VV256 Honors Calculus IV
Fall 2022 — Problem Set 7

December 14, 2022

Exercise 7.1

Given a symmetric matrix $A \in \mathrm{Mat}_n(\mathbb{R})$, show that A is positive definite iff all eigenvalues of A are positive.

Answer:

Let A be a $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$. Then A is orthogonally diagonalizable, that is, $A = QDQ^T$ for some orthogonal matrix Q and diagonal matrix D. Now if we consider $f(x) = v^T A v$, then

$$\begin{aligned} f(v) &= v^T A v \\ &= v^T Q D Q^T v \\ &= \left(Q^T v\right)^T D \left(Q^T v\right) \\ &= w^T D w \\ &= \lambda_1 w_1^2 + \ldots \lambda_n {w_n}^2 \\ &> 0. \end{aligned}$$

Exercise 7.2

Given $A, B \in M_n(\mathbb{R})$ such that $AB = BA$, suppose λ is an eigenvalue of A, show that λ is also an eigenvalue of B.

Answer:

We can view A, β as linear transformation: $\forall \xi$ from the eigen-subspace V_λ thus $B\xi = \lambda\xi$

$$B(A\xi) = AB\xi = A\lambda\xi = \lambda(A\xi)$$

thus $A\xi \in V_\lambda$, thus. V_λ is invarient subspace of $\mathbb{A}$,

We consider a constrain $A \mid V\lambda$ we have $(A \mid V_\lambda)\,\delta = A\delta$, and correspond eigenvector of δ will be in $V\lambda$ which is eigen subspace of B, if V is the eigenvector of δ which is in eigen subspace of A

$$\text{thus } V \text{ must be the common eigen vector}$$

Exercise 7.3

Let p be the characteristic polynomial for the n-the order ODE with constant coefficients

$$y^{(n)} = a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

Suppose $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are distinct roots for p, consider the following $n \times n$ matrix

$$A = \left[\begin{array}{c|c} 0 & \\ \vdots & I_{n-1} \\ \vdots & \\ \hline 0 & a_1 \cdots\cdots\cdots\cdots a_{n-1} \end{array}\right]$$

(i) Find eigenvalues and eigenvectors of A. (ii) Find invertible matrix P and diagonal matrix Λ such that $A = P\Lambda P^{-1}$.

Answer:

Exercise 7.4

Consider the circulant matrix

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

where $c_0, \dots, c_{n-1} \in \mathbb{C}$. Find the eigenvalues and associated eigenvectors of C.

Answer:
Consider the maxtrix $M \in M$.

$$M = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \\ 1 & 0 & \cdots & 1 \end{pmatrix} \quad M^k = \begin{pmatrix} 0 & I_{n-k} \\ I_k & 0 \end{pmatrix}$$

thus the circulant matrix can be written as

$$C = C_{n-1}M + C_{n-2}M^2 + \cdots + C_1 M^{n-1}$$

Let $f(x) = C_{n-1}x + C_{n-2}x^2 + \cdots + C_1 x^{n-1}$ then $C = f(M)$. We first research on the eigenvalue of M !

$$|\lambda I - M| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \ddots & -1 \\ -1 & 0 & \lambda \end{vmatrix} = \lambda^n - 1 = 0.$$

$$\lambda_k = \omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

n distinct eigenvalue the corresponding eigenvectors are

$$\begin{aligned}
\alpha_k = \quad & \left(1, w_k, {w_k}^2, \ldots {w_k}^{n-1}\right)^\top \\
& pMp^{-1} = \operatorname{diag}\left\{1, w_1 \ldots w_{n-1}\right) \\
p \quad & Cp^{-1} = pf(m)p^{-1} = f\left(pMp^{-1}\right) = \operatorname{diag} f(1), f\left(w_1\right) \ldots f\left(w_{n-1}\right)
\end{aligned}$$

which are the eigen value of C, the eigenvectors are $\alpha_k = \left(1, w_k, \ldots w_k^{n-1}\right)^\top$, the sane as M.

