VV256/MATH2560J Honors Calculus IV
Recitation Class Material

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Existence and Uniqueness Theorem

Theorem. Consider the initial value problem

$$y' = f(x, y), \ \ y(x_0) = y. \qquad ①$$

Suppose $f(x, y)$ in some closed rectangle
$R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b, a, b > 0\}$, and there exist $K, L > 0$ such that

$$③ \qquad |f(x, y)| \leq K \text{ and } \left|\frac{\partial f}{\partial y}\right| \leq L \ \forall (x, y)$$

Then the IVP has a unique solution in the interval $|x - x_0| \leq \alpha$, where
$\alpha = \min\{a, b/K\}$. ④

Remark: we don't even need the function f(x,y) to be differentiable to y, we can simply this condtion to $|f(x, y_2) - f(x, y_1)| \leq L|y_2 - y_1|$ which is called Lipschitz Condition

②

Picard Theorem.

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. ①

Rectangular Area: $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$.

f: $\forall y_1, y_2 \in R$ $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ Lipschitz

Continuity. ① has unique solution in $|x - x_0| \leq \alpha$

$\alpha = \min\{\frac{b}{K}, a\}$. Lipschitz continuity is a special continuity

In Real Analysis.

Remark

In the exploration of this theorem in the last RC, we regarded the integration equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(x))\, dt \quad (x \in I)$$

as a mapping J from a closed set D $\rightarrow$ D (itself). And we found the map has a certain property and use Banach Contraction Mapping Principle to prove this theorem. To tell the truth we did in that prove, we have constructed a **infinitely decreasing geometric sequence** to make $\{y_n\}$ converge to a fixed point which is the solution of ODE. However, if you listen to the last RC to the last minutes you will find this method add an unnecessary constrain that is $\alpha = min\{a, \frac{b}{M}, \mathbf{1/L}\}$ to the condition provided by Picard. $\forall y_1, y_2 \in D \quad |J(y_1) - J(y_2)| \leq \theta |y_1 - y_2| \quad \theta \in (0,1)$

Picard Iteration

To remove this unnecessary constrain, we must abandon the good property of map J to construct another series which is also converge in the interval I but is not the infinitely decreasing geometric sequence we construct by the compressive property of J. In a word, the new series:

- Must converge in I.
- Its ability of convergence may be worse than geometric sequence

The obvious answer is **Power Series**.

Picard Iteration

This method is based on the construction of Picard Series which is a uniformly convergent power series in the interval I:$= |x - x_0| \leq \alpha$, where $\alpha = \min\{a, b/K\}$.

Concept Picard Series: Based on the above conditions, the series $\{y_n\}$:

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt \quad (x \in I)$$

in the above equation, $n = 0, 1, 2, \cdots$ and $y_0(x) = y_0$.

The Picard Series $\{y_n\}$ has the property that it will uniformly **convergent** to the solution of ODE.

Picard Iteration

In one word, the method of Picard Iteration (Successive Approximation) is to construct a Picard Series for a given ODE and find something to which the series is convergent to.

Example

Solve the initial value problem

$$y' = 2t(1+y), \quad y(0) = 0$$

by the method of successive approximations.(This problem is from textbook Boyce.)

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$$\frac{dy}{dt} = 2t(1+y)$$

$$dy = 2t(1+y)dt$$

$$y - y_0 = \int_0^t 2x(1+y)\,dx$$

Construct a picard series of

$$y_{n+1}(t) = y_0 + \int_0^t 2x(1+y_n(x))\,dx.$$

$$y_1(t) = 0 + \int_0^t 2x\,dx = t^2$$

$$y_2(t) = \int_0^t 2x(1+x^2)\,dx = t^2 + \frac{2}{4}t^4$$

$$y_3(t) = \int_0^t 2x(1+x^2+\frac{2}{4}x^4)\,dx = t^2 + \frac{2}{4}t^4 + \frac{1}{6}t^6.$$

$$\vdots$$

$$y_n(t) = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots + \frac{1}{n!}t^{2n} \cdots$$

$$n \to \infty \quad y_n(t) = e^{t^2} - 1$$

TAA. I forget what is the series equal to, how to do?

