reduction more deeply in the chapter on laziness.

Why is β-reduction more general than our evaluation rule? Consider the expression $(\lambda x.x)\ ((\lambda y.y)\ z)$. With β-reduction, we could reduce the outer application first, obtaining $((\lambda y.y)\ z)$. Our evaluation rule would require first reducing the argument to a value—but here `z` is a free variable, not a value, so we would be stuck!

4.3 Booleans

Alonzo Church originally introduced lambda-calculus as a foundation for logic, seeking to encode logical reasoning in a purely computational form. There are multiple ways to encode various sorts of data in lambda-calculus, though not all of them work well in a typed setting—the straightforward encode/decode functions may not type-check for some encodings.

The key insight behind the **Church encoding** of booleans is to represent truth values as *selector functions*. Think about what a boolean fundamentally does: it chooses between two alternatives. So we define:

- **True** selects the first argument: `c_true` $= \lambda xy.x$
- **False** selects the second argument: `c_false` $= \lambda xy.y$

In OCaml syntax:

```
let c_true = fun x y -> x   (* "True" is projection on the first argument *)
let c_false = fun x y -> y  (* And "false" on the second argument *)
```

Once we have booleans as selectors, logical operations become elegant. Logical conjunction can be defined as:

$$\texttt{c\_and} = \lambda xy.x\ y\ \texttt{c\_false}$$

The logic behind this definition is beautifully simple: we apply `x` (which is a selector) to two arguments. If `x` is true, it selects its first argument, which is `y`—so the result is true only if both `x` and `y` are true. If `x` is false, it selects its second argument, `c_false`, and returns false immediately without even looking at `y`.

```
let c_and = fun x y -> x y c_false  (* If one is false, then return false *)
```

Let us verify this works. For `c_and c_true c_true`:

$$(\lambda xy.x\ y\ \texttt{c\_false})\ (\lambda xy.x)\ (\lambda xy.x)$$

reduces to:

$$(\lambda xy.x)\ (\lambda xy.x)\ \texttt{c\_false}$$

which gives us $\lambda xy.x =$ `c_true`. You can verify that for any other combination involving `c_false`, the result is `c_false`.

To verify our encodings in OCaml, we need encode and decode functions. The decoder works by applying our Church boolean to the actual OCaml values `true` and `false`:

```
let encode_bool b = if b then c_true else c_false
let decode_bool c = c true false  (* Test the functions in the toplevel *)
```

Exercise: Define `c_or` and `c_not` yourself! Hint: think about what `c_or` should return when the first argument is true, and when it is false. For `c_not`, consider that a boolean is a function that selects between two arguments.

4.4 If-then-else and Pairs

From now on, we will use OCaml syntax for our lambda-calculus programs. This makes it easier to experiment with our encodings in the toplevel.

An important observation is that our encoded booleans already implement conditional selection:

```
let if_then_else = fun b -> b  (* Booleans select the argument! *)
```

Wait—`if_then_else` is just the identity function? Yes! Since `c_true` returns its first argument and `c_false` returns its second, `if_then_else b then_branch else_branch` simply applies `b` to the two branches. The boolean *is* the conditional. This is one of the elegant surprises of Church encoding.

Remember to play with these functions in the toplevel to build intuition. Try expressions like `if_then_else c_true "yes" "no"` and see what happens.

Pairs Pairs (ordered tuples of two elements) can be encoded using a similar idea. The key insight is that a pair needs to "remember" two values and provide them when asked. We can achieve this by creating a function that holds onto both values and waits for a selector to choose between them:

```
let c_pair m n = fun x -> x m n  (* We couple things *)
let c_first = fun p -> p c_true  (* by passing them together *)
let c_second = fun p -> p c_false  (* Check that it works! *)
```

A pair is a function that, when given a selector, applies that selector to both components. To extract the first component, we pass `c_true` (which selects the first argument); to extract the second, we pass `c_false`. Verify for yourself that `c_first (c_pair a b)` reduces to `a`!

For verification:

```
let encode_pair enc_fst enc_snd (a, b) =
  c_pair (enc_fst a) (enc_snd b)
let decode_pair de_fst de_snd c = c (fun x y -> de_fst x, de_snd y)
let decode_bool_pair c = decode_pair decode_bool decode_bool c
```

We can define larger tuples in the same manner:

```
let c_triple l m n = fun x -> x l m n
```

4.5 Pair-Encoded Natural Numbers

Now we come to encoding numbers—a crucial test of whether functions alone can represent all data. Our first encoding of natural numbers uses nested pairs. The representation is based on the depth of nested pairs whose rightmost leaf is the identity function $\lambda x.x$ and whose left elements are `c_false`.

```
let pn0 = fun x -> x           (* Start with the identity function *)
let pn_succ n = c_pair c_false n  (* Stack another pair *)

let pn_pred = fun x -> x c_false  (* Extract the nested number *)
let pn_is_zero = fun x -> x c_true  (* Check if it's the base case *)
```

The number 0 is represented as the identity function. The number 1 is `c_pair c_false pn0`, the number 2 is `c_pair c_false (c_pair c_false pn0)`, and so on. Think of it as a stack of pairs, where the height of the stack represents the number.

How do `pn_pred` and `pn_is_zero` work? Let us think through this carefully: - The identity function `pn0`, when applied to any argument, returns that argument. - A successor `c_pair c_false n` is a function waiting for a selector; applying it to `c_false` selects the second component (the predecessor), while applying it to `c_true` selects the first component (`c_false`).

So `pn_is_zero` applies the number to `c_true`: - For `pn0`, we get `c_true` back (since `pn0` is the identity)—the number is zero! - For any successor, we get `c_false` back (the first component of the pair)—the number is not zero!

We program in untyped lambda-calculus as an exercise, and we need encoding/decoding to verify our work. Since these encodings do not type-check cleanly in OCaml, using `Obj.magic` to bypass the type system for encoding/decoding is "fair game":

```
let rec encode_pnat n =                (* We use Obj.magic to forget types *)
  if n <= 0 then Obj.magic pn0
  else pn_succ (Obj.magic (encode_pnat (n-1)))  (* Disregarding types, *)
let rec decode_pnat pn =               (* these functions are straightforward! *)
  if decode_bool (pn_is_zero pn) then 0
  else 1 + decode_pnat (pn_pred (Obj.magic pn))
```

4.6 Church Numerals

Do you remember our function `power f n` from Chapter 3 that composed a function with itself `n` times? We will use a similar idea for a different, and historically important, representation of numbers.

Church numerals represent a natural number n as a function that applies its first argument n times to its second argument:

```
let cn0 = fun f x -> x        (* The same as c_false *)
```

```
let cn1 = fun f x -> f x       (* Behaves like identity when f = id *)
let cn2 = fun f x -> f (f x)
let cn3 = fun f x -> f (f (f x))
```

This is the original Alonzo Church encoding, and it is remarkably elegant. The number n is represented as $\lambda f x. f^n(x)$, where f^n denotes n-fold composition. A number literally *is* the act of doing something n times!

Notice that `cn0` is the same as `c_false`—zero applications of `f` just returns `x`.

The successor function adds one more application of `f`:

```
let cn_succ = fun n f x -> f (n f x)
```

Exercise: Define addition, multiplication, and comparing to zero for Church numerals. Also try to define the predecessor function "-1".

It turns out even Alonzo Church could not define predecessor right away! The story goes that his student Stephen Kleene figured it out while at the dentist. Try to make some progress on addition and multiplication first (they are not too hard), and then attempt predecessor before looking at the solution below.

```
let (-|) f g x = f (g x)  (* Backward composition operator *)

let rec encode_cnat n f =
  if n <= 0 then (fun x -> x) else f -| encode_cnat (n-1) f
let decode_cnat n = n ((+) 1) 0
let cn7 f x = encode_cnat 7 f x   (* We need to eta-expand these definitions *)
let cn13 f x = encode_cnat 13 f x  (* for type-system reasons *)
                                   (* (because OCaml allows side-effects) *)
let cn_add = fun n m f x -> n f (m f x)  (* Put n of f in front *)
let cn_mult = fun n m f -> n (m f)       (* Repeat n times *)
                                          (* putting m of f in front *)
let cn_prev n =
  fun f x ->                   (* This is the "Church numeral signature" *)
    n                          (* The only thing we have is an n-step loop *)
      (fun g v -> v (g f))     (* We need sth that operates on f *)
      (fun z -> x)             (* We need to ignore the innermost step *)
      (fun z -> z)             (* We've built a "machine" not results -- start the machine *
```

Addition is intuitive: to add n and m, we first apply `f` m times (giving us `m f x`), then apply `f` n more times. Multiplication is even more clever: we apply the operation "apply `f` m times" n times, which computes $m \times n$ applications of `f`.

The predecessor function is ingenious and worth studying carefully. The challenge is that Church numerals only know how to apply `f` more times, not fewer. Kleene's insight was to build up a chain of functions that, when "started" with the identity, yields $n - 1$ applications of `f`. The key is to delay the actual application of `f` and skip the first one.

`cn_is_zero` is left as an exercise. Hint: what happens when you apply zero to a function that always returns `c_false` and start with `c_true`?

Tracing `cn_prev cn3` The predecessor function is tricky enough that it is worth tracing through a complete example. Let us trace through `decode_cnat (cn_prev cn3)` to see how it computes 2 from 3:

$$\Downarrow$$

```
(cn_prev cn3) ((+) 1) 0
```

$$\Downarrow$$

```
(fun f x ->
    cn3
      (fun g v -> v (g f))
      (fun z -> x)
      (fun z -> z)) ((+) 1) 0
```

$$\Downarrow$$

```
((fun f x -> f (f (f x)))
      (fun g v -> v (g ((+) 1)))
      (fun z -> 0)
      (fun z -> z))
```

$$\Downarrow$$

```
((fun g v -> v (g ((+) 1)))
  ((fun g v -> v (g ((+) 1)))
    ((fun g v -> v (g ((+) 1)))
      (fun z -> 0))))
  (fun z -> z))
```

$$\Downarrow$$

```
((fun z -> z)
  (((fun g v -> v (g ((+) 1)))
    ((fun g v -> v (g ((+) 1)))
      (fun z -> 0)))) ((+) 1)))
```

$$\Downarrow$$

```
(fun g v -> v (g ((+) 1)))
  ((fun g v -> v (g ((+) 1)))
    (fun z -> 0)) ((+) 1)
```

$$\Downarrow$$

```
((+) 1) ((fun g v -> v (g ((+) 1)))
          (fun z -> 0) ((+) 1))
```

$$\Downarrow$$

```
((+) 1) (((+) 1) ((fun z -> 0) ((+) 1)))
```

$$\Downarrow$$

```
((+) 1) (((+) 1) (0))
```

$$\Downarrow$$

```
((+) 1) 1
```

$$\Downarrow$$

```
2
```

4.7 Recursion: Fixpoint Combinators

We have seen how to encode data in lambda-calculus, but how do we encode *computation*, especially recursive computation? In lambda-calculus, there is no `let rec` or any built-in notion of a function referring to itself. Instead, recursion is achieved through **fixpoint combinators**—remarkable lambda terms that compute fixed points of functions.

Turing's Fixpoint Combinator

$$\Theta = (\lambda xy.y\ (x\ x\ y))\ (\lambda xy.y\ (x\ x\ y))$$

Let us verify it computes fixed points. Define $N = \Theta F$:

$$\begin{aligned} N &= \Theta F \\ &= (\lambda xy.y\ (x\ x\ y))\ (\lambda xy.y\ (x\ x\ y))\ F \\ &=_{\rightarrow\rightarrow} F\ ((\lambda xy.y\ (x\ x\ y))\ (\lambda xy.y\ (x\ x\ y))\ F) \\ &= F\ (\Theta F) = F\ N \end{aligned}$$

So $N = F\ N$, meaning N is a fixed point of F.

Curry's Fixpoint Combinator (Y Combinator)

$$\mathbf{Y} = \lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))$$

$$\begin{aligned}
N &= \mathbf{Y}F \\
&= (\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)))\ F \\
&=_{\rightarrow} (\lambda x.F\ (x\ x))\ (\lambda x.F\ (x\ x)) \\
&=_{\rightarrow} F\ ((\lambda x.F\ (x\ x))\ (\lambda x.F\ (x\ x))) \\
&=_{\leftarrow} F\ ((\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)))\ F) \\
&= F\ (\mathbf{Y}F) = F\ N
\end{aligned}$$

Call-by-Value Fixpoint Combinator

$$\texttt{fix} = \lambda f'.(\lambda fx.f'\ (f\ f)\ x)\ (\lambda fx.f'\ (f\ f)\ x)$$

$$\begin{aligned}
N &= \texttt{fix}\ F \\
&= (\lambda f'.(\lambda fx.f'\ (f\ f)\ x)\ (\lambda fx.f'\ (f\ f)\ x))\ F \\
&=_{\rightarrow} (\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x) \\
&=_{\rightarrow} \lambda x.F\ ((\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x))\ x \\
&=_{\leftarrow} \lambda x.F\ ((\lambda f'.(\lambda fx.f'\ (f\ f)\ x)\ (\lambda fx.f'\ (f\ f)\ x))\ F)\ x \\
&= \lambda x.F\ (\texttt{fix}\ F)\ x = \lambda x.F\ N\ x \\
&=_{\eta} F\ N
\end{aligned}$$

The lambda-terms we have seen above are **fixpoint combinators**—the means within lambda-calculus to perform recursion without any special recursive binding constructs.

The Problem with the First Two Combinators What is the problem with Turing's and Curry's combinators in a practical programming language? Consider what happens when we try to evaluate ΘF:

$$\begin{aligned}
\Theta F &\rightsquigarrow\rightsquigarrow F\ ((\lambda xy.y\ (x\ x\ y))\ (\lambda xy.y\ (x\ x\ y))\ F) \\
&\rightsquigarrow\rightsquigarrow F\ (F\ ((\lambda xy.y\ (x\ x\ y))\ (\lambda xy.y\ (x\ x\ y))\ F)) \\
&\rightsquigarrow\rightsquigarrow F\ (F\ (F\ ((\lambda xy.y\ (x\ x\ y))\ (\lambda xy.y\ (x\ x\ y))\ F))) \\
&\rightsquigarrow\rightsquigarrow \ldots
\end{aligned}$$

Recall the distinction between *expressions* and *values* from Chapter 3 on Computation. The reduction rule for lambda-calculus is meant to determine which expressions are considered "equal"—it is highly *non-deterministic*, while on a computer, computation needs to go one way or another.

Using the general reduction rule of lambda-calculus, for a recursive definition, it is always possible to find an infinite reduction sequence. Why? Because we can always choose to reduce the recursive call first, which generates another recursive call, and so on forever. This means a naive lambda-calculus compiler could legitimately generate infinite loops for all recursive definitions—which would not be very useful!

Therefore, we need more specific rules. Most languages use **call-by-value** (also called **eager** evaluation):

$$(\texttt{fun}\ x\ \texttt{->}\ a)\ v \rightsquigarrow a[x := v]$$

The program *eagerly* computes arguments before starting to compute the function body. This is exactly the rule we introduced in the Computation chapter.

Call-by-Value Fixpoint Combinator in Action What happens with the call-by-value fixpoint combinator?

$$\begin{aligned}\texttt{fix}\, F &\rightsquigarrow (\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x)\\ &\rightsquigarrow \lambda x.F\ ((\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x))\ x\end{aligned}$$

The computation stops because we use the rule $(\texttt{fun}\ x\ \texttt{->}\ a)\ v \rightsquigarrow a[x := v]$ rather than $(\texttt{fun}\ x\ \texttt{->}\ a_1)\ a_2 \rightsquigarrow a_1[x := a_2]$. The expression inside the lambda is not evaluated until the function is applied.

Let us compute the function on some input:

$$\begin{aligned}\texttt{fix}\, F\ v &\rightsquigarrow (\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x)\ v\\ &\rightsquigarrow (\lambda x.F\ ((\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x))\ x)\ v\\ &\rightsquigarrow F\ ((\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x))\ v\\ &\rightsquigarrow F\ (\lambda x.F\ ((\lambda fx.F\ (f\ f)\ x)\ (\lambda fx.F\ (f\ f)\ x))\ x)\ v\\ &\rightsquigarrow \text{depends on } F\end{aligned}$$

Why "Fixpoint"? If you examine our derivations, you will see they establish $x = f(x)$. Such values x are called **fixpoints** of f. An arithmetic function can have several fixpoints—for example, $f(x) = x^2$ has fixpoints 0 and 1 (since $0^2 = 0$ and $1^2 = 1$)—or no fixpoints, such as $f(x) = x + 1$ (since $x + 1 \neq x$ for all x).

When you define a function (or another object) by recursion, it has a similar meaning: the name appears on both sides of the equality. For example, `fact n = if n = 0 then 1 else n * fact (n-1)` has `fact` on both sides. In lambda-calculus, functions like Θ and $\mathbf{Y}$ take *any* function as an argument and return its fixpoint.

We turn a specification of a recursive object into a definition by solving it with respect to the recurring name: deriving $x = f(x)$ where x is the recurring name. We then have $x = \texttt{fix}(f)$.

Deriving Factorial Let us walk through this process step by step for the factorial function. This will show how to transform a recursive specification into a proper definition using `fix`. We omit the prefix `cn_` (could be `pn_` if using pair-encoded numbers) and shorten `if_then_else` to `if_t_e`:

$$
\begin{aligned}
\texttt{fact}\ n &= \texttt{if\_t\_e}\ (\texttt{is\_zero}\ n)\ \texttt{cn1}\ (\texttt{mult}\ n\ (\texttt{fact}\ (\texttt{pred}\ n)))\\
\texttt{fact} &= \lambda n.\texttt{if\_t\_e}\ (\texttt{is\_zero}\ n)\ \texttt{cn1}\ (\texttt{mult}\ n\ (\texttt{fact}\ (\texttt{pred}\ n)))\\
\texttt{fact} &= (\lambda fn.\texttt{if\_t\_e}\ (\texttt{is\_zero}\ n)\ \texttt{cn1}\ (\texttt{mult}\ n\ (f\ (\texttt{pred}\ n))))\ \texttt{fact}\\
\texttt{fact} &= \texttt{fix}\ (\lambda fn.\texttt{if\_t\_e}\ (\texttt{is\_zero}\ n)\ \texttt{cn1}\ (\texttt{mult}\ n\ (f\ (\texttt{pred}\ n))))
\end{aligned}
$$

The last line is a valid definition: we simply give a name to a *ground* (also called *closed*) expression—one with no free variables. We have already seen how `fix` works in the reduction semantics.

Exercise: Compute `fact cn2` by hand, tracing through the reduction steps.

Exercise: What does `fix (fun x -> cn_succ x)` mean? What happens if you try to evaluate it? Think about whether there is any value `x` such that `x = cn_succ x`.

4.8 Encoding Lists and Trees

Now that we have numbers and recursion, we can encode more complex data structures. The pattern we have seen with booleans and pairs extends naturally to algebraic data types like lists and trees.

A **list** is either empty (often called `Empty` or `Nil`) or consists of an element followed by another list (the "tail"), called `Cons`. Since lists have two variants, we encode them with two-argument selector functions:

- `nil` $= \lambda xy.y$ (select the second argument, like `c_false`)
- `cons` $H\ T = \lambda xy.x\ H\ T$ (apply the first argument to head and tail)

With these definitions, we can write a function to add all numbers stored inside a list:

$$\texttt{addlist}\ l = l\ (\lambda ht.\texttt{cn\_add}\ h\ (\texttt{addlist}\ t))\ \texttt{cn0}$$

To make a proper definition, we apply `fix` to the solution of the above equation:

$$\texttt{addlist} = \texttt{fix}\ (\lambda fl.l\ (\lambda ht.\texttt{cn\_add}\ h\ (f\ t))\ \texttt{cn0})$$

For **trees**, let us use a different form of binary trees than we have seen before: instead of keeping elements in inner nodes, we will keep elements in leaves. This is sometimes called an "external" tree structure.

Again, we have two variants, so we use two-argument selector functions:

- `leaf` $n = \lambda xy.x\ n$ (apply first argument to the element)
- `node` $L\ R = \lambda xy.y\ L\ R$ (apply second argument to left and right subtrees)

To add numbers stored inside a tree:

$$\texttt{addtree}\ t = t\ (\lambda n.n)\ (\lambda lr.\texttt{cn\_add}\ (\texttt{addtree}\ l)\ (\texttt{addtree}\ r))$$

And in solved form:

$$\texttt{addtree} = \texttt{fix}\ (\lambda ft.t\ (\lambda n.n)\ (\lambda lr.\texttt{cn\_add}\ (f\ l)\ (f\ r)))$$

```
let rec fix f x = f (fix f) x
let nil = fun x y -> y
let cons h t = fun x y -> x h t
let addlist l =
  fix (fun f l -> l (fun h t -> cn_add h (f t)) cn0) l
;;
decode_cnat
  (addlist (cons cn1 (cons cn2 (cons cn7 nil))));;
let leaf n = fun x y -> x n
let node l r = fun x y -> y l r
let addtree t =
  fix (fun f t ->
    t (fun n -> n) (fun l r -> cn_add (f l) (f r))
  ) t
;;
decode_cnat
  (addtree (node (node (leaf cn3) (leaf cn7))
              (leaf cn1)));;
```

The General Pattern If you look back at our encodings, you will observe a consistent pattern: when we encode a variant type with n variants, for each variant we define a function that takes n arguments.

If the kth variant C_k has m_k parameters, then the function c_k that encodes it has the form:

$$C_k(v_1, \ldots, v_{m_k}) \sim c_k\ v_1\ \ldots\ v_{m_k} = \lambda x_1 \ldots x_n.x_k\ v_1\ \ldots\ v_{m_k}$$

The encoded variants serve as shallow pattern matching with guaranteed exhaustiveness: the kth argument corresponds to the kth branch of pattern matching. This is exactly how `match` works in OCaml, but encoded purely with functions!

4.9 Looping Recursion

We have been coding in untyped lambda-calculus and verifying our code works in OCaml. But there is a subtle trap we must be aware of when combining lambda-calculus encodings with OCaml's eager evaluation.

Let us return to pair-encoded numbers and define addition:

```
let pn_add m n =
  fix (fun f m n ->
    if_then_else (pn_is_zero m)
      n (pn_succ (f (pn_pred m) n))
  ) m n;;
decode_pnat (pn_add pn3 pn3);;
```

Oops... OCaml says: `Stack overflow during evaluation (looping recursion?).`

What went wrong? Nothing as far as lambda-calculus is concerned—the definition is mathematically correct. But OCaml (and F#) always compute arguments before calling a function. This is the *eager* evaluation strategy we discussed earlier. By definition of `fix`, `f` corresponds to recursively calling `pn_add`. Therefore, `(pn_succ (f (pn_pred m) n))` will be evaluated regardless of what `(pn_is_zero m)` returns!

In other words, even when `m` is zero and we should return `n`, OCaml first tries to compute the "else" branch, which makes a recursive call, which computes its "else" branch, and so on forever.

Why do `addlist` and `addtree` work? Look at them carefully: their recursive calls are "guarded" by corresponding `fun`. The expression `(fun h t -> cn_add h (f t))` does not immediately call `f`—it creates a function that will call `f` only when that function is applied to arguments. What is inside of `fun` is not computed immediately—only when the function is applied to argument(s).

To avoid looping recursion, you need to guard all recursive calls. Besides putting them inside `fun`, in OCaml or F# you can also put them in branches of a `match` clause, as long as one of the branches does not have unguarded recursive calls.

The trick for functions like `if_then_else` is to guard their arguments with `fun x ->`, where `x` is not used, and apply the *result* of `if_then_else` to some dummy value. This delays the evaluation of both branches until the boolean has selected one of them:

```
let id x = x
let rec fix f x = f (fix f) x
```

```
let pn1 x = pn_succ pn0 x
let pn2 x = pn_succ pn1 x
let pn3 x = pn_succ pn2 x
let pn7 x = encode_pnat 7 x
let pn_add m n =
  fix (fun f m n ->
    (if_then_else (pn_is_zero m)
       (fun x -> n) (fun x -> pn_succ (f (pn_pred m) n)))
      id
  ) m n;;
decode_pnat (pn_add pn3 pn3);;
decode_pnat (pn_add pn3 pn7);;
```

Now the recursive call is wrapped in `fun x ->`, so it is not evaluated until `if_then_else` selects the second branch and applies it to `id`. When `m` is zero, the first branch `(fun x -> n)` is selected and applied to `id`, giving us `n` without ever touching the recursive call.

In OCaml or F# we would typically guard by `fun () ->` and then apply to `()`, but we do not have datatypes like `unit` in pure lambda-calculus, so we use `id` as our dummy value.

4.10 Exercises

The following exercises will help solidify your understanding of lambda-calculus encodings. For each exercise involving lambda-calculus, test your implementation by encoding some inputs, applying your function, and decoding the result.

Exercise 1: Define (implement) and test on a couple of examples functions corresponding to or computing:

1. `c_or` and `c_not`;
2. exponentiation for Church numerals;
3. is-zero predicate for Church numerals;
4. even-number predicate for Church numerals;
5. multiplication for pair-encoded natural numbers;
6. factorial $n!$ for pair-encoded natural numbers;
7. the length of a list (in Church numerals);
8. `cn_max` – maximum of two Church numerals;
9. the depth of a tree (in Church numerals).

Exercise 2: Construct lambda-terms $m_0, m_1, \ldots$ such that for all n one has:

$$
\begin{aligned}
m_0 &= x \\
m_{n+1} &= m_{n+2}\ m_n
\end{aligned}
$$

(where equality is after performing β-reductions).

Exercise 3: Representing side-effects as an explicitly "passed around" state value, write (higher-order) functions that represent the imperative constructs:

1. `for`...`to`...
2. `for`...`downto`...
3. `while`...`do`...
4. `do`...`while`...
5. `repeat`...`until`...

Rather than writing a lambda-term using the encodings that we have learnt, just implement the functions in OCaml / F#, using built-in `int` and `bool` types. You can use `let rec` instead of `fix`.

- For example, in exercise (a), write a function `let rec for_to f beg_i end_i s = ...` where `f` takes arguments `i` ranging from `beg_i` to `end_i`, state `s` at given step, and returns state `s` at next step; the `for_to` function returns the state after the last step.
- And in exercise (c), write a function `let rec while_do p f s = ...` where both `p` and `f` take state `s` at given step, and if `p s` returns true, then `f s` is computed to obtain state at next step; the `while_do` function returns the state after the last step.

Do not use the imperative features of OCaml and F#! This exercise demonstrates that imperative control flow can be encoded purely functionally by threading state through function calls.

Although we will not cover imperative features in this course, it is instructive to see the implementation using them, to better understand what is actually required of a solution to Exercise 3:

```
(* (a) *)
let for_to f beg_i end_i s =
  let s = ref s in
  for i = beg_i to end_i do
    s := f i !s
  done;
  !s

(* (b) *)
let for_downto f beg_i end_i s =
  let s = ref s in
  for i = beg_i downto end_i do
    s := f i !s
  done;
  !s

(* (c) *)
let while_do p f s =
  let s = ref s in
```

```ocaml
  while p !s do
    s := f !s
  done;
  !s

(* (d) *)
let do_while p f s =
  let s = ref (f s) in
  while p !s do
    s := f !s
  done;
  !s

(* (e) *)
let repeat_until p f s =
  let s = ref (f s) in
  while not (p !s) do
    s := f !s
  done;
  !s
```

Chapter 5: Polymorphism and Abstract Data Types

This chapter explores how OCaml's type system supports generic programming through parametric polymorphism, and how abstract data types provide clean interfaces for data structures. We begin by examining how type inference actually works – the process by which OCaml determines types for your code. Then we explore parametric types and show how they enable polymorphic functions to work with data of any shape. The second half of the chapter introduces algebraic specifications, the mathematical foundation for describing data structures, and applies these concepts to build progressively more sophisticated implementations of the map (dictionary) data structure, culminating in the elegant red-black tree.

If you see any error in this chapter, please let us know!

5.1 Type Inference

We have seen the rules that govern the assignment of types to expressions, but how does OCaml actually guess what types to use? And how does it know when no correct types exist? The answer lies in a beautiful algorithm: OCaml solves equations. When you write code, the type checker generates a set of equations that must hold for the program to be well-typed, and then it solves those equations to discover the types.

5.1.1 Variables: Unknowns and Parameters Variables in type inference play two distinct roles, and understanding this distinction is crucial for mastering

OCaml's type system. A type variable can be either an *unknown* (standing for a specific but not-yet-determined type) or a *parameter* (standing for any type whatsoever).

Consider this example:

```
# let f = List.hd;;
val f : 'a list -> 'a = <fun>
```

Here `'a` is a *parameter*: it can become any type. When you use `f` with a list of integers, `'a` becomes `int`; when you use it with a list of strings, `'a` becomes `string`. Mathematically we write: $f : \forall\alpha.\alpha \text{ list} \rightarrow \alpha$ – the quantified type is called a *type scheme*. The $\forall$ symbol indicates that this type works "for all" choices of α.

In contrast, consider this example:

```
# let x = ref [];;
val x : '_weak1 list ref = {contents = []}
```

Here `'_a` (displayed as `'_weak1` in recent OCaml versions) is an *unknown*. Unlike a parameter, it stands for a *particular* type – perhaps `float` or `int -> int` – but OCaml simply doesn't know which type yet. The underscore prefix signals this distinction. OCaml reports unknowns like `'_a` in inferred types for reasons related to mutable state (the "value restriction"), which are not relevant to purely functional programming.

When unknowns appear in inferred types against our expectations, *η-expansion* may help. This technique involves writing `let f x = expr x` instead of `let f = expr`, essentially adding an extra parameter that gets immediately applied. For example:

```
# let f = List.append [];;
val f : '_weak2 list -> '_weak2 list = <fun>
# let f l = List.append [] l;;
val f : 'a list -> 'a list = <fun>
```

In the second definition, the eta-expanded form `let f l = List.append [] l` allows full generalization, giving us a truly polymorphic function that can work with lists of any type.

5.1.2 Type Environments Before diving into the equation-solving process, we need to understand how the type checker keeps track of what names are available. A *type environment* specifies what names (corresponding to parameters and definitions) are available for an expression because they were introduced above it, and it specifies their types. Think of it as a dictionary that maps variable names to their types at any given point in your program.

5.1.3 Solving Type Equations Type inference works by solving equations over unknowns. The central question the algorithm asks is: "What has to hold so that $e : \tau$ in type environment Γ?" The answer takes the form of equations that constrain the possible types.

Let us walk through how the algorithm handles different expression forms:

- If, for example, $f : \forall\alpha.\alpha \text{ list} \to \alpha \in \Gamma$, then for $f : \tau$ we introduce $\gamma \text{ list} \to \gamma = \tau$ for some fresh unknown γ.
- For function application $e_1\ e_2 : \tau$, we introduce $\beta = \tau$ and ask for $e_1 : \gamma \to \beta$ and $e_2 : \gamma$, for some fresh unknowns β, γ.
- For a function $\text{fun } x \to e : \tau$, we introduce $\beta \to \gamma = \tau$ and ask for $e : \gamma$ in environment $\{x : \beta\} \cup \Gamma$, for some fresh unknowns β, γ.
- The case $\text{let } x = e_1 \text{ in } e_2 : \tau$ is different. One approach is to *first* solve the equations that we get by asking for $e_1 : \beta$, for some fresh unknown β. Let us say a solution $\beta = \tau_\beta$ has been found, $\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_m$ are the remaining unknowns in τ_β, and $\alpha_1 \ldots \alpha_n$ are all that do not appear in Γ. Then we ask for $e_2 : \tau$ in environment $\{x : \forall\alpha_1 \ldots \alpha_n.\tau_\beta\} \cup \Gamma$.
- Remember that whenever we establish a solution $\beta = \tau_\beta$ to an unknown β, it takes effect everywhere! The substitution propagates through all the equations, potentially triggering further unifications.
- To find a type for e (in environment Γ), we pick a fresh unknown β and ask for $e : \beta$ (in Γ). The algorithm then generates and solves equations until either a solution is found or a contradiction reveals a type error.

5.1.4 Polymorphism The "top-level" definitions for which the system infers types with variables are called *polymorphic*, which informally means "working with different shapes of data." A polymorphic function like `List.hd` can operate on lists containing any type of element – the function itself doesn't care what the elements are, only that it's working with a list.

This kind of polymorphism is called *parametric polymorphism*, since the types have parameters. The term "parametric" emphasizes that the same code works uniformly for all type instantiations. A different kind of polymorphism is provided by object-oriented programming languages (sometimes called *subtype polymorphism* or *ad-hoc polymorphism*), where different code may execute depending on the runtime type of objects.

5.2 Parametric Types

Polymorphic functions truly shine when used with polymorphic data types. The combination of the two is what makes ML-family languages so expressive. Consider this definition of our own list type:

```
type 'a my_list = Empty | Cons of 'a * 'a my_list
```

We define lists that can store elements of any type `'a`. The type parameter `'a` acts as a placeholder that gets filled in when we create actual lists. Now we can write functions that work on these lists:

```
# let tail l =
    match l with
    | Empty -> invalid_arg "tail"
    | Cons (_, tl) -> tl;;
val tail : 'a my_list -> 'a my_list = <fun>
```

This is a polymorphic function: it works for lists with elements of any type. Whether we have a list of integers, strings, or even lists of lists, the same `tail` function handles them all.

A crucial point to understand: a *parametric type* like `'a my_list` *is not* itself a data type but rather a *family* of data types. The types `bool my_list`, `int my_list`, etc. *are* different types – you cannot mix elements of different types in a single list. We say that the type `int my_list` *instantiates* the parametric type `'a my_list`.

5.2.1 Multiple Type Parameters Types can have multiple type parameters. In OCaml, the syntax might seem a bit unusual at first: type parameters precede the type name, enclosed in parentheses. For example:

```
type ('a, 'b) choice = Left of 'a | Right of 'b
```

This type has two parameters and represents a value that is either something of type `'a` (wrapped in `Left`) or something of type `'b` (wrapped in `Right`). Mathematically we would write $\text{choice}(\alpha, \beta)$.

Not all functions that use parametric types need to be polymorphic. A function may constrain the type parameters to specific types:

```
# let get_int c =
    match c with
    | Left i -> i
    | Right b -> if b then 1 else 0;;
val get_int : (int, bool) choice -> int = <fun>
```

Here, the pattern matching on `Left i` and `Right b` with arithmetic operations constrains the type to `(int, bool) choice`.

5.2.2 Syntax in Other Languages Different functional languages have different syntactic conventions for type parameters. In F#, we provide parameters (when more than one) after the type name, using angle brackets:

```
type choice<'a,'b> = Left of 'a | Right of 'b
```

In Haskell, the syntax is arguably the cleanest – we provide type parameters similarly to function arguments, separated by spaces:

```
data Choice a b = Left a | Right b
```

Despite the syntactic differences, the underlying concept of parametric polymorphism is the same across all these languages.

5.3 Type Inference, Formally

Now we present a more formal treatment of type inference. A statement that an expression has a type in an environment is called a *type judgement*. For environment $\Gamma = \{x : \forall \alpha_1 \ldots \alpha_n . \tau_x ; \ldots\}$, expression e and type τ we write:

$$\Gamma \vdash e : \tau$$

This notation reads: "In environment Γ, expression e has type τ." The turnstile symbol $\vdash$ can be thought of as "entails" or "proves."

We will derive all the constraint equations in one go using the notation $[\![\cdot]\!]$, to be solved later by unification. Besides equations we will need to manage introduced variables, using existential quantification to express that "there exists some type variable satisfying these constraints."

For local definitions we require remembering what constraints should hold when the definition is used. Therefore we extend *type schemes* in the environment to: $\Gamma = \{x : \forall \beta_1 \ldots \beta_m [\exists \alpha_1 \ldots \alpha_n . D] . \tau_x ; \ldots\}$ where D are equations – keeping the variables $\alpha_1 \ldots \alpha_n$ introduced while deriving D in front. A simpler form would be sufficient: $\Gamma = \{x : \forall \beta [\exists \alpha_1 \ldots \alpha_n . D] . \beta ; \ldots\}$

The formal constraint generation rules are:

$$[\![\Gamma \vdash x : \tau]\!] = \exists \overline{\beta' \alpha'} . (D[\overline{\beta \alpha} := \overline{\beta' \alpha'}] \wedge \tau_x [\overline{\beta \alpha} := \overline{\beta' \alpha'}] \doteq \tau)$$

where $\Gamma(x) = \forall \overline{\beta} [\exists \overline{\alpha} . D] . \tau_x$, $\overline{\beta' \alpha'} \# \mathrm{FV}(\Gamma, \tau)$

$$[\![\Gamma \vdash \textbf{fun } x \texttt{->} e : \tau]\!] = \exists \alpha_1 \alpha_2 . ([\![\Gamma \{x : \alpha_1\} \vdash e : \alpha_2]\!] \wedge \alpha_1 \rightarrow \alpha_2 \doteq \tau)$$

where $\alpha_1 \alpha_2 \# \mathrm{FV}(\Gamma, \tau)$

$$[\![\Gamma \vdash e_1 \; e_2 : \tau]\!] = \exists \alpha . ([\![\Gamma \vdash e_1 : \alpha \rightarrow \tau]\!] \wedge [\![\Gamma \vdash e_2 : \alpha]\!]), \alpha \# \mathrm{FV}(\Gamma, \tau)$$

$$[\![\Gamma \vdash K \; e_1 \ldots e_n : \tau]\!] = \exists \overline{\alpha'} . (\bigwedge_i [\![\Gamma \vdash e_i : \tau_i [\overline{\alpha} := \overline{\alpha'}]]\!] \wedge \varepsilon(\overline{\alpha'}) \doteq \tau)$$

where $K : \forall \overline{\alpha} . \tau_1 \times \ldots \times \tau_n \rightarrow \varepsilon(\overline{\alpha})$, $\overline{\alpha'} \# \mathrm{FV}(\Gamma, \tau)$

For let-expressions:

$$[\![\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : \tau]\!] = (\exists \beta . C) \wedge [\![\Gamma\{x : \forall \beta[C].\beta\} \vdash e_2 : \tau]\!]$$

where $C = [\![\Gamma \vdash e_1 : \beta]\!]$

For recursive let-expressions:

$$[\![\Gamma \vdash \textbf{letrec } x = e_1 \textbf{ in } e_2 : \tau]\!] = (\exists \beta . C) \wedge [\![\Gamma\{x : \forall \beta[C].\beta\} \vdash e_2 : \tau]\!]$$

where $C = [\![\Gamma\{x : \beta\} \vdash e_1 : \beta]\!]$

For match expressions:

$$[\![\Gamma \vdash \textbf{match } e_v \textbf{ with } \bar{c} : \tau]\!] = \exists \alpha_v . [\![\Gamma \vdash e_v : \alpha_v]\!] \bigwedge_i [\![\Gamma \vdash p_i . e_i : \alpha_v \rightarrow \tau]\!]$$

where $\bar{c} = p_1.e_1 | \ldots | p_n.e_n$, $\alpha_v \# \mathrm{FV}(\Gamma, \tau)$

For pattern clauses:

$$[\![\Gamma, \Sigma \vdash p.e : \tau_1 \rightarrow \tau_2]\!] = [\![\Sigma \vdash p \downarrow \tau_1]\!] \wedge \exists \overline{\beta} . [\![\Gamma\Gamma' \vdash e : \tau_2]\!]$$

where $\exists \overline{\beta} \Gamma'$ is $[\![\Sigma \vdash p \uparrow \tau_1]\!]$, $\overline{\beta} \# \mathrm{FV}(\Gamma, \tau_2)$

The notation $[\![\Sigma \vdash p \downarrow \tau_1]\!]$ derives constraints on the type of the matched value, while $[\![\Sigma \vdash p \uparrow \tau_1]\!]$ derives the environment for pattern variables.

By $\overline{\alpha}$ or $\overline{\alpha_i}$ we denote a sequence of some length: $\alpha_1 \ldots \alpha_n$. By $\bigwedge_i \varphi_i$ we denote a conjunction of $\overline{\varphi_i}$: $\varphi_1 \wedge \ldots \wedge \varphi_n$.

5.3.1 Polymorphic Recursion There is an interesting limitation in standard type inference for recursive functions. Note the limited polymorphism of `let rec f = ...` – we cannot use `f` polymorphically within its own definition. Why? Because when type-checking the body of a recursive definition, we don't yet know the final type of `f`, so we must treat it as having a single, unknown type.

In modern OCaml we can bypass this limitation if we provide the type of `f` upfront:

```
let rec f : 'a. 'a -> 'a list = ...
```

where `'a. 'a -> 'a list` stands for $\forall \alpha . \alpha \rightarrow \alpha$ list.

Using the recursively defined function with different types in its definition is called *polymorphic recursion*. It is most useful together with *irregular recursive datatypes* – data structures where the recursive use has different type arguments than the actual parameters. These "nested" or "non-uniform" datatypes enable some remarkably elegant data structures.

Example: A List Alternating Between Two Types of Elements Here is a fascinating example: a list that alternates between two different types of elements. Notice how the recursive occurrence swaps the type parameters:

```
type ('x, 'o) alterning =
  | Stop
  | One of 'x * ('o, 'x) alterning

let rec to_list :
    'x 'o 'a. ('x -> 'a) -> ('o -> 'a) ->
              ('x, 'o) alterning -> 'a list =
  fun x2a o2a ->
    function
    | Stop -> []
    | One (x, rest) -> x2a x :: to_list o2a x2a rest

let to_choice_list alt =
  to_list (fun x -> Left x) (fun o -> Right o) alt

let it = to_choice_list
  (One (1, One ("o", One (2, One ("oo", Stop)))))
```

Notice how the recursive call to `to_list` swaps `o2a` and `x2a` – this is necessary because the alternating structure swaps the type parameters at each level. The polymorphic recursion annotation `'x 'o 'a.` tells OCaml that we need to use `to_list` at different type instantiations within its own definition.

