VV256 Honor Calculus IV

Fall 2022 – Reference solution (Just for reference)

This worksheet is generated by ***Wei Linda***

1 Problem1

(a)

$$\begin{aligned} f(y) &= \frac{1}{2} f(y \mid \theta = 1) + \frac{1}{2} f(y \mid \theta = 2) \times \\ &= \frac{1}{\sqrt{2\pi 4}} e^{-\frac{(y-1)^2}{2\times 4}} + \frac{1}{\sqrt{2\pi} \cdot 4} \cdot e^{-\frac{(y-2)^2}{2\times 4}} \\ &= \frac{\sqrt{2\pi}}{8\pi} e^{-\frac{(y-1)^2}{8}} + \frac{\sqrt{2\pi}}{8\pi} e^{-\frac{(y-2)^2}{8}} \end{aligned} \tag{1}$$

$$\tag{2}$$

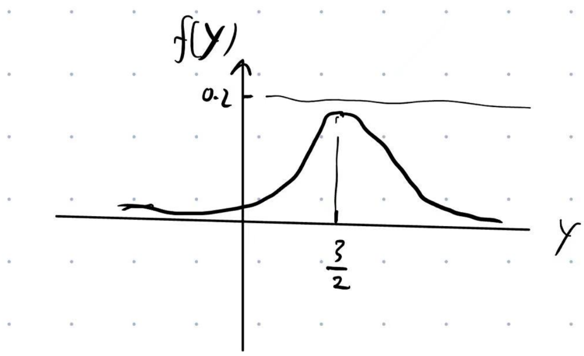

图 1: sketch the distribution

b)

$$\Pr(\theta = 1 \mid y = 1) = \frac{f(y = 1 \mid \theta = 1) \cdot p(\theta = 1)}{\frac{\sqrt{2\pi}}{4\pi}\left(1 + e^{-\frac{1}{8}}\right)} \tag{3}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2\sqrt{2\pi}}}{\frac{1}{4\sqrt{2\pi}}\left(1 + e^{-\frac{1}{8}}\right)} = 0.5312. \tag{4}$$

c)

$$\Pr(\theta \mid y) = \frac{f(y \mid \theta) P(\theta)}{f(y)}$$

To take ($\theta = 1, 1y = 1$) as example:

$$\Pr(\theta = 1 \mid y) = \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-1)^2}{\sigma^2}} \cdot \frac{1}{2}}{\frac{1}{2\sigma\sqrt{2\pi}} \left(e^{-\frac{(y-1)^2}{\sigma^2}} + e^{-\frac{(y-2)^2}{\sigma^2}} \right)}$$

$$\sigma \to \infty$$

$$\Pr(\theta = 1 \mid y) \to \frac{e^0}{e^0 + e^0} = \frac{1}{2}$$

$$\sigma \to 0$$

$$\Pr(\theta = 0) y) \to \frac{1}{1+0} = 1$$

σ tends to infinity the posterier density tends to 1/2, the prior. When σ tends to 0, the density tends to 1.

2 Problem2

(1.8) $E(u) = E(E(u \mid v))$ if u is a scalar. if $u = (u_1, u_2, \cdots u_i, \ldots)$ by (1.8) $\quad E(u_i) = E(E(u_i \mid v))$ for $\forall i$
of course $E(u) = E(E(u \mid v))$
(1.9) $\operatorname{Var}(u) = E(\operatorname{Var}(u \mid v)) + \operatorname{Var}(E(u \mid v))$
If u is a vector.

$$Var[u] = \begin{bmatrix} Var[u_1] & Cov[u_1, u_2] & \cdots & Cov[u_1, un] \\ Cov[u2, u_1] & Var[u2] & \cdots & Cov[u2, un] \\ \vdots & \vdots & \ddots & Cov[u_{n-1}, u_n] \\ Cov[u_n, u_1] & Cov[u_n, u_2] & \cdots & Var[u_n] \end{bmatrix}$$

For diagonal terms it is very obvious, for the non-diagonal terms for u_i, u_j, the corresponding right hand side will be.