↓

Solve it directly

copy to

↓

hunt for point!

No one will know how

You solve it!

$$\frac{dy}{dt} = 2t(1+y)$$

$$\frac{1}{y+1}dy = 2t\,dt$$

$$\ln|y+1| = t^2 + \ln C$$

$$|y+1| = Ce^{t^2}$$

$$|y+1| = e^{t^2} \quad (\text{initial } y=0.)$$

$$y = e^{t^2} - 1$$

Midterm 2017, Ordinary Differentiate Equation (Department of Mathematics, Fudan University) Problem 1.

Given a rectangle area $\overline{\mathcal{R}} = \left\{(x, y) \in \mathbb{R}^2; 0 \leqslant x \leqslant 1, |y| \leqslant 1\right\}$. On the area $\overline{\mathcal{R}}$, we define a function $f(x, y) = x - y^2 \sin\left(\frac{\pi}{2} y^2\right)$. Assume there is a IVP:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \\ y(0) = 0. \end{cases}$$

Use the method of successive approximation (you need to construct a picard series) to prove that the above equation has a unique solution on $x \in [0, 1]$.

Solution: $y = y_0 + \int_0^x f(t, y)\,dt$.

Construct a picard series:

$$y_{n+1}(x) = y_0 + \int_0^x t - y_n^2 \sin(\tfrac{\pi}{2} y_n^2)\,dt$$

$$|y_1(x) - y_0(x)| = \int_0^x t\,dt = \frac{1}{2}x^2.$$

$$|y_2(x) - y_1(x)| = \left|\int_0^x f(t, y_1) - f(t, y_0)\,dt\right|$$

$$\leq \int_0^x \underbrace{|f(t, y_1) - f(t, y_0)|}_{?}\,dt$$

$\forall\, y_a, y_b \in \bar{R}$

$$|f(x, y_a) - f(x, y_b)| = \left|\frac{\partial f}{\partial y}\right| |y_a - y_b|$$

$$\left|\frac{\partial f}{\partial y}\right| = \left|-2y \sin(\tfrac{\pi}{2}y^2) + y^2 \cdot \pi y \cdot \cos(\tfrac{\pi}{2}y^2)\right|$$

$$= \left|2y \sin\tfrac{\pi}{2}y^2 + y^3 \pi \cos(\tfrac{\pi}{2}y^2)\right|$$

$$\leq 2 + \pi \triangleq L. \quad \therefore |f(x, y_a) - f(x, y_b)| \leq L|y_a - y_b|.$$

$$|f(x, y)| \leq 1 \quad (x = 1, y = 0)$$

$$|y_2 - y_1| \leq L \int_0^x |y_1 - y_0|\,dt = L \int_0^x \frac{1}{2}t^2\,dt$$

$$= L \cdot \frac{1}{2\times 3} x^3.$$

$$|y_3 - y_2| \le \int_0^x |f(t, y_2) - f(t, x)| dt$$

$$\le L \int_0^x |y_2 - y_1| dt$$

$$= L \int_0^x L \cdot \frac{1}{2\times 3} t^3 dt$$

$$= L^2 \cdot \frac{1}{2\times 3\times 4} t^4$$

$|y_{n+1} - y_n| \le L^n \frac{1}{(n+2)!} t^{n+2}$ which is convergent to 0 if $n \to \infty$. in $I: |x| \le \alpha$

$\alpha = \min\{1, \frac{1}{1}\} = 1$

So. for enough large m, n.

$$|y_m - y_n| \le |y_m - y_{m-1} + y_{m-1} - y_{m-2} + \cdots + y_{n+1} - y_n|$$

$$\le |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + \cdots + |y_{n+1} - y_n|$$

$\to 0$ if $m, n \to \infty$.

