VV256/MATH2560J Honors Calculus IV
Recitation Class Material

Zhang Fan

University of Michigan - Shanghai Jiao Tong University Joint Institute

Fall Term 2022

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Abbreviation List

- **DE** Differential Equation
- **ODE** Ordinary Differential Equation
- **PDE** Partial Differential Equation
- **I.V.** Initial Value
- **IVP** Initial Value Problem
- **B.V.** Boundary Value
- **BVP** Boundary Value Problem

Beginning Examples

IVPs:

Example	Differential Equation	Initial Value	Solution
RC Circuit	$V' = -\frac{1}{RC}V$	$V(0) = V_0$	$V(t) = V_0 e^{-\frac{1}{RC}t}$
Free Fall	$mv' = mg - \gamma v$	$v(0) = 0$	$v(t) = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right) e^{-\frac{\gamma}{m}t}$
Population Growth	$P' = (\beta(t) - \delta(t))\, P$	$P(0) = P_0$	$P(t) = P_0 e^{\int_0^t (\beta(\tau) - \delta(\tau))d\tau}$
Radioactive Decay	$N' = -kN$	$N(0) = N_0$	$N(t) = N_0 e^{-kt}$
Newton's Law	$T' = -k(T - \theta)$	$T(0) = T_0$	$T = \theta + (T_0 - \theta)e^{-kt}$

Question. Which type are they? Separable, linear or some other type?

Definition. A differential equation (DE) is a relation that contains an unknown function and one or more of its derivatives.

For $y = y(x)$, the general form of an ordinary differential equation (ODE) is

$$F\left(y, y', y'', \cdots, y^{(n)}\right) = 0.$$

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For $y = y(x_1, x_2)$, the general form of a partial differential equation (PDE) is

$$T\left(y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial^2 y}{\partial x_1 \partial x_2}, \frac{\partial^2 y}{\partial x_1^2}, \frac{\partial^2 y}{\partial x_2^2}, \cdots, \frac{\partial^n y}{\partial x_1^n}, \frac{\partial^n y}{\partial x_2^2}, \frac{\partial^n y}{\partial x_1^{n-1} \partial x_2}\right) = 0.$$

Basic Concepts

Definition. A differential equation (DE) is a relation that contains an unknown function and one or more of its derivatives.

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Definition. The additional condition we used to determine the constant c is an example of initial condition. The differential equation together with its initial condition forms an initial value problem, usually the solution of an initial value problem is a continuous and differentiable function (curve) .

Exercise

Given the first order chemical reaction equation

$$2H_2O_2 = 2H_2O + O_2 \tag{1}$$

If the initial concentration of the H_2O_2 is 0.5mol/L, and $k = 0.041min^{-1}$, please find the concentration of H_2O_2 10 minutes later.

$$-dc/dt = kc^{\alpha}$$

where α is reaction order.

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where α is reaction order.

$\alpha = 1$, $\quad -\frac{dc}{c} = k\,dt$

$\frac{dc}{c} = -k\,dt$ (You should always move negative sign to a less complicated Integration Part)

$\ln|c| = -kt + \ln \text{Const}$, $\quad c > 0$ by definition.

$c = \text{Const}\, e^{-kt}$ $\qquad C(0) = 0.5$ mol/L

$C = 0.5e^{-0.041t}$ $\qquad C(10) = 0.5e^{-0.41}$ mol/L.

So. the $[H_2O_2]$ after 10 minutes is $0.5e^{-0.41}$ mol/L.

Definition. The order of a differential equation is the order of the highest derivative that appears in the equation.

Definition. The ordinary differential equation

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

is said to be linear if F is a linear function of the variables $y, y' \cdots, y^{(n)}$.

first order: $\frac{dy}{dx} + p(x)y = q(x)$

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Existence and Uniqueness Theorem

Theorem. Consider the initial value problem

$y_0.$

$$y' = f(x, y), \;\; y(x_0) = \cancel{x}.$$

Don't need. closed.

Suppose $f(x, y)$ (and $\partial f/\partial y$) are continuous in some open rectangle $R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b, a, b > 0\}$, and there exist $K, L > 0$ such that

$$|f(x, y)| < K \text{ and } \left|\frac{\partial f}{\partial y}\right| \leq L \;\forall (x, y)$$

Then the IVP has a unique solution in the interval $|x - x_0| \leq \alpha$, where $\alpha = \min\{a, b/K\}$.

