RC5 Frequentist Inference

Variable Transformation

*This note is written by **Linda***

1 Variable Transformation

In this section, I'm afraid that you must learn how to do it by hand, which is not because MMA code can not perform this job but the exam will force you to do it by hand.
Theorem:
The basic rule for transformation of densities considers an invertible, smooth mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ with inverse $f^{-1} = g$, i.e. the composition $g \circ f(\mathbf{z}) = \mathbf{z}$. If we use this mapping to transform a random variable z with distribution $q(\mathbf{z})$, the resulting random variable $\mathbf{z}' = f(\mathbf{z})$ has a distribution :

$$q(\mathbf{z}') = q(\mathbf{z}) \left| \det \frac{\partial f^{-1}}{\partial \mathbf{z}'} \right| = q(\mathbf{z}) \left| \det \frac{\partial f}{\partial \mathbf{z}} \right|^{-1}$$

where the last equality can be seen by applying the chain rule (inverse function theorem) and is a property of Jacobians of invertible functions.

1.1 Univariable Injection

Exercise 1.1 Given the random variable X whose PDF is: $f_X(x) = \frac{\beta^\alpha}{\gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$. Find the PDF of $1/X$

1.2 Univariable non Injection

Exercise 1.2.1 Given X whose PDF is

$$f_X(x) = \begin{cases} cx^2 & x \in [-1, 1] \\ 0 & \text{other wise.} \end{cases}$$

find c and $f_Y(y) \quad Y = X^2$.

Exercise 1.2.2 Let U be a standard uniform random variable. Derive a formula that will convert U

into a Pareto $(2, 5)$ random variable X with the density $f(x) = 50x^{-3}$ and the cumulative distribution function $F(x) = 1 - \left(\frac{5}{x}\right)^2, x \geq 5$. Compute X using your algorithm if our computer generated $U = 0.36$

1.3 Triple piece wise function

Let X and Y be independent random variables such that

$$f_X(x) = \begin{cases} \frac{1}{2} & 0 \leq x < 1/2 \\ 2x & 1/2 \leq x < 1 \\ 0 & \text{othereise} \end{cases} \qquad f_Y(y) = \begin{cases} 1 & 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is a and m ? Let $Z = \max\{X, Y\}$. Then

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ a \cdot z^m & 0 \leq z \leq 1/2 \\ b \cdot z^n & 1/2 \leq z \leq 1 \\ 0 & z > 1 \end{cases}$$

1.4 Multivariable function

Exercise 1.4.1 Let (X, Y) be a continuous bivariate random variable with density $f_{XY} : S \to \mathbb{R}^2$ given by

$$f_{XY}(x, y) = \begin{cases} \frac{2}{\pi}\left(x^2 + y^2\right) & \text{for } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.
(b) Find $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$.
(c) Find the correlation coefficient ϱ_{XY}.
(d) Find the density of $U = X/Y$.

Exercise 1.4.2 Let X and Y are i.i.d. random variables and their joint PDF

$$f_{XY}(x, y)$$

Find $Z = X + Y, X - Y, XY, \frac{X}{Y}$'s PDF.

Exercise 1.4.3 *From SJTU Final, Optional*
X, Y are two independent random variables, which follow $N\left(0, \sigma^2\right)$, show that the PDF of $Z = \sqrt{X^2 + Y^2}$ is

$$f_Z(z) = \begin{cases} \frac{z}{\sigma^2} \mathrm{e}^{-\frac{z^2}{2\sigma^2}}, & z \geqslant 0, \\ 0, & z < 0. \end{cases}$$

(we call Z follows a Rayleigh distribution of parameter σ $(\sigma > 0)$)

Solution:

Prove: $F_Z(z) = P\left(\sqrt{X^2 + Y^2} \leq z\right)$

$$= \begin{cases} 0, & z < 0, \\ \iint_{\sqrt{x^2+y^2 \leq z}} f(x, y) dx dy, & z \geq 0 \end{cases}$$

$\underline{x = r\cos\theta, y = r\sin\theta} \begin{cases} 0, & z < 0, \\ \int_0^{2\pi} d\theta \int_0^z f(r\cos\theta, r\sin\theta) r dr, & z \geq 0, \end{cases}$ so Z 's PDF iss

$$f_Z(z) = \begin{cases} 0, & z < 0 \\ \int_0^{2\pi} z f(z\cos\theta, z\sin\theta) d\theta, & z \geq 0 \end{cases}$$

$$= \begin{cases} 0, & z < 0 \\ \int_0^{2\pi} z \frac{1}{2\pi\sigma^2} \mathrm{e}^{-\frac{z^2}{2\sigma^2}} \mathrm{d}\theta, & z \geq 0 \end{cases} = \begin{cases} 0, & z < 0 \\ \frac{z}{\sigma^2} \mathrm{e}^{-\frac{z^2}{2\sigma^2}}, & z \geq 0 \end{cases}$$