# VV256 Honors Calculus IV
# Fall 2022 — Problem Set 6

December 5, 2022

Note: All of the solutions in this problem set is written by me on notability, but my bad handwriting may cause a lot of typo. Anyway, thanks TA Huang for turning my awful handwriting into this Latex file which is more easily for you to read.

## Exercise 6.1

Given $A' \in \mathrm{Mat}_{(n-1)\times n}(\mathbb{C}), B' \in \mathrm{Mat}_{n\times(n-1)}(\mathbb{C})$, show that

$$\det(A'B') = \sum_{j=1}^{n} \det(M_j)\det(N_j)$$

where $M_j \in \mathrm{Mat}_{n-1}(\mathbb{C})$ is $A'$ with $j$-th column deleted, and $N_j \in \mathrm{Mat}_{n-1}(\mathbb{C})$ is $B'$ with $j$-th row deleted.

**Answer:**

Note: this is the application of cauchy-Binet Theorem. Detailed proof follows:
Let $v \in \mathbb{C}^n$ given by

$$v := \begin{bmatrix} (-1)^{n+1}M_1 & \cdots & (-1)^{n+j}M_j & \cdots & (-1)^{2n}M_n \end{bmatrix}^\top$$

then $v \in \ker A'$, since for $k = 1, \ldots, n$, by Laplace expansion,

$$(A'v)_k = \sum_j a_{kj}(-1)^{n+j}M_j = \det\left(\left[\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ \hline a_{k1} & \cdots & a_{kn} \end{array}\right]\right) = 0$$

thus $A'v = 0$. If $v = 0$ (all largest minors are zero), then rank $A \le n-2$, thus

$$\mathrm{rank}(AB) \le \min\{\mathrm{rank}(A), \mathrm{rank}(B)\} \le n-2$$

Note that $AB \in M_{n-1}(\mathbb{C})$, thus $\mathrm{rank}(AB) \le n-2$ implies $\det(AB) = 0$. On the other hand, note that $v = 0$ implies that $\det(M_j) = 0$ for $j = 1, \ldots, n$. Therefore LHS = RHS = 0 in (B.66). If $v \ne 0$, let

$$A := \left[\begin{array}{c} A' \\ \hline -v^\top- \end{array}\right] \in M_n(\mathbb{C}), \quad B := \left[\begin{array}{c|c} B' & \begin{matrix} | \\ v \\ | \end{matrix} \end{array}\right] \in M_n(\mathbb{C})$$

hence $M_j$ is the $(n,j)$-minor of $A$, and $N_j$ the $(j,n)$-minor of $B$, and

$$\det(AB) = \det\left(\left[\begin{array}{c|c} A'B' & A'v \\ \hline v^*B' & v^*v \end{array}\right]\right) = \det\left(\left[\begin{array}{c|c} A'B' & 0 \\ \hline v^*B' & v^*v \end{array}\right]\right) = \|v\|^2 \det\left(A'B'\right)$$

Similarly by Laplace expansion of $\det(A)$,

$$\det(A) = \sum_{j=1}^{n} |M_j|^2 = \|v\|^2$$

and of $\det(B)$ (note that $\pm$ signs cancel)

$$\det(B) = \sum_{j=1}^{n} M_j N_j$$

The rest of the proof follows from $\det(AB) = \det(A)\det(B)$.

## Exercise 6.2

Given a matrix $A \in \mathrm{Mat}_n(\mathbb{C})$, the adjugate of $A$ denoted by adj $A \in M_n(\mathbb{C})$ is given by $(\operatorname{adj} A)_{ij} := (-1)^{i+j} \det\left(a_{mk}\right)_{m\neq j, k\neq i}$ Show that

(i) Let $A \in \mathrm{Mat}_n(\mathbb{C})$, then $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)I_n$.

(ii) Let $A, B \in \mathrm{Mat}_n(\mathbb{C})$, then $\operatorname{adj}(AB) = \operatorname{adj}(B)\operatorname{adj}(A)$.

(iii) If $X \in \mathrm{Mat}_n(\mathbb{C})$ is invertible, then $\operatorname{adj}\left(X^{-1}YX\right) = X^{-1}\operatorname{adj}(Y)X$, for all $Y \in \mathrm{Mat}_n(\mathbb{C})$.

(iv) Given $A \in \mathrm{Mat}_n(\mathbb{C})$ with $\operatorname{rank}(A) \leq n-2$, then adj $A \equiv 0 \in \mathrm{Mat}_{n-1}(\mathbb{C})$.

