STAT 4510J Bayesian Analysis

Fall 2022 – Homework4 Decision Theory

This homework is done by ***Wei Linda***

1 Problem 1

$$
\begin{aligned}
&(\mu-\mu_0)^T \Lambda_0^{-1} (\mu-\mu_0) + \sum_{i=1}^{n} (y_i-\mu)^T \Sigma^{-1} (y_i-\mu) \\
&= \mu^T \Lambda_0^{-1} \mu - 2\mu^T \Lambda_0^{-1} \mu_0 + n\mu^T \Sigma^{-1} \mu - 2\sum_{i=1}^{n} \mu^T \Sigma^{-1} y_i + \text{const} \\
&= \mu^T \left(\Lambda_0^{-1} + n\Sigma^{-1}\right) \mu - 2\mu^T \left(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y}\right) + \text{const} \\
&= \left(\mu - \left(\Lambda_0^{-1} + n\Sigma^{-1}\right)^{-1} \left(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y}\right)\right)^T \left(\Lambda_0^{-1} + n\Sigma^{-1}\right) \left(\mu - \left(\Lambda_0^{-1} + n\Sigma^{-1}\right)^{-1} \left(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y}\right)\right) \\
&= (\mu-\mu_n)^T \Lambda_n^{-1} (\mu-\mu_n)
\end{aligned}
$$

2 Problem 2

$$
\begin{aligned}
&P(\theta) = 0.1\,\mathrm{N}\left(-1, 0.5^2\right) + 0.9\,\mathrm{N}\left(1, 0.5^2\right) \\
&p_1(\theta \mid y) \propto p_1(y \mid \theta) p_1(\theta) \sim N\left(\frac{\frac{1}{10} + \bar{y}\cdot 0.5^2}{0.5^2 + \frac{1}{10}}, \frac{\frac{0.5^2}{10} \times 1}{0.5^2 + \frac{1}{10}}\right) \\
&= N\left(-\frac{13}{28}, \frac{1}{14}\right) \\
&P_2(\theta \mid y) \propto P_2(y \mid \theta) P_2(\theta) \sim N\left(\frac{u \times \frac{1}{10} + \bar{y} \times 0.5^2}{0.5^2 + 0.1}, \frac{0.5^2 + 1}{0.5^2 + 0.1}\right) \\
&= N\left(\frac{3}{28} \quad , \frac{1}{14}\right) \\
&w_1 = \frac{\lambda_1 \cdot p_1(y \mid \theta)}{\lambda_1 p_1(y \mid 0) + \lambda_2 p_2(y \mid \theta)} \\
&= \frac{0.1 \times N\left(y = -0.25 \mid -1, 0.5^2 + \frac{1}{10}\right)}{0.1 \times N(y = -0.25) - 1, 0.5^2 + \frac{1}{10}) + 0.9 N(y - 0.5) 1, 0.5^2 + \frac{1}{10})} \\
&= 0.32
\end{aligned}
$$

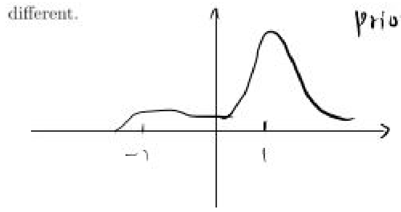

图 1: prior

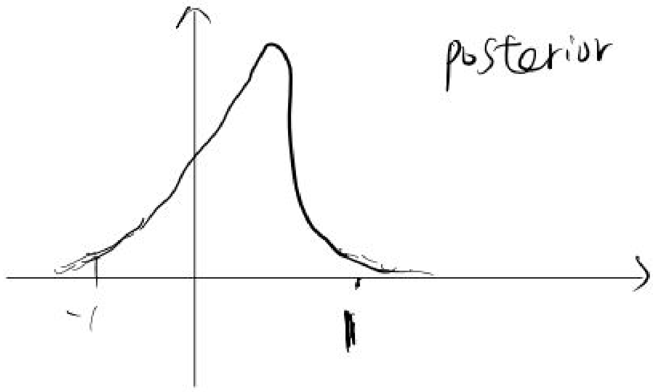

图 2: posterior

$$W_2 = 0.68.$$

$$\therefore P(\theta \mid y) = 0.32 \times N\left(-\frac{13}{28}, \frac{1}{14}\right) + 0.68N\left(\frac{3}{28}, \frac{1}{14}\right)$$

posterior

3 Problem 3

$$\begin{aligned}
& E\left[\theta_j \mid \tau, y\right] \\
=& E\left[E\left[\theta_j \mid \tau, y, n\right]\right] \\
=& E\left[\frac{\frac{1}{\sigma j^2} y_j + \frac{1}{\tau^2} u}{\frac{1}{\sigma^2} + \frac{1}{\tau^2}} \mid \tau, y\right] \\
=& \frac{\frac{1}{\sigma_j^2} y_j + \frac{1}{\tau^2} \hat{u}}{\frac{1}{\sigma_j^2} + \frac{1}{\tau^2}} \quad \hat{u} = \frac{\sum_{j+1}^{j} \frac{1}{\sigma_j^2 + \tau^2} \overline{y_j}}{\sum_{j=1}^{J} \frac{1}{\sigma_j^2 + \tau^2}}.
\end{aligned}$$

$$
\begin{aligned}
&\operatorname{Var}\left[\theta_j \mid \tau, Y\right] \\
=&E\left[\operatorname{Var}\left[\theta_j \mid \tau, y, u\right]\right]+\operatorname{Var}\left[E\left[\theta_j \mid \tau, y, u\right)\right] \\
=&E\left[\frac{\sigma_j^2 \tau^2}{\sigma_j^2+\tau^2} \mid \tau, y\right]+\left(\frac{\sigma_j^2}{\tau^2+\sigma_j^2}\right)^2 V n . \\
=&\frac{\sigma_j^2 \tau^2}{\sigma_j^2+\tau^2}+\left(\frac{\sigma_j^2}{\tau^2+\sigma_j^2}\right)^2 V u . \\
&V_n^{-1}=\sum_{j=1}^J \frac{1}{\tau^2+\sigma_j^2}
\end{aligned}
$$