Example: Data-Structural Bootstrapping Here is another powerful example of polymorphic recursion: a sequence data structure that stores elements in exponentially increasing chunks. This technique, known as *data-structural bootstrapping*, achieves logarithmic-time random access – much faster than standard lists which require linear time.

```
type 'a seq =
  | Nil
  | Zero of ('a * 'a) seq
  | One of 'a * ('a * 'a) seq
```

The key insight is that this type is *non-uniform*: the recursive occurrences use `('a * 'a) seq` rather than `'a seq`. This means that as we go deeper into the structure, elements get paired together, effectively doubling the "width" at each level. We store a list of elements in exponentially increasing chunks:

```
let example =
  One (0, One ((1,2), Zero (One ((((3,4),(5,6)), ((7,8),(9,10))), Nil))))
```

The `cons` operation adds an element to the front. Remarkably, appending an element to this data structure works exactly like adding one to a binary number:

```ocaml
let rec cons : 'a. 'a -> 'a seq -> 'a seq =  (* Appending an element to the *)
  fun x -> function                           (* datastructure is like *)
  | Nil -> One (x, Nil)                       (* adding one to a binary number: 1+0=1 *)
  | Zero ps -> One (x, ps)                    (* 1+...0=...1 *)
  | One (y, ps) -> Zero (cons (x,y) ps)       (* 1+...1=[...+1]0 *)

let rec lookup : 'a. int -> 'a seq -> 'a =
  fun i s -> match i, s with
  | _, Nil -> raise Not_found                 (* Rather than returning None : 'a option *)
  | 0, One (x, _) -> x                        (* we raise exception, for convenience. *)
  | i, One (_, ps) -> lookup (i-1) (Zero ps)
  | i, Zero ps ->                             (* Random-access lookup works *)
      let x, y = lookup (i / 2) ps in         (* in logarithmic time -- much faster than *)
      if i mod 2 = 0 then x else y            (* in standard lists. *)
```

The `Zero` and `One` constructors correspond to binary digits. A `Zero` means "no singleton element at this level," while `One` carries a singleton (or pair, or quad, etc.) before recursing. The `lookup` function exploits this structure: when looking up index `i` in a `Zero ps`, it divides by 2 and looks in the paired structure, then extracts the appropriate half of the pair.

5.4 Algebraic Specification

Now we turn to a fundamental question in computer science: how do we formally describe what a data structure *is* and what it should *do*? The mathematical answer is *algebraic specification*.

The way we introduce a data structure, like complex numbers or strings, in mathematics is by specifying an *algebraic structure*. This approach gives us a precise language for describing data structures independent of any particular implementation.

Algebraic structures consist of a set (or several sets, for so-called *multisorted* algebras) and a bunch of functions (also known as operations) over this set (or sets). Think of integers with addition and multiplication, or strings with concatenation and character access.

A *signature* is a rough description of an algebraic structure: it provides *sorts* – names for the sets (in the multisorted case) – and names of the functions-operations together with their arity (and what sorts of arguments they take). A signature tells us what operations exist, but not how they behave.

We select a class of algebraic structures by providing axioms that have to hold. We will call such classes *algebraic specifications*. In mathematics, a rusty name for some algebraic specifications is a *variety*; a more modern name is *algebraic category*.

Here is the key connection to programming: algebraic structures correspond to "implementations" and signatures to "interfaces" in programming languages. We will say that an algebraic structure *implements* an algebraic specification when all axioms of the specification hold in the structure. An important point: all algebraic specifications are implemented by multiple structures! This is precisely what we want – it gives us the freedom to choose different implementations with different performance characteristics while maintaining the same interface.

We say that an algebraic structure does not have *junk* when all its elements (i.e., elements in the sets corresponding to sorts) can be built using operations in its signature. Junk-free structures are "minimal" in some sense – they contain only the values that can be constructed using the provided operations.

We allow parametric types as sorts. In that case, strictly speaking, we define a family of algebraic specifications (a different specification for each instantiation of the parametric type).

5.4.1 Algebraic Specifications: Examples Let us look at some concrete examples to make these abstract ideas tangible. An algebraic specification can also use an earlier specification, building up complexity layer by layer. In "impure" languages like OCaml and F# we allow that the result of any operation be an error. In Haskell we would use `Maybe` to explicitly model potential failure.

Specification $\mathbf{nat}_p$ (bounded natural numbers):

This specification describes natural numbers that wrap around at some bound p (like machine integers):

nat_p
$0 : \mathrm{nat}_p$ $\mathrm{succ} : \mathrm{nat}_p \to \mathrm{nat}_p$ $+ : \mathrm{nat}_p \to \mathrm{nat}_p \to \mathrm{nat}_p$ $* : \mathrm{nat}_p \to \mathrm{nat}_p \to \mathrm{nat}_p$ Variables: $n, m : \mathrm{nat}_p$ Axioms: $0 + n = n$, $n + 0 = n$ $m + \mathrm{succ}(n) = \mathrm{succ}(m + n)$ $0 * n = 0$, $n * 0 = 0$ $m * \mathrm{succ}(n) = m + (m * n)$ $\underbrace{\mathrm{succ}(\ldots\mathrm{succ}(0))}_{\text{less than } p \text{ times}} \neq 0$ $\underbrace{\mathrm{succ}(\ldots\mathrm{succ}(0))}_{p \text{ times}} = 0$

The axioms define how addition and multiplication work recursively, and the last two axioms capture the bounded nature: applying succ less than p times never gives zero, but exactly p times wraps around to zero.

Specification string_p (bounded strings):

This specification describes strings with a maximum length p:

---

string_p

---

uses char, nat_p
`""` : string_p
`"c"` : char $\to \text{string}_p$
$\hat{}$: $\text{string}_p \to \text{string}_p \to \text{string}_p$
$\cdot[\cdot]$: $\text{string}_p \to \text{nat}_p \to \text{char}$
Variables: $s : \text{string}_p$, $c, c_1, \ldots, c_p$: char, $n : \text{nat}_p$
Axioms:
`""` $\hat{}s = s$, $s\hat{}$ `""` $= s$
$\underbrace{\text{“}c_1\text{”}\hat{}(\ldots\hat{}\text{“}c_p\text{”})}_{p \text{ times}} = \text{error}$
$r\hat{}(s\hat{}t) = (r\hat{}s)\hat{}t$
$(\text{“}c\text{”}\hat{}s)[0] = c$
$(\text{“}c\text{”}\hat{}s)[\text{succ}(n)] = s[n]$
`""`$[n] = \text{error}$

The axioms specify that concatenation is associative, that the empty string is an identity for concatenation, that exceeding the length limit produces an error, and that indexing works by stripping characters from the front.

5.5 Homomorphisms

When do two implementations of the same specification “behave the same”? The mathematical answer involves *homomorphisms* – structure-preserving mappings between algebraic structures.

Homomorphisms are mappings between algebraic structures with the same signature that preserve operations. Intuitively, if you apply an operation and then map, you get the same result as mapping first and then applying the corresponding operation.

A *homomorphism* from algebraic structure $(A, \{f^A, g^A, \ldots\})$ to $(B, \{f^B, g^B, \ldots\})$ is a function $h : A \to B$ such that: - $h(f^A(a_1, \ldots, a_{n_f})) = f^B(h(a_1), \ldots, h(a_{n_f}))$ for all $(a_1, \ldots, a_{n_f})$ - $h(g^A(a_1, \ldots, a_{n_g})) = g^B(h(a_1), \ldots, h(a_{n_g}))$ for all $(a_1, \ldots, a_{n_g})$ - and so on for all operations.

Two algebraic structures are *isomorphic* if there are homomorphisms $h_1 : A \to B$, $h_2 : B \to A$ from one to the other and back, that when composed in any order form identity: $\forall (b \in B)\ h_1(h_2(b)) = b$ and $\forall (a \in A)\ h_2(h_1(a)) = a$.

An algebraic specification whose all implementations without junk are isomorphic is called “*monomorphic*”. This means the specification pins down the structure so precisely that there's essentially only one way to implement it (up to isomorphism).

We usually only add axioms that really matter to us to the specification, so that the implementations have room for optimization. For this reason, the resulting specifications will often not be monomorphic in the above sense – and that's intentional! A non-monomorphic specification allows for multiple genuinely different implementations, which may have different performance characteristics.

5.6 Example: Maps

Now let us look at a practical example that will guide the rest of this chapter. A *map* (also called dictionary or associative array) associates keys with values. This is one of the most fundamental data structures in programming – think of Python's dictionaries, Java's `HashMap`, or OCaml's `Map` module.

Here is an algebraic specification that captures the essential behavior of maps:

(α, β) map
uses bool, type parameters α, β empty : (α, β) map member : $\alpha \to (\alpha, \beta)$ map $\to$ bool add : $\alpha \to \beta \to (\alpha, \beta)$ map $\to (\alpha, \beta)$ map remove : $\alpha \to (\alpha, \beta)$ map $\to (\alpha, \beta)$ map find : $\alpha \to (\alpha, \beta)$ map $\to \beta$ Variables: $k, k_2 : \alpha,\ v, v_2 : \beta,\ m : (\alpha, \beta)$ map Axioms: $\text{member}(k, \text{add}(k, v, m)) = \text{true}$ $\text{member}(k, \text{remove}(k, m)) = \text{false}$ $\text{member}(k, \text{add}(k_2, v, m)) = \text{true} \wedge k \neq k_2 \Leftrightarrow \text{member}(k, m) = \text{true} \wedge k \neq k_2$ $\text{member}(k, \text{remove}(k_2, m)) = \text{true} \wedge k \neq k_2 \Leftrightarrow \text{member}(k, m) = \text{true} \wedge k \neq k_2$ $\text{find}(k, \text{add}(k, v, m)) = v$ $\text{find}(k, \text{remove}(k, m)) = \text{error}$, $\text{find}(k, \text{empty}) = \text{error}$ $\text{find}(k, \text{add}(k_2, v_2, m)) = v \wedge k \neq k_2 \Leftrightarrow \text{find}(k, m) = v \wedge k \neq k_2$ $\text{find}(k, \text{remove}(k_2, m)) = v \wedge k \neq k_2 \Leftrightarrow \text{find}(k, m) = v \wedge k \neq k_2$ $\text{remove}(k, \text{empty}) = \text{empty}$

The axioms capture the intuitive behavior: adding a key-value pair makes that key findable, removing a key makes it unfindable, and operations on different keys don't interfere with each other. Notice how the specification says nothing about *how* the map is implemented – only about *what* behavior it must exhibit.

5.7 Modules and Interfaces (Signatures): Syntax

How do we express algebraic specifications in OCaml? The answer is the *module system*. In the ML family of languages, structures are given names by **module** bindings, and signatures are types of modules. From outside of a structure or signature, we refer to the values or types it provides with a dot notation: `Module.value`.

Module (and module type) names have to start with a capital letter (in ML languages). Since modules and module types have names, there is a convention to name the central type of a signature (the one that is "specified" by the signature), for brevity, `t`. Module types are often named with "all-caps" (all letters upper case).

Here is how we translate our map specification into an OCaml module signature:

```
module type MAP = sig
  type ('a, 'b) t
  val empty : ('a, 'b) t
  val member : 'a -> ('a, 'b) t -> bool
  val add : 'a -> 'b -> ('a, 'b) t -> ('a, 'b) t
  val remove : 'a -> ('a, 'b) t -> ('a, 'b) t
  val find : 'a -> ('a, 'b) t -> 'b
end

module ListMap : MAP = struct
  type ('a, 'b) t = ('a * 'b) list
  let empty = []
  let member = List.mem_assoc
  let add k v m = (k, v)::m
  let remove = List.remove_assoc
  let find = List.assoc
end
```

The `ListMap` module implements `MAP` using OCaml's built-in list functions for association lists. The type annotation `: MAP` after the module name tells OCaml to check that the implementation provides everything the signature requires, and hides any additional details.

5.8 Implementing Maps: Association Lists

Let us now build an implementation of maps from the ground up, exploring different approaches and their trade-offs. The most straightforward implementation… might not be what you expected:

```
module TrivialMap : MAP = struct
  type ('a, 'b) t =
    | Empty
    | Add of 'a * 'b * ('a, 'b) t
    | Remove of 'a * ('a, 'b) t

  let empty = Empty
```

```
  let rec member k m =
    match m with
    | Empty -> false
    | Add (k2, _, _) when k = k2 -> true
    | Remove (k2, _) when k = k2 -> false
    | Add (_, _, m2) -> member k m2
    | Remove (_, m2) -> member k m2

  let add k v m = Add (k, v, m)
  let remove k m = Remove (k, m)

  let rec find k m =
    match m with
    | Empty -> raise Not_found
    | Add (k2, v, _) when k = k2 -> v
    | Remove (k2, _) when k = k2 -> raise Not_found
    | Add (_, _, m2) -> find k m2
    | Remove (_, m2) -> find k m2
end
```

This "trivial" implementation is quite clever in its own way: it simply records all operations as a log! The data structure itself is a history of everything that has been done to it. The `add` and `remove` operations are $O(1)$ – they just prepend a new node. However, `member` and `find` must traverse the entire history to determine the current state, giving them $O(n)$ complexity where n is the number of operations performed.

This implementation illustrates an important point: there are many ways to satisfy the same specification, with very different performance characteristics.

Here is a more conventional implementation based on association lists, i.e., on lists of key-value pairs without the `Remove` constructor:

```
module MyListMap : MAP = struct
  type ('a, 'b) t = Empty | Add of 'a * 'b * ('a, 'b) t

  let empty = Empty

  let rec member k m =
    match m with
    | Empty -> false
    | Add (k2, _, _) when k = k2 -> true
    | Add (_, _, m2) -> member k m2

  let rec add k v m =
    match m with
    | Empty -> Add (k, v, Empty)
```

```
    | Add (k2, _, m) when k = k2 -> Add (k, v, m)
    | Add (k2, v2, m) -> Add (k2, v2, add k v m)

  let rec remove k m =
    match m with
    | Empty -> Empty
    | Add (k2, _, m) when k = k2 -> m
    | Add (k2, v, m) -> Add (k2, v, remove k m)

  let rec find k m =
    match m with
    | Empty -> raise Not_found
    | Add (k2, v, _) when k = k2 -> v
    | Add (_, _, m2) -> find k m2
end
```

This implementation maintains the invariant that each key appears at most once in the structure. The `add` function replaces an existing key's value rather than creating a duplicate, and `remove` actually removes the key-value pair. All operations are still $O(n)$ in the worst case, but the structure stays cleaner.

5.9 Implementing Maps: Binary Search Trees

Can we do better than linear time? Yes, by using a smarter data structure. Binary search trees are binary trees with elements stored at the interior nodes, such that elements to the left of a node are smaller than, and elements to the right bigger than, elements within a node. This ordering property is what makes them efficient.

For maps, we store key-value pairs as elements in binary search trees, and compare the elements by keys alone. The tree structure allows us to use "divide-and-conquer" to search for the value associated with a key.

On average, binary search trees are fast – $O(\log n)$ complexity for all operations. At each node, we can eliminate half the remaining elements from consideration. However, in the worst case (when keys are inserted in sorted order), the tree degenerates into a linked list and operations become $O(n)$.

A note on our design: the simple polymorphic signature for maps is only possible because OCaml provides polymorphic comparison (and equality) operators that work on elements of most types (but not on functions). These operators may not behave as you expect for all types! Our signature for polymorphic maps is not the standard approach because of this limitation; it is just to keep things simple for pedagogical purposes.

```
module BTreeMap : MAP = struct
  type ('a, 'b) t = Empty | T of ('a, 'b) t * 'a * 'b * ('a, 'b) t
```

```
  let empty = Empty

  let rec member k m =                    (* "Divide and conquer" search through the tree. *
    match m with
    | Empty -> false
    | T (_, k2, _, _) when k = k2 -> true
    | T (m1, k2, _, _) when k < k2 -> member k m1
    | T (_, _, _, m2) -> member k m2

  let rec add k v m =                     (* Searches the tree in the same way as member *)
    match m with                          (* but copies every node along the way. *)
    | Empty -> T (Empty, k, v, Empty)
    | T (m1, k2, _, m2) when k = k2 -> T (m1, k, v, m2)
    | T (m1, k2, v2, m2) when k < k2 -> T (add k v m1, k2, v2, m2)
    | T (m1, k2, v2, m2) -> T (m1, k2, v2, add k v m2)

  let rec split_rightmost m =             (* A helper function, it does not belong *)
    match m with                          (* to the "exported" signature. *)
    | Empty -> raise Not_found
    | T (Empty, k, v, Empty) -> k, v, Empty   (* We remove one element, *)
    | T (m1, k, v, m2) ->                 (* the one that is on the bottom right. *)
        let rk, rv, rm = split_rightmost m2 in
        rk, rv, T (m1, k, v, rm)

  let rec remove k m =
    match m with
    | Empty -> Empty
    | T (m1, k2, _, Empty) when k = k2 -> m1
    | T (Empty, k2, _, m2) when k = k2 -> m2
    | T (m1, k2, _, m2) when k = k2 ->
        let rk, rv, rm = split_rightmost m1 in
        T (rm, rk, rv, m2)
    | T (m1, k2, v, m2) when k < k2 -> T (remove k m1, k2, v, m2)
    | T (m1, k2, v, m2) -> T (m1, k2, v, remove k m2)

  let rec find k m =
    match m with
    | Empty -> raise Not_found
    | T (_, k2, v, _) when k = k2 -> v
    | T (m1, k2, _, _) when k < k2 -> find k m1
    | T (_, _, _, m2) -> find k m2
end
```

The `member` and `find` functions use the “divide-and-conquer” strategy: compare the target key with the key at the current node, and recursively search in the appropriate subtree. The `add` function searches the tree in the same way but copies every node along the path to create the new tree (since we're using immutable data structures).

The `remove` function is trickier. When removing a node with two children, we need to replace it with another value that maintains the ordering property. The `split_rightmost` helper function finds and removes the rightmost (largest) element from a subtree – this element is guaranteed to be smaller than everything in the right subtree and larger than everything remaining in the left subtree, making it the perfect replacement.

5.10 Implementing Maps: Red-Black Trees

The fatal weakness of ordinary binary search trees is that they can become unbalanced. If keys arrive in sorted order, each insertion adds a node at the bottom of a long chain, and we lose the logarithmic performance guarantee. How can we maintain balance automatically?

This section is based on Wikipedia's Red-black tree article, Chris Okasaki's "Purely Functional Data Structures" and Matt Might's excellent blog post on red-black tree deletion.

Binary search trees are good when we encounter keys in random order, because the cost of operations is limited by the depth of the tree which is small relative to the number of nodes… unless the tree grows unbalanced achieving large depth (which means there are sibling subtrees of vastly different sizes on some path).

To remedy this, we *rebalance* the tree while building it – i.e., while adding elements. The key insight is to detect when the tree is becoming unbalanced and perform local rotations to restore balance.

In *red-black trees* we achieve balance by: 1. Remembering one of two colors (red or black) with each node 2. Keeping the same number of black nodes on every path from the root to a leaf 3. Not allowing a red node to have a red child

These invariants together guarantee that the tree cannot become too unbalanced: the depth is at most twice the depth of a perfectly balanced tree with the same number of nodes. Why? The "black height" (number of black nodes on any root-to-leaf path) is the same everywhere, and red nodes can only appear between black nodes, so the longest path can have at most twice as many nodes as the shortest.

5.10.1 B-trees of Order 4 (2-3-4 Trees) To understand where red-black trees come from, it helps to first understand 2-3-4 trees (also known as B-trees of order 4).

How can we have perfectly balanced trees without worrying about having exactly $2^k - 1$ elements? The answer is to allow variable-width nodes. **2-3-4 trees** can store from 1 to 3 elements in each node and have 2 to 4 subtrees correspondingly. This flexibility lets us maintain perfect balance!

- A **2-node** contains one element and has two children
- A **3-node** contains two elements and has three children
- A **4-node** contains three elements and has four children

To insert into a 2-3-4 tree, we descend toward the appropriate leaf position. But if we encounter a full node (4-node) along the way, we "split" it: move the middle element up to the parent and split the remaining two elements into separate 2-nodes. This maintains perfect balance at all times – all leaves are at the same depth.

The remarkable fact is that red-black trees are just a clever way to represent 2-3-4 trees as binary trees! To represent a 2-3-4 tree as a binary tree with one element per node, we color the "primary" element of each node black (the middle element of a 4-node, or the first element of a 2-/3-node) and make it the parent of its neighbor elements colored red. The red elements then become parents of the original subtrees. This correspondence provides the deep intuition behind red-black trees: the colors encode the structure of the underlying 2-3-4 tree.

5.10.2 Red-Black Trees, Without Deletion Now let us implement red-black trees in OCaml. Red-black trees maintain two invariants:

Invariant 1. No red node has a red child. (No two consecutive red nodes on any path.)

Invariant 2. Every path from the root to an empty node contains the same number of black nodes. (The "black height" is uniform.)

For simplicity, we first implement red-black tree based *sets* (not maps) without deletion. The implementation proceeds almost exactly like for unbalanced binary search trees; we only need to add code to restore the invariants after each insertion.

The beautiful insight of Okasaki's approach is that by keeping balance at each step of constructing a node, it is enough to check *locally* (around the root of the subtree) whether a violation has occurred. We never need to examine the entire tree. For an understandable implementation of deletion, we need to introduce more colors – see Matt Might's post for details.

```
type color = R | B
type 'a t = E | T of color * 'a t * 'a * 'a t

let empty = E

let rec member x m =                    (* Like in unbalanced binary search tree. *)
  match m with
  | E -> false
```

```
  | T (_, _, y, _) when x = y -> true
  | T (_, a, y, _) when x < y -> member x a
  | T (_, _, _, b) -> member x b

let balance = function                      (* Restoring the invariants. *)
  | B, T (R, T (R,a,x,b), y, c), z, d     (* On next figure: left, *)
  | B, T (R, a, x, T (R,b,y,c)), z, d     (* top, *)
  | B, a, x, T (R, T (R,b,y,c), z, d)     (* bottom, *)
  | B, a, x, T (R, b, y, T (R,c,z,d))     (* right, *)
      -> T (R, T (B,a,x,b), y, T (B,c,z,d))     (* center tree. *)
  | color, a, x, b -> T (color, a, x, b)        (* We allow red-red violation for now. *)

let insert x s =
  let rec ins = function                    (* Like in unbalanced binary search tree, *)
    | E -> T (R, E, x, E)                   (* but fix violation above created node. *)
    | T (color, a, y, b) as s ->
        if x < y then balance (color, ins a, y, b)
        else if x > y then balance (color, a, y, ins b)
        else s
  in
  match ins s with                          (* We could still have red-red violation at root, *
  | T (_, a, y, b) -> T (B, a, y, b)        (* fixed by coloring it black. *)
  | E -> failwith "insert: impossible"
```

The `balance` function is the heart of the algorithm. It handles four cases where a red-red violation occurs (a red node with a red child). The four cases correspond to different positions of the violation: - A red left child with a red left grandchild - A red left child with a red right grandchild - A red right child with a red left grandchild - A red right child with a red right grandchild

In each case, we perform a "rotation" that restructures the tree to eliminate the violation while maintaining the binary search tree property. Remarkably, all four cases produce the same balanced result: a red root with two black children, with the subtrees `a`, `b`, `c`, `d` properly distributed.

The `insert` function works like insertion into an ordinary binary search tree, but calls `balance` after each recursive step to fix any violations that may have been introduced. New nodes are always created red (which might create a red-red violation that `balance` will fix). At the very end, we color the root black – this can never create a violation and ensures the root is always black.

Exercises

Exercise 1. Derive the equations and solve them to find the type for:

```
let cadr l = List.hd (List.tl l) in cadr (1::2::[]), cadr (true::false::[])
```

in environment $\Gamma = \{\text{List.hd} : \forall \alpha . \alpha \text{ list} \rightarrow \alpha ; \text{List.tl} : \forall \alpha . \alpha \text{ list} \rightarrow \alpha \text{ list}\}$. You can take "shortcuts" if it is too many equations to write down.

Exercise 2. *Terms* $t_1, t_2, \ldots \in T(\Sigma, X)$ are built out of variables $x, y, \ldots \in X$ and function symbols $f, g, \ldots \in \Sigma$ the way you build values out of functions:

- $X \subset T(\Sigma, X)$ – variables are terms; usually an infinite set,
- for terms $t_1, \ldots, t_n \in T(\Sigma, X)$ and a function symbol $f \in \Sigma_n$ of arity n, $f(t_1, \ldots, t_n) \in T(\Sigma, X)$ – bigger terms arise from applying function symbols to smaller terms; $\Sigma = \dot{\cup}_n \Sigma_n$ is called a signature.

In OCaml, we can define terms as: `type term = V of string | T of string * term list`, where for example `V("x")` is a variable x and `T("f", [V("x"); V("y")])` is the term $f(x, y)$.

By *substitutions* $\sigma, \rho, \ldots$ we mean finite sets of variable-term pairs which we can write as $\{x_1 \mapsto t_1, \ldots, x_k \mapsto t_k\}$ or $[x_1 := t_1; \ldots; x_k := t_k]$, but also functions from terms to terms $\sigma : T(\Sigma, X) \rightarrow T(\Sigma, X)$ related to the pairs as follows: if $\sigma = \{x_1 \mapsto t_1, \ldots, x_k \mapsto t_k\}$, then

- $\sigma(x_i) = t_i$ for $x_i \in \{x_1, \ldots, x_k\}$,
- $\sigma(x) = x$ for $x \in X \setminus \{x_1, \ldots, x_k\}$,
- $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$.

In OCaml, we can define substitutions σ as: `type subst = (string * term) list`, together with a function `apply : subst -> term -> term` which computes $\sigma(\cdot)$.

We say that a substitution σ is *more general* than all substitutions $\rho \circ \sigma$, where $(\rho \circ \sigma)(x) = \rho(\sigma(x))$. In type inference, we are interested in most general solutions.

A *unification problem* is a finite set of equations $S = \{s_1 =^? t_1, \ldots, s_n =^? t_n\}$. A solution, or *unifier* of S, is a substitution σ such that $\sigma(s_i) = \sigma(t_i)$ for $i = 1, \ldots, n$. A *most general unifier*, or *MGU*, is a most general such substitution.

1. Implement an algorithm that, given a set of equations represented as a list of pairs of terms, computes an idempotent most general unifier of the equations.

2. (Ex. 4.22 in Franz Baader and Tobias Nipkow "Term Rewriting and All That", p. 82.) Modify the implementation of unification to achieve linear space complexity by working with what could be called iterated substitutions.

Exercise 3.

1. What does it mean that an implementation has junk (as an algebraic structure for a given signature)? Is it bad?
2. Define a monomorphic algebraic specification (other than, but similar to, nat_p or string_p, some useful data type).

3. Discuss an example of a (monomorphic) algebraic specification where it would be useful to drop some axioms (giving up monomorphicity) to allow more efficient implementations.

Exercise 4.

1. Does the example `ListMap` meet the requirements of the algebraic specification for maps? Hint: here is the definition of `List.remove_assoc`; `compare a x` equals `0` if and only if `a = x`.

```
let rec remove_assoc x = function
  | [] -> []
  | (a, b as pair) :: l ->
      if compare a x = 0 then l else pair :: remove_assoc x l
```

2. Trick question: what is the computational complexity of `ListMap` or `TrivialMap`?

3. (*) The implementation `MyListMap` is inefficient: it performs a lot of copying and is not tail-recursive. Optimize it (without changing the type definition).

4. Add (and specify) $\text{isEmpty} : (\alpha, \beta)\ \text{map} \rightarrow \text{bool}$ to the example algebraic specification of maps without increasing the burden on its implementations. Hint: equational reasoning might be not enough; consider an equivalence relation $\approx$ meaning "have the same keys".

Exercise 5. Design an algebraic specification and write a signature for first-in-first-out queues. Provide two implementations: one straightforward using a list, and another one using two lists: one for freshly added elements providing efficient queueing of new elements, and "reversed" one for efficient popping of old elements.

Exercise 6. Design an algebraic specification and write a signature for sets. Provide two implementations: one straightforward using a list, and another one using a map into the unit type.

Exercise 7.

1. (Ex. 2.2 in Chris Okasaki "Purely Functional Data Structures") In the worst case, `member` performs approximately $2d$ comparisons, where d is the depth of the tree. Rewrite `member` to take no more than $d+1$ comparisons by keeping track of a candidate element that *might* be equal to the query element (say, the last element for which $<$ returned false) and checking for equality only when you hit the bottom of the tree.

2. (Ex. 3.10 in Chris Okasaki "Purely Functional Data Structures") The `balance` function currently performs several unnecessary tests: when e.g. `ins` recurses on the left child, there are no violations on the right child.

    - Split `balance` into `lbalance` and `rbalance` that test for violations of left resp. right child only. Replace calls to `balance` appropriately.

- One of the remaining tests on grandchildren is also unnecessary. Rewrite `ins` so that it never tests the color of nodes not on the search path.

Exercise 8. (*) Implement maps (i.e. write a module for the map signature) based on AVL trees. See `http://en.wikipedia.org/wiki/AVL_tree`.

Chapter 6: Folding and Backtracking

This chapter explores two fundamental programming paradigms in functional programming: **folding** (also known as reduction) and **backtracking**. We begin with the classic `map` and `fold` higher-order functions, examine how they generalize to trees and other data structures, then move on to solving puzzles using backtracking with lists.

The material in this chapter draws from Martin Odersky's "Functional Programming Fundamentals," Ralf Laemmel's "Going Bananas," Graham Hutton's "Programming in Haskell" (Chapter 11 on the Countdown Problem), and Tomasz Wierzbicki's Honey Islands Puzzle Solver.

6.1 Basic Generic List Operations

Functional programming emphasizes identifying common patterns and abstracting them into reusable higher-order functions. Rather than writing similar code repeatedly, we extract the common structure into a single generic function. Let us see how this principle works in practice through two motivating examples.

The `map` Function How do we print a comma-separated list of integers? The `String` module provides a function that joins strings with a separator:

```
val concat : string -> string list -> string
```

But `String.concat` works on strings, not integers. So first, we need to convert numbers into strings:

```
let rec strings_of_ints = function
  | [] -> []
  | hd::tl -> string_of_int hd :: strings_of_ints tl

let comma_sep_ints = String.concat ", " -| strings_of_ints
```

Here is another common task: how do we sort strings from shortest to longest? We can pair each string with its length and then sort by the first component. First, let us compute the lengths:

```
let rec strings_lengths = function
  | [] -> []
  | hd::tl -> (String.length hd, hd) :: strings_lengths tl
```

```
let by_size = List.sort compare -| strings_lengths
```

Now, look carefully at `strings_of_ints` and `strings_lengths`. Do you notice the common structure? Both functions traverse a list and transform each element independently – one applies `string_of_int`, the other applies a function that pairs a string with its length. The recursive structure is identical; only the transformation differs.

This is our cue to *extract the common pattern* into a generic higher-order function. We call it `map`:

```
let rec list_map f = function
  | [] -> []
  | hd::tl -> f hd :: list_map f tl
```

Now we can rewrite our functions more concisely:

```
let comma_sep_ints =
  String.concat ", " -| list_map string_of_int

let by_size =
  List.sort compare -| list_map (fun s -> String.length s, s)
```

The `fold` Function Now let us consider a different kind of pattern. How do we sum all the elements of a list?

```
let rec balance = function
  | [] -> 0
  | hd::tl -> hd + balance tl
```

And how do we multiply all the elements together (perhaps to compute a cumulative ratio)?

```
let rec total_ratio = function
  | [] -> 1.
  | hd::tl -> hd *. total_ratio tl
```

Again, the recursive structure is the same. In both cases, we combine each element with the result of processing the rest of the list. The differences are: (1) what we return for the empty list (the "base case" or "identity element"), and (2) how we combine the head with the recursive result. This pattern is called **folding**:

```
let rec list_fold f base = function
  | [] -> base
  | hd::tl -> f hd (list_fold f base tl)
```

Important: Note that `list_fold f base l` equals `List.fold_right f l base`. The OCaml standard library uses a different argument order, so be careful when using `List.fold_right`.

The key insight is understanding the fundamental difference between `map` and `fold`:

- `map` alters the *contents* of a data structure without changing its shape. The output list has the same length as the input; we merely transform each element.
- `fold` *collapses* a data structure down to a single value, using the structure itself as scaffolding for the computation.

Visually, consider what happens to the list `[a; b; c; d]`:

- `map f` transforms: `[a; b; c; d]` becomes `[f a; f b; f c; f d]` – same structure, different contents
- `fold f accu` collapses: `[a; b; c; d]` becomes `f a (f b (f c (f d accu)))` – structure disappears, single value remains

6.2 Making Fold Tail-Recursive

Our `list_fold` function above is not tail-recursive: it builds up a chain of deferred `f` applications on the call stack. For very long lists, this can cause stack overflow. Can we make folding tail-recursive?

Let us investigate some tail-recursive list functions to find a pattern. Consider reversing a list:

```
let rec list_rev acc = function
  | [] -> acc
  | hd::tl -> list_rev (hd::acc) tl
```

The key technique here is the *accumulator* parameter `acc`. Instead of building up work to do after the recursive call returns, we do the work *before* the recursive call and pass the intermediate result along.

Here is another example – computing an average by tracking both the running sum and the count:

```
let rec average (sum, tot) = function
  | [] when tot = 0. -> 0.
  | [] -> sum /. tot
  | hd::tl -> average (hd +. sum, 1. +. tot) tl
```

Notice how these functions process elements from left to right, threading an accumulator through the computation. This is the pattern of `fold_left`:

```
let rec fold_left f accu = function
  | [] -> accu
  | a::l -> fold_left f (f accu a) l
```

With `fold_left`, expressing our earlier functions becomes straightforward – we hide the accumulator inside the initial value:

```
let list_rev l =
  fold_left (fun t h -> h::t) [] l

let average =
  fold_left (fun (sum, tot) e -> sum +. e, 1. +. tot) (0., 0.)
```

Note that the `average` example is slightly trickier than `list_rev` because we need to track two values (sum and count) rather than one.

Why the names `fold_right` and `fold_left`? The names reflect the associativity of the combining operation:

- `fold_right f` makes `f` **right associative**, like the list constructor `::`: `List.fold_right f [a1; ...; an] b` is `f a1 (f a2 (... (f an b) ...))`
- `fold_left f` makes `f` **left associative**, like function application: `List.fold_left f a [b1; ...; bn]` is `f (... (f (f a b1) b2) ...) bn`

This "backward" structure of `fold_left` can be visualized by comparing the shape of the input list with the shape of the computation tree. The input list has a right-leaning spine (because `::` associates to the right), while `fold_left` produces a computation tree with a left-leaning spine:

```
    Input list              Result computation

        ::                          f
       /  \                        / \
      a    ::                     f   d
          /  \                   / \
         b    ::                f   c
             /  \              / \
            c    ::           f   b
                /  \         / \
               d    []   accu   a
```

This reversal of structure is why `fold_left` naturally reverses lists when the combining operation is `cons`.

Useful Derived Functions Many common list operations can be expressed elegantly using folds. List filtering selects elements satisfying a predicate – naturally expressed using `fold_right` to preserve order:

```
let list_filter p l =
  List.fold_right (fun h t -> if p h then h::t else t) l []
```

When we need a tail-recursive map and can tolerate reversed output, `fold_left` gives us `rev_map`:

```
let list_rev_map f l =
  List.fold_left (fun t h -> f h :: t) [] l
```

6.3 Map and Fold for Trees and Other Structures

The `map` and `fold` patterns are not limited to lists. They apply to any recursive data structure. The key insight is that `map` preserves structure while transforming contents, and `fold` collapses structure into a single value.

Binary Trees Mapping binary trees is straightforward:

```
type 'a btree = Empty | Node of 'a * 'a btree * 'a btree

let rec bt_map f = function
  | Empty -> Empty
  | Node (e, l, r) -> Node (f e, bt_map f l, bt_map f r)

let test = Node
  (3, Node (5, Empty, Empty), Node (7, Empty, Empty))
let _ = bt_map ((+) 1) test
```

A note on terminology: The `map` and `fold` functions we define here preserve and respect the structure of data. They are different from the `map` and `fold` operations you might find in abstract data type container libraries, which often behave more like `List.rev_map` and `List.fold_left` over container elements in arbitrary order. Here we are generalizing `List.map` and `List.fold_right` to other structures.

For binary trees, the most general form of `fold` processes each element together with the partial results already computed for its subtrees:

```
let rec bt_fold f base = function
  | Empty -> base
  | Node (e, l, r) ->
    f e (bt_fold f base l) (bt_fold f base r)
```

Here are two examples showing how `bt_fold` can compute different properties of a tree:

```
let sum_els = bt_fold (fun i l r -> i + l + r) 0
let depth t = bt_fold (fun _ l r -> 1 + max l r) 1 t
```

The first computes the sum of all elements (the combining function adds the current element to the sums of both subtrees). The second computes the depth – we ignore the element value and take the maximum depth of the subtrees, adding 1 for the current level.

More Complex Structures: Expressions Real-world data types often have more than two cases. To demonstrate map and fold for more complex structures, let us recall the expression type from Chapter 3:

```
type expression =
    Const of float
  | Var of string
  | Sum of expression * expression    (* e1 + e2 *)
  | Diff of expression * expression   (* e1 - e2 *)
  | Prod of expression * expression   (* e1 * e2 *)
  | Quot of expression * expression   (* e1 / e2 *)
```

The multitude of cases makes this datatype harder to work with than binary trees. Fortunately, OCaml's *or-patterns* help us handle multiple similar cases together:

```
let rec vars = function
  | Const _ -> []
  | Var x -> [x]
  | Sum (a,b) | Diff (a,b) | Prod (a,b) | Quot (a,b) ->
    vars a @ vars b
```

For a generic `map` and `fold` over expressions, we need to specify behavior for each case. Since there are many cases, we pack all the behaviors into records. This way, we can define default behaviors and then override just the cases we care about:

```
type expression_map = {
  map_const : float -> expression;
  map_var : string -> expression;
  map_sum : expression -> expression -> expression;
  map_diff : expression -> expression -> expression;
  map_prod : expression -> expression -> expression;
  map_quot : expression -> expression -> expression;
}

(*
   Note: In expression_fold, we use 'a instead of expression because
   fold produces values of arbitrary type, not necessarily expressions.
*)
type 'a expression_fold = {
  fold_const : float -> 'a;
  fold_var : string -> 'a;
  fold_sum : 'a -> 'a -> 'a;
  fold_diff : 'a -> 'a -> 'a;
  fold_prod : 'a -> 'a -> 'a;
  fold_quot : 'a -> 'a -> 'a;
}
```

Now we define standard "default" behaviors. The `identity_map` reconstructs the same expression (useful as a starting point when we only want to change one case), and `make_fold` creates a fold where all binary operators behave the same:

```
let identity_map = {
  map_const = (fun c -> Const c);
  map_var = (fun x -> Var x);
  map_sum = (fun a b -> Sum (a, b));
  map_diff = (fun a b -> Diff (a, b));
  map_prod = (fun a b -> Prod (a, b));
  map_quot = (fun a b -> Quot (a, b));
}

let make_fold op base = {
  fold_const = (fun _ -> base);
  fold_var = (fun _ -> base);
  fold_sum = op; fold_diff = op;
  fold_prod = op; fold_quot = op;
}
```

The actual `map` and `fold` functions:

```
let rec expr_map emap = function
  | Const c -> emap.map_const c
  | Var x -> emap.map_var x
  | Sum (a,b) -> emap.map_sum (expr_map emap a) (expr_map emap b)
  | Diff (a,b) -> emap.map_diff (expr_map emap a) (expr_map emap b)
  | Prod (a,b) -> emap.map_prod (expr_map emap a) (expr_map emap b)
  | Quot (a,b) -> emap.map_quot (expr_map emap a) (expr_map emap b)

let rec expr_fold efold = function
  | Const c -> efold.fold_const c
  | Var x -> efold.fold_var x
  | Sum (a,b) -> efold.fold_sum (expr_fold efold a) (expr_fold efold b)
  | Diff (a,b) -> efold.fold_diff (expr_fold efold a) (expr_fold efold b)
  | Prod (a,b) -> efold.fold_prod (expr_fold efold a) (expr_fold efold b)
  | Quot (a,b) -> efold.fold_quot (expr_fold efold a) (expr_fold efold b)
```

Now here is the payoff. Using OCaml's `{record with field = value}` syntax, we can easily customize behaviors for specific uses by starting from the defaults and overriding just what we need:

```
let prime_vars = expr_map
  {identity_map with map_var = fun x -> Var (x ^ "'")}

let subst s =
  let apply x = try List.assoc x s with Not_found -> Var x in
  expr_map {identity_map with map_var = apply}

let vars =
```

```
  expr_fold {(make_fold (@) []) with fold_var = fun x -> [x]}

let size = expr_fold (make_fold (fun a b -> 1 + a + b) 1)

let eval env = expr_fold {
  fold_const = id;
  fold_var = (fun x -> List.assoc x env);
  fold_sum = (+.); fold_diff = (-.);
  fold_prod = ( *.); fold_quot = (/.);
}
```

6.4 Point-Free Programming

In 1977/78, John Backus – the designer of FORTRAN and BNF notation – introduced **FP**, the first *function-level programming* language. This was a radical departure from the prevailing style: rather than manipulating variables and values, programs were built entirely by combining functions. Over the next decade, FP evolved into the **FL** language.

The philosophy behind function-level programming is captured in this quote:

> "Clarity is achieved when programs are written at the function level – that is, by putting together existing programs to form new ones, rather than by manipulating objects and then abstracting from those objects to produce programs." – *The FL Project: The Design of a Functional Language*

This style is sometimes called **point-free** or **tacit** programming, because we never mention the "points" (values) that functions operate on – we only talk about the functions themselves and how they combine.

To write in this style, we need a toolkit of **combinators** – higher-order functions that combine other functions. Here are some common ones, similar to what you will find in the *OCaml Batteries* library:

```
let const x _ = x
let ( |- ) f g x = g (f x)          (* forward composition *)
let ( -| ) f g x = f (g x)          (* backward composition *)
let flip f x y = f y x
let ( *** ) f g = fun (x,y) -> (f x, g y)
let ( &&& ) f g = fun x -> (f x, g x)
let first f x = fst (f x)
let second f x = snd (f x)
let curry f x y = f (x,y)
let uncurry f (x,y) = f x y
```

One way to understand point-free programming is to visualize the flow of computation as a circuit. Values flow through the circuit, being transformed by functions at each node. Cross-sections of the circuit can be represented as tuples of intermediate values.

Consider this simple function that converts a character and an integer to a string:

```
let print2 c i =
  let a = Char.escaped c in
  let b = string_of_int i in
  a ^ b
```

We can visualize this as a circuit: `(c, i)` enters, `c` flows through `Char.escaped`, `i` flows through `string_of_int`, and the results meet at `(^)`. In point-free style, we express this directly:

```
let print2 = curry
  ((Char.escaped *** string_of_int) |- uncurry (^))
```

Here `***` applies two functions in parallel to the components of a pair, `|-` is forward composition, `uncurry` converts a curried function to take a pair, and `curry` converts back.

Why the name "currying"? Converting a C/Pascal-style function (that takes all arguments as a tuple) into one that takes arguments one at a time is called *currying*, after the logician Haskell Brooks Curry. Since OCaml functions naturally take arguments one at a time, we often need `uncurry` to interface with tuple-based operations, and `curry` to convert back.

