ECE4010J Probabilistic Methods in Eng.

Fall 2022 – Basic Measure Theory

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PROBABLISTIC METHODS IN ENG.
(ECE4010J)

SUPPLEMENTARY MATERIAL

FUNDAMENTAL MEASURE THEORY AND PROBABILITY THEORY

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Contents

This material is indeed difficult and abstract and I promise it won't be mainly tested in the exam

1 Set Theory Review

In this section, we will introduce the concept of set sequence and the limit of it since it may not covered in your learning of naive set theory.

1.1 The limit of set sequence

Denote the set sequence $\{A_n\}$, if ∀n, we have $A_n \subset A_{n+1}$ we call the sequence $\{A_n\}$ as **non-descent**. We denote it as $A_n \uparrow$. We denote:

$$\lim_{n\to\infty} A_n \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} A_n$$

as the limit of the set sequence.
Likewise if $A_{n+1} \subset A_n$, we call the sequence A_n as **non-ascent** denoted as $A_n \downarrow$, we also have the similar definition of limit:

$$\lim_{n\to\infty} A_n \overset{\text{def}}{=} \bigcap_{n=1}^{\infty} A_n$$

Generally, the monotone set sequence will admit its limit.
Consider another 2 sequences $\{\bigcap_{k=n}^{\infty} A_k, n = 1, 2 \ldots\}$ $\{\bigcup_{k=n}^{\infty} A_k, n = 1, 2 \cdots\}$ obviously, they are monotone sequences, and they are non-descent and non-ascent respectively.Ofcourse they admit their limits. Which are:

$$\lim_{n\to\infty} \inf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \quad \lim_{n\to\infty} \sup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

You can see that $\lim_{n\to\infty} \sup A_n \subset \lim_{n\to\infty} \inf A_n$. And if they are the same, we call the set sequence $\{A_n\}$ admits it limit.

1.2 Set class

Definition.
The set which consists of the set in the space $\mathcal{X}$ is called **set class**. In this part, some of the common set class will be introduced.

1.2.1 π- class

If the set class $\mathscr{A}$ on $\mathcal{X}$ is closed about intersection. i.e.

$$A, B \in \mathscr{A} \Rightarrow A \cap B \in \mathscr{A}$$

we call the class $\mathscr{A}$ as the π class on $\mathcal{X}$.

Example: $\{(-\infty, a), a \in \mathbb{R}\}$ is the π class on $\mathbb{R}$.

1.2.2 semi-ring(note: this is not what you learned in abstract algebra)

If the π class $\mathscr{R}$ on $\mathcal{X}$ satisfies that: $A, B \in \mathscr{R}$, $B \subset A$, $A \backslash B = \bigcup_{k=1}^{n} C_k$. C_k is the pair-wise disjoint set.
For example: $\{(a, b], a, b \in \mathbb{R}\}$ is the semi-ring on $\mathbb{R}$.

1.2.3 Ring

Given a set class $\mathscr{R}$,and it is closed about union and difference. i.e.

$$A, B \in \mathscr{R} \Rightarrow A \cup B, A \backslash B \in \mathscr{R}$$

we call the $\mathscr{R}$ as a ring.
For example: The powerset of $\mathcal{X}$, which consists of finite elements is a ring.

1.2.4 Field

The π class $\mathscr{F}$ which is also close about complement. i.e.

$$A \in \mathscr{F} \Rightarrow \bar{A} \in \mathscr{F}$$

we call $\mathscr{F}$ is a field.

1.2.5 Theorem 1

A semi-ring must be a $\pi-$class, a ring must be a semi-ring, a field must be a ring.

1.2.6 monotone class

For a set class $\mathscr{A}$, for all monotone set sequence A_n,we all have $\lim_{n\to\infty} A_n \in \mathscr{A}$, we call the $\mathscr{A}$ as a monotone class.