Exercise 7.5

Exercise 7.5 Given a positive definite matrix $A \in M_n(\mathbb{R}), v \in \mathbb{R}^n$,
(i) Show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^\top Ax} \, \mathrm{d}x = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

hint: diagonalize A first.
(ii) Show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^\top Ax + v^\top x} \, \mathrm{d}x = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{\frac{1}{2}v^\top A^{-1}v}$$

(iii) Given another symmetric matrix $D \in M_n(\mathbb{R})$, show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^\top Ax + v^\top x} \left(x^\top Dx\right) \mathrm{d}x = \left[v^\top A^{-1}DA^{-1}v + \operatorname{tr}\left(DA^{-1}\right)\right] \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{\frac{1}{2}v^\top A^{-1}v}$$

hint: calculate the directional derivative $\mathbf{D}I(A) \cdot D$, where $I(A)$ is the integral in (ii).

Answer:
(i) from the perspective of the probability theory. this is just the kernel PDF of an *n-dimensional* multivariate normal distribution which is $N\left(\overline{0}, A^{-1}\right)$ the integral of all dimension from $-\infty$ to $+\infty$ will definite cause $\int_{R^n}\left((2\pi)^{-\frac{n}{2}} \det\left(A^{-1}\right)\right) e^{-(x-n)^\top A(x-u)} dx = 1$ thus $\int_{R^n} e^{-(x-\mu)^\top A(x-\mu)} dx = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det A}} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}}$

(ii) We shall complete square on the exponential part in the matrix vector form. I will apply my knowledge in Bayesian Analysis STAT4510J to solve it. You shall know that: $\nabla_x X^\top AX = 2AX$.

$$\begin{aligned}
& \nabla_x\left(-\frac{1}{2}x^\top Ax + V^\top x\right) = 0 \\
& = -Ax + V = 0, x = A^{-1}V
\end{aligned}$$

$$\text{thus} \quad \begin{aligned}
& -\frac{1}{2}x^\top Ax + V^\top x \\
= & -\frac{1}{2}\left(x - A^{-1}V\right)^\top A\left(x - A^{-1}V\right) + \frac{1}{2}V^\top A^{-1}V
\end{aligned}$$

Since

$$\begin{aligned}
& -\frac{1}{2}\left(X^{\top}-V^{\top}A^{-\top}\right)A\left(X-A^{-1}V\right) \\
= & -\frac{1}{2}\left(x^{\top}A-V^{\top}A^{-\top}A\right)\left(X-A^{-1}V\right) \\
& A^{-\top}A=I \text{ since } A \text{ is positive definite } \Rightarrow \text{ symmetric} \\
= & -\frac{1}{2}\left(x^{\top}A-V^{\top}\right)\left(x-A^{-1}V\right) \\
= & -\frac{1}{2}\left(x^{\top}Ax-V^{\top}x-x^{\top}V+V^{\top}A^{-1}V\right) \\
= & -\frac{1}{2}\left(x^{\top}Ax-2V^{\top}X+V^{\top}A^{-1}V\right) \\
= & -\frac{1}{2}x^{\top}AX+V^{\top}x-\frac{1}{2}V^{\top}A^{-1}V
\end{aligned}$$

$$\begin{aligned}
& \text{thus } \int_{\mathbb{R}^n} e^{-\frac{1}{2}x^{\top}Ax+V^{\top}x}dx \\
& \quad =\int_{R^n} e^{-\frac{1}{2}\left(x-A^{-1}V\right)^{\top}A\left(x-A^{-1}v\right)+\frac{1}{2}V^{\top}A^{-1}v}dx \\
& =\left(\int_{R^n} e^{-\frac{1}{2}\left(x-A^{-1}v\right)^{\top}A\left(x-\Delta^{-1}v\right)}dx\right)\cdot e^{\frac{1}{2}v^{\top}A^{-1}v}
\end{aligned}$$

the former one is the kenerl function of *n-dimensional* Multivariate Normal $N\left(A^{-1}v, A^{-1}\right)$

$$\begin{aligned}
& \text{thus } \int_{R} e^{-\frac{1}{2}\left(x-A^{-1}v\right)^{\top}A\left(x-A^{-1}v\right)}d\left(x-A^{-1}v\right) \\
& =(2\pi)^{\frac{n}{2}}\cdot\frac{1}{\sqrt{\det A}}
\end{aligned}$$

thus is $\frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\text{detA}}}\cdot e^{\frac{1}{2}v^{\top}A^{-1}v}$

