1. Analytic.

Given a function $f(x,y)$, we call the function $f(x,y)$ is analytic in the region $G \in R^2$, if fall $\forall (x_0, y_0) \in R^2$

$f(x,y)$ can be expressed as

$$f(x,y) = \sum_{i,j=0}^{\infty} a_{ij} (x-x_0)^i (y-y_0)^j,$$ which is coverged

in $|x-x_0| \leq a$, $|y-y_0| \leq b$.

2. Basic Series solution

$$A(x)y'' + B(x)y' + C(x) = 0.$$

$A(x), B(x), C(x)$ are analytic in x_0.

Q_1: $A(x_0) \neq 0$. whether $p(x) = \frac{B(x)}{A(x)}$, $q(x) = \frac{C(x)}{A(x)}$ Analytic in x_0?

$f(x)$ is differentiable. $g(x)$ is also ……, $g(x_0) \neq 0$.

whether $\frac{f(x)}{g(x)}$ differentiable in x_0?

A_1: Yes

$$P(x) = \frac{B(x)}{A_0 + A_1(x-x_0) + \cdots +} \qquad A_0 \neq 0.$$

$$= \frac{B(x)}{A_0 (1 + \underbrace{\frac{A_1}{A_0}(x-x_0) + \cdots +}_{\text{very small}})}$$ = covergent power series of $x-x_0$.

$$\frac{1}{1+x} = \sum (-1)^n x^n \qquad \sum (-1)^n (\quad)^n$$

related with $x-x_0$

$A(X_0)=0$?

1) $A(X) = A(x_0) + \frac{A'(x_0)}{1!}(x-x_0) + \cdots + \frac{A^{(k)}(x_0)(x-x_0)^k}{k!} + \cdots$

if not all coefficient is zero

$A(X) = \underbrace{(x-x_0)^k}_{\text{Assume this is lowest order term}} \tilde{A}(X)$, $\tilde{A}(x)$ is analytic at x_0

$\tilde{A}(x_0) \neq 0$.

$A(x_0) = \tilde{A}(x) \cdot (x-x_0)^k$ (tylor expantion) coefficient not all $=0$

$$\tilde{A}(x) = \frac{A(x_0)}{(x-x_0)^k}$$

if $(x-x_0)^k$ can be eliminate, the equation will become $\tilde{A}(x)y'' + \tilde{B}(x)y' + \tilde{C}(x)y = 0$, in which $\tilde{A}(x)$, $\tilde{B}(x)$, $\tilde{C}(x)$ is analytic near x_0, and $\tilde{A}(x_0) \neq 0$.

That's good. or. it can't be eliminated. Thus difficult, we need to use elementary transform to make remaining terms and y'' to form another term which is $\tilde{y}''$, which will cover in generalized

Series solution

2) $A(x) \equiv 0$ in the neighbor of x_0 don't need to consider

Consider the following second order ODE.

$$y'' + p(x) y' + q(x) y = 0.$$

if $p(x)$, $q(x)$ are analytic in $|x - x_0| < r$, that is to say $p(x)$, $q(x)$ can be expressed as convergent power series of $x - x_0$;

We call x_0 an ordinary point of the differentiated equation. otherwise, we call it singular point.

Eg 1. Solve the equation : analytic solution :

$$(1 - x^2) y'' - 2xy' + n(n+1) y = 0,$$ at the neighbor of $x = 0$. (Legendre Equation)

Easy to find that $x = 0$ is an ordinary point.

let $\varphi(x) = \sum_{m=0}^{\infty} a_m (x-0)^m$, substitute it

$$(1 - x^2) \sum_{m=2}^{\infty} a_m\, m(m-1)\, x^{m-2} - 2x \sum_{m=1}^{\infty} a_m\, m\, x^{m-1} + n(n+1) \sum_{m=0}^{m} a_m x^m = 0$$

Constant $a_0 n(n+1) + a_2 \times 2 = 0.$

1st: $a_1 (n(n+1) - 2) + a_3 \cdot 3 \cdot 2 = 0$

2nd $a_2 \cdot n(n+1) - 4a_2 + a_4 \times 4 \times 3 - a_2 \times 2 = 0$

m th order.

$$-a_m(m-1)m - 2m a_m + n(n+1)a_m + a_{m+2}(m+1)(m+2) = 0$$

$$\varphi(0) = a_0 \qquad \varphi'(0) = a_1$$

$$a_2 = -\frac{n(n+1)}{2} a_0$$

$$a_3 = -\frac{n(n+1)-2}{3 \times 2} a_1$$

$$a_{m+2} = -\frac{n(n+1) - m(m+1)}{(m+2)(m+1)} a_m = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m$$

You can see that: a_{odd} is controlled by a_1, a even is controlled by a_0.

Question: Why?

A: Second Order Equation has 2 independent solution.

$$a_{2k} = \frac{(-1)^k n(n-2)\cdots(n-2k+2)(n+1+0)(n+1+2)\cdots(n+1+2(k-2))}{(2k)!} a_0$$

$$a_{2k+1} = \frac{(-1)^k (n-1)(n-3)\cdots(n-2k+1)(n+1+1)\cdots(n+1+2k-1)}{(2k+1)!} a_1$$

When $n \in N^*$, if $m=n$ $a_{n+2} = 0 = a_{n+4} \cdots$

$= \cdots = 0$

let $\varphi_1(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$

$$\varphi_2(x) = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$\varphi_1, \varphi_2(x)$ must have a polynomial

call Legendre Polynomials.

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