1.2.7 λ class

the set class $\mathscr{A}$ is called a λ class on $\mathcal{X}$ if

- $\mathcal{X} \in \mathscr{A}$
- $A, B \in \mathscr{A} \Rightarrow A \backslash B \in \mathscr{A}$
- $A \in \mathscr{A}, A_n \uparrow \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$

1.2.8 σ field

Suppose that a non-empty set S is given. A σ-field $\mathcal{F}$ on S is a family of subsets of S such that

- $\emptyset \in \mathcal{F}$.
- If $A \in \mathcal{F}$, then $S \backslash A \in \mathcal{F}$.
- If $A_1, A_2, A_3, \cdots \in \mathcal{F}$ is a finite or countable sequence of subsets, then the union $\bigcup_k A_k \in \mathcal{F}$

Example about σ field:$\{\emptyset, \mathcal{X}\}, \mathscr{P}(\mathcal{X})$

1.2.9 Measurable Space

The sample space and its σ field will formulation a two-tuple ($\mathcal{X}$,$\mathscr{F}$) are called the measurable space.

1.3 Generation of σ field

Definition.We denote $\mathscr{S}$ as the (ring, monotone class, λ class or σ field) **generated by** $\mathscr{E}$ if:

1. $\mathscr{E} \subset \mathscr{S}$
2. for all ring(or monotone class or λ class or σ field) $\mathscr{S}'$, we all have

$$\mathscr{E} \subset \mathscr{S}' \Rightarrow \mathscr{S} \subset \mathscr{S}'$$

   that is to say, the $\mathscr{S}$ is the smallest ring(or monotone class or λ class or σ field) that contains $\mathscr{E}$

Theorem For any set class $\mathscr{E}$, the ring(or monotone class or λ class or σ field) that is generated by $\mathscr{E}$ always exists.
Notations: We denote $r(\mathscr{A})$,$m(\mathscr{A})$,$l(\mathscr{A})$,$\sigma(\mathscr{A})$ as the ring(or monotone class or λ class or σ field) generated by $\mathscr{A}$.

1.3.1 1st σ field generation theorem

If $\mathscr{A}$ is a field then $\sigma(\mathscr{A}) = m(\mathscr{A})$
Proof:
We consider $\sigma(\mathscr{A})$, it is a σ field which contains $\mathscr{A}$, thus it must be a monotone class that contains $\mathscr{A}$(a σ field must be monotone class), and the $m(\mathscr{A})$ is the smallest monotone class that contains $\mathscr{A}$, thus we must have $m(\mathscr{A}) \subset \sigma(\mathscr{A})$, now we just need to prove $\sigma(\mathscr{A}) \subset m(\mathscr{A})$, which means we need to prove that $m(\mathscr{A})$ is a σ field, and we just need to prove that $m(\mathscr{A})$ is a field since a field which is a monotone class must be a σ field.(That is simple, use definition).
Note that $\mathcal{X} \in \mathscr{A} \subset m(\mathscr{A})$, so if we want to prove $m(\mathscr{A})$ is a field, we can only prove it is a ring, since the complete set $\mathcal{X}$ in the set class, and the field demands closure for complement and the ring is closure about union and difference, thus they are the same.

$\forall A \in \mathscr{A}$, denote:

$$\mathscr{G}_A = \{B | B, A \cup B, A \backslash B \in m(\mathscr{A})\}$$

We first verify $\mathscr{G}_A$ is a monotone class: If there is a $B_n \uparrow$

$$\therefore \{B_n\}, \left\{A \bigcup B_n\right\}, \{A - B_n\} \subseteq m(\mathscr{A}), \text{ and } \{B_n\} \uparrow, \left\{A \bigcup B_n\right\} \uparrow, \{A - B_n\} \downarrow$$

$$\therefore \lim_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} B_n \in m(\mathscr{A})$$

$$\lim_{n\to\infty} A \bigcup B_n = \bigcup_{n=1}^{\infty} A \bigcup B_n = A \bigcup \left(\bigcup_{n=1}^{\infty} B_n\right) = A \bigcup \lim_{n\to\infty} B_n \in m(\mathscr{A})$$

$$\lim_{n\to\infty} (A - B_n) = \bigcap_{n=1}^{\infty} \left(A \cap B_n^c\right) = A \bigcap \left(\bigcap_{n=1}^{\infty} B_n^c\right) = A \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c = A$$

$$\cap \left(\lim_{n\to\infty} B_n\right)^c = A - \lim_{n\to\infty} B_n \in m(\mathscr{A})$$

$$\therefore \lim_{n\to\infty} B_n \in \mathscr{G}_A$$

If $B_n \downarrow$ the same reason.