Another approach to point-free style avoids tuples entirely, using function composition, `flip`, and the **S** combinator:

```
let s x y z = x z (y z)
```

The S combinator allows us to pass one argument to two different functions and combine their results. This can bring a particular argument of a function to the "front" and pass it to another function.

Here is an extended example showing step-by-step transformation of a filter-map function into point-free style:

```
let func2 f g l = List.filter f (List.map g l)
(* Step 1: Recognize that filter-after-map is composition *)
let func2 f g = (-|) (List.filter f) (List.map g)
(* Step 2: Eliminate l by composing with List.map *)
let func2 f = (-|) (List.filter f) -| List.map
(* Step 3: Rewrite without infix notation to see the structure *)
let func2 f = (-|) ((-|) (List.filter f)) List.map
(* Step 4: Use flip to rearrange arguments *)
let func2 f = flip (-|) List.map ((-|) (List.filter f))
(* Step 5: Factor out f using composition *)
let func2 f = (((|-) List.map) -| ((-|) -| List.filter)) f
```

```
(* Step 6: Finally, f disappears (eta-reduction) *)
let func2 = (|-) List.map -| ((-|) -| List.filter)
```

While point-free style can be elegant for simple cases, it can quickly become obscure. Use it judiciously!

6.5 Reductions and More Higher-Order Functions

Mathematics has a convenient notation for sums over intervals: $\sum_{n=a}^{b} f(n)$.

Can we express this in OCaml? The challenge is that OCaml does not have a universal addition operator – `+` works only on integers, `+.` only on floats. So we end up writing two versions:

```
let rec i_sum_fromto f a b =
  if a > b then 0
  else f a + i_sum_fromto f (a+1) b

let rec f_sum_fromto f a b =
  if a > b then 0.
  else f a +. f_sum_fromto f (a+1) b

let pi2_over6 =
  f_sum_fromto (fun i -> 1. /. float_of_int (i*i)) 1 5000
```

(The last example computes an approximation to $\pi^2/6$ using the Basel series.)

The natural generalization is to make the combining operation a parameter:

```
let rec op_fromto op base f a b =
  if a > b then base
  else op (f a) (op_fromto op base f (a+1) b)
```

Collecting Results: `concat_map` Sometimes a function produces not a single result but a *collection* of results. In mathematics, such a function is called a **multifunction** or set-valued function. If we have a multifunction f and want to apply it to every element of a set A, we take the union of all results:

$$f(A) = \bigcup_{p \in A} f(p)$$

When we represent sets as lists, "union" becomes "append". This gives us the extremely useful `concat_map` operation:

```
let rec concat_map f = function
  | [] -> []
  | a::l -> f a @ concat_map f l
```

For better efficiency on long lists, here is a tail-recursive version:

```
let concat_map f l =
  let rec cmap_f accu = function
    | [] -> accu
    | a::l -> cmap_f (List.rev_append (f a) accu) l in
  List.rev (cmap_f [] l)
```

The `concat_map` function is fundamental for backtracking algorithms. We will use it extensively in the puzzle-solving sections below.

All Subsequences of a List A classic example of a function that produces multiple results: given a list, generate all its subsequences (subsets that preserve order). The idea is simple: for each element, we either include it or exclude it.

```
let rec subseqs l =
  match l with
    | [] -> [[]]
    | x::xs ->
      let pxs = subseqs xs in
      List.map (fun px -> x::px) pxs @ pxs
```

Tail-recursively:

```
let rec rmap_append f accu = function
  | [] -> accu
  | a::l -> rmap_append f (f a :: accu) l

let rec subseqs l =
  match l with
    | [] -> [[]]
    | x::xs ->
      let pxs = subseqs xs in
      rmap_append (fun px -> x::px) pxs pxs
```

Permutations and Choices Generating all permutations of a list is another classic combinatorial problem. The key insight is the `interleave` function: given an element `x` and a list, it produces all ways of inserting `x` into the list:

```
let rec interleave x = function
  | [] -> [[x]]                   (* x can only go in one place: by itself *)
  | y::ys ->
    (x::y::ys)                    (* x goes at the front, OR *)
    :: List.map (fun zs -> y::zs) (interleave x ys)  (* x goes somewhere after y *)

let rec perms = function
  | [] -> [[]]                    (* one way to permute empty list: empty list *)
  | x::xs -> concat_map (interleave x) (perms xs)
```

For example, `interleave 1 [2;3]` produces `[[1;2;3]; [2;1;3]; [2;3;1]]` – all positions where 1 can be inserted.

For the Countdown problem below, we will need all non-empty subsequences with all their permutations – that is, all ways of choosing and ordering elements from a list:

```
let choices l = concat_map perms (List.filter ((<>) []) (subseqs l))
```

6.6 Grouping and Map-Reduce

When processing large datasets, it is often useful to organize values by some property – grouping all items with the same key together, then processing each group. This pattern is so common it has a name: **map-reduce** (popularized by Google for distributed computing).

Collecting by Key The first step is to collect elements from an association list, grouping all values that share the same key:

```
let collect l =
  match List.sort (fun x y -> compare (fst x) (fst y)) l with
  | [] -> []                                 (* Start with associations sorted by key *)
  | (k0, v0)::tl ->
    let k0, vs, l = List.fold_left
      (fun (k0, vs, l) (kn, vn) ->           (* Collect values for current key *)
        if k0 = kn then k0, vn::vs, l        (* Same key: add value to current group *)
        else kn, [vn], (k0, List.rev vs)::l) (* New key: save current group, start new *)
      (k0, [v0], []) tl in                   (* Why reverse? To preserve original order *)
    List.rev ((k0, List.rev vs)::l)
```

Now we can group elements by an arbitrary property – we just need to extract the property as the key:

```
let group_by p l = collect (List.map (fun e -> p e, e) l)
```

Reduction (Aggregation) Grouping alone is often not enough – we want to *aggregate* each group into a summary value, like SQL's `SUM`, `COUNT`, or `AVG`. This aggregation step is called **reduction**:

```
let aggregate_by p red base l =
  let ags = group_by p l in
  List.map (fun (k, vs) -> k, List.fold_right red vs base) ags
```

Using the **feed-forward** (pipe) operator `let ( |> ) x f = f x`:

```
let aggregate_by p redf base l =
  group_by p l
  |> List.map (fun (k, vs) -> k, List.fold_right redf vs base)
```

Often it is cleaner to extract both the key and the value we care about upfront, before grouping. Since we first **map** elements into key-value pairs, then group and **reduce**, we call this pattern `map_reduce`:

```
let map_reduce mapf redf base l =
  List.map mapf l
  |> collect
  |> List.map (fun (k, vs) -> k, List.fold_right redf vs base)
```

Map-Reduce Examples Sometimes our mapping function produces multiple key-value pairs per input (for example, when processing documents word by word). For this we use `concat_reduce`, which uses `concat_map` instead of `map`:

```
let concat_reduce mapf redf base l =
  concat_map mapf l
  |> collect
  |> List.map (fun (k, vs) -> k, List.fold_right redf vs base)
```

Example 1: Word histogram. Count how many times each word appears across a collection of documents:

```
let histogram documents =
  let mapf doc =
    Str.split (Str.regexp "[ \t.,;]+") doc
    |> List.map (fun word -> word, 1) in
  concat_reduce mapf (+) 0 documents
```

Example 2: Inverted index. Build an index mapping each word to the list of documents (identified by address) containing it:

```
let cons hd tl = hd::tl

let inverted_index documents =
  let mapf (addr, doc) =
    Str.split (Str.regexp "[ \t.,;]+") doc
    |> List.map (fun word -> word, addr) in
  concat_reduce mapf cons [] documents
```

Example 3: Simple search engine. Once we have an inverted index, we can search for documents containing all of a given set of words. We need set intersection – here implemented for sets represented as sorted lists:

```
let intersect xs ys =                        (* Sets as sorted lists *)
  let rec aux acc = function
    | [], _ | _, [] -> acc
    | (x::xs' as xs), (y::ys' as ys) ->
      let c = compare x y in
      if c = 0 then aux (x::acc) (xs', ys')
      else if c < 0 then aux acc (xs', ys)
```

```
      else aux acc (xs, ys') in
  List.rev (aux [] (xs, ys))
```

Now we can build a simple search function that finds all documents containing every word in a query:

```
let search index words =
  match List.map (flip List.assoc index) words with
  | [] -> []
  | idx::idcs -> List.fold_left intersect idx idcs
```

6.7 Higher-Order Functions for the Option Type

The `option` type is OCaml's way of representing values that might be absent. Rather than using null pointers (a common source of bugs), we explicitly mark possibly-missing values with `Some x` or `None`. Here are some useful higher-order functions for working with options.

First, applying a function to an optional value:

```
let map_option f = function
  | None -> None
  | Some e -> f e
```

Second, mapping a partial function over a list and keeping only the successful results:

```
let rec map_some f = function
  | [] -> []
  | e::l -> match f e with
    | None -> map_some f l
    | Some r -> r :: map_some f l
```

Tail-recursively:

```
let map_some f l =
  let rec maps_f accu = function
    | [] -> accu
    | a::l -> maps_f (match f a with None -> accu
      | Some r -> r::accu) l in
  List.rev (maps_f [] l)
```

6.8 The Countdown Problem Puzzle

Now we turn to solving puzzles, which will showcase the power of backtracking with lists. The **Countdown Problem** is a classic puzzle from a British TV game show:

- Using a given set of numbers and arithmetic operators +, -, *, /, construct an expression with a given value.
- All numbers, including intermediate results, must be positive integers.

- Each source number can be used at most once.

Example: - Source numbers: 1, 3, 7, 10, 25, 50 - Target: 765 - One possible solution: (25-10) * (50+1) = 15 * 51 = 765

This example has 780 different solutions! Changing the target to 831 gives an example with no solutions at all.

Let us develop a solver step by step, starting with the data types.

Data Types

```
type op = Add | Sub | Mul | Div

let apply op x y =
  match op with
  | Add -> x + y
  | Sub -> x - y
  | Mul -> x * y
  | Div -> x / y

let valid op x y =
  match op with
  | Add -> true
  | Sub -> x > y
  | Mul -> true
  | Div -> x mod y = 0

type expr = Val of int | App of op * expr * expr

let rec eval = function
  | Val n -> if n > 0 then Some n else None
  | App (o, l, r) ->
    eval l |> map_option (fun x ->
      eval r |> map_option (fun y ->
      if valid o x y then Some (apply o x y)
      else None))

let rec values = function
  | Val n -> [n]
  | App (_, l, r) -> values l @ values r

let solution e ns n =
  list_diff (values e) ns = [] && is_unique (values e) &&
  eval e = Some n
```

Brute Force Solution Our strategy is to generate all possible expressions from the source numbers, then filter for those that evaluate to the target. To build expressions, we need to split the numbers into two groups (for the left and right operands of an operator).

First, a helper to split a list into two non-empty parts in all possible ways:

```
let split l =
  let rec aux lhs acc = function
    | [] | [_] -> []
    | [y; z] -> (List.rev (y::lhs), [z])::acc
    | hd::rhs ->
      let lhs = hd::lhs in
      aux lhs ((List.rev lhs, rhs)::acc) rhs in
  aux [] [] l
```

We introduce a convenient operator for working with multiple data sources. The "bind" operator `|->` takes a list of values and a function that produces a list from each value, then concatenates all results:

```
let ( |-> ) x f = concat_map f x
```

Now we can generate all expressions from a list of numbers. The structure elegantly expresses the backtracking search:

```
let combine l r =                          (* Combine two expressions using each operator *
  List.map (fun o -> App (o, l, r)) [Add; Sub; Mul; Div]

let rec exprs = function
  | [] -> []                               (* No expressions from empty list *)
  | [n] -> [Val n]                         (* Single number: just Val n *)
  | ns ->
    split ns |-> (fun (ls, rs) ->          (* For each way to split numbers... *)
      exprs ls |-> (fun l ->               (* ...for each expression l from left half... *
        exprs rs |-> (fun r ->             (* ...for each expression r from right half... *
          combine l r)))                   (* ...produce all l op r combinations *)
```

Read the nested `|->` as "for each ... for each ... for each ...". This is the essence of backtracking: we explore all combinations systematically.

Finally, to find solutions, we try all choices of source numbers (all non-empty subsets with all orderings) and filter for expressions that evaluate to the target:

```
let guard n =
  List.filter (fun e -> eval e = Some n)

let solutions ns n =
  choices ns |-> (fun ns' ->
    exprs ns' |> guard n)
```

Optimization: Fuse Generation with Testing The brute force approach generates many invalid expressions (like `5 - 7` which gives a negative result, or `5 / 3` which is not an integer). We can do better by *fusing* generation with evaluation: instead of generating an expression and then checking if it is valid, we track the value alongside the expression and only generate valid subexpressions.

The key insight is to work with pairs `(e, eval e)` so that only valid subexpressions are ever generated:

```
let combine' (l, x) (r, y) =
  [Add; Sub; Mul; Div]
  |> List.filter (fun o -> valid o x y)
  |> List.map (fun o -> App (o, l, r), apply o x y)

let rec results = function
  | [] -> []
  | [n] -> if n > 0 then [Val n, n] else []
  | ns ->
    split ns |-> (fun (ls, rs) ->
      results ls |-> (fun lx ->
        results rs |-> (fun ry ->
          combine' lx ry)))

let solutions' ns n =
  choices ns |-> (fun ns' ->
    results ns'
    |> List.filter (fun (e, m) -> m = n)
    |> List.map fst)                          (* Discard memorized values *)
```

Eliminating Symmetric Cases We can further improve performance by observing that addition and multiplication are commutative: `3 + 5` and `5 + 3` give the same result. Similarly, multiplying by 1 or adding/subtracting 0 are useless. We can eliminate these redundancies by strengthening the validity predicate:

```
let valid op x y =
  match op with
  | Add -> x <= y
  | Sub -> x > y
  | Mul -> x <= y && x <> 1 && y <> 1
  | Div -> x mod y = 0 && y <> 1
```

This eliminates symmetrical solutions on the *semantic* level (based on values) rather than the *syntactic* level (based on expression structure). This approach is both easier to implement and more effective at pruning the search space.

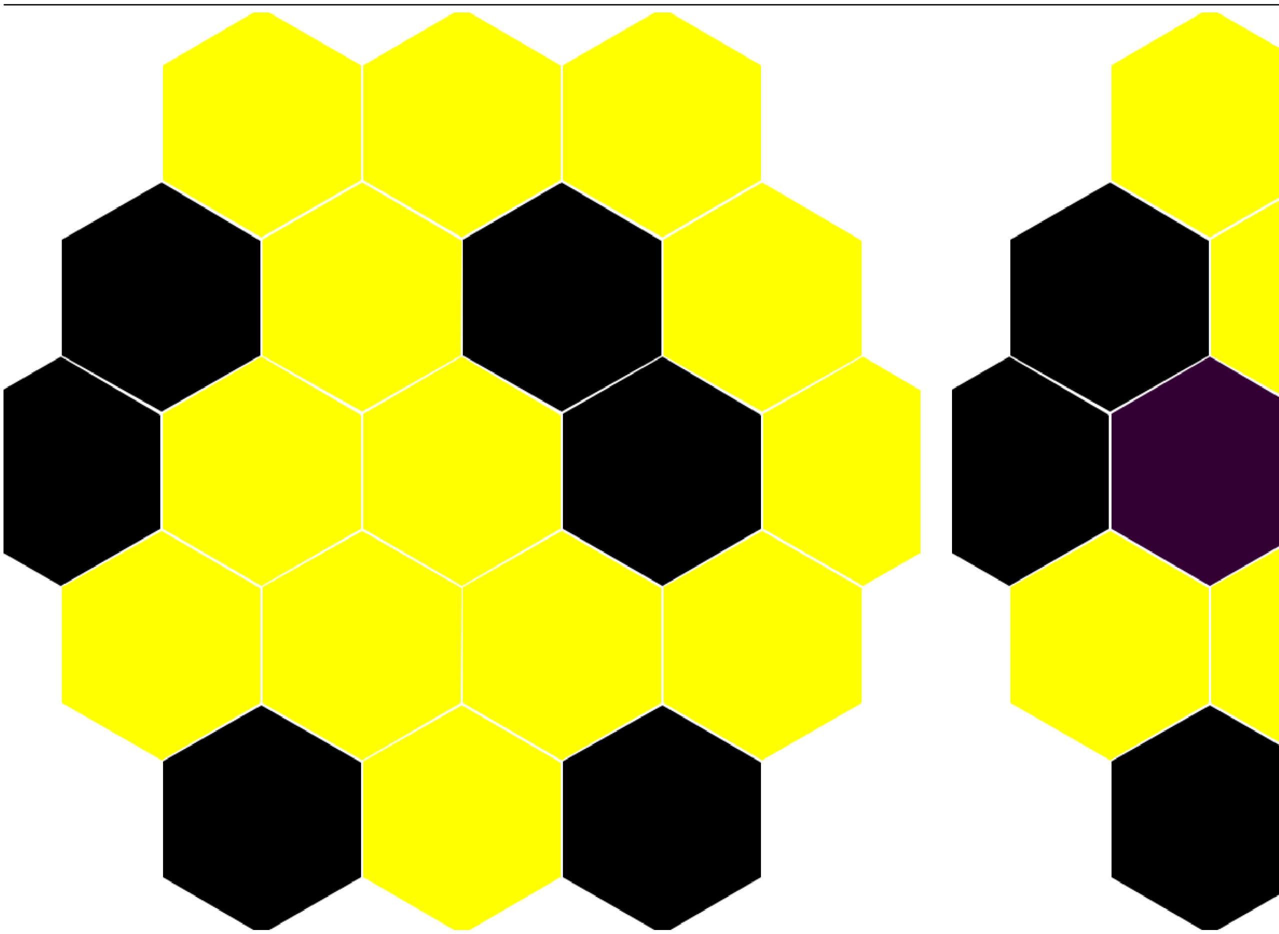
6.9 The Honey Islands Puzzle

Now let us tackle a different kind of puzzle that requires more sophisticated backtracking.

Be a bee! Imagine a honeycomb where you need to eat honey from certain cells to prevent the remaining honey from going sour. Sourness spreads through contact, so you want to divide the honey into isolated "islands" – each small enough that it will be consumed before spoiling.

More precisely: given a honeycomb with some cells initially marked black (empty), mark additional cells as empty so that the remaining (unmarked) cells form exactly `num_islands` disconnected components, each with exactly `island_size` cells.

Task: 3 islands × 3 cells



In the solution, yellow cells contain honey, black cells were initially empty, and purple cells are the newly “eaten” cells that separate the honey into 3 islands of 3 cells each.

Representing the Honeycomb We represent cells using Cartesian coordinates. The honeycomb structure means that valid cells satisfy certain parity and boundary constraints.

```
type cell = int * int                    (* Cartesian coordinates *)

module CellSet =                         (* Store cells in sets for efficient membership
  Set.Make (struct type t = cell let compare = compare end)

type task = {                            (* For board size N, coordinates *)
  board_size : int;                      (* range from (-2N, -N) to (2N, N) *)
  num_islands : int;                     (* Required number of islands *)
  island_size : int;                     (* Required cells per island *)
  empty_cells : CellSet.t;               (* Initially empty cells *)
}

let cellset_of_list l =                  (* Convert list to set (inverse of CellSet.elem
  List.fold_right CellSet.add l CellSet.empty
```

Neighborhood: In a honeycomb, each cell has up to 6 neighbors. We filter out neighbors that are outside the board or already eaten:

```
let even x = x mod 2 = 0

let inside_board n eaten (x, y) =
  even x = even y && abs y <= n &&
  abs x + abs y <= 2*n &&
  not (CellSet.mem (x, y) eaten)

let neighbors n eaten (x, y) =
  List.filter
    (inside_board n eaten)
    [x-1,y-1; x+1,y-1; x+2,y;
     x+1,y+1; x-1,y+1; x-2,y]
```

Building the honeycomb: We generate all valid honey cells by iterating over the coordinate range and filtering:

```
let honey_cells n eaten =
  fromto (-2*n) (2*n) |-> (fun x ->
    fromto (-n) n |-> (fun y ->
     pred_guard (inside_board n eaten)
        (x, y)))
```

Drawing Honeycombs To visualize the honeycomb, we generate colored polygons. Each cell is drawn as a hexagon by placing 6 points evenly spaced on a circumcircle:

```
let draw_honeycomb ~w ~h task eaten =
  let i2f = float_of_int in
  let nx = i2f (4 * task.board_size + 2) in
  let ny = i2f (2 * task.board_size + 2) in
  let radius = min (i2f w /. nx) (i2f h /. ny) in
  let x0 = w / 2 in
  let y0 = h / 2 in
  let dx = (sqrt 3. /. 2.) *. radius +. 1. in  (* Distance between *)
  let dy = (3. /. 2.) *. radius +. 2. in       (* (x,y) and (x+1,y+1) *)
  let draw_cell (x, y) =
    Array.init 7                               (* Draw a closed polygon *)
      (fun i ->                                (* with 6 points evenly *)
        let phi = float_of_int i *. pi /. 3. in   (* spaced on circumcircle *)
        x0 + int_of_float (radius *. sin phi +. float_of_int x *. dx),
        y0 + int_of_float (radius *. cos phi +. float_of_int y *. dy)) in
  let honey =
    honey_cells task.board_size (CellSet.union task.empty_cells
                                   (cellset_of_list eaten))
    |> List.map (fun p -> draw_cell p, (255, 255, 0)) in   (* Yellow cells *)
  let eaten = List.map
    (fun p -> draw_cell p, (50, 0, 50)) eaten in           (* Purple: eaten *)
  let old_empty = List.map
    (fun p -> draw_cell p, (0, 0, 0))                      (* Black: empty *)
    (CellSet.elements task.empty_cells) in
  honey @ eaten @ old_empty
```

Drawing to SVG: We can render the polygons to an SVG image file:

```
let draw_to_svg file ~w ~h ?title ?desc curves =
  let f = open_out file in
  Printf.fprintf f "<?xml version=\"1.0\" standalone=\"no\"?>
<!DOCTYPE svg PUBLIC \"-//W3C//DTD SVG 1.1//EN\"
  \"http://www.w3.org/Graphics/SVG/1.1/DTD/svg11.dtd\">
<svg width=\"%d\" height=\"%d\" viewBox=\"0 0 %d %d\"
    xmlns=\"http://www.w3.org/2000/svg\" version=\"1.1\">
" w h w h;
  (match title with None -> ()
  | Some title -> Printf.fprintf f "  <title>%s</title>\n" title);
  (match desc with None -> ()
  | Some desc -> Printf.fprintf f "  <desc>%s</desc>\n" desc);
  let draw_shape (points, (r, g, b)) =
    uncurry (Printf.fprintf f "  <path d=\"M %d %d") points.(0);
    Array.iteri (fun i (x, y) ->
```

```
      if i > 0 then Printf.fprintf f " L %d %d" x y) points;
    Printf.fprintf f "\"\n       fill=\"rgb(%d, %d, %d)\" stroke-width=\"3\" />\n"
      r g b in
  List.iter draw_shape curves;
  Printf.fprintf f "</svg>%!"
```

Drawing to screen: We can also draw interactively using the `Graphics` library. In the toplevel, load it with `#load "graphics.cma";;`. When compiling, provide `graphics.cma` to the command.

```
let draw_to_screen ~w ~h curves =
  Graphics.open_graph (" " ^ string_of_int w ^ "x" ^ string_of_int h);
  Graphics.set_color (Graphics.rgb 50 50 0);    (* Brown background *)
  Graphics.fill_rect 0 0 (Graphics.size_x ()) (Graphics.size_y ());
  List.iter (fun (points, (r, g, b)) ->
    Graphics.set_color (Graphics.rgb r g b);
    Graphics.fill_poly points) curves;
  if Graphics.read_key () = 'q'                (* Wait so solutions can be seen *)
  then failwith "User interrupted finding solutions.";
  Graphics.close_graph ()
```

Testing Correctness Before generating solutions, let us write code to *test* whether a proposed solution is correct. We walk through each island counting its cells using depth-first search: having visited everything reachable in one direction, we check whether any unvisited cells remain.

```
let check_correct n island_size num_islands empty_cells =
  let honey = honey_cells n empty_cells in

  let rec check_board been_islands unvisited visited =
    match unvisited with
    | [] -> been_islands = num_islands
    | cell::remaining when CellSet.mem cell visited ->
        check_board been_islands remaining visited     (* Keep looking *)
    | cell::remaining (* when not visited *) ->
        let (been_size, unvisited, visited) =
          check_island cell                            (* Visit another island *)
            (1, remaining, CellSet.add cell visited) in
        been_size = island_size
        && check_board (been_islands+1) unvisited visited

  and check_island current state =
    neighbors n empty_cells current
    |> List.fold_left                                  (* Walk into each direction *)
      (fun (been_size, unvisited, visited as state)
        neighbor ->
        if CellSet.mem neighbor visited then state
```

```
        else
          let unvisited = remove neighbor unvisited in
          let visited = CellSet.add neighbor visited in
          let been_size = been_size + 1 in
          check_island neighbor
            (been_size, unvisited, visited))
      state in                                        (* Initial been_size is 1 *)

  check_board 0 honey empty_cells
```

Multiple Results per Step: concat_fold When processing lists, sometimes each step can produce multiple results (not just one as in `fold_left`, or many independent ones as in `concat_map`). We need a hybrid: process elements sequentially like `fold_left`, but allow multiple results at each step, collecting all the final states.

This is `concat_fold`:

```
let rec concat_fold f a = function
  | [] -> [a]
  | x::xs ->
    f x a |-> (fun a' -> concat_fold f a' xs)
```

Generating Solutions The key insight is that we can transform the *testing* code into *generation* code by:

1. Passing around the current partial solution (the `eaten` list)
2. Returning results in a list (empty list means no solutions from this path)
3. At each neighbor cell, exploring *both* possibilities: eating it (adding to `eaten`) or keeping it as honey (continuing to walk the island)

```
let find_to_eat n island_size num_islands empty_cells =
  let honey = honey_cells n empty_cells in

  let rec find_board been_islands unvisited visited eaten =
    match unvisited with
    | [] ->
      if been_islands = num_islands then [eaten] else []
    | cell::remaining when CellSet.mem cell visited ->
      find_board been_islands remaining visited eaten
    | cell::remaining (* when not visited *) ->
      find_island cell
        (1, remaining, CellSet.add cell visited, eaten)
      |->                                             (* Concatenate solutions *)
      (fun (been_size, unvisited, visited, eaten) ->
        if been_size = island_size
        then find_board (been_islands+1)
```

```
              unvisited visited eaten
       else [])

  and find_island current state =
    neighbors n empty_cells current
    |> concat_fold                                       (* Multiple results *)
        (fun neighbor
          (been_size, unvisited, visited, eaten as state) ->
          if CellSet.mem neighbor visited then [state]
          else
            let unvisited = remove neighbor unvisited in
            let visited = CellSet.add neighbor visited in
            (been_size, unvisited, visited,
             neighbor::eaten)::
              (* solutions where neighbor is honey *)
            find_island neighbor
              (been_size+1, unvisited, visited, eaten))
        state in

  find_board 0 honey empty_cells []
```

Optimizations The brute-force generation explores far too many possibilities. The key optimization principle is: **fail (drop solution candidates) as early as possible**.

Instead of blindly exploring all choices, we add guards to prune branches that cannot lead to solutions:

- Do not try to eat more cells if we have already eaten enough
- Do not add more cells to an island that is already full
- Track exactly how many cells still need to be eaten

```
type state = {
  been_size: int;                            (* Honey cells in current island *)
  been_islands: int;                         (* Islands visited so far *)
  unvisited: cell list;                      (* Cells to visit *)
  visited: CellSet.t;                        (* Already visited *)
  eaten: cell list;                          (* Current solution candidate *)
  more_to_eat: int;                          (* Remaining cells to eat *)
}

let rec visit_cell s =
  match s.unvisited with
  | [] -> None
  | c::remaining when CellSet.mem c s.visited ->
    visit_cell {s with unvisited=remaining}
  | c::remaining (* when c not visited *) ->
```

```
    Some (c, {s with
      unvisited=remaining;
      visited = CellSet.add c s.visited})

let eat_cell c s =
  {s with eaten = c::s.eaten;
    visited = CellSet.add c s.visited;
    more_to_eat = s.more_to_eat - 1}

let keep_cell c s =                         (* c is actually unused *)
  {s with been_size = s.been_size + 1;
    visited = CellSet.add c s.visited}

let fresh_island s =                        (* Increment been_size at start of find_island *
  {s with been_size = 0;
    been_islands = s.been_islands + 1}

let init_state unvisited more_to_eat = {
  been_size = 0; been_islands = 0;
  unvisited; visited = CellSet.empty;
  eaten = []; more_to_eat;
}
```

The optimized island loop only tries actions that make sense:

```
  and find_island current s =
    let s = keep_cell current s in
    neighbors n empty_cells current
    |> concat_fold
        (fun neighbor s ->
          if CellSet.mem neighbor s.visited then [s]
          else
            let choose_eat =                (* Guard against failed actions *)
              if s.more_to_eat = 0 then []
              else [eat_cell neighbor s]
            and choose_keep =
              if s.been_size >= island_size then []
              else find_island neighbor s in
            choose_eat @ choose_keep)
        s in
  (* Finally, compute the required eaten cells and start searching *)
  let cells_to_eat =
    List.length honey - island_size * num_islands in
  find_board (init_state honey cells_to_eat)
```

6.10 Constraint-Based Puzzles

Many puzzles can be understood in terms of **constraint satisfaction**:

1. The puzzle defines the *general form* of solutions (what variables need values)
2. The puzzle specifies *constraints* that valid solutions must satisfy

For example, in Sudoku, the variables are the 81 cells, each with domain $\{1,\ldots,9\}$, and the constraints require each row, column, and 3x3 box to contain all digits exactly once.

In the Honey Islands puzzle, we could view each cell as a variable with domain {Honey, Empty}. The constraints specify which cells must be empty initially, and the requirement of forming a specific number and size of connected components.

Finite Domain Constraint Programming **Constraint propagation** is a powerful technique for solving such puzzles efficiently. The key idea is to track *sets of possible values* for each variable and systematically eliminate impossibilities:

1. **Initialize:** For each variable, start with the full set of possible values (its domain). The current "partial solution" is this collection of sets.
2. **Propagate and split:** Repeat until all variables have exactly one value:
   - (a) **Propagate constraints:** If some value for a variable is inconsistent with *all* possible values of related variables, remove it
   - (b) **Prune failures:** If any variable has an empty set of possible values, this partial solution has no completions – abandon it
   - (c) **Split:** Select a variable with multiple possible values. Create new partial solutions by partitioning its possibilities (simplest: try each value separately, or split into "this value" vs "all others")
3. **Extract solutions:** When all variables have single values, we have found a solution.

The efficiency comes from *early pruning*: constraint propagation often eliminates many possibilities without explicitly trying them, dramatically reducing the search space compared to brute-force enumeration.

6.11 Exercises

1. Recall how we generated all subsequences of a list. Find (generate) all:
   - permutations of a list
   - ways of choosing without repetition from a list
   - combinations of K distinct objects chosen from N elements of a list
2. Using folding for the `expression` data type, compute the degree of the corresponding polynomial.

3. Implement simplification of expressions using mapping for the `expression` data type.
4. Express in terms of `fold_left` or `fold_right`:
   - `indexed : 'a list -> (int * 'a) list`, which pairs elements with their indices
   - `concat_fold` as used in Honey Islands
   - Run-length encoding of a list: `encode ['a;'a;'a;'a;'b;'c;'c;'a;'a;'d] = [4,'a; 1,'b; 2,'c; 2,'a; 1,'d]`
5. Write more efficient variants:
   - `list_diff` computing difference of sets represented as sorted lists
   - `is_unique` in constant stack space
6. Write functions `compose` and `perform` that take a list of functions and return their composition:
   - `compose [f1; ...; fn] = x -> f1 (... (fn x)...)`
   - `perform [f1; ...; fn] = x -> fn (... (f1 x)...)`
7. Write a solver for the *Tents Puzzle*.
8. **Robot Squad** (harder): Given a map with walls and lidar readings (8 directions: E, NE, N, NW, W, SW, S, SE) for multiple robots, determine possible robot positions.
9. Write a solver for the *Plinx Puzzle* (does not need to solve all levels, but should handle initial ones).

Chapter 7: Laziness

"Today's lecture is about lazy evaluation. Thank you for coming, goodbye!"

Well, perhaps you have some questions? This chapter explores one of the most elegant ideas in functional programming: lazy evaluation. By deferring computation until results are actually needed, we unlock powerful techniques for working with infinite data structures, solving differential equations symbolically, and building sophisticated stream-processing pipelines.

We will examine different evaluation strategies, implement streams and lazy lists, apply them to power series computation and differential equations, build circular data structures, and develop a sophisticated pipe-based pretty-printer. Along the way, we will see how laziness transforms the way we think about computation itself.

7.1 Evaluation Strategies and Parameter Passing

Evaluation strategy is the order in which expressions are computed – primarily, when arguments are computed. Recall our problems with using *flow control* expressions like `if_then_else` in examples from the lambda-calculus lecture. There are many technical terms describing various evaluation strategies:

Strict evaluation: Arguments are always evaluated completely before the function is applied.

Non-strict evaluation: Arguments are not evaluated unless they are actually used in the evaluation of the function body.

Eager evaluation: An expression is evaluated as soon as it gets bound to a variable.

Lazy evaluation: Non-strict evaluation which avoids repeating computation.

Call-by-value: The argument expression is evaluated, and the resulting value is bound to the corresponding variable in the function (frequently by copying the value into a new memory region).

Call-by-reference: A function receives an implicit reference to a variable used as argument, rather than a copy of its value. In purely functional languages there is no difference between the two strategies, so they are typically described as call-by-value even though implementations use call-by-reference internally for efficiency. Call-by-value languages like C and OCaml support explicit references (objects that refer to other objects), and these can be used to simulate call-by-reference.

Normal order: Start computing function bodies before evaluating their arguments. Do not even wait for arguments if they are not needed.

Call-by-name: Arguments are substituted directly into the function body and then left to be evaluated whenever they appear in the function. This means an argument might be evaluated multiple times if it appears multiple times in the function body.

Call-by-need: If the function argument is evaluated, that value is stored for subsequent uses. This avoids the redundant recomputation that can occur with call-by-name.

Almost all languages do not compute inside the body of an un-applied function, but with curried functions you can pre-compute data before all arguments are provided (recall the `search_bible` example from earlier lectures, where preprocessing happened when the first argument was supplied).

In eager / call-by-value languages we can simulate call-by-name by taking a function to compute the value as an argument instead of the value directly. "Our" languages have a `unit` type with a single value `()` specifically for use as throw-away arguments – we pass `fun () -> expensive_computation` instead of `expensive_computation` directly. Scala has built-in support for call-by-name (i.e. direct, without the need to build argument functions).

ML languages have built-in support for lazy evaluation, while Haskell has built-in support for eager evaluation (to override the default laziness). This reflects the different design philosophies: OCaml defaults to strict evaluation with opt-in laziness, while Haskell defaults to lazy evaluation with opt-in strictness.

7.2 Call-by-name: Streams

Call-by-name is useful not only for implementing flow control. Remember how we struggled to define `if_then_else` as a regular function in the lambda calculus lecture? The problem was that both branches would be evaluated before the function could choose between them. With call-by-name simulation, we can finally make it work:

```
let if_then_else cond e1 e2 =
  match cond with
  | true -> e1 ()
  | false -> e2 ()
```

Here `e1` and `e2` are functions that compute their respective branches only when called. But call-by-name is useful for more than just flow control – it also enables lazy data structures.

Streams are lists with call-by-name tails:

```
type 'a stream = SNil | SCons of 'a * (unit -> 'a stream)
```

The key insight is that the tail is not a stream directly, but a *function* that produces a stream when called. This means the tail is not computed until we actually need it. Reading from a stream into a regular list forces evaluation of the requested elements:

```
let rec stake n = function
  | SCons (a, s) when n > 0 -> a :: (stake (n-1) (s ()))
  | _ -> []
```

Notice how we call `s ()` to get the next portion of the stream. This is where the "lazy" computation happens. Because of this delayed evaluation, streams can easily be infinite:

```
let rec s_ones = SCons (1, fun () -> s_ones)

let rec s_from n =
  SCons (n, fun () -> s_from (n+1))
```

The stream `s_ones` is an infinite sequence of 1s – it refers to itself as its own tail! The stream `s_from n` produces all integers starting from `n`. These definitions would cause infinite loops in a strict language, but with streams, we only compute as much as we request.

7.2.1 Stream Operations

Just as we can define higher-order functions on lists, streams admit similar operations. The key difference is that we must wrap recursive calls in functions to maintain laziness:

```ocaml
let rec smap f = function
  | SNil -> SNil
  | SCons (a, s) -> SCons (f a, fun () -> smap f (s ()))

let rec szip = function
  | SNil, SNil -> SNil
  | SCons (a1, s1), SCons (a2, s2) ->
      SCons ((a1, a2), fun () -> szip (s1 (), s2 ()))
  | _ -> raise (Invalid_argument "szip")
```

Streams can provide scaffolding for recursive algorithms, enabling elegant definitions that would be impossible with strict data structures. Consider the Fibonacci sequence:

```ocaml
let rec sfib =
  SCons (1, fun () -> smap (fun (a,b) -> a+b)
    (szip (sfib, SCons (1, fun () -> sfib))))
```

This remarkably concise definition creates a stream where each element is computed by adding pairs from the current stream and itself shifted by one position. The stream effectively "builds itself" by referring to its own earlier elements:

sfib	1	2	3	5	8	13	...
sfib	1	2	3	5	8	13	...
shifted	1	1	2	3	5	8	...

The `+` operation between corresponding elements produces the next values.

7.2.2 Streams and Input-Output Streams can be used to read from files lazily, but there is a catch – they are less functional than one might expect in the context of input-output effects:

```ocaml
let file_stream name =
  let ch = open_in name in
  let rec ch_read_line () =
    try SCons (input_line ch, ch_read_line)
    with End_of_file -> SNil in
  ch_read_line ()
```

The problem is that reading from a file is a side effect. If you traverse the stream twice, you will not get the same results – the file handle advances with each read. This is why *OCaml Batteries* uses a stream type `enum` for interfacing between various sequence-like data types, with careful documentation about when streams can be safely reused.

The safest way to use streams is in a *linear* or *ephemeral* manner: every value used only once. Streams minimize space consumption at the expense of time for recomputation – if you need to traverse the data multiple times, you will recompute it each time. For data that should be computed once and accessed multiple times, we need lazy lists.

7.3 Lazy Values

Lazy evaluation is more general than call-by-need as any value can be lazy, not only a function parameter. While streams give us call-by-name semantics (recomputing on each access), lazy values give us call-by-need semantics (computing once and caching the result).

A *lazy value* is a value that "holds" an expression until its result is needed, and from then on it "holds" the result. It is also called a *suspension*. If it holds the expression (not yet evaluated), it is called a *thunk* – a placeholder waiting to become a real value.

In OCaml, we build lazy values explicitly using the `lazy` keyword. In Haskell, all values are lazy by default, but functions can have call-by-value parameters which "need" (force evaluation of) the argument.

To create a lazy value: `lazy expr` – where `expr` is the suspended computation. The expression `expr` is not evaluated when the lazy value is created; it is stored for later.

There are two ways to use a lazy value. Be careful to understand when the result is computed! - In expressions: `Lazy.force l_expr` – explicitly forces evaluation - In patterns: `match l_expr with lazy v -> ...` – forces evaluation during pattern matching - Syntactically `lazy` behaves like a data constructor, which is why it can appear in patterns.

7.3.1 Lazy Lists

Lazy lists are the "memoizing" version of streams. Instead of a function that recomputes the tail each time, we use a lazy value that computes it once:

```
type 'a llist = LNil | LCons of 'a * 'a llist Lazy.t
```

The tail is of type `'a llist Lazy.t` – a lazy value that will produce the rest of the list when forced. Reading from a lazy list into a regular list forces evaluation of just the elements we need:

```
let rec ltake n = function
  | LCons (a, lazy l) when n > 0 -> a :: (ltake (n-1) l)
  | _ -> []
```

Notice the `lazy l` pattern – this forces evaluation of the lazy tail and binds the result to `l`. Lazy lists can easily be infinite, just like streams:

```
let rec l_ones = LCons (1, lazy l_ones)

let rec l_from n = LCons (n, lazy (l_from (n+1)))
```

The crucial difference from streams is that lazy lists support "read once, access multiple times" semantics. Once a portion of the list has been computed, subsequent accesses return the cached result:

```
let file_llist name =
  let ch = open_in name in
  let rec ch_read_line () =
    try LCons (input_line ch, lazy (ch_read_line ()))
    with End_of_file -> LNil in
  ch_read_line ()
```

With `file_llist`, you can traverse the resulting list multiple times and get the same data each time (as long as you keep a reference to the head of the list). The file is read lazily, but each line is cached after being read.

7.3.2 Lazy List Operations We can define the familiar higher-order functions on lazy lists. Notice the subtle but important difference from streams – we must use `Lazy.force` to access the lazy tail before passing it to recursive calls:

```
let rec lzip = function
  | LNil, LNil -> LNil
  | LCons (a1, ll1), LCons (a2, ll2) ->
      LCons ((a1, a2), lazy (
        lzip (Lazy.force ll1, Lazy.force ll2)))
  | _ -> raise (Invalid_argument "lzip")

let rec lmap f = function
  | LNil -> LNil
  | LCons (a, ll) ->
    LCons (f a, lazy (lmap f (Lazy.force ll)))
```

Using these operations, we can define the factorial sequence in a beautifully self-referential way:

```
let posnums = l_from 1

let rec lfact =
  LCons (1, lazy (lmap (fun (a,b) -> a*b)
                   (lzip (lfact, posnums))))
```

This produces: 1, 1, 2, 6, 24, 120, ... The definition is elegant: each factorial is the product of the previous factorial and the corresponding positive integer. The lazy list `lfact` refers to itself to get the previous factorials!

lfact	1	1	2	6	24	120	...
lfact	1	1	2	6	24	120	...
posnums	1	2	3	4	5	6	...

The `*` operation between corresponding elements produces the next values.

7.4 Power Series and Differential Equations

This section presents a fascinating application of lazy lists: computing power series and solving differential equations symbolically. The differential equations idea is due to Henning Thielemann, and demonstrates the expressive power of lazy evaluation.

The expression $P(x) = \sum_{i=0}^{n} a_i x^i$ defines a polynomial when $n < \infty$ and a power series when $n = \infty$. We can represent both as lazy lists of coefficients $[a_0; a_1; a_2; \ldots]$.