(v) Given $A \in \mathrm{Mat}_n(\mathbb{C})$ with $\operatorname{rank}(A) = n-1$, then $\operatorname{adj} A = \alpha v w^\top$ for some scalar $\alpha$, where $v \in \ker A$, and $w \in \ker A^\top$.

**Answer:**

(i) $A \cdot (\mathrm{ad}\, jA)$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

Note that the product of an element in $A$ and the cofactur of other element in other lines will always be zero.

$$= \begin{pmatrix} \sum_{i=1}^{n} a_{1i}A_{1i} & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^{n} a_{2i}A_{2i} & \cdots & c_0 \\ 0 & \cdots & \cdots & \sum_{i=1}^{n} a_{ni}A_{ni} \end{pmatrix} \quad A_{ij} \text{ is the algebraic cofactor of } a_{ij}$$

$$= \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & \\ 0 & \ldots & |A| \end{pmatrix} = |A| \cdot I_n \text{ proved}$$

You can verify $\operatorname{adj}(A) \cdot A = A \cdot \operatorname{adj}$ (A) Yourself using this property.

(ii) $\operatorname{adj}(AB) = \operatorname{adj}(B) \cdot \operatorname{adj}(A)$ if $A, B$ both invertible.( **Only do this is wrong!**)

$$\operatorname{adj}(B) = B^{-1} \det(B)$$
$$\operatorname{adj}(A) = A^{-1} \det(A)$$

$$\therefore \operatorname{adj}(B) \cdot \operatorname{adj}(A) = \left(B^{-1}A^{-1}\right) \det(AB) = (AB)^{-1} \det(AB) = adj(AB)$$

if they are both not invertible. (It means at least one of A or B is not invertable) You can refer to the brief proof in my rc slide because the following proof may have typo.

$$\operatorname{adj}(A)_{ij} = (-1)^{i+j} \det \left(\mathrm{amk}_m\right)_{m \neq j} k \neq i$$

$$\operatorname{adj}(A) = \begin{pmatrix} A_1 & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$\operatorname{adj}(A)_{ij} = Aij = (-1)^{i+j} a_{ij} M_{ij}$$

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

$$AB = \begin{pmatrix} \sum_k a_{1k}b_{k1} & \cdots \sum_k a_{nk}b_{k1} \\ \sum_k a_{1k}b_{k2} & \cdots \sum_k a_{nk}b_{k2} \\ \vdots & \vdots \\ \sum_k a_{1k}b_{kn} & \cdots \sum_k a_{nk}b_{kn} \end{pmatrix}$$

$$\operatorname{adj}(AB)_{ij} = \text{ Algebra cofactor } \left((AB)_{ij}\right)$$

$$= (-1)^{i+j} \det \left( \sum_l a_{ml} b_{lk} \right) m \neq i, k \neq j$$

Note that $(\sum_l a_{ml} b_{lk})\, m \neq i, k \neq j$ is the matrix of $n-1$ order, refer to your 6.1, You will find

$$\det\left(\sum_l a_{ml}b_{lk}\right) m \neq i, k \neq j = \sum_l \det\left(M_l\right) \cdot \det\left(N_l\right)$$

$$\begin{aligned}
&M_l = (a_{ms})\, m \neq j, s \neq l, N_l = (b_{nk}) k \neq 1, n \neq l \\
&= \sum_l \det(a_{ms})_{m \neq j} s \neq l \cdot \det\left(b_{nk}\right)_{k \neq i, n \neq l} (-1)^{i+j} \\
&= \sum_l (-1)^{j+l} \det(a_{ms})_{n \neq j} \neq s \neq l, (-1)^{i+l} \det(b_{nk})_{k \neq i} n \neq l \\
&= \sum_l A_{lj} \cdot B_{jl} \\
&= \sum_l B_{jl} \cdot A_{lj} \\
&= \sum_i (\operatorname{adj} B)(\operatorname{adj} A)\ .\ \text{Proved.}
\end{aligned}$$

(iii)

$$\begin{aligned}
&\operatorname{adj}\left(X^{-1} Y X\right) \\
&= \operatorname{adj}(Y X) \operatorname{adj}\left(X^{-1}\right) \\
&= \operatorname{adj}(X) \operatorname{adj}(Y) \operatorname{adj}\left(X^{-1}\right) \\
&= X^{-1} \det X \operatorname{adj}(Y) X \det\left(X^{-1}\right) \\
&= X^{-1} \operatorname{adj}(Y) X \quad \text{proved}
\end{aligned}$$