if $A \in \mathscr{A} \subseteq m(\mathscr{A}), \forall B \in \mathscr{A} \subseteq m(\mathscr{A})$,we have $A \cup B, A - B \in \mathscr{A} \subseteq m(\mathscr{A})$
$\therefore \forall B \in \mathscr{A}, B \in \mathscr{G}_A \therefore \mathscr{A} \subseteq \mathscr{G}_A \therefore m(\mathscr{A}) \subseteq \mathscr{G}_A$
which shows $A \in \mathscr{A}, B \in m(\mathscr{A}) \Rightarrow A \cup B, A - B \in m(\mathscr{A})$
if $A \in m(\mathscr{A}), \forall B \in \mathscr{A} \subseteq m(\mathscr{A})$, from above, $A \cup B, A - B \in m(\mathscr{A})$
$\therefore \mathscr{A} \subseteq \mathscr{G}_A, \forall A \in m(\mathscr{A}) \therefore m(\mathscr{A}) \subseteq \mathscr{G}_A, \forall A \in m(\mathscr{A})$
which shows $A \in m(\mathscr{A}), B \in m(\mathscr{A}) \Rightarrow A \cup B, A - B \in m(\mathscr{A})$
thus, $m(\mathscr{A})$ is a ring

thus proved.

1.3.2 2nd σ field generation theorem

If $\mathscr{P}$ is a π class, thus $\sigma(\mathscr{P}) = l(\mathscr{P})$ The proof thought is just like the 1st theorem.

1.4 Borel Set

Denote $\mathscr{Q}_R$ as a semi-ring on R and the $\mathscr{P}_R$ as the π class on R, we have

$$\mathscr{B}_R = \sigma(\mathscr{Q}_R) = \sigma(\mathscr{P}_R)$$

as the **Borel Set** on R. The open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \left\{y \in \mathbb{R}^d : |y - x| < r\right\}.$$

A subset $E \subset \mathbb{R}^d$ is open if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. By definition, a set is closed if its complement is open. We denote $\mathscr{O}_R$ as the open sets on R, it is easy to prove $\mathscr{B}_R = \sigma(\mathscr{O}_R)$. In a word: Borel Set is the set class consists of the open sets on R

2 Measurable Map

2.1 map and image

Definition Let X and Y be two spaces, if $\forall x \in X$, there is a unique $f(x) \in Y$, we call the relation f is a **map** from X to Y. We call $f(x)$ the value of the map f at x.

Definition $\forall B \subset Y$, we call

$$f^{-1}B = \{x \mid f(x) \in B\} \text{ as the pre-image of B under } f$$

For the set class $\mathscr{E}$ on Y, we call

$$f^{-1}\mathscr{E} = \{f^{-1}B \mid B \in \mathscr{E}\}$$

as the pre-image of $\mathscr{E}$ under f

Theorem. The pre-image of set has such properties

$$f^{-1}\varnothing = \varnothing; \quad f^{-1}Y = X$$

$$B_1 \subset B_2 \Rightarrow f^{-1}B_1 \subset f^{-1}B_2$$

$$\left(f^{-1}B\right)^c = f^{-1}B^c, \quad \forall B \subset Y$$

We take the second line as example. $f^{-1}B_1 = \{x \mid f(x) \subset B_1\}$ $\because$ $B_1 \subset B_2$ $\therefore$ $\forall x, f(x) \subset B_1 \rightarrow f(x) \subset B_2$, thus $\forall x \in f^{-1}B_1, x \in f^{-1}B_2$, thus $f^{-1}B_1 \subset f^{-1}B_2$

Theorem: For any set class $\mathscr{E}$ on Y, we have $\sigma(f^{-1}\mathscr{E}) = f^{-1}\sigma(\mathscr{E})$

