VE414 Bayesian Analysis

Fall 2022 – Homework3 Multiparameter Model

This homework is done by ***Wei Linda***

1 Problem 1

(a)

Since We don't know the variance, we shall use T distribution:

$$T_{n-1} = \frac{\bar{x} - \theta}{s/\sqrt{n}}$$

follows a T distribution, so the 95% confidence interval for θ is $\bar{x} \pm t_{0.025,n-1} \cdot S/\sqrt{n} = (7.97822, 9.11842)$

(b)

$f\left(n, \sigma^2 \mid x\right)$

$\propto f\left(x \mid n, \sigma^2\right) \cdot \pi\left(n, \sigma^2\right)$

$\propto \sigma^{-2} \prod_{i=1}^{14} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2}\frac{(x_i-\theta)^2}{\sigma^2}}$

$\propto \sigma^{-2-n} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{14}(x_i-\theta)^2\right)}$

$\propto \sigma^{-2-n} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{14}\left((n-1)S^2+n(\bar{x}-\theta)^2\right)\right.}$

$$p(\theta \mid x) = \int_0^\infty p\left(\theta, \sigma^2 \mid x\right) d\sigma^2$$

$$\propto \int_0^\infty \sigma^{-2-n} \cdot e^{-\frac{1}{2\sigma^2}\left[(n-1)s^2+n(\bar{x}-n)^2\right]}$$

$$= t_{n-1}(\theta|\bar{x}, s^2/n)$$

That is to say

$$\frac{\bar{x} - \theta}{s/\sqrt{n}} \sim T_{n-1}$$

Thus, 95% credible interval

$$\bar{x} \pm t_{0.025.13} S/ = (7.97822, 9.11842)$$

which is the same like (a).

(c) According to the BDA3 textbook.

$$p(n \mid y) \propto \left(1 + \frac{k_n(\theta - \theta n)^2}{V_n^2 \sigma_n^2}\right)^{-\frac{V_{n+1}}{2}}$$

$$= t_{vn}\left(\mu_n, \sigma_n^2 / k_n\right)$$

$$\begin{aligned}
\mu_n &= \frac{k_0}{k_0 + n} n_0 + \frac{n}{k_0 + n} \bar{x} \\
&= \frac{10}{10 + 14} 8.35 + \frac{14}{10 + 14} 8.55 \\
&= 8.4867 \\
k_n &= 24 \\
V_n &= 4 + 14 = 18 \\
18\sigma_n^2 &= 4 \times 1.5 + (14 - 1) \times 0.9758 + \frac{10 \times 14}{10 + 14} \times (8.55 - 8.35)^2 \\
\sigma_n^2 &= 1.05
\end{aligned}$$

$$P(\theta \mid x) = t_{18}\left(8.4667, \sqrt{\frac{1.05}{24}}\right)$$

$$\frac{\mu_n - 8.4667}{\sqrt{\frac{1.05^2}{24}}} \sim T_{18}.$$

- the 95% credible interval is $8.4667 \pm t_{18,0.025} \times \sqrt{\frac{1.05}{24}} = (8.02, 8.91)$

2 Problem 2

$$f(U) = \begin{cases} 1 & u \in (0, 1) \\ 0 & \text{else} \end{cases}$$

$$F(U) = \begin{cases} 0 & u \in (-\infty, 0) \\ u & u \in (0, 1) \\ 1 & u \geqslant 1 \end{cases}$$

$F[U \leq u] = u$. $F[g(u) \leqslant u] = FFU \leq g^{-1}(u)I = 1 - \frac{25}{u^2} = g^{-1}(u) \Rightarrow g(u) = \sqrt{\frac{25}{1-u}}$ propose a transformation

$$x = g(u) = \sqrt{\frac{25}{1 - u}}$$

When $n = 0.36$

$$x = \frac{25}{4}$$

3 Problem 3

$$\begin{aligned}
P(\theta \mid x) &= P(x \mid \theta) \cdot \pi(\theta) \\
&= \prod_{i=1}^{5} p\left(x_i \mid \theta\right) \pi(\theta) \\
&\propto \frac{2^4}{T(4)} \cdot \theta^3 e^{-2\theta} \prod_{i=1}^{5} \theta \cdot e^{-\theta x_i} \\
&\propto \theta^8 e^{-(2+5\bar{x})\theta}. \\
&\sim \Gamma(9, 2+5\bar{x}) \\
&\sim \Gamma(9, 10)
\end{aligned}$$

So, mean is $E = \frac{9}{10} = 0.9$ and Variance is 0.09.

4 Problem 4

In [3]:
```
import numpy as np
from scipy.stats import uniform
import seaborn as sns
import matplotlib.pyplot as plt
u = uniform(0,1).rvs(10000)
x=-np.log(1-u)/20
sns.histplot(x)
plt.title('Exponential distribution simulation')
```

Out[3]: `Text(0.5, 1.0, 'Exponential distribution simulation')`

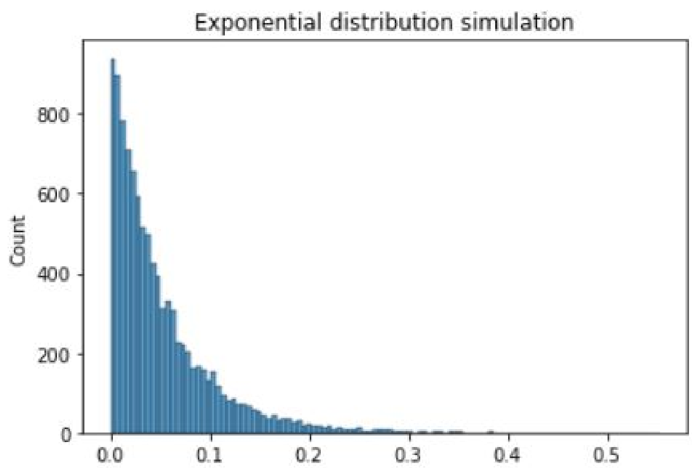

图 1: Problem 4