If we define:

```
let rec lfold_right f l base =
  match l with
    | LNil -> base
    | LCons (a, lazy l) -> f a (lfold_right f l base)
```

then we can compute polynomials using Horner's method. Horner's method evaluates polynomials efficiently by factoring out powers of x: instead of computing $a_0 + a_1 x + a_2 x^2 + \ldots$, we compute $a_0 + x(a_1 + x(a_2 + \ldots))$:

```
let horner x l =
  lfold_right (fun c sum -> c +. x *. sum) l 0.
```

But this will not work for infinite power series! Two natural questions arise: - Does it make sense to compute the value at x of a power series? - Does it make sense to fold an infinite list?

The answer to both is "yes, sometimes." If the power series converges for $x > 1$, then when the elements a_n get small, the remaining sum $\sum_{i=n}^{\infty} a_i x^i$ is also small. We can truncate the computation when the contribution becomes negligible.

The problem is that `lfold_right` falls into an infinite loop on infinite lists – it tries to reach the end before computing anything. We need call-by-name / call-by-need semantics for the argument function `f`, so it can decide to stop early:

```
let rec lazy_foldr f l base =
  match l with
    | LNil -> base
    | LCons (a, ll) ->
      f a (lazy (lazy_foldr f (Lazy.force ll) base))
```

Now we need a stopping condition in the Horner algorithm step. We stop when the coefficient becomes small enough that further terms are negligible:

```
let lhorner x l =                          (* This is a bit of a hack: *)
  let upd c sum =                          (* we hope to "hit" the interval (0, epsilon]. *)
```

```
    if c = 0. || abs_float c > epsilon_float
    then c +. x *. Lazy.force sum
    else 0. in                               (* Stop when c is tiny but nonzero. *)
  lazy_foldr upd l 0.

let inv_fact = lmap (fun n -> 1. /. float_of_int n) lfact
let e = lhorner 1. inv_fact
```

The `inv_fact` list contains $[1/0!; 1/1!; 1/2!; \ldots]$, which is the power series for e^x. Evaluating `lhorner 1. inv_fact` computes $e^1 = e$.

7.4.1 Power Series / Polynomial Operations To work with power series, we need to define arithmetic operations on lazy lists of coefficients. For floating-point coefficients, we first need a float-based version of positive numbers:

```
let rec l_from_f n = LCons (n, lazy (l_from_f (n +. 1.)))
let posnums_f = l_from_f 1.

(* Unary negation for series *)
let (~-:) = lmap (fun x -> -.x)
```

Now we can define the basic arithmetic operations on power series. Addition and subtraction work coefficient-wise:

```
let rec add xs ys =
  match xs, ys with
    | LNil, _ -> ys
    | _, LNil -> xs
    | LCons (x,xs), LCons (y,ys) ->
      LCons (x +. y, lazy (add (Lazy.force xs) (Lazy.force ys)))

let rec sub xs ys =
  match xs, ys with
    | LNil, _ -> lmap (fun x -> -.x) ys
    | _, LNil -> xs
    | LCons (x,xs), LCons (y,ys) ->
      LCons (x -. y, lazy (add (Lazy.force xs) (Lazy.force ys)))

let scale s = lmap (fun x -> s *. x)

let rec shift n xs =
  if n = 0 then xs
  else if n > 0 then LCons (0., lazy (shift (n-1) xs))  (* Multiply by x^n *)
  else match xs with                                    (* Divide by x^|n| *)
    | LNil -> LNil
    | LCons (0., lazy xs) -> shift (n+1) xs
    | _ -> failwith "shift: fractional division"
```

```
(* Multiplication uses the convolution formula *)
let rec mul xs = function
  | LNil -> LNil
  | LCons (y, ys) ->
    add (scale y xs) (LCons (0., lazy (mul xs (Lazy.force ys))))

(* Division is like long division of polynomials *)
let rec div xs ys =
  match xs, ys with
  | LNil, _ -> LNil
  | LCons (0., xs'), LCons (0., ys') ->   (* Both start with zero: cancel x *)
    div (Lazy.force xs') (Lazy.force ys')
  | LCons (x, xs'), LCons (y, ys') ->
    let q = x /. y in                     (* Leading coefficient of quotient *)
    LCons (q, lazy (div (sub (Lazy.force xs')
                            (scale q (Lazy.force ys'))) ys))
  | LCons _, LNil -> failwith "div: division by zero"

(* Integration: integral of a_0 + a_1*x + a_2*x^2 + ...
   is c + a_0*x + a_1*x^2/2 + a_2*x^3/3 + ... *)
let integrate c xs =
  LCons (c, lazy (lmap (uncurry (/.)) (lzip (xs, posnums_f))))

let ltail = function
  | LNil -> invalid_arg "ltail"
  | LCons (_, lazy tl) -> tl

(* Differentiation: derivative of a_0 + a_1*x + a_2*x^2 + ...
   is a_1 + 2*a_2*x + 3*a_3*x^2 + ... *)
let differentiate xs =
  lmap (uncurry ( *.)) (lzip (ltail xs, posnums_f))
```

7.4.2 Differential Equations Now for the remarkable part: we can solve differential equations by representing the solutions as power series! Consider the differential equations for sine and cosine:

$$\frac{d\sin x}{dx} = \cos x, \quad \frac{d\cos x}{dx} = -\sin x, \quad \sin 0 = 0, \quad \cos 0 = 1$$

We will solve the corresponding integral equations. Why integral equations rather than differential equations? Because integration gives us a way to build up the solution coefficient by coefficient, starting from the initial conditions.

Our first attempt might be to define them by direct recursion:

```
let (~-:) = lmap (fun x -> -.x)  (* Unary negation for series *)
```

```
let rec sin = integrate (of_int 0) cos
and cos = integrate (of_int 1) (~-:sin)
```

Unfortunately this fails with: `Error: This kind of expression is not allowed as right-hand side of 'let rec'`

The problem is that OCaml's `let rec` requires the right-hand side to be a "static" value – something like a function or a data constructor applied to arguments. Even changing the second argument of `integrate` to call-by-need does not help, because OCaml cannot represent the values that `sin` and `cos` refer to at the point of their definition.

The solution is to inline a bit of `integrate` so that OCaml knows how to start building the recursive structure. We provide the first coefficient explicitly:

```
let integ xs = lmap (uncurry (/.)) (lzip (xs, posnums_f))

let rec sin = LCons (of_int 0, lazy (integ cos))
and cos = LCons (of_int 1, lazy (integ (~-:sin)))
```

Now the `let rec` works because each right-hand side is just `LCons` applied to a value and a lazy expression. The lazy expressions are not evaluated during the definition, so there is no problem with the mutual recursion. When we force the lazy tails, the computation proceeds coefficient by coefficient.

The complete example would look much more elegant in Haskell, where all values are lazy by default – we would not need the explicit `LCons` and `lazy` wrappers.

Although this approach is not limited to linear equations, equations like Lotka-Volterra or Lorentz are not "solvable" this way – the computed coefficients quickly grow instead of quickly falling, so the series does not converge well.

Drawing functions work like in the previous lecture, but with open curves:

```
let plot_1D f ~w ~scale ~t_beg ~t_end =
  let dt = (t_end -. t_beg) /. of_int w in
  Array.init w (fun i ->
    let y = lhorner (dt *. of_int i) f in
    i, to_int (scale *. y))
```

7.5 Arbitrary Precision Computation

Putting together the power series computation with floating-point numbers reveals drastic numerical errors for large x. There are two problems: 1. Floating-point numbers have limited precision, so intermediate calculations accumulate errors. 2. We break out of Horner method computations too quickly – the stopping condition based on `epsilon_float` may stop before we have enough precision.

For infinite precision on rational numbers we can use the `nums` library, but it does not help by itself – the stopping condition still causes us to truncate the computation prematurely.

The key insight is that instead of computing a single approximate value, we should generate a *sequence of approximations* to the power series limit at x. Then we can watch the sequence until it converges:

```
let infhorner x l =
  let upd c sum =
    LCons (c, lazy (lmap (fun apx -> c +. x *. apx)
                      (Lazy.force sum))) in
  lazy_foldr upd l (LCons (of_int 0, lazy LNil))
```

The function `infhorner` returns a lazy list of partial sums. Each element is a better approximation than the previous one. Now we need to find where the series has converged to the precision we need:

```
let rec exact f = function           (* We arbitrarily decide that convergence is *)
  | LNil -> assert false             (* when three consecutive results are the same. *)
  | LCons (x0, lazy (LCons (x1, lazy (LCons (x2, _)))))
      when f x0 = f x1 && f x0 = f x2 -> f x0
  | LCons (_, lazy tl) -> exact f tl
```

The function `exact` applies a test function `f` to the approximations and stops when three consecutive results give the same answer. Why three? Because some power series (like those for sine and cosine) have alternating terms, and we want to be sure the result has stabilized.

Draw the pixels of the graph at exact coordinates:

```
let plot_1D f ~w ~h0 ~scale ~t_beg ~t_end =
  let dt = (t_end -. t_beg) /. of_int w in
  let eval = exact (fun y -> to_int (scale *. y)) in
  Array.init w (fun i ->
    let y = infhorner (t_beg +. dt *. of_int i) f in
    i, h0 + eval y)
```

If a power series had every third term contributing (zeros in a regular pattern), we would have to check more terms in the function `exact`. We could also use a different stopping criterion like `f x0 = f x1 && not (x0 =. x1)` (stop when the transformed values match but the raw values are still changing), similar to what we did in `lhorner`.

7.5.1 Example: Nuclear Chain Reaction Consider a nuclear chain reaction where substance A decays into B, which then decays into C. This is a classic problem in nuclear physics. The differential equations are:

$$\frac{dN_A}{dt} = -\lambda_A N_A, \quad \frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B$$

Here λ_A and λ_B are the decay constants, and N_A, N_B are the amounts of each substance. Substance A decays at a rate proportional to its amount. Substance B is produced by A's decay and itself decays into C.

We can solve these equations using the same technique as for sine and cosine:

```
let n_chain ~nA0 ~nB0 ~lA ~lB =
  let rec nA =
    LCons (nA0, lazy (integ (~-.lA *:. nA)))
  and nB =
    LCons (nB0, lazy (integ (~-.lB *:. nB +: lA *:. nA))) in
  nA, nB
```

(See Radioactive decay chain processes for more information.)

7.6 Circular Data Structures: Double-Linked Lists

Without delayed computation, the ability to define data structures with referential cycles is very limited. In a strict language, you cannot create a structure that refers to itself – the reference would have to exist before the structure is created.

Double-linked lists are a classic example of structures with inherent cycles. Even if the list itself is not circular (it has a beginning and an end), each pair of adjacent nodes forms a cycle: node A points forward to node B, and node B points backward to node A:

```
+---------+     +---------+     +---------+     +---------+     +---------+
| DLNil   | <-> |    a1   | <-> |    a2   | <-> |    a3   | <-> | DLNil   |
+---------+     +---------+     +---------+     +---------+     +---------+
```

To represent such structures in OCaml, we need to “break” the cycles by making some links lazy. The backward links will be lazy, allowing us to construct the structure one node at a time:

```
type 'a dllist =
  DLNil | DLCons of 'a dllist Lazy.t * 'a * 'a dllist
```

The type has three components: a lazy backward link, the element, and a (strict) forward link. The backward link is lazy because when we create a node, its predecessor may not exist yet.

We can navigate forward through the list, dropping elements from the front:

```
let rec dldrop n l =
  match l with
    | DLCons (_, x, xs) when n > 0 ->
       dldrop (n-1) xs
    | _ -> l
```

The tricky part is constructing a double-linked list from a regular list. Each cell must know its predecessor, but the predecessor is created first. We use a recursive lazy value to tie the knot:

```
let dllist_of_list l =
  let rec dllist prev l =
    match l with
      | [] -> DLNil
      | x::xs ->
        let rec cell =
          lazy (DLCons (prev, x, dllist cell xs)) in
        Lazy.force cell in
  dllist (lazy DLNil) l
```

The key trick is `let rec cell = lazy (DLCons (prev, x, dllist cell xs))`. The lazy value `cell` refers to itself! When we force `cell`, it creates a `DLCons` node whose forward link (`dllist cell xs`) receives `cell` as the predecessor for the next node. This is only possible because the backward link is lazy – when we create the next node, we do not need to evaluate `cell`, just store a reference to it.

Taking elements going forward is straightforward:

```
let rec dltake n l =
  match l with
    | DLCons (_, x, xs) when n > 0 ->
       x :: dltake (n-1) xs
    | _ -> []
```

Taking elements going backward shows the power of the double-linked structure – we can traverse in either direction:

```
let rec dlbackwards n l =
  match l with
    | DLCons (lazy xs, x, _) when n > 0 ->
      x :: dlbackwards (n-1) xs
    | _ -> []
```

7.7 Input-Output Streams

Let us return to streams and generalize them. The stream type we defined earlier used a throwaway argument to make a suspension:

```
type 'a stream = SNil | SCons of 'a * (unit -> 'a stream)
```

The `unit` argument serves only to delay computation. But what if we take a *real* argument – one that provides input to the stream? This leads to a more powerful abstraction:

```
type ('a, 'b) iostream =
  EOS | More of 'b * ('a -> ('a, 'b) iostream)
```

This is an *interactive* stream: it produces an output value of type `'b`, and when given an input value of type `'a`, produces the rest of the stream. The stream alternates between producing output and consuming input.

```
type 'a istream = (unit, 'a) iostream  (* Input stream produces output when "asked". *)
type 'a ostream = ('a, unit) iostream  (* Output stream consumes provided input. *)
```

The terminology can be confusing. An "input stream" (`istream`) is one that produces output when asked (like reading from a file). An "output stream" (`ostream`) is one that consumes input (like writing to a file). The confusion arises from adapting the *input file / output file* terminology.

The power of this abstraction is that we can compose streams, directing the output of one to the input of another:

```
let rec compose sf sg =
  match sg with
  | EOS -> EOS                                 (* No more output from sg. *)
  | More (z, g) ->
    match sf with
    | EOS -> More (z, fun _ -> EOS)            (* No more input from sf. *)
    | More (y, f) ->
      let update x = compose (f x) (g y) in (* Feed sf's output y to sg. *)
      More (z, update)
```

Think of it as connecting boxes with wires: every box has one incoming wire and one outgoing wire. When composing `sf` and `sg`, the output of `sf` becomes the input of `sg`. Notice that the output stream is "ahead" of the input stream – `sg` can produce its first output `z` before `sf` has produced anything.

7.8 Pipes

The `iostream` type has a limitation: it must alternate strictly between producing output and consuming input. In many real-world scenarios, we need more flexibility: - A transformation might consume several inputs before producing a single output (like computing an average). - A transformation might produce several outputs from a single input (like splitting a string). - A transformation might produce output without needing any input (like a constant source).

Following the Haskell tradition, we call this more flexible data structure a `pipe`:

```
type ('a, 'b) pipe =
  EOP                                  (* End of pipe -- done processing *)
| Yield of 'b * ('a, 'b) pipe          (* Produce output b, then continue *)
| Await of ('a -> ('a, 'b) pipe)       (* Wait for input a, then continue *)
```

A pipe can be in one of three states: finished (`EOP`), ready to produce output

(`Yield`), or waiting for input (`Await`). The key insight is that `Yield` includes the continuation pipe directly (not wrapped in a function), so multiple outputs can be produced in sequence without requiring input. For incremental processing where outputs should be lazy, you would change `Yield` to hold a lazy pipe instead.

Again, we can specialize to input-only and output-only pipes:

```
type 'a ipipe = (unit, 'a) pipe
type void
type 'a opipe = ('a, void) pipe
```

Why `void` rather than `unit`, and why only for `opipe`? Because an output pipe never yields values – if it used `unit` as the output type, it could still yield `()` values. But `void` is an abstract type with no values, making it impossible for an `opipe` to yield anything. This is a type-level guarantee that output pipes only consume.

7.8.1 Pipe Composition Composition of pipes is like "concatenating them in space" or connecting boxes. We plug the output of pipe `pf` into the input of pipe `pg`:

```
let rec compose pf pg =
  match pg with
  | EOP -> EOP                                  (* pg is done -- composition is done. *)
  | Yield (z, pg') -> Yield (z, compose pf pg')   (* pg has output ready -- pass it through
  | Await g ->                                  (* pg needs input -- try to get it from pf. *)
    match pf with
    | EOP -> EOP                                (* pf is done -- no more input for pg. *)
    | Yield (y, pf') -> compose pf' (g y)       (* pf has output -- feed it to pg. *)
    | Await f ->                                (* Both are waiting -- wait for external input.
      let update x = compose (f x) pg in
      Await update

let (>->) pf pg = compose pf pg
```

The `>->` operator lets us chain pipes together like Unix pipes: `source >-> transform >-> sink`.

Appending pipes means "concatenating them in time" rather than in space. When the first pipe finishes, we continue with the second:

```
let rec append pf pg =
  match pf with
  | EOP -> pg                                   (* pf is exhausted -- continue with pg. *)
  | Yield (z, pf') -> Yield (z, append pf' pg)  (* pf has output -- pass it through. *)
  | Await f ->                                  (* pf awaits input -- pass it through. *)
    let update x = append (f x) pg in
    Await update
```

We can also append a list of ready results in front of a pipe. This is useful for producing multiple outputs at once:

```
let rec yield_all l tail =
  match l with
  | [] -> tail
  | x::xs -> Yield (x, yield_all xs tail)
```

Finally, the `iterate` function creates a pipe that repeatedly applies a side-effecting function to its inputs. This is **not functional** (it performs side effects), but it is useful for output:

```
let rec iterate f : 'a opipe =
  Await (fun x -> let () = f x in iterate f)
```

7.9 Example: Pretty-Printing

Now let us apply pipes to a substantial example: pretty-printing. The goal is to print a hierarchically organized document with a limited line width. When a group of text fits on the current line, we keep it together; when it does not fit, we break it across multiple lines.

```
type doc =
  Text of string | Line | Cat of doc * doc | Group of doc
```

The document type has four constructors: - `Text s` – literal text - `Line` – a potential line break (rendered as a space if the group fits, or a newline if it does not) - `Cat (d1, d2)` – concatenation - `Group d` – a group that should be kept together if possible

Some convenient operators for building documents:

```
let (++) d1 d2 = Cat (d1, Cat (Line, d2))
let (!) s = Text s

let test_doc =
  Group (!"Document" ++
          Group (!"First part" ++ !"Second part"))
```

The pretty-printer should produce different outputs depending on the available width:

```
# let () = print_endline (pretty 30 test_doc);;
Document
First part Second part

# let () = print_endline (pretty 20 test_doc);;
Document
First part
Second part
```

```
# let () = print_endline (pretty 60 test_doc);;
Document First part Second part
```

7.9.1 Straightforward Solution Before diving into pipes, let us implement a straightforward recursive solution:

```
let pretty w d =                      (* Allowed width of line w. *)
  let rec width = function            (* Compute total length of subdocument. *)
    | Text z -> String.length z
    | Line -> 1                       (* A line break takes 1 character (space or newline). *
    | Cat (d1, d2) -> width d1 + width d2
    | Group d -> width d in
  let rec format f r = function       (* f: flatten (no breaks)? r: remaining space. *)
    | Text z -> z, r - String.length z
    | Line when f -> " ", r-1         (* Flatten mode: render as space. *)
    | Line -> "\n", w                 (* Break mode: newline, reset remaining to full width.
    | Cat (d1, d2) ->
      let s1, r = format f r d1 in
      let s2, r = format f r d2 in
      s1 ^ s2, r
    | Group d -> format (f || width d <= r) r d  (* Flatten if group fits. *)
  in
  fst (format false w d)              (* Start outside any group (not flattening). *)
```

The `format` function takes a boolean `f` (are we in “flatten” mode?) and the remaining space `r`. When we enter a `Group`, we check if the whole group fits in the remaining space. If so, we format it in flatten mode (all `Line`s become spaces).

7.9.2 Stream-Based Solution The straightforward solution works, but it has a problem: for each group, we compute `width` by traversing the entire subtree, potentially doing redundant work. The stream-based solution processes the document incrementally, computing positions as we go.

First, we define a type for document elements that can carry annotations:

```
type ('a, 'b) doc_e =                 (* Annotated nodes, special for group beginning. *)
  TE of 'a * string | LE of 'a | GBeg of 'b | GEnd of 'a
```

The type parameters `'a` and `'b` allow different annotations for different elements. `GBeg` (group beginning) has a different type because it will eventually carry the end position of the group.

Normalize a subdocument to remove empty groups:

```
let rec norm = function
  | Group d -> norm d
  | Text "" -> None
```

```
  | Cat (Text "", d) -> norm d
  | d -> Some d
```

Generate the stream of document elements by infix traversal:

```
let rec gen = function
  | Text z -> Yield (TE ((),z), EOP)
  | Line -> Yield (LE (), EOP)
  | Cat (d1, d2) -> append (gen d1) (gen d2)
  | Group d ->
    match norm d with
    | None -> EOP
    | Some d ->
      Yield (GBeg (),
             append (gen d) (Yield (GEnd (), EOP)))
```

The next pipe computes the position (character count from the beginning) of each element:

```
let rec docpos curpos =
  Await (function                         (* Input from a doc_e pipe, *)
  | TE (_, z) ->
    Yield (TE (curpos, z),                (* output doc_e annotated with position. *)
           docpos (curpos + String.length z))
  | LE _ ->                               (* Spaces and line breaks: 1 character. *)
    Yield (LE curpos, docpos (curpos + 1))
  | GBeg _ ->                             (* Groups themselves have no width. *)
    Yield (GBeg curpos, docpos curpos)
  | GEnd _ ->
    Yield (GEnd curpos, docpos curpos))

let docpos = docpos 0                     (* The whole document starts at position 0. *)
```

Now comes the tricky part. We want to annotate each `GBeg` with the position where the group *ends*, so we can decide whether the group fits on the line. But we see `GBeg` before we see `GEnd`! We need to buffer elements until we see the end of each group:

```
let rec grends grstack =
  Await (function
  | TE _ | LE _ as e ->
    (match grstack with
    | [] -> Yield (e, grends [])          (* No groups waiting -- yield immediately. *)
    | gr::grs -> grends ((e::gr)::grs))   (* Inside a group -- buffer the element. *)
  | GBeg _ -> grends ([]::grstack)        (* Start a new group: push empty buffer. *)
  | GEnd endp ->
    match grstack with                    (* End the group on top of stack. *)
    | [] -> failwith "grends: unmatched group end marker"
    | [gr] ->                             (* Outermost group -- yield everything now. *)
```

```
      yield_all
        (GBeg endp::List.rev (GEnd endp::gr))  (* Annotate GBeg with end position. *)
        (grends [])
    | gr::par::grs ->                        (* Nested group -- add to parent's buffer. *)
      let par = GEnd endp::gr @ [GBeg endp] @ par in
      grends (par::grs))                     (* Could use catenable lists for efficiency. *)
```

This works, but it has a problem: we wait until the entire group is processed before yielding anything. For large groups (or groups that exceed the line width), this is wasteful. We can optimize by flushing the buffer when a group clearly exceeds the line width – if we know a group will not fit, there is no need to remember where it ends:

```
type grp_pos = Pos of int | Too_far

let rec grends w grstack =
  let flush tail =                     (* When a group exceeds width w, *)
    yield_all                          (* flush the stack -- yield everything buffered. *)
      (rev_concat_map ~prep:(GBeg Too_far) snd grstack)
      tail in                          (* Mark flushed groups as Too_far. *)
  Await (function
  | TE (curp, _) | LE curp as e ->
    (match grstack with                (* Track beginning position of each group. *)
    | [] -> Yield (e, grends w [])     (* No groups -- yield immediately. *)
    | (begp, _)::_ when curp-begp > w ->
      flush (Yield (e, grends w []))   (* Group too wide -- flush and yield. *)
    | (begp, gr)::grs -> grends w ((begp, e::gr)::grs))  (* Buffer element. *)
  | GBeg begp -> grends w ((begp, [])::grstack)  (* New group: remember start position. *)
  | GEnd endp as e ->
    match grstack with                 (* No longer fail when stack is empty -- *)
    | [] -> Yield (e, grends w [])     (* could have been flushed earlier. *)
    | (begp, _)::_ when endp-begp > w ->
      flush (Yield (e, grends w []))   (* Group exceeded width -- flush. *)
    | [_, gr] ->                       (* Group fits -- annotate with end position. *)
      yield_all
        (GBeg (Pos endp)::List.rev (GEnd endp::gr))
        (grends w [])
    | (_, gr)::(par_begp, par)::grs -> (* Nested group fits -- add to parent. *)
      let par =
        GEnd endp::gr @ [GBeg (Pos endp)] @ par in
      grends w ((par_begp, par)::grs))

let grends w = grends w []             (* Initial stack is empty. *)
```

Finally, the `format` pipe produces the resulting stream of strings. It maintains a stack of booleans indicating which groups are being "flattened" (rendered inline), and the position where the current line would end:

```ocaml
let rec format w (inline, endlpos as st) =  (* inline: stack of "flatten this group?" *)
  Await (function                            (* endlpos: position where line ends *)
  | TE (_, z) -> Yield (z, format w st)      (* Text: output directly. *)
  | LE p when List.hd inline ->
    Yield (" ", format w st)                 (* In flatten mode: line break -> space. *)
  | LE p -> Yield ("\n", format w (inline, p+w))  (* Break mode: newline, update endlpos. *)
  | GBeg Too_far ->                          (* Group too wide -- don't flatten. *)
    format w (false::inline, endlpos)
  | GBeg (Pos p) ->                          (* Group fits if it ends before endlpos. *)
    format w ((p<=endlpos)::inline, endlpos)
  | GEnd _ -> format w (List.tl inline, endlpos))  (* Pop the inline stack. *)

let format w = format w ([false], w)         (* Start with no flattening, full line width. *)
```

Put the pipes together into a complete pipeline:

```
+---------+     +--------+     +----------+     +---------+     +-----------------+
| gen doc| --> |docpos | --> |grends w | --> |format w| --> |iterate print_s |
+---------+     +--------+     +----------+     +---------+     +-----------------+
```

The data flows from left to right: `gen` produces document elements, `docpos` annotates them with positions, `grends` annotates group beginnings with their end positions, `format` decides where to break lines and produces strings, and `iterate print_string` prints the strings.

7.9.3 Factored Solution For maximum flexibility, we can factorize `format` into two parts: one that decides where to break lines (producing annotated document elements), and one that converts those to strings. This allows different line breaking strategies to be plugged in:

```ocaml
(* breaks: decides where to break, outputs annotated doc_e elements *)
let rec breaks w (inline, endlpos as st) =
  Await (function
  | TE _ as e -> Yield (e, breaks w st)           (* Pass through text. *)
  | LE p when List.hd inline ->
    Yield (TE (p, " "), breaks w st)              (* Flatten: convert to space. *)
  | LE p as e -> Yield (e, breaks w (inline, p+w))  (* Break: keep as LE. *)
  | GBeg Too_far as e ->
    Yield (e, breaks w (false::inline, endlpos))
  | GBeg (Pos p) as e ->
    Yield (e, breaks w ((p<=endlpos)::inline, endlpos))
  | GEnd _ as e ->
    Yield (e, breaks w (List.tl inline, endlpos)))

let breaks w = breaks w ([false], w)

(* emit: converts doc_e elements to strings *)
```

```
let rec emit =
  Await (function
  | TE (_, z) -> Yield (z, emit)                    (* Text: output directly. *)
  | LE _ -> Yield ("\n", emit)                      (* Line break: output newline. *)
  | GBeg _ | GEnd _ -> emit)                        (* Group markers: skip. *)

let pretty_print w doc =
  gen doc >-> docpos >-> grends w >-> breaks w >->
  emit >-> iterate print_string
```

Now `breaks` can be replaced with a different strategy (for example, one that adds indentation), and `emit` stays the same. The full pipeline reads like a description of what happens: generate elements, compute positions, annotate groups with their ends, decide where to break, convert to strings, and print.

7.10 Exercises

Exercise 1: My first impulse was to define lazy list functions as follows:

```
let rec wrong_lzip = function
  | LNil, LNil -> LNil
  | LCons (a1, lazy l1), LCons (a2, lazy l2) ->
      LCons ((a1, a2), lazy (wrong_lzip (l1, l2)))
  | _ -> raise (Invalid_argument "lzip")

let rec wrong_lmap f = function
  | LNil -> LNil
  | LCons (a, lazy l) -> LCons (f a, lazy (wrong_lmap f l))
```

What is wrong with these definitions – for which edge cases do they not work as intended?

Exercise 2: Cyclic lazy lists.

1. Implement a function `cycle : 'a list -> 'a llist` that creates a lazy list with elements from a standard list, and the whole list as the tail after the last element from the input list: `[a1; a2; ...; aN]` maps to a cyclic structure where `aN` points back to `a1`. Your function `cycle` can either return `LNil` or fail for an empty list as argument.

2. Note that `inv_fact` from the lecture defines the power series for the $\exp(\cdot)$ function ($\exp(x) = e^x$). Using `cycle` and `inv_fact`, define the power series for $\sin(\cdot)$ and $\cos(\cdot)$, and draw their graphs using helper functions from the lecture script `Lec7.ml`.

Exercise 3: Modify one of the puzzle solving programs (either from the previous lecture or from your previous homework) to work with lazy lists. Implement the necessary higher-order lazy list functions. Check that indeed displaying only the first solution when there are multiple solutions in the result takes shorter than computing solutions by the original program.

Exercise 4: *Hamming's problem.* Generate in increasing order the numbers of the form $2^{a_1}3^{a_2}5^{a_3}\ldots p_k^{a_k}$, that is numbers not divisible by prime numbers greater than the kth prime number.

In the original Hamming's problem posed by Dijkstra, $k = 3$, which is related to regular numbers.

Starter code is available in the lecture script `Lec7.ml`:

```
let rec lfilter f = function
  | LNil -> LNil
  | LCons (n, ll) ->
      if f n then LCons (n, lazy (lfilter f (Lazy.force ll)))
      else lfilter f (Lazy.force ll)

let primes =
  let rec sieve = function
    | LCons(p, nf) ->
        LCons(p, lazy (sieve (sift p (Lazy.force nf))))
    | LNil -> failwith "Impossible! Internal error."
  and sift p = lfilter (fun n -> n mod p <> 0)
  in sieve (l_from 2)

let times ll n = lmap (fun i -> i * n) ll

let rec merge xs ys =
  match xs, ys with
  | LCons (x, lazy xr), LCons (y, lazy yr) ->
      if x < y then LCons (x, lazy (merge xr ys))
      else if x > y then LCons (y, lazy (merge xs yr))
      else LCons (x, lazy (merge xr yr))
  | r, LNil | LNil, r -> r

let hamming k =
  let _pr = ltake k primes in  (* TODO: use primes to generate smooth numbers *)
  let rec h = LCons (1, lazy (
     (* TODO *)h
  )) in h
```

Exercise 5: Modify `format` and/or `breaks` to use just a single number instead of a stack of booleans to keep track of what groups should be inlined.

Exercise 6: Add **indentation** to the pretty-printer for groups: if a group does not fit in a single line, its consecutive lines are indented by a given amount `tab` of spaces deeper than its parent group lines would be. For comparison, let's do several implementations.

1. Modify the straightforward implementation of `pretty`.
2. Modify the first pipe-based implementation of `pretty` by modifying the `format` function.
3. Modify the second pipe-based implementation of `pretty` by modifying the `breaks` function. Recover the positions of elements – the number of characters from the beginning of the document – by keeping track of the growing offset.
4. (Harder) Modify a pipe-based implementation to provide a different style of indentation: indent the first line of a group, when the group starts on a new line, at the same level as the consecutive lines (rather than at the parent level of indentation).

Exercise 7: Write a pipe that takes document elements annotated with linear position, and produces document elements annotated with (line, column) coordinates.

Write another pipe that takes so annotated elements and adds a line number indicator in front of each line. Do not update the column coordinate. Test the pipes by plugging them before the `emit` pipe.

```
1: first line
2: second line, etc.
```

Exercise 8: Write a pipe that consumes document elements `doc_e` and yields the toplevel subdocuments `doc` which would generate the corresponding elements.

You can modify the definition of documents to allow annotations, so that the element annotations are preserved (`gen` should ignore annotations to keep things simple):

```
type 'a doc =
  Text of 'a * string | Line of 'a | Cat of 'a doc * 'a doc | Group of 'a * 'a doc
```

Exercise 9: (Harder) Design and implement a way to duplicate arrows outgoing from a pipe-box, that would memoize the stream, i.e. not recompute everything "upstream" for the composition of pipes. Such duplicated arrows would behave nicely with pipes reading from files.

Chapter 8: Monads

This chapter explores one of functional programming's most powerful abstractions: monads. We begin with list comprehensions as a motivating example, then introduce monadic concepts and examine the monad laws. We explore the monad-plus extension that adds non-determinism, then work through various monad instances including the lazy, list, state, exception, and probability monads. We conclude with monad transformers for combining monads and cooperative lightweight threads for concurrency.

The material draws on several excellent resources: Jeff Newbern's "All About Monads," Martin Erwig and Steve Kollmansberger's "Probabilistic Functional

Programming in Haskell," and Jerome Vouillon's "Lwt: a Cooperative Thread Library."

8.1 List Comprehensions

Recall the somewhat awkward syntax we used in the Countdown Problem example from earlier chapters. The nested callback style, while functional, is hard to read and understand at a glance. The brute-force generation of expressions looked like this:

```
let combine l r =
  List.map (fun o -> App (o, l, r)) [Add; Sub; Mul; Div]

let rec exprs = function
  | [] -> []
  | [n] -> [Val n]
  | ns ->
      split ns |-> (fun (ls, rs) ->
      exprs ls |-> (fun l ->
      exprs rs |-> (fun r ->
      combine l r)))
```

Notice how the nested callbacks pile up: each `|->` introduces another level of indentation. The generate-and-test scheme used similar nesting:

```
let guard p e = if p e then [e] else []

let solutions ns n =
  choices ns |-> (fun ns' ->
  exprs ns' |->
    guard (fun e -> eval e = Some n))
```

The key insight is that we introduced the operator `|->` defined as:

```
let ( |-> ) x f = concat_map f x
```

This pattern of "for each element in a list, apply a function that returns a list, then flatten the results" is so common that many languages provide special syntax for it. We can express such computations much more elegantly with *list comprehensions*, a syntax extension that originated in languages like Haskell and Python.

In older versions of OCaml with Camlp4, list comprehensions were loaded via:

```
#load "dynlink.cma";;
#load "camlp4o.cma";;
#load "Camlp4Parsers/Camlp4ListComprehension.cmo";;
```

With list comprehensions, we can write expressions that read almost like set-builder notation in mathematics:

```
let test = [i * 2 | i <- from_to 2 22; i mod 3 = 0]
```

This reads as: "the list of `i * 2` for each `i` drawn from `from_to 2 22` where `i mod 3 = 0`." The `<-` arrow draws elements from a generator, and conditions filter which elements are kept.

The translation rules that define list comprehension semantics are straightforward:

- `[expr | ]` translates to `[expr]` – the base case, a singleton list
- `[expr | v <- generator; more]` translates to `generator |-> (fun v -> [expr | more])` – draw from a generator, then recurse
- `[expr | condition; more]` translates to `if condition then [expr | more] else []` – filter by a condition

Revisiting Countdown with List Comprehensions Now let us revisit the Countdown Problem code with list comprehensions. The brute-force generation becomes dramatically cleaner – compare this to the deeply nested version above:

```
let rec exprs = function
  | [] -> []
  | [n] -> [Val n]
  | ns ->
      [App (o, l, r) | (ls, rs) <- split ns;
       l <- exprs ls; r <- exprs rs;
       o <- [Add; Sub; Mul; Div]]
```

The intent is immediately clear: we split the numbers, recursively build expressions for left and right parts, and try each operator. The generate-and-test scheme becomes equally elegant:

```
let solutions ns n =
  [e | ns' <- choices ns;
   e <- exprs ns'; eval e = Some n]
```

The guard condition `eval e = Some n` filters out expressions that do not evaluate to the target value.

More List Comprehension Examples List comprehensions shine when expressing combinatorial algorithms. Here is computing all subsequences of a list (note that this generates some intermediate garbage, but the intent is clear):

```
let rec subseqs l =
  match l with
  | [] -> [[]]
  | x::xs -> [ys | px <- subseqs xs; ys <- [px; x::px]]
```

For each element `x`, we recursively compute subsequences of the tail, then for each such subsequence we include both the version without `x` and the version with `x` prepended.

Computing permutations can be done via insertion – inserting an element at every possible position:

```
let rec insert x = function
  | [] -> [[x]]
  | y::ys' as ys ->
      (x::ys) :: [y::zs | zs <- insert x ys']

let rec ins_perms = function
  | [] -> [[]]
  | x::xs -> [zs | ys <- ins_perms xs; zs <- insert x ys]
```

The `insert` function generates all ways to insert `x` into a list. Then `ins_perms` recursively permutes the tail and inserts the head at every position.

Alternatively, we can compute permutations via selection – repeatedly choosing which element comes first:

```
let rec select = function
  | [x] -> [x, []]
  | x::xs -> (x, xs) :: [y, x::ys | y, ys <- select xs]

let rec sel_perms = function
  | [] -> [[]]
  | xs -> [x::ys | x, xs' <- select xs; ys <- sel_perms xs']
```

The `select` function returns all ways to pick one element from a list, along with the remaining elements. Then `sel_perms` chooses a first element and recursively permutes the rest.

8.2 Generalized Comprehensions: Binding Operators

The pattern we saw with list comprehensions is remarkably general. In fact, the same `|->` pattern (applying a function that returns a container, then flattening) works for many types beyond lists. This is the essence of monads.

OCaml 5 introduced **binding operators** that provide a clean, native syntax for such computations. Instead of external syntax extensions like the old Camlp4-based `pa_monad`, we can now define custom `let*` and `let+` operators that integrate naturally with the language.

For the list monad, we define these binding operators:

```
let ( let* ) x f = concat_map f x      (* bind: sequence computations *)
let ( let+ ) x f = List.map f x        (* map: apply pure function *)
let ( and* ) x y = concat_map (fun a -> List.map (fun b -> (a, b)) y) x
let ( and+ ) = ( and* )                (* parallel binding *)
let return x = [x]                     (* inject a value into the monad *)
let fail = []                          (* the empty computation *)
```

The `let*` operator is the key: it sequences computations where each step can produce multiple results. The `and*` operator allows binding multiple values in parallel. With these operators, the expression generation code becomes:

```
let rec exprs = function
  | [] -> []
  | [n] -> [Val n]
  | ns ->
      let* (ls, rs) = split ns in
      let* l = exprs ls in
      let* r = exprs rs in
      let* o = [Add; Sub; Mul; Div] in
      [App (o, l, r)]
```

Each `let*` introduces a binding: the variable on the left is bound to each value produced by the expression on the right, and the computation continues with `in`. This is much more readable than the nested callbacks we started with.

However, the `let*` syntax does not directly support guards (conditions that filter results). If we try to write:

```
let solutions ns n =
  let* ns' = choices ns in
  let* e = exprs ns' in
  eval e = Some n;  (* Error! *)
  e
```

We get a type error: the expression expects a list, but `eval e = Some n` is a boolean. What can we do?

One approach is to explicitly decide whether to return anything:

```
let solutions ns n =
  let* ns' = choices ns in
  let* e = exprs ns' in
  if eval e = Some n then [e] else []
```

But what if we want to check a condition earlier in the computation, or check multiple conditions? We need a general "guard check" function. The key insight is that we can use the monad itself to represent success or failure:

```
let guard p = if p then [()] else []
```

When the condition `p` is true, `guard` returns `[()]` – a list with one element (the unit value). When false, it returns `[]` – an empty list. Now we can use it in a binding:

```
let solutions ns n =
  let* ns' = choices ns in
  let* e = exprs ns' in
  let* () = guard (eval e = Some n) in
```

```
  [e]
```

Why does this work? When the guard succeeds, `let* () = [()]` binds unit and continues. When it fails, `let* () = []` produces no results – the empty list – so the rest of the computation is never reached for that branch. This is exactly the filtering behavior we want!

8.3 Monads

Now we are ready to define monads properly. A **monad** is a polymorphic type `'a monad` (or `'a Monad.t`) that supports at least two operations:

- `bind : 'a monad -> ('a -> 'b monad) -> 'b monad` – sequence two computations, passing the result of the first to the second
- `return : 'a -> 'a monad` – inject a pure value into the monad
- The infix `>>=` is commonly used for `bind`: `let (>>=) a b = bind a b`

The `bind` operation is the heart of the monad: it takes a computation that produces an `'a`, and a function that takes an `'a` and produces a new computation yielding `'b`. The result is a combined computation that yields `'b`.

With OCaml 5's binding operators, we define `let*` as an alias for `bind`:

```
let bind a b = concat_map b a
let return x = [x]
let ( let* ) = bind

let solutions ns n =
  let* ns' = choices ns in
  let* e = exprs ns' in
  let* () = guard (eval e = Some n) in
  return e
```

But why does `guard` look the way it does? Let us examine more carefully:

```
let fail = []
let guard p = if p then return () else fail
```

Steps in monadic computation are composed with `let*` (or `>>=`, which is like `|->` for lists). The key insight is understanding what happens when we bind with an empty list versus a singleton:

- `let* _ = [] in ...` does not produce anything – the continuation is never called, so the computation fails (produces no results)
- `let* _ = [()] in ...` calls the continuation once with `()`, which simply continues the computation unchanged

This is why `guard` works: returning `[()]` means "succeed with unit" and returning `[]` means "fail with no results." The unit value itself is a dummy – we only care whether the list is empty or not.

Throwing away the binding argument is a common pattern. With binding operators, we use `let* () = ...` or `let* _ = ...` to indicate we do not need the bound value:

```
let (>>=) a b = bind a b
let (>>) m f = m >>= (fun _ -> f)
```

The `>>` operator (called "sequence" or "then") is useful when you want to perform a computation for its effect but discard its result.

The Binding Operator Syntax For reference, OCaml 5's binding operators translate as follows:

Source	Translation
`let* x = exp in body`	`bind exp (fun x -> body)`
`let+ x = exp in body`	`map (fun x -> body) exp`
`let* () = exp in body`	`bind exp (fun () -> body)`
`let* x = e1 and* y = e2 in body`	`bind (and* e1 e2) (fun (x, y) -> body)`

The binding operators `let*`, `let+`, `and*`, and `and+` must be defined in scope. These are regular OCaml operators and require no syntax extensions – a significant improvement over the old Camlp4 approach.

Note: For pattern matching in bindings, if the pattern is refutable (can fail to match), the monadic operation should handle the failure appropriately. For example, `let* Some x = e in body` requires a way to handle the `None` case.