(iii) from the perspective of probability theory $(2\pi)^{-\frac{n}{2}}\cdot\left(\det\left(A^{-1}\right)^{-\frac{1}{2}}\right)*\int_{R^n} e^{-\frac{1}{2}x^{\top}Ax+V^{\top}x}\left(x^{\top}Dx\right)dx$ is the expectation of

$$\begin{aligned}
& X^{\top}DX \text{ where } x\sim N\left(A^{-1}V, A^{-1}\right) \\
& \text{the } E\left[X^{\top}Dx\right]=E\left[\operatorname{tr}\left[x^{\top}DX\right]\right]=E\left[D\operatorname{tr}\left[x^{\top}x\right]\right] \\
& =\operatorname{tr}\left[DE\left[xx^{\top}\right]\right] \\
& =\operatorname{tr}\left[D\operatorname{Var}[x]+D\mu_x\cdot\mu_x^{\top}\right] \\
& =\operatorname{tr}\left[D\operatorname{Var}[x]+DA^{-1}VV^{\top}A^{-\top}\right] \\
& =\operatorname{tr}\left[D\operatorname{Var}[x]+A^{-1}VDV^{\top}A^{-1}\right] \\
& =\operatorname{tr}\left[DA^{-1}\right)+\operatorname{tr}\left(V^{\top}A^{-1}DA^{-1}V\right) \\
& =\operatorname{tr}\left(DA^{-1}\right)+V^{\top}A^{-1}DA^{-1}V
\end{aligned}$$

thus the original one is

$$\frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}}\left(\operatorname{tr}\left(DA^{-1}\right)+V^{\top}A^{-1}DA^{-1}V\right) \text{ proved}$$

Or yon can take derivative on left side and right side on (ii) for convenience, view H us D.

right:

$$D\left(\frac{(2\pi)^{n/2}}{\sqrt{\det A}}e^{\frac{1}{2}V^{\top}A^{-1}V}\right)H$$
$$=D\left(\exp\left(qoI_A\right)Dq_(I_A)\right)\cdot DI(A)H\cdot\frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

where I_A is inverse function, q is the quadratic form $\frac{1}{2}v^{\top}A^{-1}v$

$$=D(\exp(qoIn))Dq(IA)\cdot\left(-A^{-1}HA^{-1}\right)\frac{(2\pi)^{n/2}}{\sqrt{\det(\mathrm{A})}}$$
$$=D(\exp(qoIA))\cdot\left(-\frac{1}{2}V^{\top}A^{-1}HA^{-1}V\right)\frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$
$$=e^{\frac{1}{2}V^{\top}A^{-1}V}\quad\left(-\frac{1}{2}\right)V^{\top}A^{-1}HA^{-1}V\frac{(2\pi)^{n/2}}{\sqrt{\det\mathrm{A}}}$$

Left

$$DI(A)\cdot H$$
$$=\left.\frac{d}{dt}\right|_{t=0}I(A+th)$$
$$=\left.\frac{d}{dt}\right|_{t=0}\int e^{-\frac{1}{2}x^{\top}(A+tH)x+v^{\top}x}dx$$
$$=\int\left.\frac{d}{dt}\right|_{t=0}e^{-\frac{1}{2}x^{\top}Ax}\cdot e^{v^{\top}x}e^{-\frac{1}{2}tx^{\top}Hx}dx$$
$$=\int e^{-\frac{1}{2}x^{\top}Ax}\cdot e^{v^{\top}x}\left(-\frac{1}{2}x^{\top}Hx\right)dx.$$
$$=-\frac{1}{2}\int e^{-\frac{1}{2}x^{\top}Ax+v^{\top}x}x^{\top}Hxdx.$$

Left $=$ Right, proved

Exercise 7.6

Let $\Phi\left(t,t_0\right)\in\operatorname{Mat}_n(\mathbb{R})$ satisfy the IVP over $I\subseteq\mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(t,t_0\right)=A(t)\Phi\left(t,t_0\right)\text{ for all }t,t_0\in I$$
$$\Phi\left(t_0,t_0\right)=I_n\text{ for all }t_0\in I$$

where $A(t)\in\operatorname{Mat}_n(\mathbb{R})$ for all $t\in I\subseteq\mathbb{R}$. Assume all calculations are possible,

(i) Verify $\Phi\left(t_0,t\right)=\Phi\left(t,t_0\right)^{-1}$ by showing both sides satisfy the same IVP.