$$\begin{aligned} RHS =& E\left[\operatorname{cov}\left(u_i, v_j \mid v\right)\right) + \operatorname{cov}\left(E\left(u_i \mid v\right), E\left(u_j \mid v\right)\right] \\ =& E\left[E\left[u_i u_j \mid v\right] - E\left[u_i \mid v\right] E\left[u_j \mid v\right]\right] + E\left[E\left(u_i \mid v\right) \cdot E\left(u_j \mid v\right)\right] - E\left[E\left(u_i \mid v\right)\right] \cdot E\left[E\left(u_j \mid v\right)\right] \\ =& E\left[E\left[u_i u_j \mid v\right]\right] - E\left[E\left(u_i \mid v\right)\right] \cdot E\left[E\left(u_j \mid v\right)\right] \\ \because\ & E[E[u \mid v]] = E[u] \\ \therefore\ RHS =& E\left[u_i u_j\right] - E\left[u_i\right] E\left[u_j\right] = Cov[u_i, u_j] \end{aligned}$$

So for each element in the matrix, we have proved the property thus
$Var[u] = E[Var(u|v)] + Var[E(u|v)]$.

3 Problem3

P[hetero | br pand brc] $= \frac{\text{P[hetero} \cap \text{bp and brc]}}{\text{P[brp and brc]}}$
$P[\text{brp and brc }] = (1-p)^4 + (1-p)^2 2p(1-p) \times 2 + (2p(1-p))^2 \left(1 - \frac{1}{4}\right)$
P [hetero n brp and brc] $= (1-p)^2 2p(1-p) \times 2 \cdot \frac{1}{2} + (2p(1-p))^2 * 0.5$

$$\therefore p = \frac{P[hetero \cap brp\ and\ brc]}{p[\text{ brp and brc }]} = \frac{2p}{1+2p}$$

P[J is hete | n children brown and....]
$= \frac{P[J \text{ is lete and } \cdots}{P[\ldots]}$
$P[\cdots] = \frac{1}{1+2p} + \frac{2p}{1+2p} \cdot \left(\frac{3}{4}\right)^n$
P [J is hete and ...] $= \frac{2p}{1+2p} \left(\frac{3}{4}\right)^n \therefore p = \frac{\frac{2p}{1+2p}\left(\frac{3}{4}\right)\cdot}{\frac{1}{1+2p} + \frac{2p}{1+2p}\left(\frac{3}{4}\right)^n}$
$P[\text{ child is } Xx] = \left(\frac{\left(\frac{2p}{1+2p}\right)\left(\frac{3}{4}\right)^n}{\frac{1}{1+2p} + \frac{2p}{1+2p}\left(\frac{3}{4}\right)^n}\right) \times \frac{2}{3} + \left(\frac{\frac{1}{1+2p}}{\frac{1}{12p} + \frac{2p}{1+2p}\left(\frac{1}{4}\right)^n}\right) \times \frac{1}{2}$
$P[Childisxx] = P$ [child is Xx] $\times \frac{1}{2} \times \left(p^2 + 2p(1-p) \cdot \frac{1}{2}\right)$
$= \left(\frac{\frac{2}{3}\left(\frac{2p}{1+2p}\right)\left(\frac{3}{4}\right)^n + \frac{1}{2}\frac{1}{1+2p}}{\frac{1}{1+2p} + \frac{2p}{1+2p}\left(\frac{3}{4}\right)^n}\right) \frac{1}{2} p$

4 Problem4

(a)

$$P_{r_1} = \frac{8}{12} = \frac{2}{3} \quad P_{r_2} = \frac{5}{12} \quad P_{r_3} = \frac{5}{8}$$

(b)

$$\begin{aligned}
& d \sim N(0, 14) \\
& P[Y > 0] = P[d > -8] = P[d \leq 8] \\
& = 1 - P\left[\frac{d}{14} \leqslant \frac{8 + 0.5}{14}\right] \\
& = 0.728 \\
& P[Y > 8] = 1 - P[d \leqslant 0] \\
& = 1 - P\left[z \leqslant \frac{0 + 0.5}{14}\right] \\
& = 0.514 \\
& P[fwb8|fw] \\
& = \frac{P[fwb8 \cap fw]}{P[fw]} = 0.706
\end{aligned}$$

5 Problem6

Solution:

$$\frac{1/300 * 1/2}{1/2 * (1 * 2 * 1/125 + 1/300)} = \frac{5}{11}$$