8.4 Monad Laws

Not every type with `bind` and `return` operations is a proper monad. A parametric data type is a monad only if its `bind` and `return` operations meet three fundamental axioms:

$$\begin{aligned}
\text{bind } (\text{return } a)\ f &\approx f\ a && \text{(left identity)}\\
\text{bind } a\ (\lambda x.\text{return } x) &\approx a && \text{(right identity)}\\
\text{bind } (\text{bind } a\ (\lambda x.b))\ (\lambda y.c) &\approx \text{bind } a\ (\lambda x.\text{bind } b\ (\lambda y.c)) && \text{(associativity)}
\end{aligned}$$

Let us understand what these laws mean:

- **Left identity**: If you inject a value with `return` and immediately bind it to a function, you get the same result as just applying the function. The `return` operation should not add any extra "effects."
- **Right identity**: If you bind a computation to `return`, you get back the same computation. The `return` operation is neutral.

- **Associativity**: Binding is associative – it does not matter how you group nested binds. This means `let* x = (let* y = a in b) in c` is equivalent to `let* y = a in let* x = b in c` (when `x` does not appear free in `b`).

You should verify that these laws hold for our list monad:

```
let bind a b = concat_map b a
let return x = [x]
```

For example, to verify left identity: `bind (return a) f` = `bind [a] f` = `concat_map f [a]` = `f a`. The other laws can be verified similarly.

8.5 Monoid Laws and Monad-Plus

The list monad has an additional structure beyond just `bind` and `return`: it supports combining multiple computations and representing failure. This leads us to the concept of a **monoid**.

A monoid is a type with at least two operations:

- `mzero : 'a monoid` – an identity element (think: zero, or the empty container)
- `mplus : 'a monoid -> 'a monoid -> 'a monoid` – a combining operation (think: addition, or concatenation)

These operations must meet the standard monoid laws:

$$
\begin{aligned}
\text{mplus mzero } a &\approx a && \text{(left identity)} \\
\text{mplus } a \text{ mzero} &\approx a && \text{(right identity)} \\
\text{mplus } a \text{ (mplus } b\ c) &\approx \text{mplus (mplus } a\ b)\ c && \text{(associativity)}
\end{aligned}
$$

We define `fail` as a synonym for `mzero` and infix `++` for `mplus`. For lists, `mzero` is `[]` and `mplus` is `@` (append).

Fusing monads and monoids gives the most popular general flavor of monads, which we call **monad-plus** after Haskell. A monad-plus is a monad that also has monoid structure, with additional axioms relating the "addition" (`mplus`) and "multiplication" (`bind`):

$$
\begin{aligned}
\text{bind mzero } f &\approx \text{mzero} \\
\text{bind } m\ (\lambda x.\text{mzero}) &\approx \text{mzero}
\end{aligned}
$$

These laws say that `mzero` acts like a "zero" for `bind`: binding from zero produces zero, and binding to a function that always returns zero also produces zero. This is analogous to how $0 \times x = 0$ and $x \times 0 = 0$ in arithmetic.

Using infix notation with $\oplus$ for `mplus`, $\mathbf{0}$ for `mzero`, $\triangleright$ for `bind`, and $\mathbf{1}$ for `return`, the complete monad-plus axioms are:

$$
\begin{aligned}
\mathbf{0} \oplus a &\approx a \\
a \oplus \mathbf{0} &\approx a \\
a \oplus (b \oplus c) &\approx (a \oplus b) \oplus c \\
\mathbf{1}\ x \rhd f &\approx f\ x \\
a \rhd \lambda x.\mathbf{1}\ x &\approx a \\
(a \rhd \lambda x.b) \rhd \lambda y.c &\approx a \rhd (\lambda x.b \rhd \lambda y.c) \\
\mathbf{0} \rhd f &\approx \mathbf{0} \\
a \rhd (\lambda x.\mathbf{0}) &\approx \mathbf{0}
\end{aligned}
$$

The list type has a natural monad and monoid structure:

```
let mzero = []
let mplus = (@)
let bind a b = concat_map b a
let return a = [a]
```

Given any monad-plus, we can define useful derived operations:

```
let fail = mzero
let failwith _ = fail
let (++) = mplus
let (>>=) a b = bind a b
let guard p = if p then return () else fail
```

Now we can see that `guard` is defined in terms of the monad-plus structure: it returns the identity element (`return ()`) on success, or the zero element (`fail`) on failure.

8.6 Backtracking: Computation with Choice

We have seen `mzero` (i.e., `fail`) in the countdown problem – it represents a computation that produces no results. But what about `mplus`? The `mplus` operation combines two computations, giving us a way to express *choice*: try this computation, or try that one.

Here is an example from a puzzle solver where `mplus` creates a choice point:

```
let find_to_eat n island_size num_islands empty_cells =
  let honey = honey_cells n empty_cells in

  let rec find_board s =
    match visit_cell s with
    | None ->
        let* () = guard (s.been_islands = num_islands) in
        return s.eaten
    | Some (cell, s) ->
```

```
      let* s = find_island cell (fresh_island s) in
      let* () = guard (s.been_size = island_size) in
      find_board s

and find_island current s =
  let s = keep_cell current s in
  neighbors n empty_cells current
  |> foldM
       (fun neighbor s ->
         if CellSet.mem neighbor s.visited then return s
         else
           let choose_eat =
             if s.more_to_eat <= 0 then fail
             else return (eat_cell neighbor s)
           and choose_keep =
             if s.been_size >= island_size then fail
             else find_island neighbor s in
           mplus choose_eat choose_keep)  (* Choice point! *)
       s in

let cells_to_eat =
  List.length honey - island_size * num_islands in
find_board (init_state honey cells_to_eat)
```

The line `mplus choose_eat choose_keep` creates a choice point: the algorithm can either eat the cell (removing it from consideration) or keep it as part of the current island. When we use the list monad as our monad-plus, this explores *all* possible choices, collecting all solutions. The monad-plus structure handles the bookkeeping of backtracking automatically – we just express the choices declaratively.

8.7 Monad Flavors

Monads "wrap around" a type, but some monads need an additional type parameter. For example, a state monad might be parameterized by the type of state it carries. Usually the additional type does not change while within a monad, so we stick to `'a monad` rather than `('s, 'a) monad`.

As monad-plus shows, things get interesting when we add more operations to a basic monad. Different "flavors" of monads provide different capabilities. Here are the most common ones:

Monads with access:

`access : 'a monad -> 'a`

An `access` operation lets you extract the value from the monad. Not all monads support this – some only allow you to "run" the monad at the top level. Example:

the lazy monad, where `access` is `Lazy.force`.

Monad-plus (non-deterministic computation):

```
mzero : 'a monad
mplus : 'a monad -> 'a monad -> 'a monad
```

We have already seen this. The monad-plus flavor supports failure and choice, enabling backtracking search.

Monads with state (parameterized by type `store`):

```
get : store monad
put : store -> unit monad
```

These operations let you read and write a piece of state that is threaded through the computation. There is a "canonical" state monad we will examine later. Related monads include: - The **writer monad**: has `tell` (append to a log) and `listen` (read the log) - The **reader monad**: has `ask` (read an environment) and `local` to modify the environment for a sub-computation:

```
local : (store -> store) -> 'a monad -> 'a monad
```

Exception/error monads (parameterized by type `excn`):

```
throw : excn -> 'a monad
catch : 'a monad -> (excn -> 'a monad) -> 'a monad
```

These provide structured error handling within the monad. The `throw` operation raises an exception; `catch` handles it.

Continuation monad:

```
callCC : (('a -> 'b monad) -> 'a monad) -> 'a monad
```

The continuation monad gives you access to the "rest of the computation" as a first-class value. This is powerful but complex; we will not cover continuations in detail here.

Probabilistic computation:

```
choose : float -> 'a monad -> 'a monad -> 'a monad
```

The `choose p a b` operation selects `a` with probability `p` and `b` with probability `1-p`. This enables reasoning about probability distributions. The laws ensure that probability behaves correctly:

$$\begin{aligned} a \oplus_0 b &\approx b \\ a \oplus_p b &\approx b \oplus_{1-p} a \\ a \oplus_p (b \oplus_q c) &\approx (a \oplus_{\frac{p}{p+q-pq}} b) \oplus_{p+q-pq} c \\ a \oplus_p a &\approx a \end{aligned}$$

Parallel computation (monad with access and parallel bind):

```
parallel : 'a monad -> 'b monad -> ('a -> 'b -> 'c monad) -> 'c monad
```

The `parallel` operation runs two computations concurrently and combines their results. Example: lightweight threads like in the Lwt library.

8.8 Interlude: The Module System

Before we implement various monads, we need to understand OCaml's module system, which provides the infrastructure for defining monads in a reusable, generic way. This section provides a brief overview of the key concepts.

Modules collect related type definitions and operations together. Module values are introduced with `struct ... end` (called *structures*), and module types with `sig ... end` (called *signatures*). A structure is a package of definitions; a signature is an interface that specifies what a structure must provide.

A source file `source.ml` defines a module `Source`. A file `source.mli` defines its type.

In the module level, modules are defined with `module ModuleName = ...` or `module ModuleName : MODULE_TYPE = ...`, and module types with `module type MODULE_TYPE = ...`.

Locally in expressions, modules are defined with `let module M = ... in ...`.

The content of a module is made visible with `open Module`. Module `Pervasives` (now `Stdlib`) is initially visible.

Content of a module is included into another module with `include Module`.

Functors are module functions – functions from modules to modules. They are the key to writing generic code that works with any monad:

```
module Funct = functor (Arg : sig ... end) -> struct ... end
(* Or equivalently: *)
module Funct (Arg : sig ... end) = struct ... end
```

Functors can return functors, and modules can be parameterized by multiple modules. Functor application always uses parentheses: `Funct (struct ... end)`.

A signature `MODULE_TYPE with type t_name = ...` is like `MODULE_TYPE` but with `t_name` made more specific. This is useful when you want to expose the concrete type after applying a functor. We can also include signatures with `include MODULE_TYPE`.

Finally, we can pass around modules in normal functions using first-class modules:

```ocaml
module type T = sig val g : int -> int end

let f mod_v x =
```

```
  let module M = (val mod_v : T) in
  M.g x
(* val f : (module T) -> int -> int = <fun> *)

let test = f (module struct let g i = i*i end : T)
(* val test : int -> int = <fun> *)
```

8.9 The Two Metaphors

Monads are abstract, but two complementary metaphors can help build intuition for what they are and how they work.

Monads as Containers The first metaphor views a monad as a **quarantine container**. Think of it like a sealed box:

- We can put something into the container with `return` – this "seals" a pure value inside the monad
- We can operate on the contents, but the result must stay in the container – we cannot simply extract values

The `lift` function applies a pure function to the contents of a monad, keeping the result wrapped:

```
let lift f m =
  let* x = m in
  return (f x)
(* val lift : ('a -> 'b) -> 'a monad -> 'b monad *)
```

We can also "flatten" nested containers. If we have a monad containing another monad, `join` unwraps one layer – but the result is still in a monad, so the quarantine is not broken:

```
let join m =
  let* x = m in
  x
(* val join : ('a monad) monad -> 'a monad *)
```

The quarantine container for a **monad-plus** is more like a collection: it can be empty (failure), contain one element (success), or contain multiple elements (multiple solutions).

Monads with access allow us to extract the resulting element from the container. Other monads provide a `run` operation that exposes "what really happened behind the quarantine" – for example, the state monad's `run` takes an initial state and returns both the final value and the final state.

Monads as Computation The second metaphor views a monad as a way to structure computation. Each `let*` binding is a step in a sequence, and the monad controls how steps are connected. The physical metaphor is an **assembly line**:

```
let assemblyLine w =
  let* c = makeChopsticks w in    (* Worker makes chopsticks *)
  let* c' = polishChopsticks c in (* Worker polishes them *)
  let* c'' = wrapChopsticks c' in (* Worker wraps them *)
  return c''                       (* Final product goes out *)
```

Each worker (operation) takes material from the previous step and produces something for the next step. The monad defines what happens between steps – for lists, it means "do this for each element"; for state, it means "thread the state through"; for exceptions, it means "propagate errors."

Any expression can be systematically translated into a monadic form. For lambda-terms:

$$
\begin{aligned}
[\![N]\!] &= \text{return } N && \text{(constant)}\\
[\![x]\!] &= \text{return } x && \text{(variable)}\\
[\![\lambda x.a]\!] &= \text{return } (\lambda x.[\![a]\!]) && \text{(function)}\\
[\![\text{let } x = a \text{ in } b]\!] &= \text{bind } [\![a]\!] \ (\lambda x.[\![b]\!]) && \text{(local definition)}\\
[\![a \ b]\!] &= \text{bind } [\![a]\!] \ (\lambda v_a.\text{bind } [\![b]\!] \ (\lambda v_b.v_a \ v_b)) && \text{(application)}
\end{aligned}
$$

This translation inserts `bind` at every point where execution flows from one subexpression to another. The beauty of this approach is that once an expression is spread over a monad, its computation can be monitored, logged, or affected without modifying the expression itself. This is the key to implementing effects like state, exceptions, or non-determinism in a purely functional way.

8.10 Monad Classes and Instances

Now we will see how to implement monads in OCaml using the module system. To implement a monad, we need to provide the implementation type, `return`, and `bind` operations. Here is the minimal signature:

```
module type MONAD = sig
  type 'a t
  val return : 'a -> 'a t
  val bind : 'a t -> ('a -> 'b t) -> 'b t
end
```

This is the "class" that all monads must implement. Alternatively, we could start from `return`, `lift`, and `join` operations – these are mathematically equivalent starting points.

The power of functors is that we can define a suite of general-purpose functions that work for *any* monad, just based on these two operations:

```
module type MONAD_OPS = sig
  type 'a monad
  include MONAD with type 'a t := 'a monad
  val ( let* ) : 'a monad -> ('a -> 'b monad) -> 'b monad
  val ( let+ ) : 'a monad -> ('a -> 'b) -> 'b monad
  val ( >>= ) : 'a monad -> ('a -> 'b monad) -> 'b monad
  val foldM : ('a -> 'b -> 'a monad) -> 'a -> 'b list -> 'a monad
  val whenM : bool -> unit monad -> unit monad
  val lift : ('a -> 'b) -> 'a monad -> 'b monad
  val (>>|) : 'a monad -> ('a -> 'b) -> 'b monad
  val join : 'a monad monad -> 'a monad
  val ( >=>) : ('a -> 'b monad) -> ('b -> 'c monad) -> 'a -> 'c monad
end

module MonadOps (M : MONAD) = struct
  open M
  type 'a monad = 'a t
  let run x = x
  let ( let* ) a b = bind a b
  let ( let+ ) a f = bind a (fun x -> return (f x))
  let (>>=) a b = bind a b
  let rec foldM f a = function
    | [] -> return a
    | x::xs ->
        let* a' = f a x in
        foldM f a' xs
  let whenM p s = if p then s else return ()
  let lift f m =
    let* x = m in
    return (f x)
  let (>>|) a b = lift b a
  let join m =
    let* x = m in
    x
  let (>=>) f g = fun x ->
    let* y = f x in
    g y
end
```

We make the monad "safe" by keeping its type abstract. The `run` function exposes the underlying representation – "what really happened behind the scenes":

```
module Monad (M : MONAD) : sig
  include MONAD_OPS
  val run : 'a monad -> 'a M.t
end = struct
```

```
  include M
  include MonadOps(M)
end
```

The pattern here is important: we take a minimal implementation (`M : MONAD`) and produce a full-featured monad module with all the derived operations.

Monad-Plus Classes The monad-plus class extends the basic monad with failure and choice. Implementations need to provide `mzero` and `mplus` in addition to `return` and `bind`:

```
module type MONAD_PLUS = sig
  include MONAD
  val mzero : 'a t
  val mplus : 'a t -> 'a t -> 'a t
end

module type MONAD_PLUS_OPS = sig
  include MONAD_OPS
  val mzero : 'a monad
  val mplus : 'a monad -> 'a monad -> 'a monad
  val fail : 'a monad
  val (++) : 'a monad -> 'a monad -> 'a monad
  val guard : bool -> unit monad
  val msum_map : ('a -> 'b monad) -> 'a list -> 'b monad
end

module MonadPlusOps (M : MONAD_PLUS) = struct
  open M
  include MonadOps(M)
  let fail = mzero
  let (++) a b = mplus a b
  let guard p = if p then return () else fail
  let msum_map f l = List.fold_right
    (fun a acc -> mplus (f a) acc) l mzero
end

module MonadPlus (M : MONAD_PLUS) : sig
  include MONAD_PLUS_OPS
  val run : 'a monad -> 'a M.t
end = struct
  include M
  include MonadPlusOps(M)
end
```

We also need a class for computations with state. This signature will be included in state monads:

```
module type STATE = sig
  type store
  type 'a t
  val get : store t
  val put : store -> unit t
end
```

8.11 Monad Instances

Now let us see concrete implementations of various monads.

The Lazy Monad If you find OCaml's laziness notation (with `lazy` and `Lazy.force` everywhere) too heavy, you can use a monad! The lazy monad wraps lazy computations:

```
module LazyM = Monad (struct
  type 'a t = 'a Lazy.t
  let bind a b = lazy (Lazy.force (b (Lazy.force a)))
  let return a = lazy a
end)

let laccess m = Lazy.force (LazyM.run m)
```

The `bind` operation creates a new lazy value that, when forced, forces `a`, passes the result to `b`, and forces the result. The `laccess` function forces the final lazy value to get the result.

The List Monad Our familiar list monad is a monad-plus, supporting non-deterministic computation:

```
module ListM = MonadPlus (struct
  type 'a t = 'a list
  let bind a b = concat_map b a
  let return a = [a]
  let mzero = []
  let mplus = List.append
end)
```

Backtracking Parameterized by Monad-Plus Here is the power of abstraction: we can write the Countdown solver parameterized by *any* monad-plus. The same code works with lists (exploring all solutions), lazy lists (computing solutions on demand), or any other monad-plus implementation:

```
module Countdown (M : MONAD_PLUS_OPS) = struct
  open M  (* Open the module to make monad operations visible *)

  let rec insert x = function  (* All choice-introducing operations *)
```

```
  | [] -> return [x]          (* need to happen in the monad *)
  | y::ys as xs ->
      let* xys = insert x ys in
      return (x::xs) ++ return (y::xys)

let rec choices = function
  | [] -> return []
  | x::xs ->
      let* cxs = choices xs in            (* Choosing which numbers in what order *)
      return cxs ++ insert x cxs          (* and now whether with or without x *)

type op = Add | Sub | Mul | Div

let apply op x y =
  match op with
  | Add -> x + y
  | Sub -> x - y
  | Mul -> x * y
  | Div -> x / y

let valid op x y =
  match op with
  | Add -> x <= y
  | Sub -> x > y
  | Mul -> x <= y && x <> 1 && y <> 1
  | Div -> x mod y = 0 && y <> 1

type expr = Val of int | App of op * expr * expr

let op2str = function
  | Add -> "+" | Sub -> "-" | Mul -> "*" | Div -> "/"

let rec expr2str = function  (* We will provide solutions as strings *)
  | Val n -> string_of_int n
  | App (op, l, r) -> "(" ^ expr2str l ^ op2str op ^ expr2str r ^ ")"

let combine (l, x) (r, y) o =  (* Try out an operator *)
  let* () = guard (valid o x y) in
  return (App (o, l, r), apply o x y)

let split l =  (* Another choice: which numbers go into which argument *)
  let rec aux lhs = function
    | [] | [_] -> fail                    (* Both arguments need numbers *)
    | [y; z] -> return (List.rev (y::lhs), [z])
    | hd::rhs ->
        let lhs = hd::lhs in
```

```
        return (List.rev lhs, rhs)
          ++ aux lhs rhs in
    aux [] l

  let rec results = function  (* Build possible expressions once numbers *)
    | [] -> fail               (* have been picked *)
    | [n] ->
        let* () = guard (n > 0) in
        return (Val n, n)
    | ns ->
        let* (ls, rs) = split ns in
        let* lx = results ls in
        let* ly = results rs in  (* Collect solutions using each operator *)
        msum_map (combine lx ly) [Add; Sub; Mul; Div]

  let solutions ns n =  (* Solve the problem: *)
    let* ns' = choices ns in         (* pick numbers and their order, *)
    let* (e, m) = results ns' in     (* build possible expressions, *)
    let* () = guard (m = n) in       (* check if the expression gives target value, *)
    return (expr2str e)              (* "print" the solution *)
end
```

Understanding Laziness Now let us explore a practical question: what if we only want *one* solution, not all of them? With the list monad, we compute all solutions even if we only look at the first one. Can laziness help?

Let us measure execution times to find out:

```
let time f =
  let tbeg = Unix.gettimeofday () in
  let res = f () in
  let tend = Unix.gettimeofday () in
  tend -. tbeg, res
```

With the list monad:

```
module ListCountdown = Countdown (ListM)
let test1 () = ListM.run (ListCountdown.solutions [1;3;7;10;25;50] 765)
let t1, sol1 = time test1
(* val t1 : float = 2.28... *)
(* val sol1 : string list = ["((25-(3+7))*(1+50))"; "(((25-3)-7)*(1+50))"; ...] *)
```

Finding all 49 solutions takes about 2.3 seconds. What if we want only one solution? Laziness to the rescue!

Our first attempt uses an “odd lazy list” – a list where the tail is lazy but the head is strict:

```
type 'a llist = LNil | LCons of 'a * 'a llist Lazy.t
```

```
let rec ltake n = function
  | LCons (a, lazy l) when n > 0 -> a::(ltake (n-1) l)
  | _ -> []

let rec lappend l1 l2 =
  match l1 with
  | LNil -> l2
  | LCons (hd, tl) ->
      LCons (hd, lazy (lappend (Lazy.force tl) l2))

let rec lconcat_map f = function
  | LNil -> LNil
  | LCons (a, lazy l) ->
      lappend (f a) (lconcat_map f l)

module LListM = MonadPlus (struct
  type 'a t = 'a llist
  let bind a b = lconcat_map b a
  let return a = LCons (a, lazy LNil)
  let mzero = LNil
  let mplus = lappend
end)
```

But testing shows disappointing results: the odd lazy list still takes about 2.5 seconds just to create the lazy list! The elements are almost all computed by the time we get the first one.

Why? Because whenever we pattern match on `LCons (hd, tl)`, we have already evaluated the head. And when building lists with `mplus`, the head of the first list is computed immediately.

What about using the **option monad** to find just the first solution?

```
module OptionM = MonadPlus (struct
  type 'a t = 'a option
  let bind a b =
    match a with None -> None | Some x -> b x
  let return a = Some a
  let mzero = None
  let mplus a b = match a with None -> b | Some _ -> a
end)
```

This very quickly computes… nothing! The option monad returns `None`.

Why? The `OptionM` monad (Haskell's `Maybe` monad) is good for computations that might fail, but it does not *search* – its `mplus` just picks the first non-`None` value. Since our search often needs to backtrack when a choice leads to failure, option gives up too early.

Our odd lazy list type is not lazy *enough*. Whenever we "make" a choice with `a ++ b` or `msum_map`, it computes the first candidate for each choice path immediately. We need **even lazy lists** – lists where even the outermost constructor is wrapped in `lazy`:

```
type 'a lazy_list = 'a lazy_list_ Lazy.t
and 'a lazy_list_ = LazNil | LazCons of 'a * 'a lazy_list

let rec laztake n = function
  | lazy (LazCons (a, l)) when n > 0 -> a::(laztake (n-1) l)
  | _ -> []

let rec append_aux l1 l2 =
  match l1 with
  | lazy LazNil -> Lazy.force l2
  | lazy (LazCons (hd, tl)) ->
      LazCons (hd, lazy (append_aux tl l2))

let lazappend l1 l2 = lazy (append_aux l1 l2)

let rec concat_map_aux f = function
  | lazy LazNil -> LazNil
  | lazy (LazCons (a, l)) ->
      append_aux (f a) (lazy (concat_map_aux f l))

let lazconcat_map f l = lazy (concat_map_aux f l)

module LazyListM = MonadPlus (struct
  type 'a t = 'a lazy_list
  let bind a b = lazconcat_map b a
  let return a = lazy (LazCons (a, lazy LazNil))
  let mzero = lazy LazNil
  let mplus = lazappend
end)
```

Now the first solution takes only about 0.37 seconds – considerably less time than the 2.3 seconds for all solutions! The next 9 solutions are almost computed once the first one is (just 0.23 seconds more). But computing all 49 solutions takes about 4 seconds – nearly twice as long as without laziness. This is the price we pay for lazy computation: overhead when we do need all results.

The lesson: even lazy lists enable true lazy search, but they come with overhead. Choose the right monad for your use case.

The Exception Monad OCaml has built-in exceptions that are efficient and flexible. However, monadic exceptions have advantages in certain situations:

- They are safer in multi-threading contexts (no risk of unhandled exceptions escaping)
- They compose well with other monads (via monad transformers)
- They make the possibility of failure explicit in the type

The monadic lightweight-thread library Lwt has `throw` (called `fail` there) and `catch` operations in its monad for exactly these reasons.

```
module ExceptionM (Excn : sig type t end) : sig
  type excn = Excn.t
  type 'a t = OK of 'a | Bad of excn
  include MONAD_OPS
  val run : 'a monad -> 'a t
  val throw : excn -> 'a monad
  val catch : 'a monad -> (excn -> 'a monad) -> 'a monad
end = struct
  type excn = Excn.t
  module M = struct
    type 'a t = OK of 'a | Bad of excn
    let return a = OK a
    let bind m b = match m with
      | OK a -> b a
      | Bad e -> Bad e
  end
  include M
  include MonadOps(M)
  let throw e = Bad e
  let catch m handler = match m with
    | OK _ -> m
    | Bad e -> handler e
end
```

The State Monad The state monad threads a piece of mutable state through a computation without actually using mutation. The key insight is that a stateful computation can be represented as a *function* from the current state to a pair of (result, new state):

```
module StateM (Store : sig type t end) : sig
  type store = Store.t
  type 'a t = store -> 'a * store  (* A stateful computation *)
  include MONAD_OPS
  include STATE with type 'a t := 'a monad
                 and type store := store
  val run : 'a monad -> 'a t
end = struct
  type store = Store.t
  module M = struct
```

```
    type 'a t = store -> 'a * store
    let return a = fun s -> a, s      (* Return value, keep state unchanged *)
    let bind m b = fun s -> let a, s' = m s in b a s'
  end                              (* Run m, then pass result and new state to b *)
  include M
  include MonadOps(M)
  let get = fun s -> s, s            (* Return the current state *)
  let put s' = fun _ -> (), s'       (* Replace the state, return unit *)
end
```

The bind operation sequences two stateful computations: it runs the first one with the initial state, then passes both the result and the new state to the second computation.

The state monad is useful to hide the threading of a "current" value through a computation. Here is an example that renames variables in lambda-terms to eliminate potential name clashes (alpha-conversion):

```
type term =
  | Var of string
  | Lam of string * term
  | App of term * term

let (!) x = Var x
let (|->) x t = Lam (x, t)
let (@) t1 t2 = App (t1, t2)
let test = "x" |-> ("x" |-> !"y" @ !"x") @ !"x"

module S = StateM (struct type t = int * (string * string) list end)
open S

let rec alpha_conv = function
  | Var x as v ->                       (* Function from terms to StateM monad *)
      let* (_, env) = get in            (* Seeing a variable does not change state *)
      let v = try Var (List.assoc x env)  (* but we need its new name *)
        with Not_found -> v in          (* Free variables don't change name *)
      return v
  | Lam (x, t) ->                       (* We rename each bound variable *)
      let* (fresh, env) = get in        (* We need a fresh number *)
      let x' = x ^ string_of_int fresh in
      let* () = put (fresh+1, (x, x')::env) in  (* Remember new name, update number *)
      let* t' = alpha_conv t in
      let* (fresh', _) = get in         (* We need to restore names, *)
      let* () = put (fresh', env) in    (* but keep the number fresh *)
      return (Lam (x', t'))
  | App (t1, t2) ->
      let* t1 = alpha_conv t1 in        (* Passing around of names *)
```

```
    let* t2 = alpha_conv t2 in       (* and the currently fresh number *)
    return (App (t1, t2))            (* is done by the monad *)

(* val test : term = Lam ("x", App (Lam ("x", App (Var "y", Var "x")), Var "x")) *)
(* # StateM.run (alpha_conv test) (5, []);;
   - : term * (int * (string * string) list) =
   (Lam ("x5", App (Lam ("x6", App (Var "y", Var "x6")), Var "x5")), (7, [])) *)
```

The state consists of a fresh counter and an environment mapping old names to new names. The `get` and `put` operations access and modify this state, while `let*` sequences the operations. Without the state monad, we would have to explicitly pass the state through every recursive call – tedious and error-prone.

Note: This alpha-conversion does not make a lambda-term safe for multiple steps of beta-reduction. Can you find a counter-example?

8.12 Monad Transformers

Sometimes we need the capabilities of multiple monads at the same time. For example, we might want both state (to track information) and non-determinism (to explore choices). The straightforward idea is to nest one monad within another: either `'a AM.monad BM.monad` or `'a BM.monad AM.monad`. But this does not work well – we want a single monad that has operations of *both* `AM` and `BM`.

The solution is a **monad transformer**. A monad transformer `AT` takes a monad `BM` and produces a new monad `AT(BM)` that has operations of both. The transformed monad wraps around `BM` in a specific way to make the operations interact correctly.

We will develop a monad transformer `StateT` which adds state to any monad-plus. The resulting monad has all the operations: `return`, `bind`, `mzero`, `mplus`, `put`, `get`, and all their derived functions.

Why do we need monad transformers in OCaml? Because "monads are contagious": although we have built-in state and exceptions, we need to use *monadic* state and exceptions when we are inside a monad. For example, using OCaml's native `ref` cells inside a list monad would give the wrong semantics for backtracking. This is also why Lwt is both a concurrency monad and an exception monad – it needs monadic exceptions to interact correctly with its concurrency model.

To understand how the transformer works, let us compare the regular state monad with the transformed version. The regular state monad uses ordinary OCaml binding:

```
type 'a state = store -> ('a * store)

let return (a : 'a) : 'a state =
```

```
  fun s -> (a, s)

let bind (u : 'a state) (f : 'a -> 'b state) : 'b state =
  fun s -> let (a, s') = u s in f a s'
```

The transformed version wraps everything in the underlying monad `M`:

```
(* Monad M transformed to add state, in pseudo-code: *)
type 'a stateT(M) = store -> ('a * store) M
(* Note: this is store -> ('a * store) M, not ('a M) state *)

let return (a : 'a) : 'a stateT(M) =
  fun s -> M.return (a, s)          (* Use M.return instead of just returning *)

let bind (u : 'a stateT(M)) (f : 'a -> 'b stateT(M)) : 'b stateT(M) =
  fun s -> M.bind (u s) (fun (a, s') -> f a s')  (* Use M.bind instead of let *)
```

The key insight is that the result type is `('a * store) M` – the result and state are wrapped *together* in the underlying monad. This ensures that backtracking (in a monad-plus) correctly restores the state.

State Transformer Implementation

```
module StateT (MP : MONAD_PLUS_OPS) (Store : sig type t end) : sig
  type store = Store.t
  type 'a t = store -> ('a * store) MP.monad
  include MONAD_PLUS_OPS         (* Exporting all monad-plus operations *)
  include STATE with type 'a t := 'a monad
                and type store := store  (* and state operations *)
  val run : 'a monad -> 'a t     (* Expose "what happened" -- resulting states *)
  val runT : 'a monad -> store -> 'a MP.monad
end = struct                     (* Run the state transformer -- get resulting values *)
  type store = Store.t
  module M = struct
    type 'a t = store -> ('a * store) MP.monad
    let return a = fun s -> MP.return (a, s)
    let bind m b = fun s ->
      MP.bind (m s) (fun (a, s') -> b a s')
    let mzero = fun _ -> MP.mzero            (* Lift the monad-plus operations *)
    let mplus ma mb = fun s -> MP.mplus (ma s) (mb s)
  end
  include M
  include MonadPlusOps(M)
  let get = fun s -> MP.return (s, s)        (* Instead of just returning, *)
  let put s' = fun _ -> MP.return ((), s')   (* MP.return *)
  let runT m s = MP.lift fst (m s)
end
```

Backtracking with State Now we can combine backtracking with state for our puzzle solver. The state tracks which cells have been visited, eaten, and how many islands we have found. The monad-plus structure handles the backtracking when a choice leads to a dead end:

```
module HoneyIslands (M : MONAD_PLUS_OPS) = struct
  type state = {
    been_size : int;
    been_islands : int;
    unvisited : cell list;
    visited : CellSet.t;
    eaten : cell list;
    more_to_eat : int;
  }

  let init_state unvisited more_to_eat = {
    been_size = 0;
    been_islands = 0;
    unvisited;
    visited = CellSet.empty;
    eaten = [];
    more_to_eat;
  }

  module BacktrackingM = StateT (M) (struct type t = state end)
  open BacktrackingM

  let rec visit_cell () =           (* State update actions *)
    let* s = get in
    match s.unvisited with
    | [] -> return None
    | c::remaining when CellSet.mem c s.visited ->
        let* () = put {s with unvisited=remaining} in
        visit_cell ()               (* Throwaway argument because of recursion *)
    | c::remaining ->
        let* () = put {s with
          unvisited=remaining;
          visited = CellSet.add c s.visited} in
        return (Some c)             (* This action returns a value *)

  let eat_cell c =
    let* s = get in
    let* () = put {s with eaten = c::s.eaten;
         visited = CellSet.add c s.visited;
         more_to_eat = s.more_to_eat - 1} in
    return ()                       (* Remaining state update actions just affect the state
```

```
let keep_cell c =
  let* s = get in
  let* () = put {s with
    visited = CellSet.add c s.visited;
    been_size = s.been_size + 1} in
  return ()

let fresh_island =
  let* s = get in
  let* () = put {s with been_size = 0;
      been_islands = s.been_islands + 1} in
  return ()

let find_to_eat n island_size num_islands empty_cells =
  let honey = honey_cells n empty_cells in
  let rec find_board () =
    let* cell = visit_cell () in
    match cell with
    | None ->
        let* s = get in
        let* () = guard (s.been_islands = num_islands) in
        return s.eaten
    | Some cell ->
        let* () = fresh_island in
        let* () = find_island cell in
        let* s = get in
        let* () = guard (s.been_size = island_size) in
        find_board ()

  and find_island current =
    let* () = keep_cell current in
    neighbors n empty_cells current
    |> foldM
         (fun () neighbor ->
            let* s = get in
            whenM (not (CellSet.mem neighbor s.visited))
              (let choose_eat =
                 let* () = guard (s.more_to_eat > 0) in
                 eat_cell neighbor
               and choose_keep =
                 let* () = guard (s.been_size < island_size) in
                 find_island neighbor in
               choose_eat ++ choose_keep)) () in

  let cells_to_eat =
```

```
      List.length honey - island_size * num_islands in
    init_state honey cells_to_eat
    |> runT (find_board ())
end

module HoneyL = HoneyIslands (ListM)
let find_to_eat a b c d =
  ListM.run (HoneyL.find_to_eat a b c d)
```

8.13 Probabilistic Programming

Using a random number generator, we can define procedures that produce various outputs. This is **not functional** in the mathematical sense – mathematical functions have deterministic results for fixed arguments.

Just as we can "simulate" mutable variables with the state monad and non-determinism with the list monad, we can "simulate" random computation with a **probability monad**. But the probability monad is more than just randomized computation – it lets us *reason* about probabilities. We can ask questions like "what is the probability of this outcome?" or "what is the distribution of possible results?"

Different monad implementations make different tradeoffs: - **Exact distribution**: Track all possible outcomes and their probabilities precisely - **Sampling (Monte Carlo)**: Approximate probabilities by running many random trials

The Probability Monad The essential functions for the probability monad class are `choose` (for making probabilistic choices) and `distrib` (for extracting the probability distribution). Other operations could be defined in terms of these but are provided by each instance for efficiency.

Inside-monad operations (building probabilistic computations):

- `choose : float -> 'a monad -> 'a monad -> 'a monad`: `choose p a b` represents an event which is `a` with probability p and `b` with probability $1 - p$.
- `pick : ('a * float) list -> 'a monad`: Draw a result from a given probability distribution. The argument must be a valid distribution: positive probabilities summing to 1.
- `uniform : 'a list -> 'a monad`: Uniform distribution – each element equally likely.
- `flip : float -> bool monad`: A biased coin: `true` with probability `p`, `false` otherwise.
- `coin : bool monad`: A fair coin: `flip 0.5`.

Outside-monad operations (querying probabilistic computations):

- `prob : ('a -> bool) -> 'a monad -> float`: Returns the probability that a predicate holds.

- `distrib : 'a monad -> ('a * float) list`: Returns the full distribution of probabilities over outcomes.
- `access : 'a monad -> 'a`: Samples a random result from the distribution – this is **non-functional** behavior (different calls may return different results).

```
module type PROBABILITY = sig
  include MONAD_OPS
  val choose : float -> 'a monad -> 'a monad -> 'a monad
  val pick : ('a * float) list -> 'a monad
  val uniform : 'a list -> 'a monad
  val coin : bool monad
  val flip : float -> bool monad
  val prob : ('a -> bool) -> 'a monad -> float
  val distrib : 'a monad -> ('a * float) list
  val access : 'a monad -> 'a
end
```

Helper functions:

```
let total dist =
  List.fold_left (fun a (_,b) -> a +. b) 0. dist

let merge dist = map_reduce (fun x -> x) (+.) 0. dist  (* Merge repeating elements *)

let normalize dist =                   (* Normalize a measure into a distribution *)
  let tot = total dist in
  if tot = 0. then dist
  else List.map (fun (e,w) -> e, w /. tot) dist

let roulette dist =                    (* Roulette wheel from a distribution/measure *)
  let tot = total dist in
  let rec aux r = function
    | [] -> assert false
    | (e, w)::_ when w <= r -> e
    | (_, w)::tl -> aux (r -. w) tl in
  aux (Random.float tot) dist
```

Exact Distribution Monad

```
module DistribM : PROBABILITY = struct
  module M = struct           (* Exact probability distribution -- naive implementation *)
    type 'a t = ('a * float) list
    let bind a b = merge               (* x w.p. p and then y w.p. q happens = *)
      (List.concat_map (fun (x, p) ->
        List.map (fun (y, q) -> (y, q *. p)) (b x)) a)  (* y results w.p. p*q *)
    let return a = [a, 1.]             (* Certainly a *)
```

```
  end
  include M
  include MonadOps (M)
  let choose p a b =
    List.map (fun (e,w) -> e, p *. w) a @
      List.map (fun (e,w) -> e, (1. -. p) *. w) b
  let pick dist = dist
  let uniform elems = normalize
    (List.map (fun e -> e, 1.) elems)
  let coin = [true, 0.5; false, 0.5]
  let flip p = [true, p; false, 1. -. p]
  let prob p m = m
    |> List.filter (fun (e,_) -> p e)     (* All cases where p holds, *)
    |> List.map snd |> List.fold_left (+.) 0.  (* add up *)
  let distrib m = m
  let access m = roulette m
end
```

Sampling Monad

```
module SamplingM (S : sig val samples : int end) : PROBABILITY = struct
  module M = struct                          (* Parameterized by how many samples *)
    type 'a t = unit -> 'a                   (* used to approximate prob or distrib *)
    let bind a b () = b (a ()) ()            (* Randomized computation -- each call a() *)
    let return a = fun () -> a               (* is an independent sample. Always a. *)
  end
  include M
  include MonadOps (M)
  let choose p a b () =
    if Random.float 1. <= p then a () else b ()
  let pick dist = fun () -> roulette dist
  let uniform elems =
    let n = List.length elems in
    fun () -> List.nth elems (Random.int n)
  let coin = Random.bool
  let flip p = choose p (return true) (return false)
  let prob p m =
    let count = ref 0 in
    for i = 1 to S.samples do
      if p (m ()) then incr count
    done;
    float_of_int !count /. float_of_int S.samples
  let distrib m =
    let dist = ref [] in
    for i = 1 to S.samples do
      dist := (m (), 1.) :: !dist done;
```

```
    normalize (merge !dist)
  let access m = m ()
end
```

Example: The Monty Hall Problem The Monty Hall problem is a famous probability puzzle. In search of a new car, the player picks a door, say 1. The game host (who knows what is behind each door) then opens one of the other doors, say 3, to reveal a goat and offers to let the player switch to door 2 instead of door 1. Should the player switch?

Most people's intuition says it does not matter, but let us compute the actual probabilities:

```
module MontyHall (P : PROBABILITY) = struct
  open P
  type door = A | B | C
  let doors = [A; B; C]

  let monty_win switch =
    let* prize = uniform doors in
    let* chosen = uniform doors in
    let* opened = uniform (list_diff doors [prize; chosen]) in
    let final =
      if switch then List.hd (list_diff doors [opened; chosen])
      else chosen in
    return (final = prize)
end

module MontyExact = MontyHall (DistribM)
module Sampling1000 =
  SamplingM (struct let samples = 1000 end)
module MontySimul = MontyHall (Sampling1000)

(* DistribM.distrib (MontyExact.monty_win false);;
   - : (bool * float) list = [(true, 0.333...); (false, 0.666...)]

   DistribM.distrib (MontyExact.monty_win true);;
   - : (bool * float) list = [(true, 0.666...); (false, 0.333...)] *)
```

The famous result: switching doubles your chances of winning! Counter-intuitively, the host's choice of which door to open gives you information – by switching, you are betting that your initial choice was wrong (which it is 2/3 of the time).

Conditional Probabilities So far we have computed unconditional probabilities. But what if we want to answer questions like "given that X happened, what is the probability of Y?" This is a conditional probability $P(Y|X)$.

Wouldn't it be nice to have a monad-plus rather than just a monad? Then we could use `guard` for conditional probabilities!

