Convex Optimization

Theory part - Convex Set

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CONVEX OPTIMIZATION
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READING NOTES

CONVEX SET

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1 Affine Set and Convex Set

In this section, a special definition of line and segment will be talked about.

1.1 Line and Segment

Definition 1.1
Let $x_1 \neq x_2$ are two different point in a set $\mathbf{R}^n$, if for any $\theta \in \mathbf{R}$, we have the point of the format $y = \theta x_1 + (1-\theta)x_2$, then those y compose a **line** through x_1 and x_2. If we constrain that the θ should range from 0 to 1, we call this a **segment**. The line and the segment are shown in the following figure. From the figure (the deep dark means segment) follows we can view y as the sum of the **base point** x_2 and the product of parameter θ and direction $(x_1 - x_2)$.

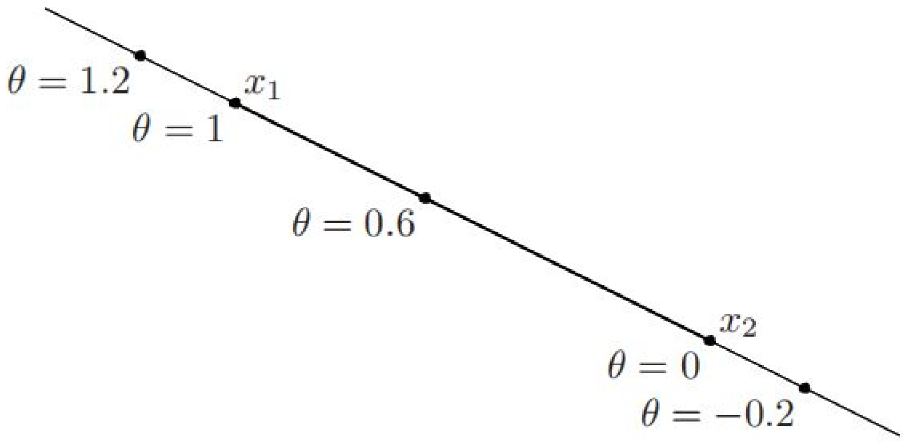

Figure 1: Figure of a Line and Segment

1.2 Affine Set

Definition 1.2
Given a set $C \subset R^n$, if $\forall$ two points $x_1, x_2 \in C$, the line go through x_1, x_2 are still in C, we call C is an **Affine Set**. In mathematical languae. Given a set C, $\forall\, x_1, x_2 \in C, \theta \in R.$ we have $\theta x_1 + (1-\theta)x_2 \in C$.
Remark: C is a set contains all the linear combination of x_1, x_2.

Definition 1.3
The linear combination we mentioned above can be extended to multiple points. Given n points $x_1, x_2, \cdots x_n \in C$, and $\theta_1, \theta_2, \cdots, \theta_n \in R$. if $\sum_{i=1}^{n} \theta_i = 1$, we call $\sum_{i=1}^{n} \theta_i x_i$ as the **affine combination** of $x_1, x_2, \cdots x_n$.

Remark: Using mathematical Induction, we can conclude that: an affine set will include all of the affine combination of the points in it, i.e. $\sum_{i=1}^{n} \theta_i x_i \in C$.

Definition 1.4
If C is an affine set, and $x_0 \in C$, then the set $V = \{v \mid x - x_0, x \in C\}$ will be a subspace. We call it the **subspace** of affine set C. And we define the **dimension of the affine set** as the dimension of the subspace V,i.e. $dim(C) = dim(V)$.

Proof 1.4
For $a, b \in R, a + b = 1$ and $v_1, v_2 \in V$. We have:

$$\begin{aligned} av_1 + bv_2 &= a(x_1 - x_0) + b(x_2 - x_0) \\ &= ax_1 + bx_2 - x_0 \\ &\because ax_1 + bx_2 \in C \\ &\therefore ax_1 + bx_2 - x_0 \in V. \end{aligned}$$

thus V is a subspace.

Exercise 1.1 Show that the solution set of an equation $C = \{x \mid Ax = b\}$, in which $A \in R^{m \times n}$ and $b \in R^m$, is an affine set.

Solution

First we must know that what we need to do is to show that $\forall x_1, x_2 \in C \Rightarrow \theta x_1 + (1 - \theta)x_2 \in C$.So we choose $Ax_1 = b, Ax_2 = b$.

$$A(\theta x_1 + (1 - \theta)x_2) = \theta(Ax_1 - Ax_2) + Ax_2 = b$$

Proved.

Definition 1.5
We call all the affine combination in $C \subset R^n$ the **affine hull** of C. That is

$$\operatorname{aff} C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \ldots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$$

Note that: the affine hull is the smallest affine set that contains C.

2 Affine Dimension and Relative Interior

We define the affine dimension of a set C as the dimension of its affine hull. Affine dimension is useful in the context of convex analysis and optimization, but is not always consistent with other definitions of dimension. As an example consider the unit circle in $\mathbf{R}^2$, i.e., $\left\{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 = 1\right\}$. Its affine hull is all of $\mathbf{R}^2$, so its affine dimension is two. By most definitions of dimension, however, the unit circle in $\mathbf{R}^2$ has dimension one.

If the affine dimension of a set $C \subseteq \mathbf{R}^n$ is less than n, then the set lies in the affine set $\operatorname{aff} C \neq \mathbf{R}^n$. We define the relative interior of the set C, denoted relint C, as its interior relative to $\operatorname{aff} C$:

$$\operatorname{relint} C = \{x \in C \mid B(x, r) \cap \operatorname{aff} C \subseteq C \text{ for some } r > 0\},$$

where $B(x, r) = \{y \mid \|y - x\| < r\}$, the ball of radius r and center x in the norm $\|\cdot\|$. (Here $\|\cdot\|$ is any norm; all norms define the same relative interior.) We can then define the relative boundary of a set C as $\operatorname{cl} C \setminus \operatorname{relint} C$, where $\operatorname{cl} C$ is the closure of C.