(ii) Show that

$$\det\Phi\left(t,t_0\right)=\exp\left(\int_{t_0}^{t}\operatorname{tr}A(s)\mathrm{d}s\right)$$

Answer:

First show $\Phi\left(t,t_0\right)=\Phi\left(t,t_1\right)\Phi\left(t_1,t_0\right)$

Note $\cdot\frac{d}{dt}\Phi\left(t,t_0\right)=A(t)\Phi\left(t,t_0\right)\&\Phi\left(t_1\right)=\Phi\left(t_1,t_0\right)$ and

- $\frac{d}{dt}\left[\Phi\left(t,t_1\right)\Phi\left(t,t_0\right)\right]=A(t)\left[\Phi\left(t,t_1\right)\Phi\left(t,t_0\right)\right]$

$\&$ $\Phi\left(t_1,t_1\right)\Phi\left(t_1,t_0\right)=\Phi\left(t_1,t_0\right)$

So $\Phi\left(t_0,t\right)\Phi\left(t,t_0\right)=\Phi\left(t_0,t_0\right)=I$

$\Rightarrow\Phi\left(t_0,t\right)=\Phi\left(t,t_0\right)^{-1}$

Exercise 7.7

Find the general solution to the ODE $\dot{x} = Ax$ following matrice A. Express the final results in real functions.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$

Answer:

(a) The eigenvalues and eigenvectors of the coefficient

$$\begin{pmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ The

determinant of coefficients reduces to $(1-r)(r^2 - 2r + 5)$ so the eigenvalues are $r_1 = 1, r_2 = 1 + 2i$, and $r_3 =$ 1 - $2i$. The eigenvector corresponding to r_1 satisfies $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; hence $\xi_1 - \xi_3 = 0$ and $3\xi_1 + 2\xi_2 = 0$. If we let $\xi_2 = -3$ then $\xi_1 = 2$ and $\xi_3 = 2$, so one solution of the D.E. is $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t$. The eigenvector corresponding to r_2 satisfies $\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence $\xi_1 = 0$ and $i\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_3 = -i$. Thus a complex-valued solution is $\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^t(\cos 2t + i \sin 2t)$. Taking the real and imaginary parts, see prob. 1, we obtain $\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} e^t$ and $\begin{pmatrix} 0 \\ \sin 2t \\ \text{-cos2t} \end{pmatrix} e^t$, respectively. Thus the general solution is x $= c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2 e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3 e^t \begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix}$, which spirals to ∞ about the x_1 axis in the $x_1 x_2 x_3$ space as $t \to \infty$.

(b) The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The characteristic equation of the coefficient matrix is $r^3 + 4r^2 + 7r + 6 = 0$, with roots $r_1 = -2, r_2 = -1 - \sqrt{2}i$ and $r_3 = -1 + \sqrt{2}i$. Setting $r = -2$, the equations reduce to

$$\begin{aligned} -\xi_1 + 2\xi_3 &= 0 \\ \xi_1 + \xi_2 &= 0. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2, -2, 1)^T$. With $r = -1 - \sqrt{2}i$, the system of equations is equivalent to

$$\begin{aligned} (2 - i\sqrt{2})\xi_1 - 2\xi_3 &= 0 \\ \xi_1 + i\sqrt{2}\xi_2 &= 0. \end{aligned}$$

An eigenvector is given by $\boldsymbol{\xi}^{(2)} = (-i\sqrt{2}, 1, -1 - i\sqrt{2})^T$. Hence one of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-(1+i\sqrt{2})it} \\ &= \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1 - i\sqrt{2} \end{pmatrix} e^{-t}(\cos\sqrt{2}t - i\sin\sqrt{2}t) \\ &= e^{-t} \begin{pmatrix} -\sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \\ -\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + ie^{-t} \begin{pmatrix} -\sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix} \end{aligned}$$

The other complex-valued solution is $\mathbf{x}^{(3)} = \overline{\boldsymbol{\xi}^{(2)}} e^{r_3 t}$. The general solution is

$$\begin{aligned} \mathbf{x} = c_1 &\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + \\ &+ c_2 e^{-t} \begin{pmatrix} \sqrt{2}\sin\sqrt{2}t \\ -\cos\sqrt{2}t \\ \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} \sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t + \sin\sqrt{2}t \end{pmatrix} \end{aligned}$$

It is easy to see that all solutions converge to the equilibrium point $(0, 0, 0)$.