Proof: Consider the LHS, $\sigma(f^{-1}\mathscr{E})$ is the smallest σ field which contains $\mathscr{E}$. And the RHS, by the property of the pre-image it must be a sigma field. Since $\mathscr{E} \subset \sigma(\mathscr{E})$ so $f^{-1}\mathscr{E} \subset f^{-1}\sigma(\mathscr{E})$, thus $f^{-1}\sigma(\mathscr{E})$ is a σ field that contains $f^{-1}\mathscr{E}$, thus $\sigma(f^{-1}\mathscr{E}) \subset f^{-1}\sigma(\mathscr{E})$. Then we shall prove $\sigma(f^{-1}\mathscr{E}) \supset f^{-1}\sigma(\mathscr{E})$. Let,

$$\mathscr{G} = \{B \subset Y \mid f^{-1}B \in \sigma(f^{-1}\mathscr{E})\}$$

Since the pre-image of B belongs to a σ field, thus the set of B is a σ field. What's more, the $\sigma(f^{-1}\mathscr{E}$ contains all the pre-image of $\mathscr{E}$, thus the $\mathscr{G}$ must contains $\mathscr{E}$, which is $\mathscr{E} \subset \mathscr{G}$, thus $\mathscr{G}$ contains the $\sigma(\mathscr{E})$ we take the pre-image on both side and we get $\sigma(f^{-1}\mathscr{E}) \supset f^{-1}\sigma(\mathscr{E})$.
Hence we get $\sigma(f^{-1}\mathscr{E}) = f^{-1}\sigma(\mathscr{E})$ Q.E.D

2.2 Random Unit

Given the measurable spaces $(X, \mathscr{F})$, $(Y, \mathscr{T})$ and a map f from X to Y, if

$$f^{-1}\mathscr{T} \subset \mathscr{F},$$

we call f as the **Random Unit** from $(X, \mathscr{F})$ to $(Y, \mathscr{T})$, we call $\sigma(f) = f^{-1}\mathscr{T}$ as the **the smallest σ field that makes f measurable.** We shall see that for B in $\mathscr{T}$, if the measure of B equals to the measure of $f^{-1}B$ that is $f^{-1}B \in \mathscr{F}$(the pre-image of B is in the σ field of X) (we can define the measure only in the σ field). Thus we call the map f as measurable.

Theorem Let $\mathscr{E}$ be a set class on Y, then f is the measurable map from $(X, \mathscr{F}), (Y, \sigma(\mathscr{E}))$ iff $f^{-1}\mathscr{E} \subset \mathscr{F}$
Proof The key of this theorem is that: we only need to verify $f^{-1}\mathscr{E}$ instead of $f^{-1}\sigma(\mathscr{E})$. By the definition, we just want to verify $f^{-1}\sigma(\mathscr{E}) \subset \mathscr{F}$.Note the following equation:

$$f^{-1}\mathscr{E} \subset \mathscr{F}$$

we perform the σ generation on both side, and we get

$$\sigma(f^{-1}\mathscr{E}) \subset \sigma(\mathscr{F})$$

since $\mathscr{F}$ is just a σ field, so the equation is equivalent to

$$f^{-1}\sigma(\mathscr{E}) \subset \mathscr{F}$$

which is just the definition, thus we proved this theorem.

Theorem Given g as a measurable map from the measurable space $(X, \mathscr{F})$ to $(Y, \mathscr{T})$ and the measurable map from the space $(Y, \mathscr{T})$ to $(Z, \mathscr{Z})$. The map f o g is a measurable map from the $(X, \mathscr{F})$ to $(Z, \mathscr{Z})$.
Proof:We just want to show that $\forall C \in \mathscr{Z}$, we have $g^{-1}(f^{-1}C) \in \mathscr{F}$

$$\begin{aligned}(f \circ g)^{-1}C &= \{x \in X : f(g(x)) \in C\} \\ &= \left\{x \in X : g(x) \in f^{-1}C\right\} \\ &= g^{-1}\left(f^{-1}C\right)\end{aligned}$$

Since we know that f is measurable, so $f^{-1}C \in \mathscr{T}$, and g is measurable so $g^{-1}(f^{-1}C) \in \mathscr{F}$ thus proved.

2.3 Measurable function

In order to deal with the concept *function*, we need to discuss the $\mathbb{R}$ furthermore, here, we introduce **Generalized Real Number**.

