Scalable Partitioning of Large Complex Networks

Luce le Gorrec

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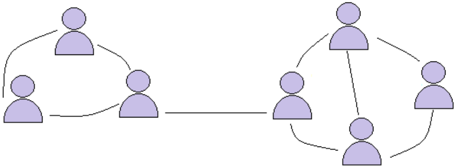
Glasgow, United Kingdom

Graph Partitioning

Graph (or network): entities (nodes) connected by relations (edges).



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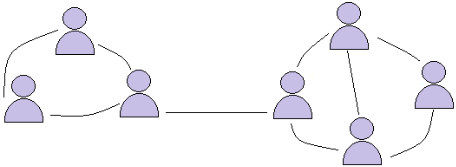
Graph (or network): entities (nodes) connected by relations (edges).
Different kinds of graphs: directed, weighted, evolving...



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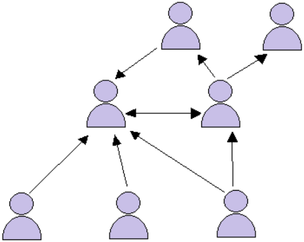

E.g. Who-follows-who network.

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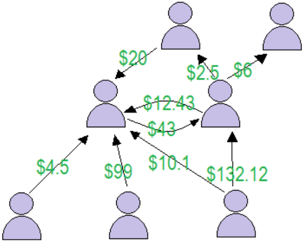

E.g. Money transfer network.

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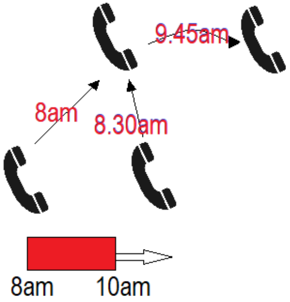

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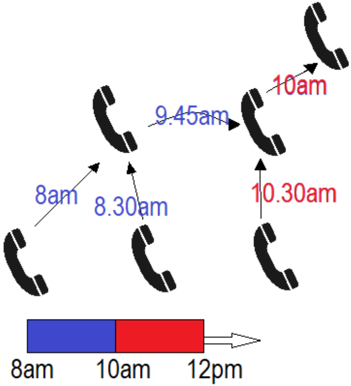

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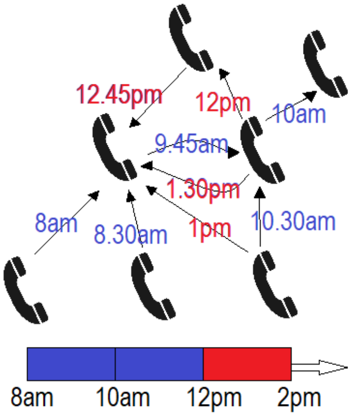

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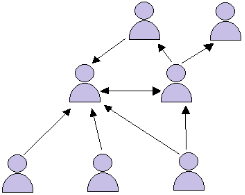

$\Longrightarrow$ **Our study case : directed, unweighted, static.**

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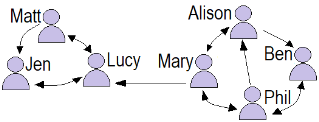
Graph Partitioning

What : A partition of nodes with high (low) density of edges within (between) the groups: a **community structure**.

A Network



Its Adjacency Matrix

	Alison	Ben	Jen	Lucy	Mary	Matt	Phil
Alison	♪	x			x		
Ben		♪					x
Jen			♪	x			
Lucy			x	♪		x	
Mary	x			x	♪		x
Matt			x	x		♪	
Phil	x	x			x		♪

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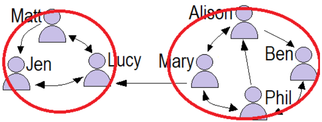

Its Adjacency Matrix

	Jen	Matt	Lucy	Mary	Alison	Ben	Phil
Jen	♪		x				
Matt	x	♪	x				
Lucy	x	x	♪				
Mary			x	♪	x		x
Alison				x	♪	x	
Ben						♪	x
Phil				x	x	x	♪

