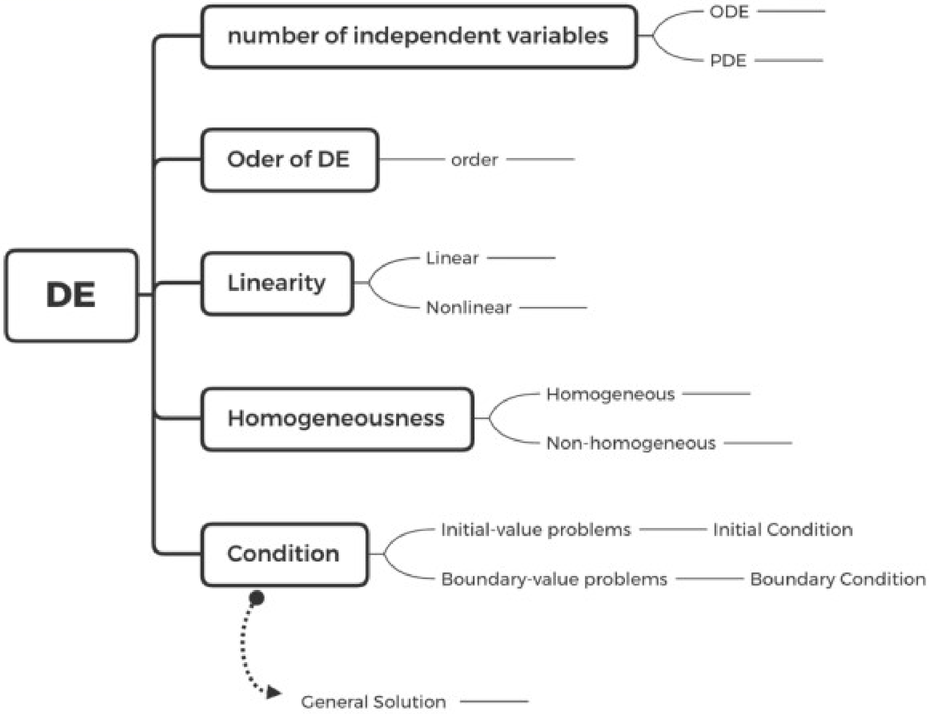
VV256 RC First Order ODE Basic

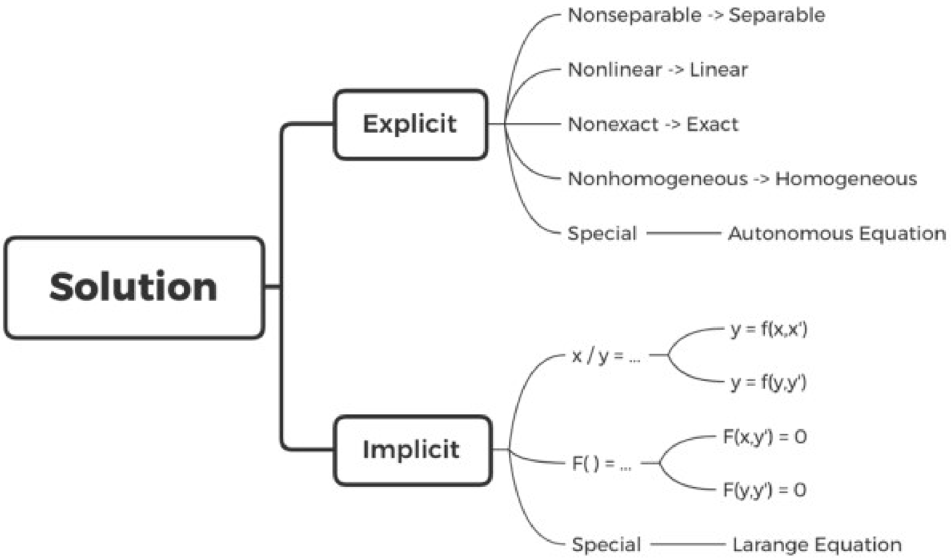
I. Elementary Integral Methods.