Exercise 7.8

Given $A \in M_n(\mathbb{R})$, consider the IVP

$$\dot{x} = Ax + g(t), \quad x(0) = x_0$$

where $g : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function. Show that

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}g(s)\mathrm{d}s$$

Answer:
recall the first order linear equation.

$$y' + p(x)y = q(x).$$
$$\tfrac{dx}{dt} - Ax = g(t)$$

We multiply the integrating factor $e^{-\int Adt}$ on both side.

$$e^{-At}\frac{dx}{dt} - Ae^{-At}x = e^{-At}g(t)$$

$$\frac{d\left(e^{-At}x\right)}{dt} = e^{-At}g(t)$$

$$e^{-At}x = \int e^{-At}g(t)dt + C$$

$$x = e^{At}c + \int_0^t e^{A(t-s)}g(s)ds.$$

$$t = 0$$

$$x_0 = c$$

$$x = e^{At}x_0 + \int_0^t e^{A(t-s)}g(s)ds$$

Exercise 7.9

Solve the following IVP with initial condition $x(0) = \begin{bmatrix} -1 & 2 & -30 \end{bmatrix}^\top$

(a) $\dot{x} = \begin{bmatrix} -4 & 1 & 0 \\ 3 & 6 & 2 \\ 1 & 0 & 0 \end{bmatrix} x$

(b) $\dot{x} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} x$

(c) $\dot{x} = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} x$

hint: one does not need matrix theory to solve (a).

Answer:
a) The eigenvalues are $r = 1, 1, 2$.

For $r = 2$, we have

$$\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

, which yields

$$\boldsymbol{\xi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

, so one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.$$

For $r = 1$, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

, which yields the second solution

$$\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^t$$

.

The third solution is of the form

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \eta e^t$$

, where

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \boldsymbol{n} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$$

and thus $\boldsymbol{\eta_1} = -1/4$ and $6\boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 = -21/4$.

Choosing $\boldsymbol{\eta}_2 = 0$ gives $\boldsymbol{\eta}_3 = -21/4$ and

$$\text{hence } \mathbf{x}(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t \right] + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

The I.C. then yield $c_1 = 2, c_2 = 4$ and $c_3 = 3$ and hence $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix} e^t +$

$4 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$, which become unbounded as $t \to \infty$.

(b)

The eigenvalues and eigenvectors of the coefficient matrix satisfy

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of coefficients is $8 - 12r + 6r^2 - r^3 = (2-r)^3$, so the eigenvalues are $r_1 = r_2 = r_3 = 2$. The eigenvectors corresponding to this triple eigenvalue satisfy

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Using row reduction we can reduce this to the equivalent system $\xi_1 - \xi_2 - \xi_3 = 0$, and $\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_3 = -1$ and $\xi_1 = 0$, so the only eigenvectors are multiples of $\boldsymbol{\xi} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

From previous part, one solution of the given D.E. is

$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathrm{e}^{2t}$, but there are no other linearly independent solutions of this form.

We now seek a second solution of the form $\mathbf{x} = \boldsymbol{\xi} t\mathrm{e}^{2\mathrm{t}} + \boldsymbol{\eta}\mathrm{e}^{2\mathrm{t}}$. Thus $\mathbf{Ax} = \mathbf{A}\boldsymbol{\xi} t\mathrm{e}^{2\mathrm{t}} + \mathbf{A}\boldsymbol{\eta}\mathrm{e}^{2t}$ and $\mathbf{x}' = 2\boldsymbol{\xi} te^{2t} + \boldsymbol{\xi} e^{2t} + 2\boldsymbol{\eta} e^{2t}$. Equating like terms, we then have $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = 0$ and $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$. Thus $\boldsymbol{\xi}$ is as in part a and the second equation yields $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} =$ $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. By row reduction this is equivalent to the system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} =$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. If we choose $\eta_3 = 0$, then $\eta_2 = 1$ and $\eta_1 = 1$, so $\eta = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence a second solution of the D.E. is $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$