To compute $P(A|B)$: 1. Compute what is needed for both A and B 2. Guard B 3. Return A

For the exact distribution monad, we allow intermediate distributions to be *unnormalized* (probabilities sum to less than 1) and normalize at the end. For the sampling monad, we use *rejection sampling*: generate samples and discard those that do not satisfy the condition (though `mplus` has no straightforward correct implementation in this approach).

```
module type COND_PROBAB = sig
  include PROBABILITY
  include MONAD_PLUS_OPS with type 'a monad := 'a monad
end

module DistribMP : COND_PROBAB = struct
  module MP = struct
    type 'a t = ('a * float) list      (* Measures no longer restricted to *)
    let bind a b = merge               (* probability distributions *)
      (List.concat_map (fun (x, p) ->
        List.map (fun (y, q) -> (y, q *. p)) (b x)) a)
    let return a = [a, 1.]
    let mzero = []                     (* Measure equal 0 everywhere is OK *)
    let mplus = List.append
  end
  include MP
  include MonadPlusOps (MP)
  let choose p a b =                   (* It isn't a w.p. p & b w.p. (1-p) since a and b *)
    List.map (fun (e,w) -> e, p *. w) a @  (* are not normalized! *)
      List.map (fun (e,w) -> e, (1. -. p) *. w) b
  let pick dist = dist
  let uniform elems = normalize
    (List.map (fun e -> e, 1.) elems)
  let coin = [true, 0.5; false, 0.5]
  let flip p = [true, p; false, 1. -. p]
  let prob p m = normalize m           (* Final normalization step *)
    |> List.filter (fun (e,_) -> p e)
    |> List.map snd |> List.fold_left (+.) 0.
  let distrib m = normalize m
  let access m = roulette m
end

module SamplingMP (S : sig val samples : int end) : COND_PROBAB = struct
  exception Rejected                   (* For rejecting current sample *)
  module MP = struct                   (* Monad operations are exactly as for SamplingM *)
```

```
    type 'a t = unit -> 'a
    let bind a b () = b (a ()) ()
    let return a = fun () -> a
    let mzero = fun () -> raise Rejected  (* but now we can fail *)
    let mplus a b = fun () ->
      failwith "SamplingMP.mplus not implemented"
  end
  include MP
  include MonadPlusOps (MP)
  let choose p a b () =              (* Inside-monad operations don't change *)
    if Random.float 1. <= p then a () else b ()
  let pick dist = fun () -> roulette dist
  let uniform elems =
    let n = List.length elems in
    fun () -> List.nth elems (Random.int n)
  let coin = Random.bool
  let flip p = choose p (return true) (return false)
  let prob p m =                     (* Getting out of monad: handle rejected samples *)
    let count = ref 0 and tot = ref 0 in
    while !tot < S.samples do        (* Count up to the required *)
      try                            (* number of samples *)
        if p (m ()) then incr count; (* m() can fail *)
        incr tot                     (* But if we got here it hasn't *)
      with Rejected -> ()            (* Rejected, keep sampling *)
    done;
    float_of_int !count /. float_of_int S.samples
  let distrib m =
    let dist = ref [] and tot = ref 0 in
    while !tot < S.samples do
      try
        dist := (m (), 1.) :: !dist;
        incr tot
      with Rejected -> ()
    done;
    normalize (merge !dist)
  let rec access m =
    try m () with Rejected -> access m
end
```

Burglary Example: Encoding a Bayes Net Consider a problem with this dependency structure:

- An alarm can be due to either a burglary or an earthquake
- You are on vacation and have asked neighbors John and Mary to call if the alarm rings
- Mary only calls when she is really sure about the alarm, but John has

better hearing
- Earthquakes are twice as probable as burglaries
- The alarm has about 30% chance of going off during an earthquake
- You can check on the radio if there was an earthquake, but you might miss the news

Probability tables:

- $P(\text{Burglary}) = 0.001$
- $P(\text{Earthquake}) = 0.002$
- $P(\text{Alarm}|\text{B}, \text{E})$ varies (0.001 for FF, 0.29 for FT, 0.94 for TF, 0.95 for TT)
- $P(\text{John calls}|\text{Alarm})$ is 0.9 if alarm, 0.05 otherwise
- $P(\text{Mary calls}|\text{Alarm})$ is 0.7 if alarm, 0.01 otherwise

```
module Burglary (P : COND_PROBAB) = struct
  open P
  type what_happened =
    | Safe | Burgl | Earthq | Burgl_n_earthq

  let check ~john_called ~mary_called ~radio =
    let* earthquake = flip 0.002 in
    let* () = guard (radio = None || radio = Some earthquake) in
    let* burglary = flip 0.001 in
    let alarm_p =
      match burglary, earthquake with
      | false, false -> 0.001
      | false, true -> 0.29
      | true, false -> 0.94
      | true, true -> 0.95 in
    let* alarm = flip alarm_p in
    let john_p = if alarm then 0.9 else 0.05 in
    let* john_calls = flip john_p in
    let* () = guard (john_calls = john_called) in
    let mary_p = if alarm then 0.7 else 0.01 in
    let* mary_calls = flip mary_p in
    let* () = guard (mary_calls = mary_called) in
    match burglary, earthquake with
    | false, false -> return Safe
    | true, false -> return Burgl
    | false, true -> return Earthq
    | true, true -> return Burgl_n_earthq
end

module BurglaryExact = Burglary (DistribMP)
module Sampling2000 =
  SamplingMP (struct let samples = 2000 end)
module BurglarySimul = Burglary (Sampling2000)
```

```
(* DistribMP.distrib
     (BurglaryExact.check ~john_called:true ~mary_called:true ~radio:None);;
   - : (BurglaryExact.what_happened * float) list =
   [(Burgl_n_earthq, 0.000574...); (Earthq, 0.175...);
    (Burgl, 0.283...); (Safe, 0.540...)] *)
```

8.14 Lightweight Cooperative Threads

Running multiple tasks asynchronously can hide I/O latency and utilize multi-core architectures. Traditional operating system threads are "heavyweight" – they have significant overhead for context switching and memory. **Lightweight threads** are managed by the application rather than the OS, allowing many concurrent tasks with lower overhead.

Lightweight threads can be: - **Preemptive**: The scheduler interrupts running threads to switch between them - **Cooperative**: Threads voluntarily give up control at specific points (like I/O operations)

Lwt is a popular OCaml library for lightweight cooperative threads, implemented as a monad. The monadic structure ensures that thread switching happens at well-defined points (whenever you use `let*`), making the code easier to reason about.

The `bind` operation is inherently sequential: `bind a (fun x -> b)` computes `a`, and only resumes computing `b` once the result `x` is known.

For concurrency, we need to "suppress" this sequentiality. We introduce a parallel bind:

```
parallel : 'a monad -> 'b monad -> ('a -> 'b -> 'c monad) -> 'c monad
```

With `parallel a b (fun x y -> c)`, computations `a` and `b` can proceed concurrently. The continuation `c` runs once both results are available.

If the monad starts computing right away (as in the Lwt library), `parallel ea eb c` is equivalent to:

```
let a = ea in
let b = eb in
let* x = a in
let* y = b in
c x y
```

Fine-Grained vs. Coarse-Grained Concurrency There are two approaches to when threads switch:

Fine-grained concurrency suspends at every `bind`. The scheduler runs other threads and comes back to complete the `bind` before running threads created since the suspension. This gives maximum interleaving but has higher overhead.

Coarse-grained concurrency only suspends when explicitly requested via a `suspend` (often called `yield`) operation. Library operations that need to wait for I/O should call `suspend` internally. This is more efficient but requires careful placement of suspension points.

Thread Monad Signatures The thread monad extends the basic monad with parallel composition:

```
module type THREADS = sig
  include MONAD
  val parallel :
    'a t -> 'b t -> ('a -> 'b -> 'c t) -> 'c t
end

module type THREAD_OPS = sig
  include MONAD_OPS
  include THREADS with type 'a t := 'a monad
  val parallel_map :
    'a list -> ('a -> 'b monad) -> 'b list monad
  val (>||=) :
    'a monad -> 'b monad -> ('a -> 'b -> 'c monad) -> 'c monad
  val (>||) :
    'a monad -> 'b monad -> (unit -> 'c monad) -> 'c monad
end

module type THREADSYS = sig
  include THREADS
  val access : 'a t -> 'a
  val kill_threads : unit -> unit
end

module ThreadOps (M : THREADS) = struct
  open M
  include MonadOps (M)
  let parallel_map l f =
    List.fold_right (fun a bs ->
      parallel (f a) bs
        (fun a bs -> return (a::bs))) l (return [])
  let (>||=) = parallel
  let (>||) a b c = parallel a b (fun _ _ -> c ())
end

module Threads (M : THREADSYS) : sig
  include THREAD_OPS
  val access : 'a monad -> 'a
  val kill_threads : unit -> unit
```

```
end = struct
  include M
  include ThreadOps(M)
end
```

Cooperative Thread Implementation The implementation uses a mutable state to track thread progress. Each thread is in one of three states: completed (`Return`), waiting (`Sleep` with a list of callbacks to invoke when done), or forwarded to another thread (`Link`):

```
module Cooperative = Threads(struct
  type 'a state =
    | Return of 'a                    (* The thread has returned *)
    | Sleep of ('a -> unit) list      (* When thread returns, wake up waiters *)
    | Link of 'a t                    (* A link to the actual thread *)
  and 'a t = {mutable state : 'a state}  (* State of the thread can change *)
                                      (* -- it can return, or more waiters added *)
  let rec find t =                    (* Union-find style link chasing *)
    match t.state with
    | Link t -> find t
    | _ -> t

  let jobs = Queue.create ()          (* Work queue -- will store unit -> unit procedures *

  let wakeup m a =                    (* Thread m has actually finished -- *)
    let m = find m in                 (* updating its state *)
    match m.state with
    | Return _ -> assert false
    | Sleep waiters ->
        m.state <- Return a;          (* Set the state, and only then *)
        List.iter ((|>) a) waiters    (* wake up the waiters *)
    | Link _ -> assert false

  let return a = {state = Return a}

  let connect t t' =                  (* t was a placeholder for t' *)
    let t' = find t' in
    match t'.state with
    | Sleep waiters' ->
        let t = find t in
        (match t.state with
        | Sleep waiters ->            (* If both sleep, collect their waiters *)
            t.state <- Sleep (waiters' @ waiters);
            t'.state <- Link t        (* and link one to the other *)
        | _ -> assert false)
    | Return x -> wakeup t x          (* If t' returned, wake up the placeholder *)
```

```
    | Link _ -> assert false

  let rec bind a b =
    let a = find a in
    let m = {state = Sleep []} in      (* The resulting monad *)
    (match a.state with
    | Return x ->                      (* If a returned, we suspend further work *)
        let job () = connect m (b x) in  (* (In exercise 11, this should *)
        Queue.push job jobs            (* only happen after suspend) *)
    | Sleep waiters ->                 (* If a sleeps, we wait for it to return *)
        let job x = connect m (b x) in
        a.state <- Sleep (job::waiters)
    | Link _ -> assert false);
    m

  let parallel a b c =                 (* Since in our implementation *)
    bind a (fun x ->                   (* the threads run as soon as they are created, *)
    bind b (fun y ->                   (* parallel is redundant *)
    c x y))

  let rec access m =                   (* Accessing not only gets the result of m, *)
    let m = find m in                  (* but spins the thread loop till m terminates *)
    match m.state with
    | Return x -> x                    (* No further work *)
    | Sleep _ ->
        (try Queue.pop jobs ()         (* Perform suspended work *)
         with Queue.Empty ->
           failwith "access: result not available");
        access m
    | Link _ -> assert false

  let kill_threads () = Queue.clear jobs  (* Remove pending work *)
end)
```

Testing the Thread Implementation Let us test the implementation with two threads that each print a sequence of numbers:

```
module TTest (T : THREAD_OPS) = struct
  open T
  let rec loop s n =
    let* () = return (Printf.printf "-- %s(%d)\n%!" s n) in
    if n > 0 then loop s (n-1)         (* We cannot use whenM because the thread *)
    else return ()                     (* would be created regardless of condition *)
end

module TT = TTest (Cooperative)
```

```
let test =
  Cooperative.kill_threads ();          (* Clean-up after previous tests *)
  let thread1 = TT.loop "A" 5 in
  let thread2 = TT.loop "B" 4 in
  Cooperative.access thread1;           (* We ensure threads finish computing *)
  Cooperative.access thread2            (* before we proceed *)

(* Output:
   -- A(5)
   -- B(4)
   -- A(4)
   -- B(3)
   -- A(3)
   -- B(2)
   -- A(2)
   -- B(1)
   -- A(1)
   -- B(0)
   -- A(0)
   val test : unit = () *)
```

The output shows that the threads interleave their execution beautifully: A(5), B(4), A(4), B(3), and so on. Each `bind` (the `let*`) causes a context switch to the other thread. This is fine-grained concurrency in action.

The key insight is that monadic structure gives us precise control over concurrency. Every `let*` is a potential suspension point, making the code's behavior predictable and debuggable – a significant advantage over preemptive threading where context switches can happen anywhere.

8.15 Exercises

Exercise 1. (Puzzle via Oleg Kiselyov)

"U2" has a concert that starts in 17 minutes and they must all cross a bridge to get there. All four men begin on the same side of the bridge. It is night. There is one flashlight. A maximum of two people can cross at one time. Any party who crosses, either 1 or 2 people, must have the flashlight with them. The flashlight must be walked back and forth, it cannot be thrown, etc. Each band member walks at a different speed. A pair must walk together at the rate of the slower man's pace:

- Bono: 1 minute to cross
- Edge: 2 minutes to cross
- Adam: 5 minutes to cross
- Larry: 10 minutes to cross

For example: if Bono and Larry walk across first, 10 minutes have elapsed when they get to the other side of the bridge. If Larry then returns with the flashlight, a total of 20 minutes have passed and you have failed the mission.

Find all answers to the puzzle using a list comprehension. The comprehension will be a bit long but recursion is not needed.

Exercise 2. Assume `concat_map` as defined in lecture 6 and the binding operators defined above. What will the following expressions return? Why?

1. `let* _ = return 5 in return 7`
2. `let guard p = if p then [()] else [] in let* () = guard false in return 7`
3. `let* _ = return 5 in let* () = guard false in return 7`

Exercise 3. Define `bind` in terms of `lift` and `join`.

Exercise 4. Define a monad-plus implementation based on binary trees, with constant-time `mzero` and `mplus`. Starter code:

```
type 'a tree = Empty | Leaf of 'a | T of 'a tree * 'a tree

module TreeM = MonadPlus (struct
  type 'a t = 'a tree
  let bind a b = (* TODO *)
  let return a = (* TODO *)
  let mzero = (* TODO *)
  let mplus a b = (* TODO *)
end)
```

Exercise 5. Show the monad-plus laws for one of: 1. `TreeM` from your solution of exercise 4 2. `ListM` from lecture

Exercise 6. Why is the following monad-plus not lazy enough?

```
let rec badappend l1 l2 =
  match l1 with lazy LazNil -> l2
  | lazy (LazCons (hd, tl)) ->
      lazy (LazCons (hd, badappend tl l2))

let rec badconcatmap f = function
  | lazy LazNil -> lazy LazNil
  | lazy (LazCons (a, l)) ->
      badappend (f a) (badconcatmap f l)

module BadyListM = MonadPlus (struct
  type 'a t = 'a lazylist
  let bind a b = badconcatmap b a
  let return a = lazy (LazCons (a, lazy LazNil))
  let mzero = lazy LazNil
```

```
  let mplus = badappend
end)
```

Exercise 7. Convert a "rectangular" list of lists of strings, representing a matrix with inner lists being rows, into a string, where elements are column-aligned. (Exercise not related to monads.)

Exercise 8. Recall the enriched monad signature with `('s, 'a) t` type. Design the signatures for the exception monad operations to provide more flexibility than our exception monad. Does the implementation need to change?

Exercise 9. Implement the following constructs for *all* monads:

1. `for...to...`
2. `for...downto...`
3. `while...do...`
4. `do...while...`
5. `repeat...until...`

Explain how, when your implementation is instantiated with the StateM monad, we get the solution to exercise 2 from lecture 4.

Exercise 10. A canonical example of a probabilistic model is that of a lawn whose grass may be wet because it rained, because the sprinkler was on, or for some other reason. The probability tables are:

$$\begin{aligned}
P(\text{cloudy}) &= 0.5\\
P(\text{rain}|\text{cloudy}) &= 0.8\\
P(\text{rain}|\neg\text{cloudy}) &= 0.2\\
P(\text{sprinkler}|\text{cloudy}) &= 0.1\\
P(\text{sprinkler}|\neg\text{cloudy}) &= 0.5\\
P(\text{wet_roof}|\neg\text{rain}) &= 0\\
P(\text{wet_roof}|\text{rain}) &= 0.7\\
P(\text{wet_grass}|\text{rain}\wedge\neg\text{sprinkler}) &= 0.9\\
P(\text{wet_grass}|\text{sprinkler}\wedge\neg\text{rain}) &= 0.9
\end{aligned}$$

We observe whether the grass is wet and whether the roof is wet. What is the probability that it rained?

Exercise 11. Implement the coarse-grained concurrency model:

- Modify `bind` to compute the resulting monad straight away if the input monad has returned.
- Introduce `suspend` to do what in the fine-grained model was the effect of `bind (return a) b`, i.e., suspend the work although it could already be started.

- One possibility is to introduce `suspend` of type `unit monad`, introduce a "dummy" monadic value `Suspend` (besides `Return` and `Sleep`), and define `bind suspend b` to do what `bind (return ()) b` would formerly do.

Chapter 9: Algebraic Effects

This chapter replaces the chapter *Compilation, Runtime, Optimization, and Parsing* from the old lectures.

TODO

Chapter 10: Functional Reactive Programming

How do we deal with change and interaction in functional programming? This is one of the most challenging questions in the field, and over the years programmers have developed increasingly sophisticated answers. This chapter explores a progression of techniques: we begin with *zippers*, a clever data structure for navigating and modifying positions within larger structures. We then advance to *adaptive programming* (also known as incremental computing), which automatically propagates changes through computations. Finally, we arrive at *Functional Reactive Programming* (FRP), a declarative approach to handling time-varying values and event streams. We conclude with practical examples including graphical user interfaces.

Recommended Reading:

- *"Zipper"* in Haskell Wikibook and *"The Zipper"* by Gerard Huet
- *"How `froc` works"* by Jacob Donham
- *"The Haskell School of Expression"* by Paul Hudak
- *"Deprecating the Observer Pattern with `Scala.React`"* by Ingo Maier, Martin Odersky

10.1 Zippers

Imagine you are editing a document, a tree structure, or navigating through a file system. You need to keep track of where you are, easily access and modify the data at that location, and move around efficiently. This is exactly the problem that zippers solve.

Recall from earlier chapters how we defined *context types* for datatypes – types that represent a data structure with one of its elements missing. We discovered that taking the derivative of an algebraic datatype gives us exactly this context type. Now we will put this theory to practical use.

Consider binary trees:

```
type btree = Tip | Node of int * btree * btree
```

Using our algebraic datatype calculus, where T represents the tree type:

$$
\begin{array}{rcl}
T & = & 1 + xT^2 \\
\frac{\partial T}{\partial x} & = & 0 + T^2 + 2xT\frac{\partial T}{\partial x} = TT + 2xT\frac{\partial T}{\partial x}
\end{array}
$$

This derivative gives us the context type:

```
type btree_dir = LeftBranch | RightBranch
type btree_deriv =
  | Here of btree * btree
  | Below of btree_dir * int * btree * btree_deriv
```

The key insight is that **Location = context + subtree**! A location in a data structure consists of two parts: the context (everything around the focused element) and the subtree (what we are currently looking at).

However, there is a problem with the representation above: we cannot easily move the location if `Here` is at the bottom of our context representation. Think about it: if we want to move up from our current position, we need to access the innermost layer of the context first. The part closest to the location should be on top, not buried at the bottom.

Revisiting the Equations Let us revisit the equations for trees and lists:

$$
\begin{array}{rcl}
T & = & 1 + xT^2 \\
\frac{\partial T}{\partial x} & = & 0 + T^2 + 2xT\frac{\partial T}{\partial x} \\
\frac{\partial T}{\partial x} & = & \frac{T^2}{1-2xT} \\
L(y) & = & 1 + yL(y) \\
L(y) & = & \frac{1}{1-y} \\
\frac{\partial T}{\partial x} & = & T^2 L(2xT)
\end{array}
$$

This algebraic manipulation reveals something beautiful: the context can be stored as a list with the root as the last node. The $L(2xT)$ factor tells us that we have a list where each element consists of $2xT$ – that is, a direction indicator (left or right, hence the factor of 2), the element at that node (x), and the sibling subtree (T).

It does not matter whether we use built-in OCaml lists or define a custom type with `Above` and `Root` variants – the structure is the same.

In practice, contexts of subtrees are more useful than contexts of single elements. Rather than tracking where a single value lives, we track the position of an entire subtree within the larger structure:

```
type 'a tree = Tip | Node of 'a tree * 'a * 'a tree
type tree_dir = Left_br | Right_br
type 'a context = (tree_dir * 'a * 'a tree) list
type 'a location = {sub: 'a tree; ctx: 'a context}
```

```
let access {sub} = sub       (* Get the current subtree *)
let change {ctx} sub = {sub; ctx}  (* Replace the subtree, keep context *)
let modify f {sub; ctx} = {sub = f sub; ctx}  (* Transform the subtree *)
```

There is a wonderful visual intuition for zippers: imagine taking a tree and pinning it at one of its nodes, then letting it hang down under gravity. The pinned node becomes "the current focus," and all the other parts of the tree dangle from it. This mental picture helps understand how movement works: moving to a child means letting a new node become the pin point, with the old parent now hanging above. For excellent visualizations, see http://en.wikibooks.org/wiki/Haskell/Zippers.

Moving Around Navigation functions allow us to traverse the structure. Each movement operation restructures the zipper: what was context becomes part of the subtree, and vice versa. Watch how ascending rebuilds a parent node from the context, while descending breaks apart a node to create new context:

```
let ascend loc =
  match loc.ctx with
  | [] -> loc  (* At root already, or raise exception *)
  | (Left_br, n, l) :: up_ctx ->
    (* We were in the right subtree; rebuild the parent node *)
    {sub = Node (l, n, loc.sub); ctx = up_ctx}
  | (Right_br, n, r) :: up_ctx ->
    (* We were in the left subtree; rebuild the parent node *)
    {sub = Node (loc.sub, n, r); ctx = up_ctx}

let desc_left loc =
  match loc.sub with
  | Tip -> loc  (* Cannot descend into a tip, or raise exception *)
  | Node (l, n, r) ->
    (* Focus on left child; right sibling goes into context *)
    {sub = l; ctx = (Right_br, n, r) :: loc.ctx}

let desc_right loc =
  match loc.sub with
  | Tip -> loc  (* Cannot descend into a tip, or raise exception *)
  | Node (l, n, r) ->
    (* Focus on right child; left sibling goes into context *)
    {sub = r; ctx = (Left_br, n, l) :: loc.ctx}
```

Trees with Arbitrary Branching Following *The Zipper* by Gerard Huet, let us look at a tree with an arbitrary number of branches. This is particularly useful for representing document structures where a group can contain any number of children:

```
type doc = Text of string | Line | Group of doc list
type context = (doc list * doc list) list  (* left siblings, right siblings *)
```

```
type location = {sub: doc; ctx: context}
```

In this design, the context at each level stores two lists: the siblings to the left of our current position (in reverse order for efficient access) and the siblings to the right. This allows us to move not just up and down, but also left and right among siblings.

The navigation functions for this more complex structure show how we reconstruct the parent when going up, and how we split the sibling list when going down:

```
let go_up loc =
  match loc.ctx with
  | [] -> invalid_arg "go_up: at top"
  | (left, right) :: up_ctx ->
    (* Reconstruct the Group: reverse left siblings, add current, then right *)
    {sub = Group (List.rev left @ loc.sub :: right); ctx = up_ctx}

let go_left loc =
  match loc.ctx with
  | [] -> invalid_arg "go_left: at top"
  | (l :: left, right) :: up_ctx ->
    (* Move to left sibling; current element moves to right siblings *)
    {sub = l; ctx = (left, loc.sub :: right) :: up_ctx}
  | ([], _) :: _ -> invalid_arg "go_left: at first"

let go_right loc =
  match loc.ctx with
  | [] -> invalid_arg "go_right: at top"
  | (left, r :: right) :: up_ctx ->
    (* Move to right sibling; current element moves to left siblings *)
    {sub = r; ctx = (loc.sub :: left, right) :: up_ctx}
  | (_, []) :: _ -> invalid_arg "go_right: at last"

let go_down loc =
  (* Go to the first (i.e. leftmost) subdocument *)
  match loc.sub with
  | Text _ -> invalid_arg "go_down: at text"
  | Line -> invalid_arg "go_down: at line"
  | Group [] -> invalid_arg "go_down: at empty"
  | Group (doc :: docs) ->
    (* First child becomes focus; rest become right siblings *)
    {sub = doc; ctx = ([], docs) :: loc.ctx}
```

10.2 Example: Context Rewriting

Let us put zippers to work on a real problem. Imagine a friend working on string theory asks us for help simplifying equations. The task is to pull out particular subexpressions as far to the left as possible, while changing the whole expression as little as possible. This kind of algebraic manipulation is common in symbolic computation.

We can illustrate our algorithm using mathematical notation. Let: - x be the thing we pull out - $C[e]$ and $D[e]$ be big expressions with subexpression e - operator $\circ$ stand for one of: $*, +$

The rewriting rules are:

$$
\begin{array}{ccc}
D[(C[x] \circ e_1) \circ e_2] & \Rightarrow & D[C[x] \circ (e_1 \circ e_2)] \\
D[e_2 \circ (C[x] \circ e_1)] & \Rightarrow & D[C[x] \circ (e_1 \circ e_2)] \\
D[(C[x] + e_1)e_2] & \Rightarrow & D[C[x]e_2 + e_1e_2] \\
D[e_2(C[x] + e_1)] & \Rightarrow & D[C[x]e_2 + e_1e_2] \\
D[e \circ C[x]] & \Rightarrow & D[C[x] \circ e]
\end{array}
$$

These rules encode the algebraic properties we need: associativity (first two rules), distributivity of multiplication over addition (third and fourth rules), and commutativity (last rule, which lets us swap operands). The key insight is that we can implement these transformations efficiently using a zipper, since each rule only needs to look at a small neighborhood of the current position.

First, the groundwork. We define expression types and a zipper for navigating them:

```
type op = Add | Mul
type expr = Val of int | Var of string | App of expr * op * expr
type expr_dir = Left_arg | Right_arg
type context = (expr_dir * op * expr) list
type location = {sub: expr; ctx: context}
```

To locate the subexpression described by predicate `p`, we search the expression tree and build up the context as we go. Notice that we build the context in reverse order during the search, then reverse it at the end so the innermost context comes first (as required for efficient navigation):

```
let rec find_aux p e =
  if p e then Some (e, [])
  else match e with
  | Val _ | Var _ -> None
  | App (l, op, r) ->
    match find_aux p l with
    | Some (sub, up_ctx) ->
      Some (sub, (Right_arg, op, r) :: up_ctx)
    | None ->
      match find_aux p r with
      | Some (sub, up_ctx) ->
        Some (sub, (Left_arg, op, l) :: up_ctx)
```

```
      | None -> None

let find p e =
  match find_aux p e with
  | None -> None
  | Some (sub, ctx) -> Some {sub; ctx = List.rev ctx}
```

Now we can implement the pull-out transformation. This is where the zipper shines: we pattern match on the context to decide which rewriting rule to apply, then modify the context directly. The function recursively moves the target subexpression outward until it reaches the root:

```
let rec pull_out loc =
  match loc.ctx with
  | [] -> loc  (* Done: reached the root *)
  | (Left_arg, op, l) :: up_ctx ->
    (* D[e . C[x]] => D[C[x] . e] -- use commutativity to swap sides *)
    pull_out {loc with ctx = (Right_arg, op, l) :: up_ctx}
  | (Right_arg, op1, e1) :: (_, op2, e2) :: up_ctx
      when op1 = op2 ->
    (* D[(C[x] . e1) . e2] => D[C[x] . (e1 . e2)] -- associativity *)
    pull_out {loc with ctx = (Right_arg, op1, App(e1, op1, e2)) :: up_ctx}
  | (Right_arg, Add, e1) :: (_, Mul, e2) :: up_ctx ->
    (* D[(C[x] + e1) * e2] => D[C[x] * e2 + e1 * e2] -- distributivity *)
    pull_out {loc with ctx =
        (Right_arg, Mul, e2) ::
          (Right_arg, Add, App(e1, Mul, e2)) :: up_ctx}
  | (Right_arg, op, r) :: up_ctx ->
    (* No rule applies: move up by incorporating current context *)
    pull_out {sub = App(loc.sub, op, r); ctx = up_ctx}
```

Since we assume operators are commutative, we can ignore the direction for the second piece of context above – both `(C[x] . e1) . e2` and `e2 . (C[x] . e1)` are handled by the same associativity rule.

Let us test the implementation with a concrete example:

```
let (+) a b = App (a, Add, b)  (* Convenient syntax for building expressions *)
let ( * ) a b = App (a, Mul, b)
let (!) a = Val a
let x = Var "x"
let y = Var "y"

(* Original: 5 + y * (7 + x) * (3 + y) -- we want to pull x to the front *)
let ex = !5 + y * (!7 + x) * (!3 + y)
let loc = find (fun e -> e = x) ex
let sol =
  match loc with
```

```
  | None -> raise Not_found
  | Some loc -> pull_out loc
(* Result: "(((x*y)*(3+y))+(((7*y)*(3+y))+5))" *)
(* The x has been pulled out to the leftmost position! *)
```

The transformation successfully pulled `x` from deep inside the expression to the outermost left position. For best results on complex expressions, we can iterate the `pull_out` function until a fixpoint is reached, ensuring all instances of the target are pulled out as far as possible.

10.3 Adaptive Programming (Incremental Computing)

While zippers are elegant for navigating and modifying data structures, they are somewhat unnatural for general-purpose programming. The fundamental problem is this: once we change something using a zipper, how do we propagate those changes through all the computations that depend on the modified data? We would need to rewrite all our algorithms to explicitly work with context changes, which defeats the purpose of clean functional programming.

Adaptive Programming, also known as *incremental computation* or *self-adjusting computation*, offers a more elegant solution. The idea is beautifully simple: we write programs in a straightforward functional manner, but the runtime system tracks dependencies between computations. When we later modify any input data, only the minimal amount of work required to update the results is performed – everything else is reused from before.

The functional description of computation lives within a monad. We can change monadic values – for example, parts of input – from outside the computation, and the changes automatically propagate to all dependent results. In the *Froc* library by Jake Donham, the monadic *changeables* are represented by type `'a Froc_sa.t`, and the ability to modify them from outside is exposed by type `'a Froc_sa.u` – the *writeables*.

Dependency Graphs The key to making incremental computation work is tracking *how* a result was computed, not just *what* the result is. The monadic value `'a changeable` stores the *dependency graph* of the computation of the represented value `'a`.

Consider a simple computation:

```
let u = v / w + x * y + z
```

This creates a dependency graph where `u` depends on intermediate results (let us call them `n0 = v/w`, `n1 = x*y`, `n2 = n0+n1`), which in turn depend on the input variables. When we modify inputs – say, both `v` and `z` simultaneously – the runtime needs to update intermediate nodes in the correct order. Since `n2` depends on `n0`, we must update `n0` before `n2`, and both must be updated before `u`.

The order in which the computation was originally performed determines the order of updates. We record timestamps for each computation, and updates follow this timestamp order. Similar to `parallel` in the concurrency monad from Chapter 8, we provide `bind2`, `bind3`, etc., and corresponding `lift2`, `lift3`, etc., to introduce nodes that depend on several children simultaneously:

```
let n0 = bind2 v w (fun v w -> return (v / w))
let n1 = bind2 x y (fun x y -> return (x * y))
let n2 = bind2 n0 n1 (fun n0 n1 -> return (n0 + n1))
let u = bind2 n2 z (fun n2 z -> return (n2 + z))
```

The beauty of lifting is that we can make our code look almost identical to ordinary arithmetic. Do-notation is not necessary to have readable expressions:

```
let (/) = lift2 (/)
let ( * ) = lift2 ( * )
let (+) = lift2 (+)
let u = v / w + x * y + z  (* Looks like normal code, but tracks dependencies! *)
```

As in other monads, we can decrease overhead by combining multiple operations into bigger chunks. Instead of creating a dependency node for every single operation, we can batch several operations together:

```
let n0 = blift2 v w (fun v w -> v / w)
let n2 = blift3 n0 x y (fun n0 x y -> n0 + x * y)
let u = blift2 n2 z (fun n2 z -> n2 + z)
```

Handling Conditional Dependencies There is a subtlety that arises with conditionals. Consider this example:

```
let b = x >>= fun x -> return (x = 0)
let n0 = x >>= fun x -> return (100 / x)
let y = bind2 b n0 (fun b n0 -> if b then return 0 else n0)
```

If we blindly recompute all nodes in their original order when `x` changes, we have a problem. If `x` becomes 0, we would compute `n0 = 100 / 0` and crash – even though the conditional in `y` would never use that result!

The solution is to use *time intervals* rather than single timestamps. Each computation records when it began and when it ended. When updating the `y` node, we first *detach* all nodes in its time range (let us say 4-9) from the graph. The conditional is then recomputed, and it will re-attach only the nodes it actually needs. If `b` is true, the `n0` computation is never re-attached and thus never re-executed.

What if the value of `b` does not change? Then we can skip updating `y` entirely and proceed directly to updating `n0`. Since `y` contains a link to the value of `n0`, the final result of `y` will still reflect any changes to `n0`.

We also need *memoization* to efficiently re-attach the same nodes when they do not need updating. When should a detached node be considered up-to-date? When the update process has progressed past that node's timestamp range, it is safe to re-attach it unchanged.

Example Using Froc Let us see adaptive programming in action with a concrete example: incrementally growing and displaying a tree. The `Froc_sa` module (for *self-adjusting*) exports the monadic type `t` for changeable computation, and a handle type `u` for updating the computation from outside.

We define a binary tree where each node stores its screen location. Crucially, the children are wrapped in the `t` type, making them changeable:

```
open Froc_sa

type tree =
  | Leaf of int * int                (* A leaf stores its x,y position *)
  | Node of int * int * tree t * tree t  (* Children are changeable! *)
```

Displaying the tree is itself a changeable effect. Whenever the tree changes, the display will be automatically updated. The key insight is that only *new* nodes will be drawn after an update – unchanged parts of the tree do not trigger any drawing:

```
let rec display px py t =  (* px, py = parent position for drawing line *)
  match t with
  | Leaf (x, y) ->
    return
      (Graphics.draw_poly_line [|px, py; x, y|];  (* Draw line to parent *)
       Graphics.draw_circle x y 3)  (* Draw the leaf node *)
  | Node (x, y, l, r) ->
    return (Graphics.draw_poly_line [|px, py; x, y|])
    >>= fun _ -> l >>= display x y  (* Recursively display left child *)
    >>= fun _ -> r >>= display x y  (* Recursively display right child *)
```

Now the interesting part: growing the tree. The `grow_at` function replaces a leaf with a new internal node that has two leaf children. The crucial operations are `changeable` (which creates a new changeable value with a writeable handle) and `write` (which updates a changeable from outside):

```
let grow_at (x, depth, upd) =
  (* Calculate positions for left and right children *)
  let x_l = x - f2i (width *. (2.0 ** (~-. (i2f (depth + 1))))) in
  let l, upd_l = changeable (Leaf (x_l, (depth + 1) * 20)) in
  let x_r = x + f2i (width *. (2.0 ** (~-. (i2f (depth + 1))))) in
  let r, upd_r = changeable (Leaf (x_r, (depth + 1) * 20)) in
  (* Replace the old leaf with a new internal node *)
  write upd (Node (x, depth * 20, l, r));
  propagate ();  (* Trigger update propagation! *)
```

```
  (* Return handles for future growth at the new leaves *)
  [x_l, depth + 1, upd_l; x_r, depth + 1, upd_r]
```

The main loop grows the tree level by level, calling `grow_at` for every leaf at the current frontier:

```
let rec loop t subts steps =
  if steps <= 0 then ()
  else loop t (concat_map grow_at subts) (steps - 1)

let incremental steps () =
  Graphics.open_graph " 1024x600";
  let t, u = changeable (Leaf (512, 20)) in
  (* Set up the display ONCE -- it will update automatically! *)
  let d = t >>= display (f2i (width /. 2.)) 0 in
  loop t [512, 1, u] steps;  (* New nodes will be drawn automatically *)
  Graphics.close_graph ()
```

Notice the elegance: we set up the display computation once, and then as we grow the tree by writing to changeable leaves, the display automatically updates to show only the new nodes. The dependency tracking ensures that only the affected parts of the display computation are re-executed.

However, there is a practical caveat: the overhead of incremental computation is quite large. Comparing byte code execution times for growing and displaying trees of various depths:

depth	12	13	14	15	16	17	18	19	20
incremental	0.66s	1s	2.2s	4.4s	9.3s	21s	50s	140s	255s
rebuilding	0.5s	0.63s	1.3s	3s	5.3s	13s	39s	190s	–

Rebuilding the entire tree from scratch is actually faster for smaller depths! Incremental computation only wins when changes are small relative to the total computation. The moral: use incremental computation when you expect to make many small updates to a large structure, not when building something from scratch.

10.4 Functional Reactive Programming

We have seen how zippers let us navigate structures and how adaptive programming propagates changes. But what about programs that must respond to *time* itself – animations, games, interactive applications? This is the domain of *Functional Reactive Programming* (FRP).

FRP is an attempt to declaratively deal with time. The key insight is to distinguish two kinds of time-varying values:

- *Behaviors* are continuous functions of time. A behavior has a specific value at every instant. Think of a mouse position, window size, or the current frame of an animation.
- *Events* are discrete occurrences. An event is a set of (time, value) pairs, organized into streams of actions. Think of mouse clicks, key presses, or timer ticks.

Two fundamental problems arise in FRP:

1. **Causality**: Behaviors and events must be well-defined, which means they cannot depend on future values. A behavior at time t can only depend on events that have already occurred.
2. **Efficiency**: We need to minimize the overhead of tracking time and dependencies, especially for real-time applications like games.

FRP is *synchronous*: it is possible to set up multiple events to happen at exactly the same time, and the system handles this correctly. It is also *continuous*: behaviors can have details at arbitrary time resolution. Although the actual results are *sampled* at discrete moments, there is no fixed (minimal) time step for specifying behavior – you describe what the behavior *should be* at any time, and the system samples it as needed.

(Note: "Asynchrony" in reactive programming refers to various different ideas depending on context, so always ask what people mean when they use the term.)

Idealized Definitions Let us start with the idealized, mathematical definitions and then see how practical considerations force us to refine them.

In the purest form, we would define:

```
type time = float
type 'a behavior = time -> 'a  (* Arbitrary function of time *)
type 'a event = ('a, time) stream  (* Stream of values at increasing time instants *)
```

This is mathematically elegant: a behavior is literally a function from time to values, and events are a lazy stream of timestamped occurrences. Forcing the stream would block until the next event arrives.

But this idealized view has problems. Behaviors need to react to external events – the position of a paddle should follow the mouse, not just be a predetermined function of time:

```
type user_action =
  | Key of char * bool
  | Button of int * int * bool * bool
  | MouseMove of int * int
  | Resize of int * int

type 'a behavior = user_action event -> time -> 'a
```

Now a behavior takes both the event history and the current time. But this leads to an efficiency problem: every time we evaluate a behavior, we would need to scan through all events from the beginning of time up to the current moment. This is wasteful in both time and space.

The solution is to turn behaviors into stream transformers. Instead of a function that answers "what is the value at time t?", we produce a stream of values, one for each sampling time. This allows us to forget about events that are already in the past:

```
type 'a behavior =
  user_action event -> time stream -> 'a stream
```

The next optimization is to combine the user actions and sampling times into a single stream. At each sampling moment, we either have a user action or nothing happened:

```
type 'a behavior =
  (user_action option * time) stream -> 'a stream
```

The `None` action corresponds to a sampling moment when nothing happened – we still need to produce a value for the behavior at that time, even if no event triggered it.

This transformation from functions-of-time to stream transformers is analogous to a classic algorithm optimization: computing the intersection of two sorted lists. The naive approach checks every pair, giving $O(mn)$ time. The smart approach walks through both lists simultaneously, giving $O(m+n)$ time. Similarly, our stream-based behaviors process time and events together in a single pass.

With behaviors as stream transformers, we can elegantly define events in terms of behaviors:

```
type 'a event = 'a option behavior
```

An event is simply a behavior that produces `None` at most sampling times and `Some value` when the event actually occurs. This unifies our treatment of behaviors and events, although it somewhat betrays the discrete character of events (they conceptually happen at points in time, not vary over intervals).

We have now arrived at something very close to the *stream processing* we discussed in Chapter 7. Recall the incremental pretty-printing example that could "react" to more input being added. The stream combinators we developed there, along with *fork* (from the exercises) and a corresponding *merge*, turn stream processing into *synchronous discrete reactive programming*. FRP is, in a sense, stream processing with explicit time.

Behaviors as Monads Behaviors form a monad – at least in the original, idealized specification. Looking at the simple definition `type 'a behavior = time -> 'a`, we can define:

```
type 'a behavior = time -> 'a

val return : 'a -> 'a behavior
let return a = fun _ -> a  (* Constant behavior: same value at all times *)

val bind : 'a behavior -> ('a -> 'b behavior) -> 'b behavior
let bind a f = fun t -> f (a t) t  (* Sample 'a' at time t, then sample the result *)
```

The `return` function creates a constant behavior that has the same value at all times. The `bind` function samples the first behavior at the current time, uses that value to select a second behavior, and samples *that* at the current time.

In practice, as we saw with changeables, we mostly use *lifting* rather than full monadic bind. In the Haskell world, behaviors are often called *applicative* rather than monadic. We can build our own lifting functions from the applicative `ap` combinator:

```
val ap : ('a -> 'b) monad -> 'a monad -> 'b monad
let ap fm am =
  let* f = fm in
  let* a = am in
  return (f a)
```

A word of caution: for changeables and other incremental systems, this naive implementation of `ap` will introduce unnecessary dependencies in the computation graph. If `fm` changes, we would unnecessarily recompute everything even if only `am` matters for the result. Good FRP and incremental computing libraries provide optimized variants that track dependencies more precisely. This is analogous to how we needed `parallel` (rather than sequential bind) for concurrent computing in Chapter 8.

Converting Between Events and Behaviors One of the most important operations in FRP is converting between events and behaviors. Going from events to behaviors, the key combinators `until` and `switch` have type:

```
'a behavior -> 'a behavior event -> 'a behavior
```

while `step` has type:

```
'a -> 'a event -> 'a behavior
```

Here is what each does:

- `until b es` behaves as `b` until the first event in `es` occurs, then permanently switches to behaving as the behavior carried by that event. This is “one-shot” switching.
- `switch b es` behaves as the behavior from the *most recent* event in `es` (prior to current time), if any event has occurred, otherwise it behaves as `b`. Unlike `until`, this keeps switching whenever a new event arrives.

- `step a es` is the simplest: it starts as a constant behavior returning `a`, and then switches to returning the value of the most recent event in `es`. This creates a *step function* – a behavior that jumps from value to value at discrete times.

We will use the term "*signal*" to refer to either a behavior or an event. Be aware that terminology varies across FRP libraries: some use "signal" to mean specifically what we call a behavior. Always check the documentation when working with a new FRP library.