2.3.1 Generalized Real Number

Definition We define the generalized real number $\bar{R} = R \cup \{\infty\} \cup \{-\infty\}$ We also define the order of the $\bar{R}$

$$-\infty < a < \infty, a \in R$$

$\forall a \in \bar{R}$ we denote

$$a^+ = max(a, 0), \text{ and } a^- = min(-a, 0)$$

we call them the **positive negative part** of a, note that these are non-negative.

Definition We define the Borel Set on the generalized real number, that is

$$\mathscr{B}_{\bar{\mathbb{R}}} = \sigma(\mathscr{B}_R, \{\infty\}, \{-\infty\})$$

Theorem

$$\begin{aligned}\mathscr{B}_{\overline{\boldsymbol{R}}} &= \sigma([-\infty, a) : a \in \boldsymbol{R}) \\ &= \sigma([-\infty, a] : a \in \boldsymbol{R}) \\ &= \sigma((a, \infty] : a \in \boldsymbol{R}) \\ &= \sigma([a, \infty]; a \in \boldsymbol{R})\end{aligned}$$

Proof:

We Prove the first one:

$$\because [-\infty, a] = (-\infty, a) \cup \{-\infty\} \in \mathscr{B}_{\bar{R}}$$
$$\therefore \sigma([-\infty, a]) \subset \sigma(\mathscr{B}_{\bar{R}}) = \mathscr{B}_{\bar{R}}$$

since the $\mathscr{B}_{\bar{R}}$ is a σ field. Next we prove that $\mathscr{B}_{\bar{R}} \supset \sigma([-\infty, a])$
We consider the three key elements that are needed to generate the $\mathscr{B}_{\bar{R}}$, $\sigma(\mathscr{B}_R, \{\infty\}, \{-\infty\})$.

First, we consider the $\{-\infty\}$
$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n) \in \sigma([-\infty, a))$, it is because the σ field is closure on the listable intersection(σ field is always a π class).
$\{\infty\} = \bigcap_{n=1}^{\infty} [n, \infty] = \bigcap_{n=1}^{\infty} [-\infty, n)^c \in \sigma([-\infty, a))$ since the σ field is also closure on the complement.
$(-\infty, a) = [-\infty, a) \backslash \{-\infty\} \in \sigma(-\infty, a)$ since σ field is also closure on the real difference.
$\mathscr{B}_R = \sigma(-\infty, a) \subset \sigma([-\infty, a))$
thus $\sigma(\mathscr{B}_R, \{-\infty\}, \{\infty\}) \subset \sigma([-\infty, a))$, which means $\mathscr{B}_{\bar{R}} \subset \sigma([-\infty, a))$. Thus $\mathscr{B}_{\bar{R}} = \sigma([-\infty, a))$, proved.

2.3.2 Random Variable

Definition: The map from the measurable space $(\mathcal{X}, \mathscr{F})$ to $(\bar{R}, \mathscr{B}_{\bar{R}})$ is called a **measurable function** on $(\mathcal{X}, \mathscr{F})$. Especially The map from the measurable space $(\mathcal{X}, \mathscr{F})$ to $(R, \mathscr{B}_R)$ is called a **Random Variable** on $(\mathcal{X}, \mathscr{F})$.

Theorem T. F. R. E.:

- f is a random variable on $(X, \mathscr{F})$
- $\{f < a\} \in \mathscr{F}, \forall a \in \boldsymbol{R}$;
- $\{f \leqslant a\} \in \mathscr{F}, \forall a \in \boldsymbol{R}$;
- $\{f > a\} \in \mathscr{F}, \forall a \in \boldsymbol{R}$;
- $\{f \geqslant a\} \in \mathscr{F}, \forall a \in \boldsymbol{R}$.

Note that $\{f < a\}$ is $\{x \mid f(x) < a\}$ This is quite obvious.