- Classification

  basic concepts



- Solution



1. Exact Equations

$P(x,y)+Q(x,y)y'=0 \quad \rightarrow \quad P(x,y)\,dx+Q(x,y)\,dy=0$

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ exact equations (sufficient and necessary condition)

Basic principle … **detailed derivation**

$$P(x,y)\,dx+Q(x,y)\,dy=df(x,y)$$

General solution

a. $f_x(x,y)=P(x,y) \qquad f_y(x,y)=Q(x,y)$

b. Integrate $f_x(x,y)$ or $f_y(x,y)$ (choose an easy one to integrate)
   Take integrating the partial derivative of x as example, then notice that $f(x,y)=\cdots+g(y)$

c. Calculate the derivative of the result in the previous step to find g(y)

d. General solution: $f(x,y)=C$

Exact equation table (textbook)

a. $ydx+xdy=d(xy),$

b. $\frac{ydx-xdy}{y^2}=d\left(\frac{x}{y}\right) \longrightarrow \frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$

c. $\frac{ydx-xdy}{xy}=d\left(\ln\left|\frac{x}{y}\right|\right) \longrightarrow \frac{xdy-ydx}{xy}=d\left(\ln\left|\frac{y}{x}\right|\right)$

d. $\frac{ydx-xdy}{x^2+y^2}=d\left(\arctan\frac{x}{y}\right) \longrightarrow \frac{xdy-ydx}{x^2+y^2}=d\left(\arctan\frac{y}{x}\right)$

e. $\frac{ydx-xdy}{x^2-y^2}=\frac{1}{2}d\left(\ln\left|\frac{x-y}{x+y}\right|\right), \ \frac{ydy+xdx}{x^2+y^2}=d\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right)$

Exercise2

Please judge whether the following equations are exact equation or not, if it is, please find its general solution.

1. $(\mathrm{ye^x}+2\mathrm{e^x}+\mathrm{y^2})\mathrm{dx} + (\mathrm{e^x}+2\mathrm{xy})\mathrm{dy}=0$

2. Separable Equations

$$\frac{dy}{dx} = f(x)g(y)$$

a. Discuss the condition of $g(y_0) = 0 \rightarrow$ Find the equilibrium solution

b. When $g(y_0) \neq 0 \rightarrow \frac{dy}{g(y)} = f(x)\,dx$

c. Integrate $\rightarrow G(y) = F(x) + C$

Conversion

a. Homogenous Polar Equation

i) Form

$$y' = f(\frac{y}{x})$$

ii) Step

Step I. Let $y = xv(x)$ to transform it into the seperable DE

Step II. Chek whether the equilibrium solution exists. If it exists, check if it satisfies Initial conditions.

Step III. If $f(v) - v = 0$, the singular solution happens, check if it satisfies Initial conditions.

Step IV. If $f(v) - v \neq 0$, integrate both sides.

Exercise3

1. $\sqrt{1+x^2}\frac{dy}{dx} = xy^3, y(0) = 1.$
2. $x^2 - y^2 + 2xyy' = 0$.

3. First Order Linear ODE.

$$y' + p(t)y = q(t)$$

a. Without initial condition

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)\,dt + C\right), where\ \mu(t) = e^{\int p\,dt}$$

b. With initial condition

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(\tau)q(\tau)\,d\tau + \mu(t_0)y(t_0)\right)$$

Exercise4

1. Transform the following equations to linear ODE

(1) $\frac{dy}{dx} = \frac{x^2+y^2}{2y}$

(2) $\frac{dy}{dx} = \frac{1}{cosy} +$ xtany

4. Elementary Transformation Method

a. Bernoulli Equation

i) Form

$$y' + p(t)y = q(t)y^n \quad n \neq 1$$

ii) Step

$$Let\ y = w^{\frac{1}{1-n}} \rightarrow w' + (1-n)p(t)w = (1-n)q(t)$$

After calculating w, don't forget to calculate y!

Exercise5

$y' - \frac{y}{x} = y^9$

b. Riccati Equation

i) Form

$$y' = q_0(t) + q_1(t)y + q_2(t)y^2 \quad q_0, q_2 \neq 0$$

ii) Step

Find a particular solution $y_1 \rightarrow y = y_1 + \frac{1}{w}$

$\rightarrow w' + (q_1 + 2q_2y_1)w = -q_2$

Possible choice for y_1

Exercise6

$y' = 1 + (y-x)^2,\ y(0) = \frac{1}{2}$

5. Method of Integrating Factor

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ non-exact equations $\rightarrow$ integrating factors: $\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$

a. μ is only related to x

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \phi(x)$$

The integrating factor is

$$\mu = e^{\int \phi(x)dx}$$

b. μ is only related to y

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P} = \phi(y)$$

The integrating factor is

$$\mu = e^{\int \phi(y)dy}$$

Exercise7

ydx + (y-x)dy=0

Reference:

1. VV256 RC Slide FA 2021, Zhou Zixi
2. Ordinate Differentiate Equation, Ding Tongren