10.5 Reactivity by Stream Processing

Now let us implement FRP using the stream processing techniques from Chapter 7. The infrastructure should be familiar:

```
type 'a stream = 'a stream_ Lazy.t
and 'a stream_ = Cons of 'a * 'a stream

let rec lmap f l = lazy (
  let Cons (x, xs) = Lazy.force l in
  Cons (f x, lmap f xs))

let rec liter (f : 'a -> unit) (l : 'a stream) : unit =
  let Cons (x, xs) = Lazy.force l in
  f x; liter f xs

let rec lmap2 f xs ys = lazy (
  let Cons (x, xs) = Lazy.force xs in
  let Cons (y, ys) = Lazy.force ys in
  Cons (f x y, lmap2 f xs ys))

let rec lmap3 f xs ys zs = lazy (
  let Cons (x, xs) = Lazy.force xs in
  let Cons (y, ys) = Lazy.force ys in
  let Cons (z, zs) = Lazy.force zs in
  Cons (f x y z, lmap3 f xs ys zs))

let rec lfold acc f (l : 'a stream) = lazy (
  let Cons (x, xs) = Lazy.force l in  (* Fold a function over the stream *)
  let acc = f acc x in  (* producing a stream of partial results *)
  Cons (acc, lfold acc f xs))
```

Since a behavior is a function from the input stream to an output stream, we face a subtle sharing problem: if we apply the same behavior function twice to the "same" input, we might create two separate streams that diverge. We need to ensure that for any actual input stream, each behavior creates exactly one output stream. This requires memoization:

```ocaml
type ('a, 'b) memo1 =
  {memo_f : 'a -> 'b; mutable memo_r : ('a * 'b) option}

let memo1 f = {memo_f = f; memo_r = None}

let memo1_app f x =
  match f.memo_r with
  | Some (y, res) when x == y -> res  (* Physical equality check *)
  | _ ->
    let res = f.memo_f x in
    f.memo_r <- Some (x, res);  (* Cache for next call *)
    res

let ($) = memo1_app  (* Convenient infix for memoized application *)

type 'a behavior =
  ((user_action option * time) stream, 'a stream) memo1
```

We use physical equality (`==`) rather than structural equality (`=`) because the external input stream is a single physical object – if we see the same pointer, we know it is the same stream. During debugging, we can verify that `memo_r` is `None` before the first call and `Some` afterwards.

Building Complex Behaviors Now we can build the monadic/applicative functions for composing behaviors. A practical tip: when working with these higher-order types, type annotations are essential. If you do not provide type annotations in `.ml` files, work together with an `.mli` interface file to catch type problems early.

```ocaml
(* A constant behavior: returns the same value at all times *)
let returnB x : 'a behavior =
  let rec xs = lazy (Cons (x, xs)) in  (* Infinite stream of x *)
  memo1 (fun _ -> xs)

let ( !* ) = returnB  (* Convenient prefix operator for constants *)

(* Lift a unary function to work on behaviors *)
let liftB f fb = memo1 (fun uts -> lmap f (fb $ uts))

(* Lift binary and ternary functions similarly *)
let liftB2 f fb1 fb2 = memo1
  (fun uts -> lmap2 f (fb1 $ uts) (fb2 $ uts))

let liftB3 f fb1 fb2 fb3 = memo1
  (fun uts -> lmap3 f (fb1 $ uts) (fb2 $ uts) (fb3 $ uts))
```

```
(* Lift a function to work on events (None -> None, Some e -> Some (f e)) *)
let liftE f (fe : 'a event) : 'b event = memo1
  (fun uts -> lmap
    (function Some e -> Some (f e) | None -> None)
    (fe $ uts))

let (=>>) fe f = liftE f fe  (* Map over events, infix style *)
let (->>) e v = e =>> fun _ -> v  (* Replace event value with constant *)
```

We also need to create events from behaviors and vice versa. Creating events out of behaviors:

```
(* whileB: produces an event at every moment the behavior is true *)
let whileB (fb : bool behavior) : unit event =
  memo1 (fun uts ->
    lmap (function true -> Some () | false -> None)
      (fb $ uts))

(* unique: filters out duplicate consecutive events *)
let unique fe : 'a event =
  memo1 (fun uts ->
    let xs = fe $ uts in
    lmap2 (fun x y -> if x = y then None else y)
      (lazy (Cons (None, xs))) xs)  (* Compare with previous value *)

(* whenB: produces an event when the behavior becomes true (edge detection) *)
let whenB fb =
  memo1 (fun uts -> unique (whileB fb) $ uts)

(* snapshot: when an event occurs, capture both the event value and current behavior value
let snapshot fe fb : ('a * 'b) event =
  memo1 (fun uts -> lmap2
    (fun x -> function Some y -> Some (y, x) | None -> None)
      (fb $ uts) (fe $ uts))
```

Creating behaviors out of events:

```
(* step: holds the value of the most recent event, starting with 'acc' *)
let step acc fe =
  memo1 (fun uts -> lfold acc
    (fun acc -> function None -> acc | Some v -> v)
    (fe $ uts))

(* step_accum: accumulates by applying functions from events to current value *)
let step_accum acc ff =
  memo1 (fun uts ->
    lfold acc (fun acc -> function
      | None -> acc | Some f -> f acc)
```

```
      (ff $ uts))
```

For physics simulations like our upcoming paddle game, we need to integrate behaviors over time. This requires access to the sampling timestamps:

```
let integral fb =
  let rec loop t0 acc uts bs =
    let Cons ((_, t1), uts) = Lazy.force uts in
    let Cons (b, bs) = Lazy.force bs in
    (* Rectangle rule: b is fb(t1), acc approximates integral up to t0 *)
    let acc = acc +. (t1 -. t0) *. b in
    Cons (acc, lazy (loop t1 acc uts bs)) in
  memo1 (fun uts -> lazy (
    let Cons ((_, t), uts') = Lazy.force uts in
    Cons (0., lazy (loop t 0. uts' (fb $ uts)))))
```

In our upcoming *paddle game* example, we will express position and velocity in a mutually recursive manner – position is the integral of velocity, but velocity changes when position hits a wall. This seems paradoxical: how can we define position in terms of velocity if velocity depends on position?

The trick is the same as we saw in Chapter 7: integration introduces one step of delay. The integral at time t depends on velocities at times *before* t, while the bounce detection at time t uses the position at time t. This breaks the cyclic dependency and makes the recursion well-founded.

We define behaviors for user actions by extracting them from the input stream:

```
(* Left button press event *)
let lbp : unit event =
  memo1 (fun uts -> lmap
    (function Some(Button(_,_)), _ -> Some() | _ -> None)
    uts)

(* Mouse movement event (carries coordinates) *)
let mm : (int * int) event =
  memo1 (fun uts -> lmap
    (function Some(MouseMove(x, y)), _ -> Some(x, y) | _ -> None)
    uts)

(* Window resize event *)
let screen : (int * int) event =
  memo1 (fun uts -> lmap
    (function Some(Resize(x, y)), _ -> Some(x, y) | _ -> None)
    uts)

(* Behaviors derived from events using step *)
let mouse_x : int behavior = step 0 (liftE fst mm)   (* Current mouse X *)
let mouse_y : int behavior = step 0 (liftE snd mm)   (* Current mouse Y *)
```

```
let width : int behavior = step 640 (liftE fst screen)  (* Window width *)
let height : int behavior = step 512 (liftE snd screen) (* Window height *)
```

The Paddle Game Example Now let us put all these pieces together to build a classic paddle game (similar to Pong). A ball bounces around the screen, and the player controls a paddle at the bottom to prevent the ball from falling.

First, we define a *scene graph*, a data structure that represents a "world" which can be drawn on screen:

```
type scene =
  | Rect of int * int * int * int  (* position, width, height *)
  | Circle of int * int * int  (* position, radius *)
  | Group of scene list
  | Color of Graphics.color * scene  (* color of subscene objects *)
  | Translate of float * float * scene  (* additional offset of origin *)
```

The drawing function interprets the scene graph, accumulating translations as it traverses:

```
let draw sc =
  let f2i = int_of_float in
  let open Graphics in
  let rec aux t_x t_y = function  (* t_x, t_y accumulate translations *)
    | Rect (x, y, w, h) ->
      fill_rect (f2i t_x + x) (f2i t_y + y) w h
    | Circle (x, y, r) ->
      fill_circle (f2i t_x + x) (f2i t_y + y) r
    | Group scs ->
      List.iter (aux t_x t_y) scs
    | Color (c, sc) ->
      set_color c; aux t_x t_y sc  (* Set color, then draw *)
    | Translate (x, y, sc) ->
      aux (t_x +. x) (t_y +. y) sc in  (* Add to accumulated offset *)
  clear_graph ();  (* Clear the back buffer *)
  aux 0. 0. sc;
  synchronize ()  (* Swap buffers -- this avoids flickering *)
```

An *animation* is simply a scene behavior – a time-varying scene. The `reactimate` function runs the animation loop: it creates the input stream (user actions paired with sampling times), feeds it to the scene behavior to get a stream of scenes, and draws each scene. We use double buffering to avoid flickering.

For the game logic, we define lifted operators so we can write behavior expressions naturally:

```
let (+*) = liftB2 (+)     (* Addition on behaviors *)
let (-*) = liftB2 (-)     (* Subtraction on behaviors *)
let ( *** ) = liftB2 ( * )  (* Multiplication on behaviors *)
```

```
let (/*) = liftB2 (/)     (* Division on behaviors *)
let (&&*) = liftB2 (&&)  (* Logical AND on behaviors *)
let (||*) = liftB2 (||)  (* Logical OR on behaviors *)
let (<*) = liftB2 (<)     (* Less-than on behaviors *)
let (>*) = liftB2 (>)     (* Greater-than on behaviors *)
```

Now we can define the game elements. The walls are drawn on the left, top and right borders of the window:

```
let walls =
  liftB2 (fun w h -> Color (Graphics.blue, Group
    [Rect (0, 0, 20, h-1); Rect (0, h-21, w-1, 20);
     Rect (w-21, 0, 20, h-1)]))
    width height
```

The paddle is tied to the mouse at the bottom border of the window:

```
let paddle = liftB (fun mx ->
  Color (Graphics.black, Rect (mx, 0, 50, 10))) mouse_x
```

The ball has a velocity in pixels per second and bounces from the walls. Unfortunately, OCaml being an eager language does not let us encode mutually recursive behaviors as elegantly as we might in a lazy language like Haskell. We need to unpack behaviors and events as explicit functions of the input stream and tie the knot manually using mutable record fields.

The key ideas in the ball implementation:

- `xbounce ->> (~-.)` – When an `xbounce` event fires, emit the negation function `(~-.)`. This will be used to flip the velocity sign.
- `step_accum vel (xbounce ->> (~-.))` – Start with velocity `vel`, and whenever a bounce event occurs, apply the negation function to flip the sign. This creates a velocity that bounces back and forth.
- `liftB int_of_float (integral xvel) +* width /* !*2` – Integrate velocity to get position (as a float), truncate to integers, and offset to center the ball in the window.
- `whenB ((xpos >* width -* !*27) ||* (xpos <* !*27))` – Fire an event the *first* time the position exceeds the wall boundaries (27 pixels from edges, accounting for wall thickness and ball radius). The `whenB` combinator produces an event only on the *transition* from false to true, ensuring we do not keep bouncing while inside the wall.

10.6 Reactivity by Incremental Computing

In the previous section, we implemented FRP using lazy streams. An alternative approach is to use the incremental computing infrastructure from Section 10.3. The *Froc* library takes this approach.

In *Froc*, both behaviors and events are implemented as changeables, but they have different lifetimes. Behaviors *persist* – they always have a current value. Events are *instantaneous* – they fire, propagate their values, and then are removed from the dependency graph. This captures an intuitive distinction: a behavior like “current mouse position” always exists, while an event like “mouse button pressed” happens at a moment and is gone.

Behaviors are composed out of constants and prior events, capturing the “changeable” aspect. Events capture the “writeable” aspect – they are how external inputs enter the system. Together, events and behaviors are called *signals*.

One important design choice in *Froc*: it does not explicitly represent time. Instead, it provides the function `changes : 'a behavior -> 'a event`, which fires an event whenever a behavior changes. This violates the continuous semantics we discussed earlier – it breaks the illusion that behaviors vary continuously rather than at discrete points. But it simplifies the implementation by avoiding the need to synchronize global time samples with events. The result is “less continuous but more dense” in the sense that updates happen exactly when needed, not at fixed intervals.

Sending an event using `send` starts an *update cycle*. During an update cycle, all dependent signals are brought up to date. Signals themselves cannot call `send` (that would create unpredictable cascades), but they can call `send_deferred`, which schedules an event for the *next* update cycle. Things that happen in the same update cycle are considered *simultaneous*.

Froc provides `fix_b` and `fix_e` functions to define signals recursively. The “current value” in a recursive definition refers to the value from the *previous* update cycle, and each recursive step is deferred to the next cycle, until values converge.

Update cycles can happen “back-to-back” via `send_deferred` and `fix_b`/`fix_e`, or can be triggered from outside *Froc* by sending events at arbitrary times. With a `time` behavior that tracks a clock event, events from back-to-back update cycles can have the same clock time even though they are not simultaneous in the FRP sense. This architecture prevents *glitches*, where an outdated signal value is accidentally used before it has been updated.

Pure vs. Impure Style *Froc* supports two programming styles. A behavior is written in *pure style* when its definition does not use `send`, `send_deferred`, `notify_e`, `notify_b`, or `sample`. In pure style:

- `sample`, `notify_e`, `notify_b` are used only from *outside* the behavior (from its “environment”) – analogous to observing the result of a function after it completes
- `send`, `send_deferred` are used only from outside – analogous to providing input to a function before it runs

In *impure style*, we can freely mix signal definitions with imperative notifications and samples. This is more flexible but has an important pitfall: we must ensure that all pieces of our behavior are *referred to* from somewhere, otherwise the garbage collector will reclaim them and our behavior will mysteriously stop working!

A value is "referred to" when it has a name in the global environment, or is stored as part of a larger value that is referred to. Signals are also referred to when they are part of the dependency graph. If you define a signal, attach a notification to it, but do not keep the signal itself alive, the notification may stop working when the signal is garbage collected.

Reimplementing the Paddle Game Example Let us reimplement the paddle game using *Froc* instead of lazy streams. We will follow the same structure as our stream-based FRP example: a scene behavior that represents the complete game state at each moment.

First, we introduce time explicitly (since *Froc* does not track it automatically):

```
open Froc
let clock, tick = make_event ()  (* clock event, tick to send it *)
let time = hold (Unix.gettimeofday ()) clock  (* Behavior: current time *)
```

The main loop will call `send tick current_time` at each frame. Now we can define integration. Note the use of `sample` to read the current value of a behavior – this is the impure style:

```
let integral fb =
  let aux (sum, t0) t1 =
    sum +. (t1 -. t0) *. sample fb, t1 in
  collect_b aux (0., sample time) clock
```

For convenience, the integral remembers the current upper limit of integration. It will be useful to get the integer part:

```
let integ_res fb =
  lift (fun (v, _) -> int_of_float v) (integral fb)
```

We can also define integration in *pure style*, which avoids calling `sample` inside the behavior definition:

```
let pair fa fb = lift2 (fun x y -> x, y) fa fb

let integral_nice fb =
  let samples = changes (pair fb time) in  (* Event when either changes *)
  let aux (sum, t0) (fv, t1) =
    sum +. (t1 -. t0) *. fv, t1 in
  collect_b aux (0., sample time) samples
```

The initial value `(0., sample time)` uses `sample`, but this is evaluated *once* when setting up the behavior, not inside the behavior definition itself, so it does not spoil the pure style.

10.7 Direct Control

The declarative style of FRP is elegant for continuous behaviors, but real-world interactions are often *state machines* that proceed through distinct stages. Consider a recipe: *1. Preheat the oven. 2. Put flour, sugar, eggs into a bowl. 3. Mix well. 4. Pour into pan.* Each step must complete before the next begins. How do we express this kind of sequential, staged behavior in FRP?

We want a *flow* that can proceed through events in sequence: when the first event arrives, we remember its result, and then wait for the next event. Crucially, we *ignore* any further occurrences of the first event after we have moved on. Standard FRP constructs like mapping events or attaching notifications do not give us this "move forward and never look back" semantics.

We also want to be able to *repeat* or *loop* a flow. But the loop should restart from the notification of the first event that arrives *after* the previous iteration completed – not from events that happened during the previous iteration.

The key primitive is `next e`, an event that propagates only the *first* occurrence of `e` and then goes silent. This will be the basis of our `await` function.

Additionally, the whole flow should be *cancellable* from outside at any time – for instance, when the user quits the application.

If this sounds familiar, it should: a flow is essentially a *lightweight thread* as we discussed at the end of Chapter 8. We will make it a monad. Unlike general threads, a flow only "stores" a non-unit value when it is suspended waiting for an event (via `await`). But it has a primitive to `emit` values. We are actually implementing *coarse-grained* threads (Chapter 8 exercise 11), with `await` playing the role of `suspend`.

We build a module `Flow` with monadic type `('a, 'b) flow`. The type has two parameters: `'a` is the type of values we emit (output), and `'b` is the type of values we store (the result of awaited events):

```
type ('a, 'b) flow
type cancellable  (* Handle to cancel a flow and stop further computation *)

val noop_flow : ('a, unit) flow  (* Do nothing, same as return () *)
val return : 'b -> ('a, 'b) flow  (* Immediately completed flow with result 'b *)
val await : 'b Froc.event -> ('a, 'b) flow  (* Suspend until event fires *)
val bind :   (* Sequential composition of flows *)
  ('a, 'b) flow -> ('b -> ('a, 'c) flow) -> ('a, 'c) flow
val emit : 'a -> ('a, unit) flow  (* Output a value *)
val cancel : cancellable -> unit  (* Cancel a running flow *)
val repeat :  (* Loop until the 'until' event fires; return that event's value *)
  ?until:'a Froc.event -> ('b, unit) flow -> ('b, 'a) flow
```

```
val event_flow :    (* Turn a flow into an event that fires on each emit *)
  ('a, unit) flow -> 'a Froc.event * cancellable
val behavior_flow :  (* Turn a flow into a behavior; initial value + flow to update *)
  'a -> ('a, unit) flow -> 'a Froc.behavior * cancellable
val is_cancelled : cancellable -> bool  (* Check if flow was cancelled *)
```

Implementation Details The implementation follows our lightweight threads from Chapter 8 (or the *Lwt* library), adapted for the needs of cancellation:

```
module F = Froc
type 'a result =
  | Return of 'a  (* Completed with value *)
  | Sleep of ('a -> unit) list * F.cancel ref list  (* Waiting for wakeup *)
  | Cancelled  (* Flow was cancelled *)
  | Link of 'a state  (* Indirection to another state *)
and 'a state = {mutable state : 'a result}
type cancellable = unit state  (* Handle to check/trigger cancellation *)
```

The `Sleep` state holds both waiters (callbacks to invoke when a result arrives) and a list of *Froc* cancel handles (for cancelling event notifications if the flow is cancelled).

Functions `find`, `wakeup`, `connect` are similar to Chapter 8, with the addition that connecting to a cancelled flow cancels the other flow as well.

The key insight is that our flow monad is actually a *reader monad* layered over the state. The reader environment supplies the `emit` function:

```
type ('a, 'b) flow = ('a -> unit) -> 'b state
```

The `return` and `bind` functions are as in our lightweight threads, but we need to handle cancelled flows: for `m = bind a b`, if `a` is cancelled then `m` is cancelled, and if `m` is cancelled then we do not wake up `b`:

```
let waiter x =
  if not (is_cancelled m)
  then connect m (b x emit) in
  ...
```

`await` is implemented like `next`, but it wakes up a flow:

```
let await t = fun emit ->
  let c = ref F.no_cancel in
  let m = {state = Sleep ([], [c])} in
  c :=
    F.notify_e_cancel t begin fun r ->
      F.cancel !c;
      c := F.no_cancel;
      wakeup m r
```

```
    end;
  m
```

`repeat` attaches the whole loop as a waiter for the loop body.

Example: Drawing Shapes Let us see flows in action with a simple drawing program. The user draws shapes by pressing and dragging the mouse; releasing the mouse closes the current shape and starts a new one.

The scene is a list of shapes, where the first shape is "open" (still being drawn) and the rest are closed:

```
type scene = (int * int) list list  (* First element is the open shape *)

let draw sc =
  let open Graphics in
  clear_graph ();
  (match sc with
  | [] -> ()
  | opn :: cld ->
    draw_poly_line (Array.of_list opn);  (* Draw open shape as line *)
    List.iter (fill_poly -| Array.of_list) cld);  (* Fill closed shapes *)
  synchronize ()
```

Now we build the drawing flow. Notice how naturally we can express the sequential logic: wait for button press, then repeatedly add points until button release, then start over:

```
let painter =
  let cld = ref [] in  (* Accumulated closed shapes *)
  repeat (perform  (* Outer loop: one shape per iteration *)
      await mbutton_pressed;  (* Wait for mouse button down *)
      let opn = ref [] in     (* Points in current shape *)
      repeat (perform  (* Inner loop: points in one shape *)
          mpos <-- await mouse_move;  (* Wait for mouse movement *)
          emit (opn := mpos :: !opn; !opn :: !cld))  (* Emit updated scene *)
        ~until:mbutton_released;  (* Exit inner loop on button release *)
      emit (cld := !opn :: !cld; opn := []; [] :: !cld))  (* Close shape *)

let painter, cancel_painter = behavior_flow [] painter
let () = reactimate painter  (* Run the animation *)
```

Flows and State Global state and thread-local state can both be used with flows, but you must pay careful attention to *when* expressions are evaluated. The key question is: is this computation *inside* the monad (executed when the flow runs), or is it executed *while building* the initial monadic value (executed once at setup time)?

Side effects hidden in `return` and `emit` *arguments* are evaluated immediately when constructing the flow, not when the flow runs. This leads to a subtle distinction:

```
let f =
  repeat (
      let* () = emit (Printf.printf "[0]\n%!"; '0') in  (* The printf runs NOW *)
      let* () = await aas in  (* Suspend until 'a' event *)
      let* () = emit (Printf.printf "[1]\n%!"; '1') in  (* Printf after resume *)
      let* () = await bs in
      let* () = emit (Printf.printf "[2]\n%!"; '2') in
      let* () = await cs in
      let* () = emit (Printf.printf "[3]\n%!"; '3') in
      let* () = await ds in
      emit (Printf.printf "[4]\n%!"; '4'))

let e, cancel_e = event_flow f
let () =
  F.notify_e e (fun c -> Printf.printf "flow: %c\n%!" c);
  Printf.printf "notification installed\n%!"

let () =
  F.send a (); F.send b (); F.send c (); F.send d ();
  F.send a (); F.send b (); F.send c (); F.send d ()
```

The output demonstrates this subtle timing:

- `[0]` – Printed only *once*, when building the loop (not inside the monad!)
- `notification installed` – Notification set up
- `event: a` – First event fires
- `[1]` – Now inside the monad, after first await returns
- `flow: 1` – Emitted value
- … continues through the remaining events and loop iterations

The key insight: `[0]` is in the *first line* of the loop before any `await`, so it is evaluated when constructing the `repeat` expression. The `Printf.printf` in subsequent `emit` calls is after a bind (after an `await`), so it runs each time that point in the flow is reached.

10.8 Graphical User Interfaces

An in-depth discussion of GUIs is beyond the scope of this course. However, GUIs are a natural application of FRP and flows, so we will cover enough to build a complete example: a calculator.

We demonstrate two OCaml GUI libraries. *LablTk* (based on the Tk toolkit from Tcl) uses optional labelled arguments (discussed in Chapter 2 exercise 2) and polymorphic variants. *LablGTk* (based on GTK+) additionally uses objects.

We will learn more about objects and polymorphic variants in the next chapter.

Calculator Flow The calculator is a perfect example of a state machine with sequential stages. We represent its mechanics directly as a flow:

```
let digits, digit = F.make_event ()  (* Events for digit button presses *)
let ops, op = F.make_event ()        (* Events for operator button presses *)
let dots, dot = F.make_event ()      (* Event for decimal point (exercise) *)

let calc =
  (* Two state variables: current number and pending operation *)
  let f = ref (fun x -> x) and now = ref 0.0 in
  repeat (
      (* Phase 1: Enter digits of a number *)
      let* op = repeat (
            let* d = await digits in  (* Wait for digit press *)
            emit (now := 10. *. !now +. d; !now))  (* Build up number *)
        ~until:ops in  (* Until operator button is pressed *)
      (* Phase 2: Apply pending operation, store new operator *)
      let* () = emit (now := !f !now; f := op !now; !now) in
      (* Phase 3: Allow user to change operator before entering next number *)
      let* d = repeat
        (let* op = await ops in return (f := op !now))
        ~until:digits in  (* Until they start entering the next number *)
      (* Phase 4: Reset for the new number *)
      emit (now := d; !now))

let calc_e, cancel_calc = event_flow calc  (* Event notifies display update *)
```

Notice how the flow structure directly mirrors the user interaction pattern: enter a number, press an operator, optionally change your mind about the operator, enter another number, and so on.

Tk: LablTk The *Tk* widget toolkit originated with the Tcl language and is known for its simplicity. *LablTk* provides OCaml bindings using labelled arguments.

First, we define the layout of our calculator buttons – this part is the same regardless of which GUI toolkit we use:

```
let layout = [|
  [|"7", `Di 7.; "8", `Di 8.; "9", `Di 9.; "+", `O (+.)|];
  [|"4", `Di 4.; "5", `Di 5.; "6", `Di 6.; "-", `O (-.)|];
  [|"1", `Di 1.; "2", `Di 2.; "3", `Di 3.; "*", `O ( *.)|];
  [|"0", `Di 0.; ".", `Dot;   "=",  `O sk; "/", `O (/.)|]
|]
```

Each entry is a pair of the button label and its action: `` `Di d `` means send digit `d`, `` `O f `` means send operator function `f`, and `` `Dot `` means send the decimal point event (handling this is left as an exercise).

Key GUI concepts in Tk:

- Every *widget* (window gadget) has a *parent* widget in which it is located
- *Buttons* have an action (callback function) invoked when pressed; *labels* just display information; *entries* (text fields) allow keyboard input
- Actions are *callback* functions passed as the `~command` argument
- *Frames* group related widgets together
- The parent widget is passed as the last argument, after optional labelled arguments

```
let top = Tk.openTk ()  (* Open the main window *)

let btn_frame =
  Frame.create ~relief:`Groove ~borderwidth:2 top  (* Container for buttons *)

let buttons =
  Array.map (Array.map (function
    | text, `Dot ->
      Button.create ~text
        ~command:(fun () -> F.send dot ()) btn_frame
    | text, `Di d ->
      Button.create ~text
        ~command:(fun () -> F.send digit d) btn_frame  (* Send digit event *)
    | text, `O f ->
      Button.create ~text
        ~command:(fun () -> F.send op f) btn_frame))  (* Send operator event *)
    layout

let result = Label.create ~text:"0" ~relief:`Sunken top  (* Result display *)
```

GUI toolkits provide layout algorithms, so we only specify *which* widgets go together and *how* they should fill space. Tk offers `pack` for sequential layout and `grid` for table-like organization:

Common layout options:

- `~fill:` how the widget fills allocated space: `` `X ``, `` `Y ``, `` `Both `` or `` `None ``
- `~expand:` whether to request extra space (`true`) or only what is needed (`false`)
- `~anchor:` glue the widget to a direction: `` `Center ``, `` `E ``, `` `Ne ``, etc.
- `grid` also supports `~columnspan` and `~rowspan` for multi-cell widgets
- `configure` functions change existing widgets using the same arguments as `create`

```
let () =
```

```
  Wm.title_set top "Calculator";
  Tk.pack [result] ~side:`Top ~fill:`X;  (* Result at top, fills width *)
  Tk.pack [btn_frame] ~side:`Bottom ~expand:true;  (* Buttons below *)
  Array.iteri (fun column -> Array.iteri (fun row button ->
    Tk.grid ~column ~row [button])) buttons;  (* Grid layout for buttons *)
  Wm.geometry_set top "200x200";
  (* Connect Froc event to update the display *)
  F.notify_e calc_e
    (fun now ->
      Label.configure ~text:(string_of_float now) result);
  Tk.mainLoop ()  (* Enter the GUI event loop *)
```

GTk+: LablGTk *LablGTk* provides OCaml bindings for the *GTk+* library (written in C). Unlike LablTk, it uses an object-oriented interface: widgets are objects, and operations are method calls.

In OCaml's object system, fields are only visible to the object's own methods, and methods are called with `#` syntax: e.g., `window#show ()`.

GTk+ has its own reactive event system (confusingly, GTk+ uses "signal" where we say "event"):

- Registering a callback is called *connecting a signal handler*: `button#connect#clicked ~callback:hello` takes `~callback:(unit -> unit)` and returns a `GtkSignal.id`
- Multiple handlers can be attached to the same signal, just like *Froc* notifications
- GTk+ *events* (note: different from signals) relate to window-system events: `window#event#connect#delete ~callback:delete_event`
- Event callbacks receive more information: `~callback:(event -> unit)` for some event type

GTk+ layout is simpler than Tk's:

- Only horizontal (`hbox`) and vertical (`vbox`) boxes are available
- Grid layout is called `table`, with `~fill` and `~expand` taking `` `X``, `` `Y``, `` `BOTH``, `` `NONE``

A few API differences: `coerce` is a method that casts widget types (Tk uses a `coe` function). Labels do not have a dedicated module. Widget properties are set via `widget#set_X` methods rather than a single `configure` function.

Here is the GTk+ version of our calculator. First, setting up the window and layout:

```
let _ = GtkMain.Main.init ()  (* Initialize GTk+ *)
let window =
  GWindow.window ~width:200 ~height:200 ~title:"Calculator" ()
let top = GPack.vbox ~packing:window#add ()  (* Vertical box container *)
```

```
let result = GMisc.label ~text:"0" ~packing:top#add ()  (* Result display *)
let btn_frame =
  GPack.table ~rows:(Array.length layout)
   ~columns:(Array.length layout.(0)) ~packing:top#add ()  (* Button grid *)
```

Creating the buttons and connecting their click handlers to *Froc* events:

```
let buttons =
  Array.map (Array.map (function
    | label, `Dot ->
      let b = GButton.button ~label () in
      let _ = b#connect#clicked
        ~callback:(fun () -> F.send dot ()) in b
    | label, `Di d ->
      let b = GButton.button ~label () in
      let _ = b#connect#clicked
        ~callback:(fun () -> F.send digit d) in b
    | label, `O f ->
      let b = GButton.button ~label () in
      let _ = b#connect#clicked
        ~callback:(fun () -> F.send op f) in b)) layout
```

Finally, we attach buttons to the grid, connect the result notification, and start the application:

```
let delete_event _ = GMain.Main.quit (); false  (* Handle window close *)

let () =
  let _ = window#event#connect#delete ~callback:delete_event in
  Array.iteri (fun column -> Array.iteri (fun row button ->
    btn_frame#attach ~left:column ~top:row
      ~fill:`BOTH ~expand:`BOTH (button#coerce))  (* Attach to grid *)
  ) buttons;
  (* Connect Froc event to update the display *)
  F.notify_e calc_e
    (fun now -> result#set_label (string_of_float now));
  window#show ();  (* Make window visible *)
  GMain.Main.main ()  (* Enter the GTk+ event loop *)
```

10.9 Exercises

Exercise 1: Introduce operators $-, /$ into the context rewriting "pull out subexpression" example. Remember that they are not commutative.

Exercise 2: Add to the *paddle game* example: 1. game restart, 2. score keeping, 3. game quitting (in more-or-less elegant way).

Exercise 3: Our numerical integration function roughly corresponds to the rectangle rule. Modify the rule and write a test for the accuracy of: 1. the trapezoidal rule; 2. the Simpson's rule. See http://en.wikipedia.org/wiki/Simpson%27s_rule

Exercise 4: Explain the recursive behavior of integration: 1. In *paddle game* implemented by stream processing (`Lec10b.ml`), do we look at past velocity to determine current position, at past position to determine current velocity, both, or neither? 2. What is the difference between `integral` and `integral_nice` in `Lec10c.ml`, what happens when we replace the former with the latter in the `pbal` function? How about after rewriting `pbal` into pure style as in the following exercise?

Exercise 5: Reimplement the *Froc* based paddle ball example in a pure style: rewrite the `pbal` function to not use `notify_e`.

Exercise 6: Our implementation of flows is a bit heavy. One alternative approach is to use continuations, as in `Scala.React`. OCaml has a continuations library *Delimcc*; for how it can cooperate with *Froc*, see http://ambassadortothecomputers.blogspot.com/2010/08/mixing-monadic-and-direct-style-code.html

Exercise 7: Implement `parallel` for flows, retaining coarse-grained implementation and using the event queue from *Froc* somehow (instead of introducing a new job queue).

Exercise 8: Add quitting, e.g. via a `'q'` key press, to the *painter* example. Use the `is_cancelled` function.

Exercise 9: Our calculator example is not finished. Implement entering decimal fractions: add handling of the `dots` event.

Exercise 10: The Flow module has reader monad functions that have not been discussed in this chapter:

```
let local f m = fun emit -> m (fun x -> emit (f x))
let local_opt f m = fun emit ->
  m (fun x -> match f x with None -> () | Some y -> emit y)

val local : ('a -> 'b) -> ('a, 'c) flow -> ('b, 'c) flow
val local_opt : ('a -> 'b option) -> ('a, 'c) flow -> ('b, 'c) flow
```

Implement an example that uses this compositionality-increasing capability.

Chapter 11: The Expression Problem

This chapter explores **the expression problem**, a classic challenge in software engineering that addresses how to design systems that can be extended with both new data variants and new operations without modifying existing code, while maintaining static type safety. The expression problem lies at the heart of code organization, extensibility, and reuse, so understanding the various solutions helps us write more maintainable and flexible software.

We will examine multiple approaches in OCaml, ranging from algebraic data types through object-oriented programming to polymorphic variants with recursive modules. Each approach has different trade-offs in terms of type safety, code organization, and ease of use. The chapter concludes with a practical application: parser combinators with dynamic code loading, demonstrating how these techniques apply to real-world problems.

11.1 The Expression Problem: Definition

The **Expression Problem** concerns the design of an implementation for expressions where:

- **Datatype extensibility**: New variants of expressions can be added
- **Functional extensibility**: New operations on expressions can be added

By *extensibility* we mean three conditions:

1. **Code-level modularization**: The new datatype variants and new operations are in separate files
2. **Separate compilation**: The files can be compiled and distributed separately
3. **Static type safety**: We do not lose type checking help and guarantees

The name comes from a classic example: extending a language of expressions with new constructs. Consider two sub-languages:

- **Lambda calculus**: variables `Var`, λ-abstractions `Abs`, function applications `App`
- **Arithmetic**: variables `Var`, constants `Num`, addition `Add`, multiplication `Mult`

And operations we want to support:

- Evaluation `eval`
- Pretty-printing to strings `string_of`
- Free variables computation `free_vars`

The challenge is to combine these sub-languages and add new operations without breaking existing code or sacrificing type safety. This is a fundamental tension in programming language design: functional languages typically make it easy to add new operations (just write a new function with pattern matching), while object-oriented languages typically make it easy to add new data variants (just add a new subclass). Finding a solution that provides both kinds of extensibility simultaneously, with static type safety and separate compilation, is the essence of the expression problem.

References

- Ralf Lammel lectures on MSDN's Channel 9: The Expression Problem, Haskell's Type Classes

- The book *Developing Applications with Objective Caml*: Comparison of Modules and Objects, Extending Components
- *Real World OCaml*: Chapter 11: Objects, Chapter 12: Classes
- Jacques Garrigue's Code reuse through polymorphic variants, and Recursive Modules for Programming with Keiko Nakata
- Extensible variant types
- Graham Hutton's and Erik Meijer's Monadic Parser Combinators

11.2 Functional Programming Non-Solution: Ordinary Algebraic Datatypes

Pattern matching makes **functional extensibility** easy in functional programming. When we want to add a new operation, we simply write a new function that pattern-matches on the existing datatype. However, ensuring **datatype extensibility** is complicated when using standard variant types, because adding a new variant requires modifying the type definition and all functions that pattern-match on it.

For brevity, we place examples in a single file, but the component type and function definitions are not mutually recursive, so they can be put in separate modules for separate compilation.

Non-solution penalty points:

- Functions implemented for a broader language (e.g., `lexpr_t`) cannot be used with a value from a narrower language (e.g., `expr_t`). This breaks the intuition that a smaller language should be usable wherever a larger one is expected.
- Significant memory (and some time) overhead due to *tagging*: the work of the `wrap` and `unwrap` functions, adding tags such as `Lambda` and `Expr` to distinguish which sub-language an expression belongs to.
- Some code bloat due to tagging. For example, deep pattern matching needs to be manually unrolled and interspersed with calls to `unwrap`, making the code harder to read and maintain.

Verdict: Non-solution, but better than the extensible variant types-based approach and the direct OOP approach.

Here is the implementation. Note how we use type parameters and wrap/unwrap functions to achieve a form of extensibility:

```
type var = string  (* Variables constitute a sub-language of its own *)
                   (* We treat this sub-language slightly differently --
                      no need for a dedicated variant *)

let eval_var wrap sub (s : var) =
  try List.assoc s sub with Not_found -> wrap s

type 'a lambda =  (* Here we define the sub-language of lambda-expressions *)
```

```
  VarL of var | Abs of string * 'a | App of 'a * 'a

(* During evaluation, we need to freshen variables to avoid capture *)
(* (mistaking distinct variables with the same name) *)
let gensym = let n = ref 0 in fun () -> incr n; "_" ^ string_of_int !n

let eval_lambda eval_rec wrap unwrap subst e =
  match unwrap e with  (* Alternatively, unwrapping could use an exception *)
  | Some (VarL v) -> eval_var (fun v -> wrap (VarL v)) subst v
  | Some (App (l1, l2)) ->  (* but we use the option type as it is safer *)
    let l1' = eval_rec subst l1  (* and more flexible in this context *)
    and l2' = eval_rec subst l2 in  (* Recursive processing function returns expression *)
    (match unwrap l1' with  (* of the completed language, we need *)
    | Some (Abs (s, body)) ->  (* to unwrap it into the current sub-language *)
      eval_rec [s, l2'] body  (* The recursive call is already wrapped *)
    | _ -> wrap (App (l1', l2')))  (* Wrap into the completed language *)
  | Some (Abs (s, l1)) ->
    let s' = gensym () in  (* Rename variable to avoid capture (alpha-equivalence) *)
    wrap (Abs (s', eval_rec ((s, wrap (VarL s'))::subst) l1))
  | None -> e  (* Falling-through when not in the current sub-language *)

type lambda_t = Lambda_t of lambda_t lambda  (* Defining lambdas as the completed language *

let rec eval1 subst =  (* and the corresponding eval function *)
  eval_lambda eval1
    (fun e -> Lambda_t e) (fun (Lambda_t e) -> Some e) subst
```

Now we define the arithmetic sub-language:

```
type 'a expr =  (* The sub-language of arithmetic expressions *)
  VarE of var | Num of int | Add of 'a * 'a | Mult of 'a * 'a

let eval_expr eval_rec wrap unwrap subst e =
  match unwrap e with
  | Some (Num _) -> e
  | Some (VarE v) ->
    eval_var (fun x -> wrap (VarE x)) subst v
  | Some (Add (m, n)) ->
    let m' = eval_rec subst m
    and n' = eval_rec subst n in
    (match unwrap m', unwrap n' with  (* Unwrapping to check if the subexpressions *)
    | Some (Num m'), Some (Num n') ->  (* got computed to values *)
      wrap (Num (m' + n'))
    | _ -> wrap (Add (m', n')))  (* Here m' and n' are wrapped *)
  | Some (Mult (m, n)) ->
    let m' = eval_rec subst m
    and n' = eval_rec subst n in
```

```
    (match unwrap m', unwrap n' with
    | Some (Num m'), Some (Num n') ->
      wrap (Num (m' * n'))
    | _ -> wrap (Mult (m', n')))
  | None -> e

type expr_t = Expr_t of expr_t expr  (* Defining arithmetic as the completed language *)

let rec eval2 subst =  (* aka "tying the recursive knot" *)
  eval_expr eval2
    (fun e -> Expr_t e) (fun (Expr_t e) -> Some e) subst
```

Finally, we merge the two sub-languages. The key insight is that we can compose evaluators by using the "fall-through" property: when one evaluator does not recognize an expression (returning it unchanged via the `None` case), we pass it to the next evaluator:

```
type 'a lexpr =  (* The language merging lambda-expressions and arithmetic expressions *)
  Lambda of 'a lambda | Expr of 'a expr  (* can also be used in further extensions *)

let eval_lexpr eval_rec wrap unwrap subst e =
  eval_lambda eval_rec
    (fun e -> wrap (Lambda e))
    (fun e ->
      match unwrap e with
      | Some (Lambda e) -> Some e
      | _ -> None)
    subst
    (eval_expr eval_rec  (* We use the "fall-through" property of eval_expr *)
       (fun e -> wrap (Expr e))  (* to combine the evaluators *)
       (fun e ->
         match unwrap e with
         | Some (Expr e) -> Some e
         | _ -> None)
       subst e)

type lexpr_t = LExpr_t of lexpr_t lexpr  (* Tying the recursive knot one last time *)

let rec eval3 subst =
  eval_lexpr eval3
    (fun e -> LExpr_t e)
    (fun (LExpr_t e) -> Some e) subst
```

11.3 Lightweight FP Non-Solution: Extensible Variant Types

Exceptions have always formed an extensible variant type in OCaml, whose pattern matching is done using the `try...with` syntax. Since recently, new extensible variant types can be defined using the `type t = ..` syntax. This augments the normal function extensibility of FP with straightforward data extensibility, providing a seemingly elegant solution.

The syntax is simple: `type expr = ..` declares an extensible type, and `type expr += Var of string` adds a new variant case to it. This mirrors how exceptions work in OCaml, but for arbitrary types.

Non-solution penalty points:

- **Giving up exhaustivity checking**, which is an important aspect of static type safety. The compiler cannot warn you when you forget to handle a case, because new cases can be added at any time.
- More natural with "single inheritance" extension chains, although merging is possible and demonstrated in our example. The sub-languages are not differentiated by types, which is a significant shortcoming.
- Requires "tying the recursive knot" for functions, similar to the previous approach.