Assuming $\mathbf{x} = \boldsymbol{\xi}\,(\mathrm{t}^2/2)\,\mathrm{e}^{2\mathrm{t}} + \boldsymbol{\eta}\mathrm{e}^{2\mathrm{t}} + \boldsymbol{\zeta}\mathrm{e}^{2\mathrm{t}}$, we have $\mathbf{A}\mathbf{x} = \mathbf{A}\boldsymbol{\xi}\,(\mathrm{t}^2/2)\,\mathrm{e}^{2\mathrm{t}} + \mathbf{A}\boldsymbol{\eta}\mathrm{te}^{2\mathrm{t}} + \mathbf{A}\boldsymbol{\xi}\mathrm{e}^{2\mathrm{t}}$ and

$\mathbf{x}' = \boldsymbol{\xi}\mathrm{te}^{2t} + 2\boldsymbol{\xi}\left(\mathrm{t}^2/2\right)\mathrm{e}^{2t} + \boldsymbol{\eta}^{2\mathrm{t}} + 2\boldsymbol{\eta}\mathrm{e}^{2\mathrm{t}} + 2\boldsymbol{\xi}\mathrm{e}^{2t}$ and thus

$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, $\quad(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ and $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$. Again, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are as found previously and the last equation is equivalent to

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. By row reduction we find the equivalent system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$. If we let

$\zeta_2 = 0$, then $\zeta_3 = 3$ and $\zeta_1 = 2$, so $\zeta = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (t^2/2)\,e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mathrm{te}^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \mathrm{e}^{2t}$.

$\mathbf{T} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and using row operations on $\mathbf{T}$ and $\mathbf{I}$, or a computer algebra system, $\mathbf{T}^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$ and thus

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{J}$$

(Initial Problem:)
Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}$$

(a) Show that $r = 2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix A and that there is only one corresponding eigenvector, namely,

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(b) Using the information in part (a), write down one solution $\mathbf{x}^{(1)}(t)$ of the system (i). There is no other solution of the purely exponential form $\mathbf{x} = \xi e^{nt}$.

(c) To find a second solution, assume that $\mathbf{x} = \xi te^{2t} + \eta e^{2t}$. Show that ξ and η satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\xi = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\eta = \boldsymbol{\xi}.$$

Since ξ has already been found in part (a), solve the second equation for η. Neglect the multiple of $\xi^{(1)}$ that appears in η, since it leads only to a multiple of the first solution $\mathbf{x}^{(1)}$. Then write down a second solution $\mathbf{x}^{(2)}(t)$ of the system (i).

(d) To find a third solution, assume that $\mathbf{x} = \xi\,(t^2/2)\,e^{2t} + \eta te^{2t} + \zeta e^{2t}$. Show that $\boldsymbol{\xi}, \boldsymbol{\eta}$, and ζ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\xi = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\eta = \xi, \quad (\mathbf{A} - 2\mathbf{I})\zeta = \eta.$$

The first two equations are the same as in part (c), so solve the third equation for ζ, again neglecting the multiple of $\xi^{(1)}$ that appears. Then write down a third solution $\mathbf{x}^{(3)}(t)$ of the system (i).

(e) Write down a fundamental matrix $\Psi(t)$ for the system (i).

(f) Form a matrix $\mathbf{T}$ with the eigenvector $\xi^{(1)}$ in the first column and the generalized eigenvectors η and ζ in the second and third columns. Then find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}$ T. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

(c)

$$\begin{pmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r^2 + 3r - 1 = 0$, with a single root of multiplicity three, $r = 1$. Setting $r = 1$, we have

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system of algebraic equations reduce to a single equation

$$4\xi_1 - 3\xi_2 - 2\xi_3 = 0.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(1)} = (1, 0, 2)^T$. Since the last equation has two free variables, a second linearly independent eigenvector (associated with $r = 1$) is $\boldsymbol{\xi}^{(2)} =$

$(0, 2, -3)^T$. Therefore two solutions are obtained as

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t.$$

Note that a linear combination of two eigenvectors, associated with the same eigenvalue, is also an eigenvector. Consider the equation $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}$. The augmented matrix is

$$\left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 8 & -6 & -4 & 2c_2 \\ -4 & 3 & 2 & 2c_1 - 3c_2 \end{array}\right).$$

Using elementary row operations, we obtain

$$\left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 0 & 0 & 0 & -2c_1 + 2c_2 \\ 0 & 0 & 0 & 3c_1 - 3c_2 \end{array}\right).$$

It is evident that a solution exists provided $c_1 = c_2$.