that is to say $\hat{y_n} = \lim_{n\to\infty} y_n(x)$

$$y_n = y_0 + \int_0^x f(t, y_{n-1}) dt$$

$$\lim_{n\to\infty} y_n = y_0 + \lim_{n\to\infty} \int_0^x f(t, y_{n-1}) dt$$

$$= y_0 + \int_0^x \lim_{n\to\infty} f(t, y_{n-1}) dt$$

$$\tilde{y_n} = y_0 + \int_0^x f(t, \tilde{y_n}) dt.$$

$\tilde{y_n}$ is solution to ODE in

$x \in I$, $I = |x| \leq 1$ ∵ $x \in [0,1]$ $I = [0,1]$

Next prove the uniqueness.

if $u(x)$, $v(x)$ are two solutions.

$$|u(x) - v(x)| = \left| \int_0^x f(t, u(t)) dt - f(t, v(t)) dt \right|$$

$$\leq \int_0^x | f(t, u(t)) dt - f(t, v(t)) dt |$$

$$\leq L \int_0^x | u(t) - v(t) | dt$$

$$\leq L \cdot 2b \, (x - 0)$$

$$\leq L \int_0^x L 2b t \, dt$$

$$= L^2 \cdot \frac{2b}{2} x^2$$

$$\le L^3 \cdot \frac{2b}{2\times 3} x^3$$

$$\le \cdots$$

$$\le \underbrace{L^n \frac{2b}{n!}} x^n \quad n \to \infty,\ x \in [0,1]$$

$$\longrightarrow 0.$$

$\therefore u(x) = v(x)$ uniqueness proved.

Separable Equation

Definition. A separable ODE is an equation of the form

$$y' = f(t)g(y),$$

where f and g are given functions.

Remark. If there's an value y_0 such that $g(y_0) = 0$, then $y = y_0$ is a solution (equilibrium solution). To find other solutions, we assume that $g(y) \neq 0$.

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- Method: Direct Integration

$$dy = y'(t)dt = \frac{dy}{dt}dt = f(t)g(y)dt \Rightarrow \frac{1}{g(y)}dy = f(t)dt$$

$$\Rightarrow \int \frac{1}{g(y)}dy = \int f(t)dt \Rightarrow G(y) = F(t) + c, \ \ c = \text{const}\,.$$

Exercise 2.1 Solve the equations $\frac{dy}{dx} = \frac{x^2}{1-y^2}$ and $y' + 8xy = 0$.

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Solution 2.1

(1) There's no equilibrium solution. For other solutions,

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \Rightarrow (1-y^2)dy = x^2dx \Rightarrow \int(1-y^2)dy = \int x^2dx \Rightarrow x^3 - 3y + y^3 = c.$$

(2) The equilibrium solution is $y = 0$. For other solutions,

$$y' + 8xy = 0 \Rightarrow \frac{dy}{dx} = -8xy \Rightarrow \frac{dy}{y} = -9xdx \Rightarrow \ln|y| = -4x^2 + c \Rightarrow y = \tilde{c}e^{-4x^2}.$$

Remark. The differential form of separable equation is

$$M(x)dx + N(y)dy = 0.$$

Exercise

Given the separable ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y),$$

In which $f(y)$ is continuous at a certain neighborhood of $y = a$ (for example: $|y - a| \leqslant \varepsilon$) , and $f(y) = 0$,if and only if $y = a$. then prove that on every point of the line $y = a$, the solution to the ODE is partly unique if and only if

$$\left| \int_a^{a\pm\varepsilon} \frac{\mathrm{d}y}{f(y)} \right| = \infty \quad \text{(divergent)}.$$

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In this part I will briefly talk about logistic model but I will not cover Predator-Prey and SIR model because they need you to have knowledge about linear system.

Definition. An autonomous equation is represented in the form

$$y' = f(y),$$

where the function f does not depend explicitly on t.

Exponential Growth.

Let $y = y(t)$ be the population of the given species at time t. The simplest hypothesis concerning the variation of population is that the rate of change of y is proportional to the current value of y, *i.e.*,

$$y' = ry, \; y(0) = y_0,$$

where the constant of proportionality r is called the rate of growth or decline, depending on whether r is positive or negative.