Verdict: Pleasant-looking, but arguably the worst approach because of possible bugginess. The loss of exhaustivity checking means that bugs from unhandled cases will only be discovered at runtime. However, if bug-proneness is not a concern (e.g., for rapid prototyping), this is actually the most concise approach.

```
type expr = ..  (* This is how extensible variant types are defined *)

type var_name = string
type expr += Var of string  (* We add a variant case *)

let eval_var sub = function
  | Var s as v -> (try List.assoc s sub with Not_found -> v)
  | e -> e

let gensym = let n = ref 0 in fun () -> incr n; "_" ^ string_of_int !n

type expr += Abs of string * expr | App of expr * expr
(* The sub-languages are not differentiated by types, a shortcoming of this non-solution *)

let eval_lambda eval_rec subst = function
  | Var _ as v -> eval_var subst v
  | App (l1, l2) ->
    let l2' = eval_rec subst l2 in
    (match eval_rec subst l1 with
    | Abs (s, body) ->
      eval_rec [s, l2'] body
    | l1' -> App (l1', l2'))
  | Abs (s, l1) ->
    let s' = gensym () in
```

```
    Abs (s', eval_rec ((s, Var s')::subst) l1)
  | e -> e

let freevars_lambda freevars_rec = function
  | Var v -> [v]
  | App (l1, l2) -> freevars_rec l1 @ freevars_rec l2
  | Abs (s, l1) ->
    List.filter (fun v -> v <> s) (freevars_rec l1)
  | _ -> []

let rec eval1 subst e = eval_lambda eval1 subst e
let rec freevars1 e = freevars_lambda freevars1 e

let test1 = App (Abs ("x", Var "x"), Var "y")
let e_test = eval1 [] test1
let fv_test = freevars1 test1
```

Now we extend with arithmetic:

```
type expr += Num of int | Add of expr * expr | Mult of expr * expr

let map_expr f = function
  | Add (e1, e2) -> Add (f e1, f e2)
  | Mult (e1, e2) -> Mult (f e1, f e2)
  | e -> e

let eval_expr eval_rec subst e =
  match map_expr (eval_rec subst) e with
  | Add (Num m, Num n) -> Num (m + n)
  | Mult (Num m, Num n) -> Num (m * n)
  | (Num _ | Add _ | Mult _) as e -> e
  | e -> e

let freevars_expr freevars_rec = function
  | Num _ -> []
  | Add (e1, e2) | Mult (e1, e2) -> freevars_rec e1 @ freevars_rec e2
  | _ -> []

let rec eval2 subst e = eval_expr eval2 subst e
let rec freevars2 e = freevars_expr freevars2 e

let test2 = Add (Mult (Num 3, Var "x"), Num 1)
let e_test2 = eval2 [] test2
let fv_test2 = freevars2 test2
```

Merging the sub-languages:

```
let eval_lexpr eval_rec subst e =
```

```
  eval_expr eval_rec subst (eval_lambda eval_rec subst e)

let freevars_lexpr freevars_rec e =
  freevars_lambda freevars_rec e @ freevars_expr freevars_rec e

let rec eval3 subst e = eval_lexpr eval3 subst e
let rec freevars3 e = freevars_lexpr freevars3 e

let test3 =
  App (Abs ("x", Add (Mult (Num 3, Var "x"), Num 1)),
       Num 2)
let e_test3 = eval3 [] test3
let fv_test3 = freevars3 test3
```

11.4 Object-Oriented Programming: Subtyping

Before examining OOP solutions to the expression problem, let us understand OCaml's object system.

OCaml's **objects** are values, somewhat similar to records. Viewed from the outside, an OCaml object has only **methods**, identifying the code with which to respond to messages (method invocations). All methods are **late-bound**; the object determines what code is run (i.e., *virtual* in C++ parlance). This is in contrast to records, where field access is resolved at compile time.

Subtyping determines if an object can be used in some context. OCaml has **structural subtyping**: the content of the types concerned (the methods they provide) decides if an object can be used, not the name of the type or class. Parametric polymorphism can be used to infer if an object has the required methods.

```
let f x = x#m  (* Method invocation: object#method *)
(* val f : < m : 'a; .. > -> 'a *)
(* Type polymorphic in two ways: 'a is the method type, *)
(* .. means that objects with more methods will be accepted *)
```

Methods are computed when they are invoked, even if they do not take arguments (unlike record fields, which are computed once when the record is created). We define objects inside `object...end` (compare: records `{...}`) using keywords:

- `method` for methods (always late-bound)
- `val` for constant fields (only accessible within the object)
- `val mutable` for mutable fields

Constructor arguments can often be used instead of constant fields. Here is a simple example:

```
let square w = object
  method area = float_of_int (w * w)
```

```
  method width = w
end
```

Subtyping often needs to be explicit: we write `(object :> supertype)` or in more complex cases `(object : type :> supertype)`.

Technically speaking, subtyping in OCaml always is explicit, and *open types*, containing `..`, use **row polymorphism** rather than subtyping.

```
let a = object method m = 7  method x = "a" end  (* Toy example: object types *)
let b = object method m = 42 method y = "b" end  (* share some but not all methods *)

(* let l = [a; b]  -- Error: the exact types of the objects do not agree *)
(* Error: This expression has type < m : int; y : string >
         but an expression was expected of type < m : int; x : string >
         The second object type has no method y *)

let l = [(a :> <m : 'a>); (b :> <m : 'a>)]  (* But the types share a supertype *)
(* val l : < m : int > list *)
```

Object-Oriented Programming: Inheritance The system of object classes in OCaml is similar to the module system. Object classes are not types; rather, classes are a way to build object *constructors*, which are functions that return objects. Classes have their types, called class types (compare: modules and signatures).

In OCaml parlance:

- **Late binding** is not called anything special, since all methods are late-bound (called *virtual* in C++)
- A method or field declared to be defined in sub-classes is called **virtual** (called *abstract* in C++); classes that use virtual methods or fields are also called virtual
- A method that is only visible in sub-classes is called **private** (called *protected* in C++)
- A method not visible outside the class is achieved by omitting it from the class type (called *private* in C++) – you provide the type for the class and omit the method in the class type, similar to module signatures and `.mli` files

OCaml allows **multiple inheritance**, which can be used to implement *mixins* as virtual/abstract classes. Inheritance works somewhat similarly to textual inclusion: the inherited class's methods and fields are copied into the inheriting class, but with late binding preserved.

The `{< ... >}` syntax creates a *clone* of the current object with some fields changed. This is essential for functional-style object programming, where we create new objects rather than mutating existing ones.

11.5 Direct Object-Oriented Non-Solution

It turns out that although object-oriented programming was designed with data extensibility in mind, it is a bad fit for recursive types like those in the expression problem. Below is an attempt at solving our problem using classes.

We can try to solve the expression problem using objects directly. However, adding new functionality still requires modifying old code, so this approach does not fully solve the expression problem.

Non-solution penalty points:

- No way to add functionality without modifying old code (in particular, the abstract class and all concrete classes must be extended with new methods)
- Functions implemented for a broader language cannot handle values from a narrower one
- No deep pattern matching: we cannot examine the structure of nested expressions

Verdict: Non-solution, and probably the worst approach.

Here is an implementation using objects. The abstract class `evaluable` defines the interface that all expression objects must implement. For lambda calculus, we need helper methods: `rename` for renaming free variables (needed for alpha-conversion), and `apply` for beta-reduction when possible:

```
type var_name = string

let gensym = let n = ref 0 in fun () -> incr n; "_" ^ string_of_int !n

class virtual ['lang] evaluable =
object
  method virtual eval : (var_name * 'lang) list -> 'lang
  method virtual rename : var_name -> var_name -> 'lang
  method apply (_arg : 'lang)
    (fallback : unit -> 'lang) (_subst : (var_name * 'lang) list) =
    fallback ()
end

class ['lang] var (v : var_name) =
object (self)  (* We name the current object `self` for later reference *)
  inherit ['lang] evaluable
  val v = v
  method eval subst =
    try List.assoc v subst with Not_found -> self
  method rename v1 v2 =  (* Renaming a variable: *)
    if v = v1 then {< v = v2 >} else self  (* clone with new name if matched *)
end
```

```
class ['lang] abs (v : var_name) (body : 'lang) =
object (self)
  inherit ['lang] evaluable
  val v = v
  val body = body
  method eval subst =  (* We do alpha-conversion prior to evaluation *)
    let v' = gensym () in  (* Generate fresh name to avoid capture *)
    {< v = v'; body = (body#rename v v')#eval subst >}
  method rename v1 v2 =  (* Renaming the free variable v1 *)
    if v = v1 then self  (* If v=v1, then v1 is bound here, not free -- no work *)
    else {< body = body#rename v1 v2 >}
  method apply arg _ subst =  (* Beta-reduction: substitute arg for v in body *)
    body#eval ((v, arg)::subst)
end

class ['lang] app (f : 'lang) (arg : 'lang) =
object (self)
  inherit ['lang] evaluable
  val f = f
  val arg = arg
  method eval subst =  (* We use `apply` to differentiate between f=abs *)
    let arg' = arg#eval subst in  (* (beta-redexes) and f<>abs *)
    f#apply arg' (fun () -> {< f = f#eval subst; arg = arg' >}) subst
  method rename v1 v2 =  (* Cloning ensures result is subtype of 'lang *)
    {< f = f#rename v1 v2; arg = arg#rename v1 v2 >}  (* not just 'lang app *)
end

type evaluable_t = evaluable_t evaluable
let new_var1 v : evaluable_t = new var v
let new_abs1 v (body : evaluable_t) : evaluable_t = new abs v body
let new_app1 (arg1 : evaluable_t) (arg2 : evaluable_t) : evaluable_t =
  new app arg1 arg2

let test1 = new_app1 (new_abs1 "x" (new_var1 "x")) (new_var1 "y")
let e_test1 = test1#eval []
```

Extending with arithmetic requires additional mixins. To use lambda-expressions together with arithmetic expressions, we need to upgrade them with a helper method `compute` that returns the numeric value if one exists:

```
class virtual compute_mixin = object
  method compute : int option = None
end

class ['lang] var_c v = object
  inherit ['lang] var v
  inherit compute_mixin
```

```
end

class ['lang] abs_c v body = object
  inherit ['lang] abs v body
  inherit compute_mixin
end

class ['lang] app_c f arg = object
  inherit ['lang] app f arg
  inherit compute_mixin
end

class ['lang] num (i : int) =
object (self)
  inherit ['lang] evaluable
  val i = i
  method eval _subst = self
  method rename _ _ = self
  method compute = Some i
end

class virtual ['lang] operation
    (num_inst : int -> 'lang) (n1 : 'lang) (n2 : 'lang) =
object (self)
  inherit ['lang] evaluable
  val n1 = n1
  val n2 = n2
  method eval subst =
    let self' = {< n1 = n1#eval subst; n2 = n2#eval subst >} in
    match self'#compute with
    | Some i -> num_inst i
    | _ -> self'
  method rename v1 v2 = {< n1 = n1#rename v1 v2; n2 = n2#rename v1 v2 >}
end

class ['lang] add num_inst n1 n2 =
object (self)
  inherit ['lang] operation num_inst n1 n2
  method compute =
    match n1#compute, n2#compute with
    | Some i1, Some i2 -> Some (i1 + i2)
    | _ -> None
end

class ['lang] mult num_inst n1 n2 =
object (self)
```

```
  inherit ['lang] operation num_inst n1 n2
  method compute =
    match n1#compute, n2#compute with
    | Some i1, Some i2 -> Some (i1 * i2)
    | _ -> None
end

class virtual ['lang] computable =
object
  inherit ['lang] evaluable
  inherit compute_mixin
end

type computable_t = computable_t computable
let new_var2 v : computable_t = new var_c v
let new_abs2 v (body : computable_t) : computable_t = new abs_c v body
let new_app2 v (body : computable_t) : computable_t = new app_c v body
let new_num2 i : computable_t = new num i
let new_add2 (n1 : computable_t) (n2 : computable_t) : computable_t =
  new add new_num2 n1 n2
let new_mult2 (n1 : computable_t) (n2 : computable_t) : computable_t =
  new mult new_num2 n1 n2

let test2 =
  new_app2 (new_abs2 "x" (new_add2 (new_mult2 (new_num2 3) (new_var2 "x"))
                              (new_num2 1)))
    (new_num2 2)
let e_test2 = test2#eval []
```

11.6 OOP Non-Solution: The Visitor Pattern

The **visitor pattern** is an object-oriented programming pattern for turning objects into variants with shallow pattern-matching (i.e., dispatch based on which variant a value is). It effectively replaces data extensibility with operation extensibility: instead of being able to add new data variants easily, we can add new operations easily.

The key idea is that each data variant has an `accept` method that takes a visitor object and calls the appropriate `visit` method on it. This inverts the usual pattern matching: instead of the function choosing which branch to take based on the data, the data chooses which method to call on the visitor.

Non-solution penalty points:

- Adding new functionality requires modifying old code (the abstract visitor class must declare new `visit` methods)
- Heavy code bloat compared to pattern matching

- No deep pattern matching: we can only dispatch on the outermost constructor
- Side-effects appear to be required for returning results (we store computation results in mutable fields because keeping the visitor polymorphic while having the result type depend on the visitor is difficult)

Verdict: Poor solution, better than approaches we considered so far, and worse than approaches we consider next.

```
type 'visitor visitable = < accept : 'visitor -> unit >
(* The variants need be visitable *)
(* We store the computation as side effect because of the difficulty *)
(* to keep the visitor polymorphic but have the result type depend on the visitor *)

type var_name = string

class ['visitor] var (v : var_name) =
object (self)  (* The 'visitor will determine the (sub)language *)
               (* to which a given var variant belongs *)
  method v = v
  method accept : 'visitor -> unit =  (* The visitor pattern inverts the way *)
    fun visitor -> visitor#visitVar self  (* pattern matching proceeds: the variant *)
end  (* selects the computation *)
let new_var v = (new var v :> 'a visitable)

class ['visitor] abs (v : var_name) (body : 'visitor visitable) =
object (self)
  method v = v
  method body = body
  method accept : 'visitor -> unit =
    fun visitor -> visitor#visitAbs self
end
let new_abs v body = (new abs v body :> 'a visitable)

class ['visitor] app (f : 'visitor visitable) (arg : 'visitor visitable) =
object (self)
  method f = f
  method arg = arg
  method accept : 'visitor -> unit =
    fun visitor -> visitor#visitApp self
end
let new_app f arg = (new app f arg :> 'a visitable)

class virtual ['visitor] lambda_visit =
object
  method virtual visitVar : 'visitor var -> unit
  method virtual visitAbs : 'visitor abs -> unit
```

```
  method virtual visitApp : 'visitor app -> unit
end

let gensym = let n = ref 0 in fun () -> incr n; "_" ^ string_of_int !n

class ['visitor] eval_lambda
  (subst : (var_name * 'visitor visitable) list)
  (result : 'visitor visitable ref) =
object (self)
  inherit ['visitor] lambda_visit
  val mutable subst = subst
  val mutable beta_redex : (var_name * 'visitor visitable) option = None
  method visitVar var =
    beta_redex <- None;
    try result := List.assoc var#v subst
    with Not_found -> result := (var :> 'visitor visitable)
  method visitAbs abs =
    let v' = gensym () in
    let orig_subst = subst in
    subst <- (abs#v, new_var v')::subst;
    (abs#body)#accept self;
    let body' = !result in
    subst <- orig_subst;
    beta_redex <- Some (v', body');
    result := new_abs v' body'
  method visitApp app =
    app#arg#accept self;
    let arg' = !result in
    app#f#accept self;
    let f' = !result in
    match beta_redex with
    | Some (v', body') ->
      beta_redex <- None;
      let orig_subst = subst in
      subst <- (v', arg')::subst;
      body'#accept self;
      subst <- orig_subst
    | None -> result := new_app f' arg'
end

class ['visitor] freevars_lambda (result : var_name list ref) =
object (self)
  inherit ['visitor] lambda_visit
  method visitVar var =
    result := var#v :: !result
  method visitAbs abs =
```

```
    (abs#body)#accept self;
    result := List.filter (fun v' -> v' <> abs#v) !result
  method visitApp app =
    app#arg#accept self; app#f#accept self
end

type lambda_visit_t = lambda_visit_t lambda_visit
type lambda_t = lambda_visit_t visitable

let eval1 (e : lambda_t) subst : lambda_t =
  let result = ref (new_var "") in
  e#accept (new eval_lambda subst result :> lambda_visit_t);
  !result

let freevars1 (e : lambda_t) =
  let result = ref [] in
  e#accept (new freevars_lambda result);
  !result

let test1 =
  (new_app (new_abs "x" (new_var "x")) (new_var "y") :> lambda_t)
let e_test = eval1 test1 []
let fv_test = freevars1 test1
```

Extending with arithmetic expressions follows a similar pattern, and the merged language visitor inherits from both `lambda_visit` and `expr_visit`.

11.7 Polymorphic Variants

Polymorphic variants provide a flexible alternative to standard variants. They are to ordinary variants as objects are to records: both enable *open types* and subtyping, both allow different types to share the same components.

Interestingly, they are *dual* concepts: if we replace "product" of records/objects by "sum" (as we discussed in earlier chapters), we get variants/polymorphic variants. This duality implies many behaviors are opposite. For example:

- While object subtypes have *more* methods, polymorphic variant subtypes have *fewer* tags
- The `>` sign means "these tags or more" (open for adding tags)
- The `<` sign means "these tags or less" (closed to these tags only)
- No sign means a closed type

Because distinct polymorphic variant types can share the same tags, the solution to the Expression Problem becomes straightforward: we can define sub-languages with overlapping tags and compose them.

Penalty points:

- Requires explicit type annotations more often than regular variants
- Requires "tying the recursive knots" for types, e.g., `type lambda_t = lambda_t lambda`
- The need to tie the recursive knot separately at both the type level and the function level. At the function level, an eta-expansion is sometimes required due to the *value recursion* problem
- There can be a slight time cost compared to the visitor pattern: additional dispatch at each level of type aggregation (i.e., merging sub-languages)

Verdict: A flexible and concise solution, second-best place overall.

```
type var = [`Var of string]

let eval_var sub (`Var s as v : var) =
  try List.assoc s sub with Not_found -> v

type 'a lambda =
  [`Var of string | `Abs of string * 'a | `App of 'a * 'a]

let gensym = let n = ref 0 in fun () -> incr n; "_" ^ string_of_int !n

let eval_lambda eval_rec subst : 'a lambda -> 'a = function
  | #var as v -> eval_var subst v  (* We could also leave the type open *)
  | `App (l1, l2) ->               (* rather than closing it to `lambda` *)
    let l2' = eval_rec subst l2 in
    (match eval_rec subst l1 with
    | `Abs (s, body) ->
      eval_rec [s, l2'] body
    | l1' -> `App (l1', l2'))
  | `Abs (s, l1) ->
    let s' = gensym () in
    `Abs (s', eval_rec ((s, `Var s')::subst) l1)

let freevars_lambda freevars_rec : 'a lambda -> 'b = function
  | `Var v -> [v]
  | `App (l1, l2) -> freevars_rec l1 @ freevars_rec l2
  | `Abs (s, l1) ->
    List.filter (fun v -> v <> s) (freevars_rec l1)

type lambda_t = lambda_t lambda

let rec eval1 subst e : lambda_t = eval_lambda eval1 subst e
let rec freevars1 (e : lambda_t) = freevars_lambda freevars1 e

let test1 = (`App (`Abs ("x", `Var "x"), `Var "y") :> lambda_t)
let e_test = eval1 [] test1
let fv_test = freevars1 test1
```

The arithmetic expression sub-language:

```
type 'a expr =
  [`Var of string | `Num of int | `Add of 'a * 'a | `Mult of 'a * 'a]

let map_expr (f : _ -> 'a) : 'a expr -> 'a = function
  | #var as v -> v
  | `Num _ as n -> n
  | `Add (e1, e2) -> `Add (f e1, f e2)
  | `Mult (e1, e2) -> `Mult (f e1, f e2)

let eval_expr eval_rec subst (e : 'a expr) : 'a =
  match map_expr (eval_rec subst) e with
  | #var as v -> eval_var subst v  (* Here and elsewhere, we could also *)
  | `Add (`Num m, `Num n) -> `Num (m + n)  (* factor-out the sub-language *)
  | `Mult (`Num m, `Num n) -> `Num (m * n)  (* of variables *)
  | e -> e

let freevars_expr freevars_rec : 'a expr -> 'b = function
  | `Var v -> [v]
  | `Num _ -> []
  | `Add (e1, e2) | `Mult (e1, e2) -> freevars_rec e1 @ freevars_rec e2

type expr_t = expr_t expr

let rec eval2 subst e : expr_t = eval_expr eval2 subst e
let rec freevars2 (e : expr_t) = freevars_expr freevars2 e

let test2 = (`Add (`Mult (`Num 3, `Var "x"), `Num 1) : expr_t)
let e_test2 = eval2 ["x", `Num 2] test2
let fv_test2 = freevars2 test2
```

Merging the sub-languages:

```
type 'a lexpr = ['a lambda | 'a expr]

let eval_lexpr eval_rec subst : 'a lexpr -> 'a = function
  | #lambda as x -> eval_lambda eval_rec subst x
  | #expr as x -> eval_expr eval_rec subst x

let freevars_lexpr freevars_rec : 'a lexpr -> 'b = function
  | #lambda as x -> freevars_lambda freevars_rec x
  | #expr as x -> freevars_expr freevars_rec x

type lexpr_t = lexpr_t lexpr

let rec eval3 subst e : lexpr_t = eval_lexpr eval3 subst e
```

```
let rec freevars3 (e : lexpr_t) = freevars_lexpr freevars3 e

let test3 =
  (`App (`Abs ("x", `Add (`Mult (`Num 3, `Var "x"), `Num 1)),
         `Num 2) : lexpr_t)
let e_test3 = eval3 [] test3
let fv_test3 = freevars3 test3
let e_old_test = eval3 [] (test2 :> lexpr_t)
let fv_old_test = freevars3 (test2 :> lexpr_t)
```

11.8 Polymorphic Variants with Recursive Modules

Using recursive modules, we can clean up the confusing or cluttering aspects of tying the recursive knots: type variables and recursive call arguments. The module system handles the recursion for us, making the code cleaner and more modular.

We need **private types**, which for objects and polymorphic variants means *private rows*. We can conceive of open row types, e.g., `[>` \Int of int | ‘String of string]`as using a *row variable*, e.g.,`'a‘:

```
[`Int of int | `String of string | 'a]
```

and then of private row types as abstracting the row variable:

```
type 'row t = [`Int of int | `String of string | 'row]
```

But the actual formalization of private row types is more complex. The key point is that private row types allow us to specify that a type is "at least" a certain set of variants, while still being extensible.

Penalty points:

- We still need to tie the recursive knots for types, for example `private [> 'a lambda] as 'a`
- There can be slight time costs due to the use of functors and dispatch on merging of sub-languages

Verdict: A clean solution, best place. The recursive module approach is the most elegant solution we have seen so far.

```
type var = [`Var of string]

let eval_var subst (`Var s as v : var) =
  try List.assoc s subst with Not_found -> v

type 'a lambda =
  [`Var of string | `Abs of string * 'a | `App of 'a * 'a]

module type Eval =
```

```
sig type exp val eval : (string * exp) list -> exp -> exp end

module LF(X : Eval with type exp = private [> 'a lambda] as 'a) =
struct
  type exp = X.exp lambda

  let gensym = let n = ref 0 in fun () -> incr n; "_" ^ string_of_int !n

  let eval subst : exp -> X.exp = function
    | #var as v -> eval_var subst v
    | `App (l1, l2) ->
      let l2' = X.eval subst l2 in
      (match X.eval subst l1 with
      | `Abs (s, body) ->
        X.eval [s, l2'] body
      | l1' -> `App (l1', l2'))
    | `Abs (s, l1) ->
      let s' = gensym () in
      `Abs (s', X.eval ((s, `Var s')::subst) l1)
end
module rec Lambda : (Eval with type exp = Lambda.exp lambda) =
  LF(Lambda)

module type FreeVars =
sig type exp val freevars : exp -> string list end

module LFVF(X : FreeVars with type exp = private [> 'a lambda] as 'a) =
struct
  type exp = X.exp lambda

  let freevars : exp -> 'b = function
    | `Var v -> [v]
    | `App (l1, l2) -> X.freevars l1 @ X.freevars l2
    | `Abs (s, l1) ->
      List.filter (fun v -> v <> s) (X.freevars l1)
end
module rec LambdaFV : (FreeVars with type exp = LambdaFV.exp lambda) =
  LFVF(LambdaFV)

let test1 = (`App (`Abs ("x", `Var "x"), `Var "y") : Lambda.exp)
let e_test = Lambda.eval [] test1
let fv_test = LambdaFV.freevars test1
```

The arithmetic expression sub-language:

```
type 'a expr =
  [`Var of string | `Num of int | `Add of 'a * 'a | `Mult of 'a * 'a]
```

```
module type Operations =
sig include Eval include FreeVars with type exp := exp end

module EF(X : Operations with type exp = private [> 'a expr] as 'a) =
struct
  type exp = X.exp expr

  let map_expr f = function
    | #var as v -> v
    | `Num _ as n -> n
    | `Add (e1, e2) -> `Add (f e1, f e2)
    | `Mult (e1, e2) -> `Mult (f e1, f e2)

  let eval subst (e : exp) : X.exp =
    match map_expr (X.eval subst) e with
    | #var as v -> eval_var subst v
    | `Add (`Num m, `Num n) -> `Num (m + n)
    | `Mult (`Num m, `Num n) -> `Num (m * n)
    | e -> e

  let freevars : exp -> 'b = function
    | `Var v -> [v]
    | `Num _ -> []
    | `Add (e1, e2) | `Mult (e1, e2) -> X.freevars e1 @ X.freevars e2
end
module rec Expr : (Operations with type exp = Expr.exp expr) =
  EF(Expr)

let test2 = (`Add (`Mult (`Num 3, `Var "x"), `Num 1) : Expr.exp)
let e_test2 = Expr.eval ["x", `Num 2] test2
let fvs_test2 = Expr.freevars test2
```

Merging the sub-languages:

```
type 'a lexpr = ['a lambda | 'a expr]

module LEF(X : Operations with type exp = private [> 'a lexpr] as 'a) =
struct
  type exp = X.exp lexpr
  module LambdaX = LF(X)
  module LambdaFVX = LFVF(X)
  module ExprX = EF(X)

  let eval subst : exp -> X.exp = function
    | #LambdaX.exp as x -> LambdaX.eval subst x
    | #ExprX.exp as x -> ExprX.eval subst x
```

```
  let freevars : exp -> 'b = function
    | #lambda as x -> LambdaFVX.freevars x  (* Either of #lambda or #LambdaX.exp is fine *)
    | #expr as x -> ExprX.freevars x  (* Either of #expr or #ExprX.exp is fine *)
end
module rec LExpr : (Operations with type exp = LExpr.exp lexpr) =
  LEF(LExpr)

let test3 =
  (`App (`Abs ("x", `Add (`Mult (`Num 3, `Var "x"), `Num 1)),
         `Num 2) : LExpr.exp)
let e_test3 = LExpr.eval [] test3
let fv_test3 = LExpr.freevars test3
let e_old_test = LExpr.eval [] (test2 :> LExpr.exp)
let fv_old_test = LExpr.freevars (test2 :> LExpr.exp)
```

11.9 Parser Combinators

We now turn to an application that demonstrates the extensibility concepts we have been discussing. Large-scale parsing in OCaml is typically done using external languages like OCamlLex and Menhir, which generate efficient parsers from grammar specifications. But it is often convenient to have parsers written directly in OCaml, especially for smaller grammars or when we want to extend the parser dynamically.

Language **combinators** are ways of defining languages by composing definitions of smaller languages. This is exactly the kind of compositional, extensible design we have been exploring with the expression problem. For example, the combinators of the **Extended Backus-Naur Form** notation are:

- **Concatenation**: $S = A, B$ stands for $S = \{ab \mid a \in A, b \in B\}$
- **Alternation**: $S = A \mid B$ stands for $S = \{a \mid a \in A \vee a \in B\}$
- **Option**: $S = [A]$ stands for $S = \{\epsilon\} \cup A$, where ϵ is an empty string
- **Repetition**: $S = \{A\}$ stands for $S = \{\epsilon\} \cup \{as \mid a \in A, s \in S\}$
- **Terminal string**: $S = \text{“}a\text{”}$ stands for $S = \{a\}$

Parsers implemented directly in a functional programming paradigm are functions from character streams to the parsed values. Algorithmically they are **recursive descent parsers**.

Parser combinators approach builds parsers as **monad plus** values:

- **Bind**: `val (>>=) : 'a parser -> ('a -> 'b parser) -> 'b parser`
  - `p >>= f` is a parser that first parses `p`, and makes the result available for parsing `f`
- **Return**: `val return : 'a -> 'a parser`
  - `return x` parses an empty string, symbolically $S = \{\epsilon\}$, and returns `x`

- **MZero**: `val fail : 'a parser`
  - `fail` fails to parse anything, symbolically $S = \varnothing = \{\}$
- **MPlus**: `val (<|>) : 'a parser -> 'a parser -> 'a parser`
  - `p <|> q` tries `p`, and if `p` succeeds, its result is returned, otherwise the parser `q` is used

The only non-monad-plus operation that has to be built into the monad is some way to consume a single character from the input stream, for example:

- `val satisfy : (char -> bool) -> char parser`
  - `satisfy (fun c -> c = 'a')` consumes the character "a" from the input stream and returns it; if the input stream starts with a different character, this parser fails

Ordinary monadic recursive descent parsers **do not allow** *left-recursion*: if a cycle of calls not consuming any character can be entered when a parse failure should occur, the cycle will keep repeating indefinitely.

For example, if we define numbers $N := D \mid ND$, where D stands for digits, then a stack of uses of the rule $N \rightarrow ND$ will build up when the next character is not a digit. The parser will try to match N, which requires matching ND, which requires matching N again, leading to infinite recursion.

On the other hand, rules can share common prefixes, and the backtracking monad will handle trying alternatives correctly.

11.10 Parser Combinators: Implementation

The parser monad is actually a composition of two monads:

- The **state monad** for storing the stream of characters that remain to be parsed (specifically, the current position in the input string)
- The **backtracking monad** for handling parse failures and ambiguities (allowing us to try alternatives when one parse fails)

Alternatively, one can split the state monad into a reader monad with the parsed string, and a state monad with the parsing position. This is the approach we take here.

We experiment with a different approach to monad-plus: **lazy-monad-plus**. The difference from regular monad-plus is that the second argument to `mplus` is lazy:

```
val mplus : 'a monad -> 'a monad Lazy.t -> 'a monad
```

This laziness prevents the second alternative from being evaluated until it is actually needed, which is important for avoiding infinite recursion in some parsing scenarios.

Implementation of lazy-monad-plus First a brief reminder about monads with backtracking. Starting with an operation from `MonadPlusOps`:

```ocaml
let msum_map f l =
  List.fold_left  (* Folding left reverses the apparent order of composition *)
    (fun acc a -> mplus acc (lazy (f a))) mzero l  (* order from l is preserved *)
```

The implementation of the lazy-monad-plus using lazy lists:

```ocaml
type 'a llist = LNil | LCons of 'a * 'a llist Lazy.t

let rec ltake n = function
  | LCons (a, l) when n > 1 -> a::(ltake (n-1) (Lazy.force l))
  | LCons (a, l) when n = 1 -> [a]  (* Avoid forcing the tail if not needed *)
  | _ -> []

let rec lappend l1 l2 =
  match l1 with LNil -> Lazy.force l2
  | LCons (hd, tl) -> LCons (hd, lazy (lappend (Lazy.force tl) l2))

let rec lconcat_map f = function
  | LNil -> LNil
  | LCons (a, l) -> lappend (f a) (lazy (lconcat_map f (Lazy.force l)))

module LListM = MonadPlus (struct
  type 'a t = 'a llist
  let bind a b = lconcat_map b a
  let return a = LCons (a, lazy LNil)
  let mzero = LNil
  let mplus = lappend
end)
```

The Parsec Monad File `Parsec.ml`:

```ocaml
module type PARSE = sig
  type 'a backtracking_monad  (* Name for the underlying monad-plus *)
  type 'a parsing_state = int -> ('a * int) backtracking_monad  (* State: position *)
  type 'a t = string -> 'a parsing_state  (* Reader for the parsed text *)
  include MONAD_PLUS_OPS
  val (<|>) : 'a monad -> 'a monad Lazy.t -> 'a monad  (* A synonym for mplus *)
  val run : 'a monad -> 'a t
  val runT : 'a monad -> string -> int -> 'a backtracking_monad
  val satisfy : (char -> bool) -> char monad  (* Consume a character of the class *)
  val end_of_text : unit monad  (* Check for end of the processed text *)
end

module ParseT (MP : MONAD_PLUS_OPS) :
  PARSE with type 'a backtracking_monad := 'a MP.monad =
struct
  type 'a backtracking_monad = 'a MP.monad
```

```
  type 'a parsing_state = int -> ('a * int) MP.monad
  module M = struct
    type 'a t = string -> 'a parsing_state
    let return a = fun s p -> MP.return (a, p)
    let bind m b = fun s p ->
      MP.bind (m s p) (fun (a, p') -> b a s p')
    let mzero = fun _ p -> MP.mzero
    let mplus ma mb = fun s p ->
      MP.mplus (ma s p) (lazy (Lazy.force mb s p))
  end
  include M
  include MonadPlusOps(M)
  let (<|>) ma mb = mplus ma mb
  let runT m s p = MP.lift fst (m s p)
  let satisfy f s p =
    if p < String.length s && f s.[p]  (* Consuming a character means accessing it *)
    then MP.return (s.[p], p + 1) else MP.mzero  (* and advancing the parsing position *)
  let end_of_text s p =
    if p >= String.length s then MP.return ((), p) else MP.mzero
end
```

Additional Parser Operations

```
module type PARSE_OPS = sig
  include PARSE
  val many : 'a monad -> 'a list monad
  val opt : 'a monad -> 'a option monad
  val (?|) : 'a monad -> 'a option monad
  val seq : 'a monad -> 'b monad Lazy.t -> ('a * 'b) monad  (* Exercise: why lazy here? *)
  val (<*>) : 'a monad -> 'b monad Lazy.t -> ('a * 'b) monad  (* Synonym for seq *)
  val lowercase : char monad
  val uppercase : char monad
  val digit : char monad
  val alpha : char monad
  val alphanum : char monad
  val literal : string -> unit monad  (* Consume characters of the given string *)
  val (<<>) : string -> 'a monad -> 'a monad  (* Prefix and postfix keywords *)
  val (<>>) : 'a monad -> string -> 'a monad
end

module ParseOps (R : MONAD_PLUS_OPS)
  (P : PARSE with type 'a backtracking_monad := 'a R.monad) :
  PARSE_OPS with type 'a backtracking_monad := 'a R.monad =
struct
  include P
  let rec many p =
```

```
    (let* r = p in
     let* rs = many p in
     return (r::rs))
    ++ lazy (return [])
  let opt p = (let* x = p in return (Some x)) ++ lazy (return None)
  let (?|) p = opt p
  let seq p q =
    let* x = p in
    let* y = Lazy.force q in
    return (x, y)
  let (<*>) p q = seq p q
  let lowercase = satisfy (fun c -> c >= 'a' && c <= 'z')
  let uppercase = satisfy (fun c -> c >= 'A' && c <= 'Z')
  let digit = satisfy (fun c -> c >= '0' && c <= '9')
  let alpha = lowercase ++ lazy uppercase
  let alphanum = alpha ++ lazy digit
  let literal l =
    let rec loop pos =
      if pos = String.length l then return ()
      else satisfy (fun c -> c = l.[pos]) >>- loop (pos + 1) in
    loop 0
  let (<<>) bra p = literal bra >>- p
  let (<>>) p ket =
    let* x = p in
    literal ket >>- return x
end
```

11.11 Parser Combinators: Tying the Recursive Knot

Now we come to the key insight connecting parser combinators to the expression problem: how do we allow the grammar to be extended dynamically? The answer is to use a mutable reference holding a list of grammar rules, and tie the recursive knot lazily.

File `PluginBase.ml`:

```
module ParseM = ParseOps (LListM) (ParseT (LListM))
open ParseM

let grammar_rules : (int monad -> int monad) list ref = ref []

let get_language () : int monad =
  let rec result =
    lazy
      (List.fold_left
         (fun acc lang -> acc <|> lazy (lang (Lazy.force result)))
          mzero !grammar_rules) in
```

```
  let* r = Lazy.force result in
  let* () = end_of_text in return r  (* Ensure we parse the whole text *)
```

11.12 Parser Combinators: Dynamic Code Loading

OCaml supports dynamic code loading through the `Dynlink` module. This allows us to load compiled modules at runtime, which can register new grammar rules by mutating the `grammar_rules` reference. This is a powerful form of extensibility: we can add new syntax to our language without recompiling the main program.

File `PluginRun.ml`:

```
let load_plug fname : unit =
  let fname = Dynlink.adapt_filename fname in
  if Sys.file_exists fname then
    try Dynlink.loadfile fname
    with
    | (Dynlink.Error err) as e ->
      Printf.printf "\nERROR loading plugin: %s\n%!"
        (Dynlink.error_message err);
      raise e
    | e -> Printf.printf "\nUnknow error while loading plugin\n%!"
  else (
    Printf.printf "\nPlugin file %s does not exist\n%!" fname;
    exit (-1))

let () =
  for i = 2 to Array.length Sys.argv - 1 do
    load_plug Sys.argv.(i) done;
  let lang = PluginBase.get_language () in
  let result =
    Monad.LListM.run
      (PluginBase.ParseM.runT lang Sys.argv.(1) 0) in
  match Monad.ltake 1 result with
  | [] -> Printf.printf "\nParse error\n%!"
  | r::_ -> Printf.printf "\nResult: %d\n%!" r
```

11.13 Parser Combinators: Toy Example

Let us see how this works with a concrete example. We will define two plugins: one for parsing numbers and addition, and another for parsing multiplication. Each plugin registers its grammar rules by appending to the `grammar_rules` list.

File `Plugin1.ml`:

```
open ParseM
```

```
let digit_of_char d = int_of_char d - int_of_char '0'

let number _ =  (* Numbers: N := D N | D where D is digits *)
  let rec num =  (* Note: we avoid left-recursion by having the digit first *)
    lazy ((let* d = digit in
           let* (n, b) = Lazy.force num in
           return (digit_of_char d * b + n, b * 10))
      <|> lazy (let* d = digit in return (digit_of_char d, 10))) in
  Lazy.force num >>| fst

let addition lang =  (* Addition rule: S -> (S + S) *)
  (* Requiring a parenthesis '(' turns the rule into non-left-recursive *)
  (* because we consume a character before recursing *)
  let* () = literal "(" in
  let* n1 = lang in
  let* () = literal "+" in
  let* n2 = lang in
  let* () = literal ")" in
  return (n1 + n2)

let () = grammar_rules := number :: addition :: !grammar_rules
```

File `Plugin2.ml` adds multiplication to the language. Notice how we can add this functionality without modifying any existing code:

```
open ParseM

let multiplication lang =  (* Multiplication rule: S -> (S * S) *)
  let* () = literal "(" in
  let* n1 = lang in
  let* () = literal "*" in
  let* n2 = lang in
  let* () = literal ")" in
  return (n1 * n2)

let () = grammar_rules := multiplication :: !grammar_rules
```

11.14 Exercises

The following exercises will help you deepen your understanding of the expression problem and the various solutions we have explored. They range from implementing additional operations to refactoring the code for better organization.

Exercise 1: Implement the `string_of_` functions or methods, covering all data cases, corresponding to the `eval_` functions in at least two examples from the lecture, including both an object-based example and a variant-based example (either standard, or polymorphic, or extensible variants). This will help you understand how functional extensibility works in each approach.

Exercise 2: Split at least one of the examples from the previous exercise into multiple files and demonstrate separate compilation.

Exercise 3: Can we drop the tags `Lambda_t`, `Expr_t` and `LExpr_t` used in the examples based on standard variants (file `FP_ADT.ml`)? When using polymorphic variants, such tags are not needed.

Exercise 4: Factor-out the sub-language consisting only of variables, thus eliminating the duplication of tags `VarL`, `VarE` in the examples based on standard variants (file `FP_ADT.ml`).

Exercise 5: Come up with a scenario where the extensible variant types-based solution leads to a non-obvious or hard to locate bug. This exercise illustrates why exhaustivity checking is so valuable for static type safety.

Exercise 6: Re-implement the direct object-based solution to the expression problem (file `Objects.ml`) to make it more satisfying. For example, eliminate the need for some of the `rename`, `apply`, `compute` methods.

Exercise 7: Re-implement the visitor pattern-based solution to the expression problem (file `Visitor.ml`) in a functional way, i.e., replace the mutable fields `subst` and `beta_redex` in the `eval_lambda` class with a different solution to the problem of treating `abs` and non-`abs` expressions differently.

Exercise 8: Extend the sub-language `expr_visit` with variables, and add to arguments of the evaluation constructor `eval_expr` the substitution. Handle the problem of potentially duplicate fields `subst`. (One approach might be to use ideas from exercise 6.)

Exercise 9: Implement the following modifications to the example from the file `PolyV.ml`:

1. Factor-out the sub-language of variables, around the already present `var` type.
2. Open the types of functions `eval3`, `freevars3` and other functions as required, so that explicit subtyping, e.g., in `eval3 [] (test2 :> lexpr_t)`, is not necessary.
3. Remove the double-dispatch currently in `eval_lexpr` and `freevars_lexpr`, by implementing a cascading design rather than a "divide-and-conquer" design.

Exercise 10: Streamline the solution `PolyRecM.ml` by extending the language of λ-expressions with arithmetic expressions, rather than defining the sub-languages separately and then merging them. See slide on page 15 of Jacques Garrigue *Structural Types, Recursive Modules, and the Expression Problem.*

Exercise 11: Transform a parser monad, or rewrite the parser monad transformer, by adding state for the line and column numbers.

Exercise 12: Implement `_of_string` functions as parser combinators on top of the example `PolyRecM.ml`. Sections 4.3 and 6.2 of *Monadic Parser Combinators* by Graham Hutton and Erik Meijer might be helpful. Split the result into multiple files as in Exercise 2 and demonstrate dynamic loading of code.

Exercise 13: What are the benefits and drawbacks of our lazy-monad-plus (built on top of *odd lazy lists*) approach, as compared to regular monad-plus built on top of *even lazy lists*? To additionally illustrate your answer:

1. Rewrite the parser combinators example to use regular monad-plus and even lazy lists.
2. Select one example from Lecture 8 and rewrite it using lazy-monad-plus and odd lazy lists.

(In an "odd" lazy list, the first element is strict and only the tail is lazy. In an "even" lazy list, the entire list is wrapped in laziness. The choice affects when computation happens and how infinite structures are handled.)