Theorem If f, g are measurable functions, then $\{f < g\}, \{f \leqslant g\}, \{f = g\} \in \mathscr{F}$.
We denote the $\mathbb{Q}$ as the set of the rational number on R. Thus
$\{f < g\} = \{f < g\} = \bigcup_{\gamma \in Q}\{\{f < \gamma\} \cap \{g > \gamma\}\}$ by the above theorem, we know that $\{f < \gamma\} \in \mathscr{F}$, and $\{g > \gamma\} \in \mathscr{F}$, since $\mathscr{F}$ is a σ field thus a π class thus $\{f < \gamma\} \cap \{g > \gamma\} \in \mathscr{F}$, what's more, σ field is closure on listable union, since the Rational number is dense and it is listable (the cardinality of rational number is the same as natural number), thus $\bigcup_{\gamma \in Q}\{\{f < \gamma\} \cap \{g > \gamma\}\} \in \mathscr{F}$. Proved.
For $\{f \leqslant g\}$, it is equivalent to $\{g \leqslant f\}^c$, since σ field is closure on complement, thus proved.
For the third one, $\{f = g\} = \{f \leqslant g\} \backslash \{f < g\}$ since σ field is closure on real difference, thus proved.
Examples The indicator function I_A, $A \in \mathscr{F}$, in the measurable space $(\mathcal{X}, \mathscr{F})$ is measurable.
Explanation:
Note that the indicator function $I_A = 0$ if $x \notin A I_A = 1$ else , thus we assume the image of I_A is C, if $0, 1 \notin C$, we know $f^{-1}C = \{\varnothing \subset \mathscr{F}\}$,by definition of σ field, if $0, 1 \in C$, we know that $f^{-1}C = \mathcal{X} \subset \mathscr{F}$, thus I_A is obviously measurable.
Likewise, we can prove that if A,B are two sets in $\mathscr{F}$, and $a, b \in \bar{R}$, we know that $aI_A + bI_B$ is measurable function.

3 Random Variable Operation

In this part, we will discuss some operation on measurable function (random variable).

Theorem Given the measurable function f, g, then

- $\forall a \in \bar{R}$, af is a measurable function.
- if $\forall x \in \mathcal{X}$, $f(x) + g(x)$ exist, then it will be measurable function.
- fg is measurable function
- if $g(x) \neq 0$, then f/g is measurable function.

Proof
(1). We first consider $a = -\infty, 0, \infty$, in which cases, $af = -\infty \cdot I_{\{f>0\}} + \infty \cdot I_{\{f<0\}}, 0, -\infty \cdot I_{\{f<0\}} + \infty \cdot I_{\{f>0\}}$ since the linear combination of the indicator functions and const function are measurable so, when $a = -\infty, 0, \infty$, the function af is measurable, then we consider the other cases:
if $a \in R^+$ $\forall b \in R$, we have $\{af \leq b\} = \{f \leqslant b/a\}$ since f is measurable, thus $\{f \leqslant b/a\} \in \mathscr{F}$, thus af is measurable.
if $a \in R^-$ $\forall b \in R$, we have $\{af \leq b\} = \{f \geq b/a\}$ since f is measurable, thus $\{f \geqslant b/a\} \in \mathscr{F}$, thus af is measurable.
All in all $\forall a \in \bar{R}$, af is measurable if f is measurable.
(2) All we need to consider is such thing, whether $\{f + g < a\} \in \mathscr{F}$ or not. We can denote $\{f + g < a\} = A_1 \cup A_2 \cup A_3$

$$\begin{aligned} A_1 &= \{f + g < a\}, min(f, g) = -\infty, max(f, g) < \infty\} \\ &= \{f = -\infty, g < \infty\} \cup \{f < \infty, g = -\infty\} \in \mathscr{F} \end{aligned}$$

the reason the above theorem If f, g are measurable functions, then $\{f < g\}, \{f \leqslant g\}, \{f = g\} \in \mathscr{F}$.

$$A_2 = \{f + g < a, max(f, g) = \infty\} = \varnothing \in \mathscr{F}$$

Then we finally consider A_3.

$$A_3 = \{f + g < a, max(f, g) < \infty, min(f, g) > -\infty\} \qquad = \{f < a - g\} \cap \{-\infty < f, g < \infty\}$$
$$= \bigcup_{\gamma \in \mathbb{Q}} [\{f < \gamma\} \cap \{g < a - \gamma\}] \cap \{-\infty < f, g < \infty\}$$
$$\in \mathscr{F}$$