Exponential Growth.

Solving the IVP $y' = ry,\ y(0) = y_0$, we have $y = y_0 e^{rt}$.

- It is observed to be reasonably accurate for many populations, at least for limited periods of time.
- Such ideal conditions cannot continue indefinitely.
- Limitations on space, food supply, or other resources will reduce the growth rate and stop the uninhibited exponential growth.

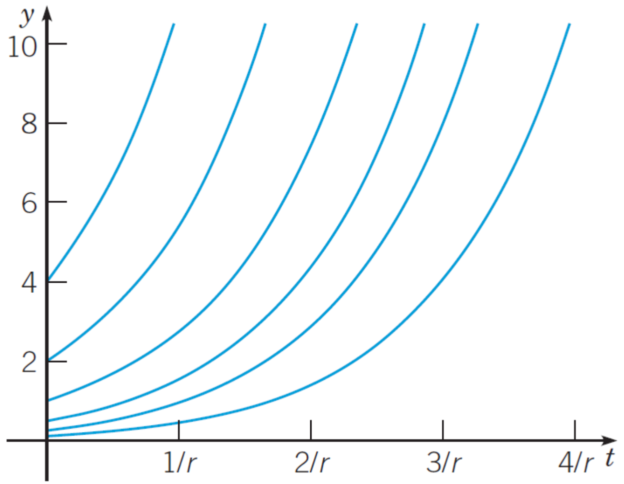

Figure: Exponential Growth.

This model is not very realistic.

Logistic Growth.

In this model, we replace the constant rate r by a function $u(y)$, which should satisfy

$$u(y) \approx r \text{ for small } y,\ u(y) \text{ decreases as } y \text{ increases, and } u(y) < 0 \text{ for large } y.$$

The simplest function $u(y)$ is $u(y) = r - ay$.

Therefore, we obtain the logistic equation

$$y' = (r - ay)y = r\left(1 - \frac{y}{K}\right)y, \quad K = r/a.$$

Equilibrium Solutions. $y' = 0 \Rightarrow y = 0$ or $y = K$. (Constant solutions, critical points).

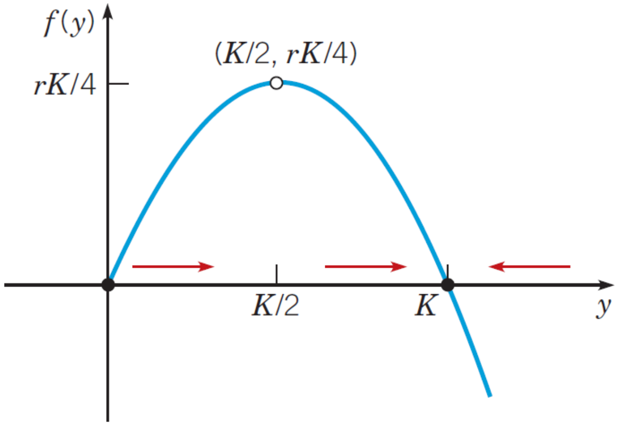

Figure: Logistic Growth: $f(y)$.