Remark: we don't even need the function f(x,y) to be differentiable to y, we can simply this condtion to $|f(x, y_2) - f(x, y_1)| \leq L|y_2 - y_1|$ which is called Lipschitz Condition

We change IVP to a Integration equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = x_0 \quad ①$$

$$dy = f(x,y)\,dx.$$

$$y - y_0 = \int_{x_0}^{x} f(t,y)\,dt. \quad ②$$

Do ①, ② really equivalent?

for ②. let $x = x_0$, $0 = 0$ ✓

derivative on both side:

$$\frac{d(y-y_0)}{dx} = \frac{d}{dx}\int_{x_0}^{x} f(t,y)\,dt$$

$$\Downarrow$$

$$\frac{dy}{dx} = f(x,y) \implies \frac{d\int_{x_0}^{x} f(t)dt}{dx} = f(x)$$

$$\Downarrow$$

$$\lim_{\Delta x \to 0} \frac{\int_{x_0}^{x+\Delta x} f(t)dt - \int_{x_0}^{x} f(t)dt}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_{x}^{x+\Delta x} f(t)dt}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\xi)\cdot(x+\Delta x - x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(\xi) = f(x)$$

So. ①, ② are equivalent.

$$y(x) = y_0 + \int_{x_0}^{x} f(t,y)\,dt$$

y related, y related.

map J: $y \mapsto y_0 + \int_{x_0}^{x} f(t,y)\,dt$

$y = J(y)$. y is a fixed point of map J.

The existence and uniqueness of y is the .... of fixed point of map J. for the EAU of fixed point. we easily think of theorem in Mathematical Analysis.

Banach Contraction Mapping Principle

Complete Space The Space that any Cauchy Sequence defined in the space must be converged to the space.
Theorem (Have time, look it, no time, pass it). Given a space $\mathbb{R}^n$ (this theorem is true as long as the space is a complete space,the absolute value afterwards should be changed to the correspond norm in the space), and a closed set D inside the space. If for any two points y_1, y_2 in D, we have a map $J : D \to D$ which satisfies $|J(y_2) - J(y_1)| \leq \theta |y_2 - y_1|$ in which $\theta \in (0, 1)$, then the map J has a unique fixed point. That is to say there exist a unique point X which makes $J(X) = X$

a kind of
squeeze, decrense
the distance of
2 point, finally
get 1 point.

Prove this as a recover for your freshman year calculus

We randomly choose a point y_0 in the ~~space~~ $\mathbb{R}^n$, set D.

and define a sequence $y_1 = J(y_0)$ $y_2 = J(y_1) \cdots y_n = J(y_{n-1})$

We tends to get one point so we may prove that

The sequence is covergent (cauchy sequence), select $m, n \in \mathbb{N}$

$$|y_m - y_n| = |y_m - y_{m-1} + y_{m-1} - y_{m-2} + \cdots + y_{n+1} - y_n|$$

$$\leq |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + \cdots + |y_{n+1} - y_n|$$

$$= \sum_{N=n}^{m} |y_N - y_{N-1}|$$

$$|y_N - y_{N-1}| = |J(y_{N-1}) - J(y_{N-2})|$$

$$\leq \theta\, |y_{N-1} - y_{N-2}|$$

$$\vdots$$

$$\leq \theta^{N-1} |y_1 - y_0|$$

$$\leq \sum_{N=n}^{m} \underbrace{|y_1 - y_0|}_{\text{const.}} \cdot \underbrace{\theta^{N-1}}_{0 < \text{const} < 1}$$

infinite decreasing geometric sequence $N \to \infty$

$\to 0$

$\lim_{n\to\infty} y_n = \hat{y}_n$, prove $J(\hat{y}_n) = \hat{y}_n$.

$$y_n = J(y_{n-1})$$

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} J(y_{n-1})$ J continous

$$\hat{y}_n = J(\lim_{n\to\infty} y_{n-1}) = J(\hat{y}_n) \text{ proved.}$$

let another fixed point $\tilde{y}_n'$

$$J(\hat{y}_n) = \hat{y}_n$$

$$J(\tilde{y}_n') = \tilde{y}_n'$$

$$\Rightarrow \hat{y}_n - \tilde{y}_n' = J(\hat{y}_n) - J(\tilde{y}_n')$$

$<$ $\leq$ $\underbrace{\theta}_{(0,1)} \underbrace{|\hat{y}_n - \tilde{y}_n'|}$

contradict.

So we need this theorem to prove the EAU of ODE Solution. We needs:

1. complete space ② closed set ③ Contraction map.