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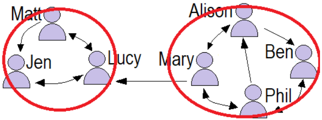

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	Jen	Matt	Lucy	Mary	Alison	Ben	Phil
Jen	♪		x				
Matt	x	♪	x				
Lucy	x	x	♪				
Mary			x	♪	x		x
Alison				x	♪	x	
Ben						♪	x
Phil				x	x	x	♪

Why :

- **Data analysis**: Users spreading terrorist propaganda on Twitter[1].
- **Numerical efficiency**: Analysis of a hundred-million-node network[2].

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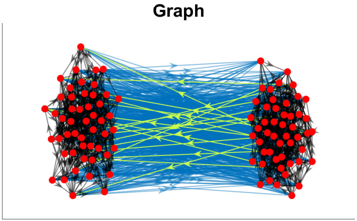
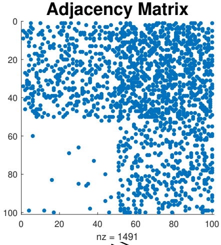
Graph Partitioning

How: Partitioning a graph: NP-hard:

- But for undirected graphs: Efficient, simple and well-established heuristic (Louvain [3], Metis [4], ...).
- For directed graphs: Nothing as simple or well-established ([5]).

$\Longrightarrow$ "Forgetting" edge directions to get undirected networks (suboptimal).





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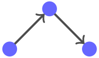
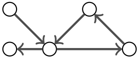
Motifs

Recent works [6,7,8] focus on **motifs** to partition directed networks.

Motif: A small **induced** subgraph of a certain kind.

Looking for motifs $\mathcal{M}$ *in a graph*





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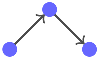

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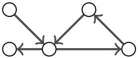
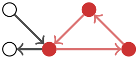


$\Longrightarrow$

not a motif $\mathcal{M}$



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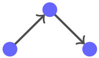

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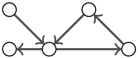

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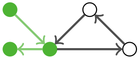
a motif $\mathcal{M}$



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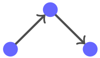

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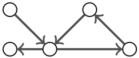
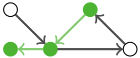


$\Longrightarrow$

another motif $\mathcal{M}$



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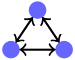
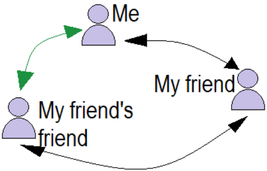
Motif: A small **induced** subgraph of a certain kind.

Motifs express complex notions in networks:

Friends of my friends are my friends:





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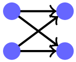
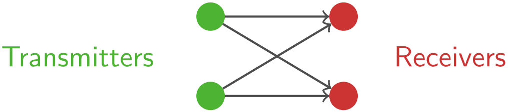
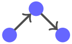
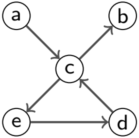
Cooperative propagation of information: 



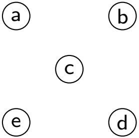
Motifs

An application of motifs: an undirected network induced from a directed one:

Looking for motifs  *in the graph*



provides the graph:



whose adjacency matrix is:

	a	b	c	d	e
a					
b					
c					
d					
e					

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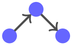
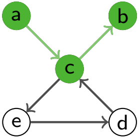
Looking for motifs  *in the graph*



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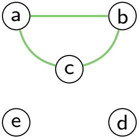

whose adjacency matrix is:

	a	b	c	d	e
a		1	1		
b	1		1		
c	1	1			
d					
e					

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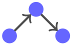
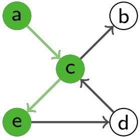
Looking for motifs  *in the graph* 