Autonomous Equations

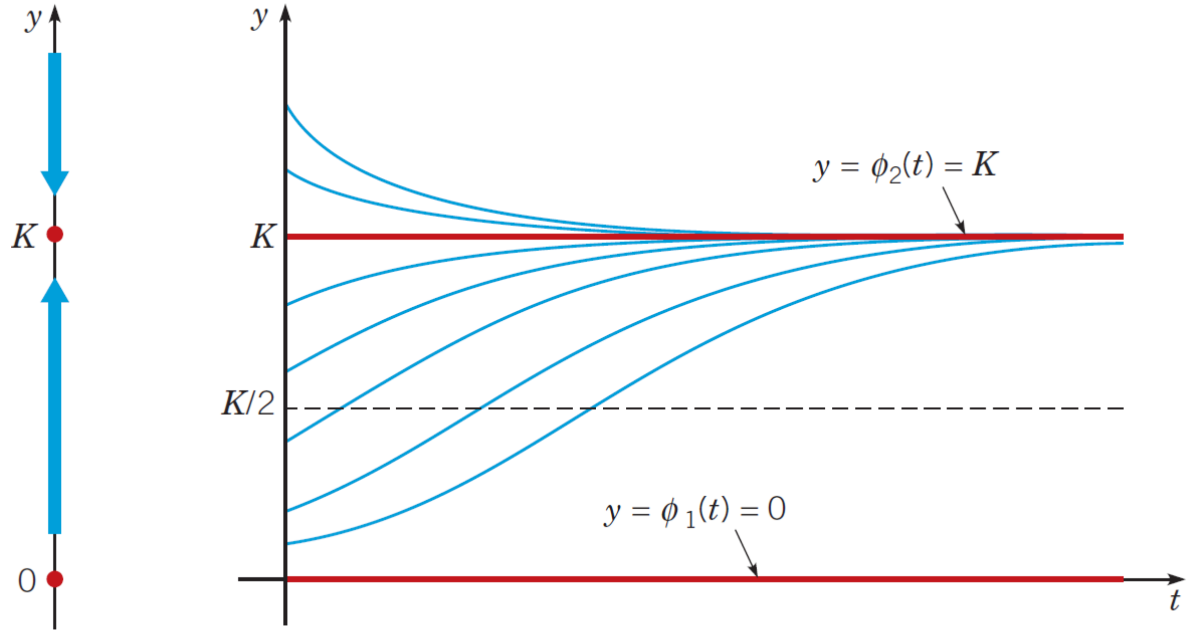

Figure: Logistic Growth: $y(t)$.

Remark. According to the uniqueness theorem, the curves cannot intersect!

Autonomous Equations

The solution of the logistic equation is

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + c, \;\; c = \text{const}\,.$$

For $0 < y < K$, y will remain in the same interval $\Rightarrow$ we can get

$$\frac{y}{K-y} = ce^{rt}, \;\; c = \text{const} \neq 0.$$

Including the IV $y(0) = y_0$, we have

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}.$$

If we take the limit, we will get

$$\lim_{t\to\infty} y(t) = \frac{y_0 K}{y_0} = K \text{ (Environmental Carrying Capacity)}.$$

Autonomous Equations

Definition. An equilibrium solution $y = y_0$ is called

- stable if any other solution close to y_0 remains close to y_0 for all time.
- asymptotically stable if it is stable and any solution starting close to y_0 becomes arbitrarily close to y_0 as t increases.

An equilibrium solution that is not stable is called unstable.

Obviously, here $y = K$ is asymptotically stable and $y = 0$ is unstable.

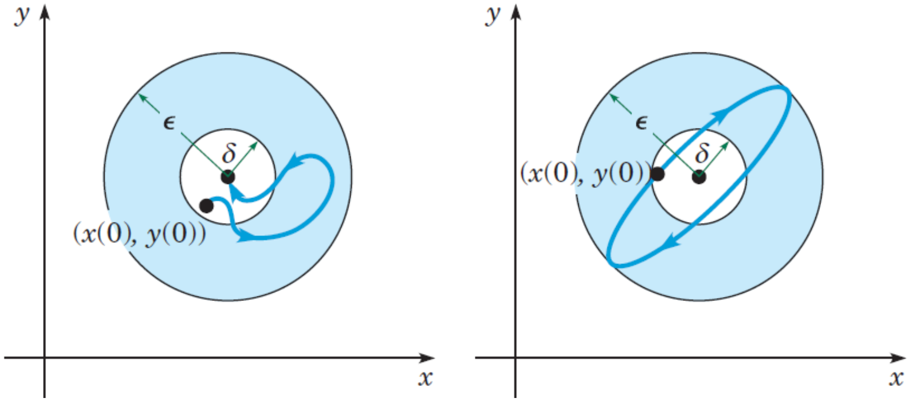

Figure: Asymptotically Stable and Stable.