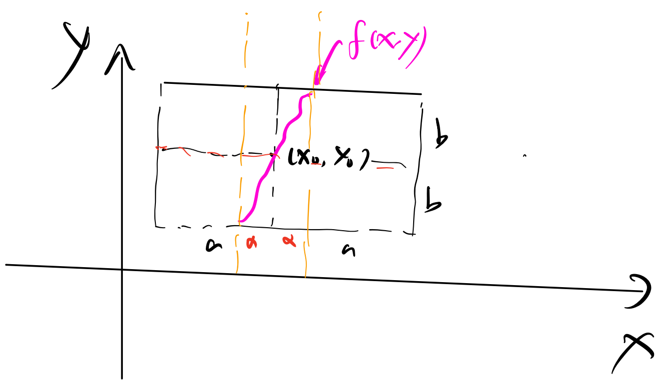
$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0. \end{cases}$$

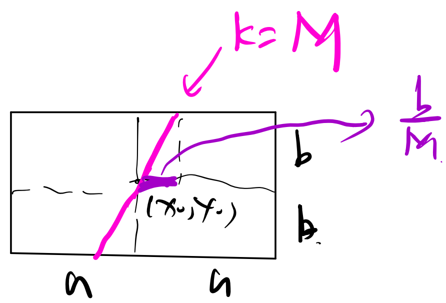
$\mathbb{R}^n \longrightarrow C$ continuous function.

$D \longrightarrow D$. rectangular Area.

$J \longrightarrow J$ how to construct a sequence?

How to select D?





where to define the solution.

①. it can be not one of the rec.

thus $h \le a$. next it can't out of the orange range. we suppose.

$|f(x,y)| \le M \quad \forall (x,y) \in D$ (rectangle), $\frac{dy}{dx} = f(x,y)$, $f(x,y)$ means the value of the slope, extreme condition let $\frac{dy}{dx}$ always be M, we get $\alpha \le \frac{b}{M}$ $\therefore \alpha = \min\{\frac{b}{M}, a\}$.

How to construct the J. condition, ie. to prove J has a similar property that we discussed in part of Banach Contraction Mapping principle.

① $J: D \to D$ ② $|J(y_2) - J(y_1)| \le \theta\, |y_2 - y_1|$.
$\forall\, y_2, y_1$ in D

① Abitarily take $y(x) \in D$

$$J(y(x)) = y_0 + \int_{x_0}^{x} f(t,y)\,dt.$$

$$|J(y(x)) - y_0| = \left| \int_{x_0}^{x} f(t,y)\,dt \right|$$
$$\le M\, |x - x_0|$$
$$\le M\, \delta$$
$$\le b. \quad \therefore \text{ in } D.$$

②. Abitarily take $y_1, y_2 \in D$. J has a certainty property?

$$|J(y_2) - J(y_1)| = \left| \int_{x_0}^{x} f(t,y_2) - f(t,y_1)\,dt \right|$$
$$\le \int_{x_0}^{x} |f(t,y_2) - f(t,y_1)|\,dt$$
$$= \int_{x_0}^{x} \left|\frac{\partial f}{\partial y}\right| |y_2 - y_1|\,dt \qquad 0 < L \cdot (x - x_0) < 1$$
$$\le \underbrace{\left\|\frac{\partial f}{\partial y}\right\|}_{=L} |y_2 - y_1|\,(x - x_0) \qquad 0 < d < \frac{1}{L}.$$

Theorem 1: The Existence and Uniqueness theorem of Cauchy,

Given IVP: $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$, f is continous on

$[x_0-a, x_0+a] \times [y_0-b, y_0+b]$ and $\left|\frac{\partial f}{\partial y}\right| \leq L$

Then the IVP has a unique solution on $x \in [x_0-h, x_0+h]$

$h = \min\{\frac{b}{M}, a, \underline{\frac{1}{L}}\}$ You may find

in slide, there is no $\frac{1}{L}$, it is because

we use geometry series to prove convergent

You can use power series, which is

less intensitive to remove $\frac{1}{L}$, that's main

contribution of picards for those prove.

You can refer to text book, I only tell you

how the theory comes from vaih.

Summary for this proof.

① equivalence between IVP and Integration Equation

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases} \Longleftrightarrow \quad y(x) = y_0 + \int_{x_0}^{x} f(t,y)\,dt.$$

② Construct a $\{y_n(x)\}$ series, In above example.

We construct infinitely degreasing geometry sequence $\Rightarrow$ power Series.

③ Such sequence covergent: Existence of the fixed point.

④ Prove the uniqueness of that fixed point.

To remove the condition $\frac{1}{L}$, we can no longer use geometry sequence to show that $\{y_n(x)\}$ in a cauchy sequence in the space of continous function. Thus we must abandon the Principle above.

first step remain, which tell us to research on $y(x) = y_0 + \int_{x_0}^{x} f(t, y) dt$.