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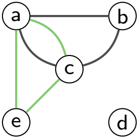

whose adjacency matrix is:

	a	b	c	d	e
a		1	2		1
b	1		1		
c	2	1			1
d					
e	1		1		

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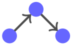
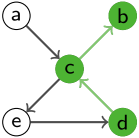
Looking for motifs  *in the graph*



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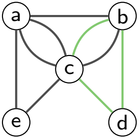

whose adjacency matrix is:

	a	b	c	d	e
a		1	2		1
b	1		2	1	
c	2	2		1	1
d		1	1		
e	1		1		

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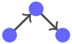
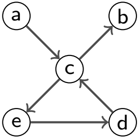
Looking for motifs  *in the graph* 

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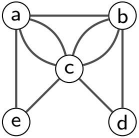

whose adjacency matrix is:

	a	b	c	d	e
a		**1**	**2**		**1**
b	**1**		**2**	**1**	
c	**2**	**2**		**1**	**1**
d		**1**	**1**		
e	**1**		**1**		

$\Longrightarrow$ the **Benson Graph** of the initial network [6].

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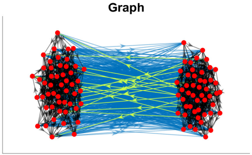
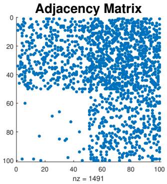
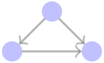
Motifs and Benson Graphs





Adjacency Matrix
of the Benson graph
of motif:



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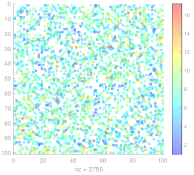

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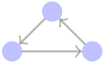

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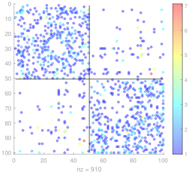
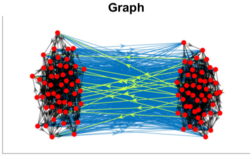
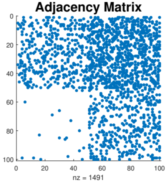


Motifs and Benson Graphs





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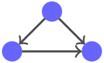

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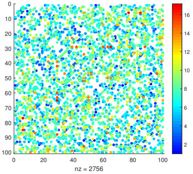

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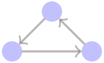

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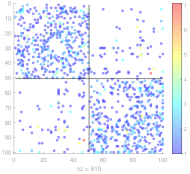

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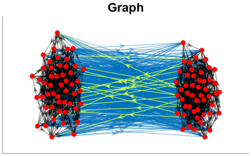

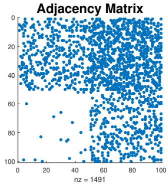

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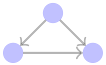
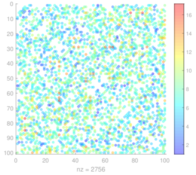
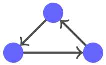


$\Downarrow$



Adjacency Matrix of the Benson graph of motif:



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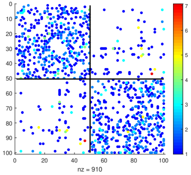

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Which motif(s) should we use?

Assessing a motif significance:

- With inferred knowledge.
- ✗ What if no available knowledge?

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- With inferred knowledge.
- ✗ What if no available knowledge?
- Statistical significance (ZScore) [9]: *'Significant motifs appear more often than "by chance".'*
  A network G, a motif $\mathcal{M}$, a sequence of random networks $\{H_1, ..., H_k\}$:

$$ZScore(\mathcal{M}) = \frac{\#\text{motifs } \mathcal{M} \text{ in } G - mean(\#\text{motifs } \mathcal{M} \text{ in } H_i)}{std(\#\text{motifs } \mathcal{M} \text{ in } H_i) + \varepsilon}.$$

- ✗ Expensive, no consensus about random model.