Let $c_1 = c_2 = 2$. The components of the generalized eigenvector must satisfy

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$$

Based on Part (c), the equations reduce to the single equation $4\eta_1 - 3\eta_2 - 2\eta_3 = 2$. Let $\eta_1 = \alpha$ and $\eta_2 = 2\beta$, where α and β are arbitrary constants. We then have

$$\eta_3 = -1 + 2\alpha - 3\beta,$$

so that

$$\boldsymbol{\eta} = \begin{pmatrix} \alpha \\ 2\beta \\ -1 + 2\alpha - 3\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

Observe that $\boldsymbol{\eta} = \alpha\boldsymbol{\xi}^{(1)} + \beta\boldsymbol{\xi}^{(2)}$. Hence a third linearly independent solution is

$$\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} e^t$$

Given the three linearly independent solutions, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^t & 0 & 2te^t \\ 0 & 2e^t & 4te^t \\ 2e^t & -3e^t & -2te^t - e^t \end{pmatrix}.$$

Given the three linearly independent solutions, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^t & 0 & 2te^t \\ 0 & 2e^t & 4te^t \\ 2e^t & -3e^t & -2te^t - e^t \end{pmatrix}$$

We construct the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix}$$

with inverse

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 2 & -3/2 & -1 \end{pmatrix}.$$

The Jordan form of the matrix $\mathbf{A}$ is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(Initial Problem:)
Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}\mathbf{x}$$

(a) Show that $r = 1$ is a triple eigenvalue of the coefficient matrix $\mathbf{A}$ and that there are only two linearly independent eigenvectors, which we may take as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

Write down two linearly independent solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ of Eq. (i).

(b) To find a third solution, assume that $\mathbf{x} = \xi te^t + \eta e^t$; then show that ξ and η must satisfy

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0},$$
$$(\mathbf{A} - \mathbf{I})\eta = \boldsymbol{\xi}.$$

(c) Equation (iii) is satisfied if ξ is an eigenvector, so one way to proceed is to choose ξ to be a suitable linear combination of $\xi^{(1)}$ and $\xi^{(2)}$ so that Eq. (iv) is solvable, and then to solve that equation for η. However, let us proceed in a different way and follow the pattern of Problem 17. First, show that η satisfies

$$(\mathbf{A}-\mathbf{I})^2\eta = \mathbf{0}.$$

Further, show that $(\mathbf{A}-\mathbf{I})^2 = \mathbf{0}$. Thus η can be chosen arbitrarily, except that it must be independent of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.

(d) A convenient choice for η is $\eta = (0,0,1)^T$. Find the corresponding ξ from Eq. (iv). Verify that ξ is an eigenvector.

(e) Write down a fundamental matrix $\mathbf{\Psi}(t)$ for the system (i).

(f) Form a matrix $\mathbf{T}$ with the eigenvector $\xi^{(1)}$ in the first column and with the eigenvector ξ from part (d) and the generalized eigenvector η in the other two columns. Find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

Exercise 7.10

Let

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

(i) Show that for $n \in \mathbb{N}$,

$$J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

(ii) Determinine $\exp(tJ)$

Answer:

(a). Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\mathbf{J}^{n+1} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda\cdot\lambda^n & \lambda^n + n\lambda\cdot\lambda^{n-1} & n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2} \\ 0 & \lambda\cdot\lambda^n & \lambda^n + n\lambda\cdot\lambda^{n-1} \\ 0 & 0 & \lambda\cdot\lambda^n \end{pmatrix}.$$

The result follows by noting that

$$\begin{aligned} n\lambda^{n-1} + \frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2} &= \left[n + \frac{n(n-1)}{2}\right]\lambda^{n-1} \\ &= \frac{n^2+n}{2}\lambda^{n-1}. \end{aligned}$$