Thus $A_1 \cup A_2 \cup A_3 \in \mathscr{F}$

(3)We also apply the same method. Denote $\{fg < a\} = A_1 \cup A_2$.

$$A_1 \overset{\text{def}}{=} \{fg < a\} \cap \{g = 0\}$$
$$= \begin{cases} \varnothing \in \mathscr{F}, & a \leqslant 0 \\ \{g = 0\} \in \mathscr{F}, & a > 0 \end{cases}$$

$$A_2 \overset{\text{def}}{=} \{fg < a\} \cap \{g \neq 0\}$$
$$= [\{f < ag^{-1}\} \cap \{g > 0\}] \cup [\{f > ag^{-1}\} \cap \{g < 0\}] \text{ since g is not equal to 0}$$
$$= \left[\{g > 0\} \cap \bigcup_{\gamma \in Q} \{f < \gamma, \gamma g < a\}\right] \cup \left[\{g < 0\} \cap \bigcup_{\gamma \in Q} \{f > \gamma, \gamma g < a\}\right] \in .\mathscr{F},$$

Thus fg is measurable.

(4) Omit.

4 Probability Space

In this section, we will introduce the concept about measure and measure space.

Given a space $\mathcal{X}$ and a set class $\mathscr{E}$ on it. We call the function whose range is $[0, \infty]$ as **non-negative set function**, and we denoted it as μ.

Now, let μ is a non-negative set function on $\mathscr{E}$, if for $A_1 \cdots A_n$ in $\mathscr{E}$, which are mutually not intersected, and we have:

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu\left(A_n\right)$$

we call the function μ is **countably additive**

Given a set class $\mathscr{E}$ on the space $\mathcal{X}$, if $\varnothing \in \mathscr{E}$ and the non-negative set function μ on $\mathscr{E}$ which is countably additive, and $\mu(\varnothing) = 0$, we call μ a **measure** on $\mathscr{E}$. In addition if $\forall A \in \mathscr{E}$, $\mu(A) < \infty$, we call the measure is **finite**.

Theorem Given a sample space $\mathcal{X}$ and a corresponding semi-ring $\mathcal{Q}_R$ on it. Recall that $\mathcal{Q}_R = \{(a, b], a, b \in R\}$. If there is a function F, F is a non-descent and continuous real function. We can let:

$$\mu((a, b]) = \begin{cases} F(b) - F(a), & a < b, \\ 0, & a \geqslant b, \end{cases}$$

then μ is a measure on $\mathcal{Q}_R$. We call this F as **quasi-distribution function**, we will learn **Cumulative Distribution Function(CDF)** in later part of this course, which is something like the quasi-distribution function.
Now, we need to define the measure on σ field.

Definition Given a sample space $\mathcal{X}$, a σ field $\mathscr{F}$ on this space, and a measure μ on $\mathscr{F}$, if $N \in \mathscr{F}$, and $\mu(N) = 0$,we call N a set of **measure zero** on $\mathscr{F}$, we denote the three-element tuple $(\mathcal{X}, \mathscr{F}, \mu)$ as measure space if P is a measure on $\mathscr{F}$, and $P(\mathcal{X}) = 1$, then we call P a **probability measure** on $\mathscr{F}$,here we denote the three-element tuple $(\mathcal{X}, \mathscr{F}, P)$ as the probability space. The set A in $\mathscr{F}$ is called **event**, and $P(A)$ is called **the probability of event A happens**.And finally we get the measure theory definition of probability.

5 Closing Remark

I think now, you have got a rough idea about what probability is, then you can continue with your study of ECE4010J, good luck! If you find it is interesting, you can refer to real analysis to know further things about measure theory, such as **exterior measure**, **expansion and completeness of measure space**. But if you just want to apply elementary probability theory and statistic, you don't have to know anything about measure theory.

6 Exercises

Can we randomly and equal likely pick one number from $\mathbb{N}$, why?

Solution: That is impossible. Since randomly and equal likely, we know that $P(N) = 1$(P must be a probability measure), and N can be divided as $\{0\}, \{1\}, \cdots$, which is countable and infinite, obviously 1 can not be written as the sum of equal countable non negative values.

7 Reference

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