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Assessing a motif significance:

- With inferred knowledge.
- ✗ What if no available knowledge?
- Statistical significance (ZScore) [9]: *'Significant motifs appear more often than "by chance".'*
- ✗ Expensive, no consensus about random model.
- Our proposal : Assessing the **discriminatory capacity of motifs on a dataset**: a measure (γ-score) derived from a feature selection process based on Principal Component Analysis. *(A preprint submitted).*

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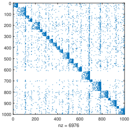
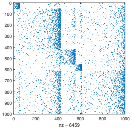
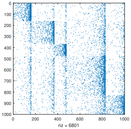
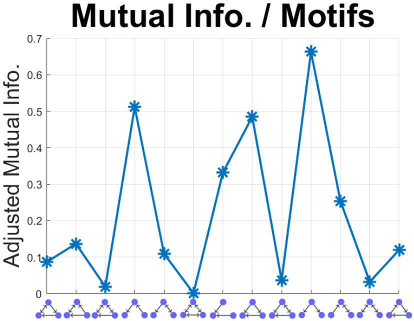
Which motif(s) should we use? Work in progress

Detecting communities in modular networks [10] using Louvain applied on the Benson graphs.









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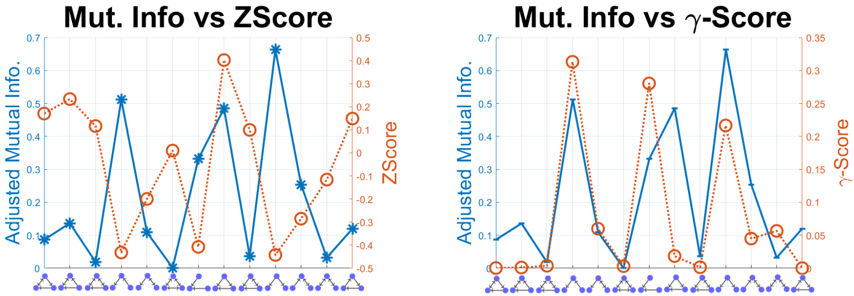

$\Longrightarrow$ Motifs with highest γ-score $\sim$ Motifs with highest Adj. Mut. Info.

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Which motif(s) should we use? TO DO

- Expand the preliminary study to confirm/dismiss the correlation high γ-score/well-detected blocks.
- Focus on local community detection: different motifs may help to detect different communities [11].

Building the Benson Graphs

- Naive : Whole decomposition of the network (finding all the Benson graphs): FanMod [12].

✗ Prohibitive complexity.

- Some efficient techniques exist [6].

✗ For certain kinds of motifs only.

- We have derived generic formulas to directly compute the Benson adjacency matrices of 3- and 4-node motifs.

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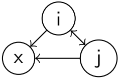
Building the Benson Graphs: Our formulas

A graph G with n nodes, A of dim $n \times n$ its adjacency matrix.
Two observations:
Observation 1: 3 matrices B, U, N of dim $n \times n$ based on A s.t.:

- $B(i,j) = 1 \iff$ i $\leftrightarrow$ j in G.
- $U(i,j) = 1 \iff$ i $\rightarrow$ j in G.
- $N(i,j) = 1 \iff$ i j in G.

Observation 2: Given 2 nodes i, j in G, number of x s.t. 

is:

$$\begin{array}{ccc} (i \longleftrightarrow j) & \text{and} & i \rightarrow x \leftarrow j \\ B(i,j) & \times & \sum_{x=1}^{n} U(i,x).U(x,j) \end{array}$$

Building the Benson Graphs:TO DO

- We are working on an efficient implementation of our formulas.
- Extension to larger motifs ?

Take Home Messages

- **Partitioning directed networks** : not as "simple" as for undirected case.
- **Motifs** express **complex notions in networks** .
- The **Benson Graph** provides an undirected representation of the network.
- Without other knowledge, γ-**score** seems to provide good **indication about which motifs used to partition the graph**.
- With linear algebra, the **Benson adjacency matrix** can be **directly built for 3-and 4-node motifs**.

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Thank you for your attention

Some codes and the slides are available on `github.com/luleg/`

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