Convex Optimization

Theory part - Convex Set

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CONVEX OPTIMIZATION
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READING NOTES

CONVEX SET

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1 Affine Set and Convex Set

In this section, a special definition of line and segment will be talked about.

1.1 Line and Segment

Definition 1.1
Let $x_1 \neq x_2$ are two different point in a set $\mathbf{R}^n$, if for any $\theta \in \mathbf{R}$, we have the point of the format $y = \theta x_1 + (1-\theta)x_2$, then those y compose a **line** through x_1 and x_2. If we constrain that the θ should range from 0 to 1, we call this a **segment**. The line and the segment are shown in the following figure. From the figure (the deep dark means segment) follows we can view y as the sum of the **base point** x_2 and the product of parameter θ and direction $(x_1 - x_2)$.

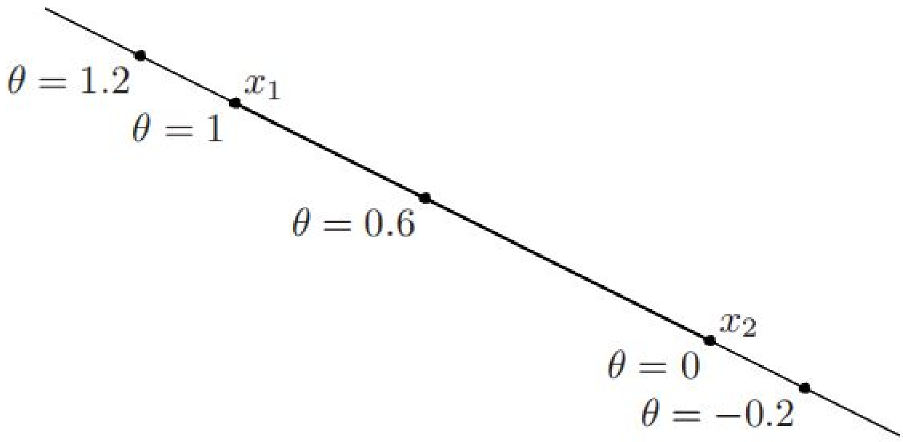

Figure 1: Figure of a Line and Segment

1.2 Affine Set

Definition 1.2
Given a set $C \subset R^n$, if $\forall$ two points $x_1, x_2 \in C$, the line go through x_1, x_2 are still in C, we call C is an **Affine Set**. In mathematical languae. Given a set C, $\forall\, x_1, x_2 \in C, \theta \in R$. we have $\theta x_1 + (1-\theta)x_2 \in C$.
Remark: C is a set contains all the linear combination of x_1, x_2.

Definition 1.3
The linear combination we mentioned above can be extended to multiple points. Given n points $x_1, x_2, \cdots x_n \in C$, and $\theta_1, \theta_2, \cdots, \theta_n \in R$. if $\sum_{i=1}^{n} \theta_i = 1$, we call $\sum_{i=1}^{n} \theta_i x_i$ as the **affine combination** of $x_1, x_2, \cdots x_n$.

Remark: Using mathematical Induction, we can conclude that: an affine set will include all of the affine combination of the points in it, i.e. $\sum_{i=1}^{n} \theta_i x_i \in C$.

Definition 1.4
If C is an affine set, and $x_0 \in C$, then the set $V = \{v \mid x - x_0, x \in C\}$ will be a subspace. We call it the **subspace** of affine set C. And we define the **dimension of the affine set** as the dimension of the subspace V,i.e. $dim(C) = dim(V)$.

Proof 1.4
For $a, b \in R, a + b = 1$ and $v_1, v_2 \in V$. We have:

$$\begin{aligned} av_1 + bv_2 &= a(x_1 - x_0) + b(x_2 - x_0) \\ &= ax_1 + bx_2 - x_0 \\ &\because ax_1 + bx_2 \in C \\ &\therefore ax_1 + bx_2 - x_0 \in V. \end{aligned}$$

thus V is a subspace.