The rectangle D remain, what we need to do is to change $\{y_n(x)\}$. In the above proof, we take $y_0(x)$ as an abitrary curve in D but this time we (Picard) takes $y_0(x) \equiv y_0$.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0) dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1) dt$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}) dt$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n) dt$$

we want to prove $\{y_n(x)\}$ is a Cauchy sequence

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_0) dt\right| \le \int_{x_0}^{x} |f(t, y_0)| dt \le M \cdot |x - x_0|$$

$$|y_2(x) - y_1(x)| \le \int_{x_0}^{x} |f(t, y_1) - f(t, y_0)| dt \le L \int_{x_0}^{x} |y_1 - y_0| dt$$

$$|y_3(x) - y_2(x)| \le \int_{x_0}^{x} |f(t, y_2) - f(t, y_1)| dt \qquad \le L \int_{x_0}^{x} M |t - x_0| dt$$

$$\le L \int_{x_0}^{x} |y_2 - y_1| dt \qquad = L \cdot \frac{1}{2} M (x - x_0)^2.$$

$$= L^2 \cdot \frac{1}{2} M \int_{x_0}^{x} (t - x_0)^2 dt = L^2 \cdot \frac{1}{2 \times 3} M (x - x_0)^3.$$

$$|y_{n+1}(x) - y_n(x)| \le \cdots \le L^n \cdot \frac{1}{(n+1)!} M (x - x_0)^{n+1}$$

the series $\sum_{n=1}^{\infty} L^n \frac{1}{(n+1)!} (x - x_0)^{n+1}$ must covergent. by your knowledge in W156

So for enough large m, n, $|y_m(x) - y_n(x)| < \varepsilon$. $\forall \varepsilon > 0$.

let $\tilde{y_n}(x) = \lim_{n\to\infty} y_n(x)$, thus for $y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}) dt$

limit both side we know $\tilde{y_n}(x) = y_0 + \int_{x_0}^{x} f(t, \hat{y_n}(x)) dt$

so $\hat{y_n}(x)$ is a solution, existence proved.

Now prove uniqueness,

let $\varphi(x)$, $\psi(x)$ are two different solution

$$|\varphi(x) - \psi(x)| = \left| \int_{x_0}^{x} f(t, \varphi(t)) dt - \int_{x_0}^{x} f(t, \psi(t)) dt \right|$$

$$\leq \int_{x_0}^{x} |f(t, \varphi(t)) - f(t, \psi(t))| dt.$$

$$\leq L \int_{x_0}^{x} |\varphi(t) - \psi(t)| dt$$

$$\leq L \cdot 2b |x - x_0|$$

$$\leq L \int_{x_0}^{x} L \cdot 2b |t - x_0| dt$$

$$= L^2 \cdot \frac{2b}{2} |t - x_0|^2$$

$$\leq L \int_{x_0}^{x} L^2 \cdot \frac{2b}{2} |t - x_0|^2 dt$$

$$= L^3 \cdot \frac{2b}{2\times 3} |x - x_0|^3$$

$$\leq \frac{L^n \cdot 2b}{1 \times 2 \times \cdots n} |x - x_0|^n$$

$n \to \infty$ $\quad y \longrightarrow 0$

$\therefore$ $\psi(x) = \varphi(x)$ uniqueness proved

Peano Existence Theorem

Theorem. Consider the initial value problem

$$y' = f(x, y), \; y(x_0) = y.$$

If $f(x, y)$ continuous in some open rectangle
$R = \{(x, y) : |x - x_0| < a, |y - y_0| < b, a, b > 0\}$ then IVP admits a solution (may be not unique) in the interval $|x - x_0| \leq \alpha$, where $\alpha = \min\{a, b/K\}$, K is the upper bound of f in the rectangle area.
The proof of this theorem is a little bit complex, I guess you don't need to know this, if you want to know, you may have to know about **Ascoli Theorem**(which is a development of the Bolzano-Weierstrass theorem in a infinite dimension space) and Euler Sequence.

Remark on these theorem

Acknowledge to the question on Piazza by Hanyu Gan.
You need to know:

- A continued first order (explicit) ODE not satisfying Lipschitz Continuity $\nRightarrow$ existence and non-uniqueness of solution, there exist weaker condition than Lipschitz Continuity. For example: Osgood Condition.
- A continued first order (explicit) ODE satisfying Lipschitz Continuity $\Rightarrow$ existence and ~~non-~~uniqueness of solution
- A continued first order (explicit) ODE $\Rightarrow$ existence of solution (Peano Theorem)