(b)

$$\sum_{n=0}^{\infty} \lambda^n \frac{t^n}{n!} = e^{\lambda t}$$

$$\sum_{n=0}^{\infty} n\lambda^{n-1} \frac{t^n}{n!} = t \sum_{n=1}^{\infty} \lambda^{n-1} \frac{t^{n-1}}{(n-1)!} = te^{\lambda t}$$

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2} \lambda^{n-2} \frac{t^n}{n!} = \frac{t^2}{2} \sum_{n=2}^{\infty} \lambda^{n-2} \frac{t^{n-2}}{(n-2)!} = \frac{t^2}{2} e^{\lambda t}.$$

Therefore

$$\exp(\mathbf{J}t) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

Exercise 7.11

Find the Fourier series, Fourier sine series, and Fourier cosine series for the given functions

(a) $f(x) = x^2/2, -2 \leq x \leq 2$

(b) $f(x) = \begin{cases} x+2, & -2 \leq x < 0, \\ 2-2x, & 0 \leq x < 2; \end{cases}$

Answer:

Exercise 7.12

Find the solution of the heat conduction problem

$$u_{xx} = 4u_t, \quad 0 < x < 2, \quad t > 0$$
$$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0$$
$$u(x,0) = 2\sin(\pi x/2) - \sin \pi x + 4 \sin 2\pi x, \quad 0 \leq x \leq 2$$

Answer:

Following the procedures of Eqs, we set $u(x,y) = x(x)T(t)$ in the P.D.E. to obtain $x''T = 4XT'$, or $\mathrm{X''X} = 4\ \mathrm{T'/T}$, which must be a constant. As stated in the text this separation constant must be $-\lambda^2$ (we choose $-\lambda^2$ so that when a square root is used later, the symbols are simpler) and thus $\mathrm{X''} + \lambda^2\mathrm{X} = 0$ and $\mathrm{T'} + (\lambda^2/4)\,\mathrm{T} = 0$. Now $\mathrm{u(0,t)} = X(0)T(t) = 0$, for all $t > 0$, yields $X(0) = 0$, as discussed after Eq. (11) and similarly $u(2,t) = X(2)T(t) = 0$, for all $t > 0$, implies $X(2) = 0$. The D.E. for X has the solution $\mathrm{X(x)} = \mathrm{C_1 \cos \lambda x + C_2 \sin \lambda x}$ and $\mathrm{X(0)} = 0$ yields $\mathrm{C_1} = 0$. Setting $\mathrm{x} = 2$ in the remaining form of x yields $\mathrm{x(2)} = \mathrm{C_2 \sin 2\lambda} = 0$, which has the solutions $2\lambda = n\pi$ or $\lambda = \mathrm{n}\pi/2, \mathrm{n} = 1, 2, \ldots$. Note that we exclude $n = 0$ since then $\lambda = 0$ would yield $\mathrm{X(x)} = 0$, which is unacceptable. Hence $\mathrm{X(x)} = \sin(\mathrm{n}\pi\mathrm{x}/2), \mathrm{n} = 1, 2, \ldots$. Finally, the solution of the D.E. for T yields $T(t) = \exp(-\lambda^2 t/4) = \exp(-n^2\pi^2 t/16)$. Thus we have found $u_n(x,t) = \exp(-n^2\pi^2 t/16) \sin(n\pi x/2)$. Setting $t = 0$ in this last expression indicates that $u_n(x,0)$ has, for the correct choices of n, the same form as the terms in $u(x,0)$, the initial condition. Using the principle of superposition we know that $u(x,t) =$

$c_1u_1(x,t)+c_2u_2(x,t)+c_4u_4(x,t)$ satisfies the P.D.E. and the B.C. and hence we let $t=0$ to obtain $u(x,0) = c_1u_1(x,0)+c_2u_2(x,0)+c_4u_4(x,0) = c_1 \sin \pi x/2+c_2 \sin \pi x+c_4 \sin 2\pi x$. If we choose $c_1 = 2$, $c_2 = -1$ and $c_4 = 4$ then $u(x,0)$ here will match the given initial condition, and hence substituting these values in $u(x,t)$ above then gives the desired solution.

Reference

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