(iv)

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & \cdots \end{pmatrix}$$

with rank $n-2$, we can perform

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ A_{12} & & \vdots \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix}$$

the first minors will be 0 thus $\operatorname{adj} A \equiv 0$. Note, the rank of $n$ matrix determine the order of the highest order nun-zero minor, when $r \leqslant n-2$, the highest order non-zero minor will be at must $n-2$, thus the adj matrix consist of $n-1$ order minus will be the zero matrix

(v) $A(\operatorname{adj} A) = |A| \cdot In$ by $(i)$ since $\operatorname{ran} k(A) = n-1, \quad |A| = 0. \quad (\operatorname{rank}(A) = n \Leftrightarrow |A| \neq 0)$

$$\begin{aligned}
&\therefore A(\operatorname{adj} A) = 0 \\
&\therefore \operatorname{adj} A \in \ker A.
\end{aligned}$$

By rank-nullity theorem.

$$\dim \ker A + \operatorname{rank}(A) = n$$

$$\operatorname{dimker} A = 1$$

$$\text{So}, \ker A = \operatorname{range}(\operatorname{adj} A)$$

$$\because v \in \ker A$$

$$\therefore \quad \operatorname{range}(\operatorname{adj} A) = \operatorname{span}(v)$$

$$A^\top (\operatorname{adj} A)^\top = ((\operatorname{adj} A)A)^\top = 0$$

$$\therefore (\operatorname{adj}(A))^\top \in \ker A^\top$$

Also by rank-nullity theorem

$$\ker A^\top = \operatorname{range}\left(\operatorname{adj} A^\top\right)$$

$$\because w \in \ker A^\top$$

$$\therefore \operatorname{range}\left(\operatorname{adj}(A)^\top\right) = \operatorname{span}(w)$$

Since $\operatorname{rank}(A) = n-1$, So, adj $A \neq 0$.

so. $\operatorname{rank}(\operatorname{adj}(A)) \geqslant 1$

$$\underbrace{\operatorname{rank}(A)}_{n-1} + \underbrace{\operatorname{rank}(\operatorname{adj}(n))}_{?} \leqslant n - \operatorname{rank}(A \cdot \operatorname{adj}(A))$$

$\operatorname{rank}(\operatorname{adj}(A)) \leqslant 1$.

$$\therefore \operatorname{rank}(\operatorname{adj}(A)) = 1$$

$\therefore$ It can be expressed as $\alpha v w^T$, no $\alpha$ also ok.

# Exercise 6.3

Given an invertible matrix function $A(t) \in \mathrm{Mat}_n(\mathbb{R})$ with differentiable entries, verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[A(t)^{-1}\right] = -A(t)^{-1}\frac{\mathrm{d}A(t)}{\mathrm{d}t}A(t)^{-1}$$

(i) by using the identity $\mathbf{D}\iota(A) \cdot H = -A^{-1}HA^{-1}$, where $\iota(A) = A^{-1}$.

(ii) by differentiating $A(t)A(t)^{-1} = I_n$.

**Answer:**

(i) set the direction matrix to the direction of $t$ we can set it triuclly.

(ii)

$$\frac{d}{dt}\left[A(t)A(t)^{-1}\right] = \frac{d}{dt}\left(I_n\right)$$

$$\left(\frac{d}{dt}A(t)\right) \cdot A(t)^{-1} + \frac{d}{dt} \cdot A(t)^{-1} \cdot A(t) = 0$$

$$\frac{d}{dt}A^{-1}(t) = -A(t)^{-1} \cdot \frac{d}{dt}A(t)A(t)^{-1}$$

proved.

# Exercise 6.4

Let $\Phi(t) \in \mathrm{Mat}_n(\mathbb{R})$ satisfy the IVP

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = A(t)\Phi(t), \quad \Phi(0) = I_n$$

where $A(t) \in \mathrm{Mat}_n(\mathbb{R})$ for all $t \in I \subseteq \mathbb{R}$. Assume all calculations are possible,

(i) Verify $\Phi(-t) = \Phi(t)^{-1}$ by showing both sides satisfy the same IVP.

(ii) Show that

$$\det \Phi(t) = \exp\left(\int_0^t \operatorname{tr} A(s)\mathrm{d}s\right)$$

**Answer:**

(i)

**This problem is wrong. Refer to the problem in hmwk7**

(ii) directly proved by Liouville formula. or by example 4.11

$$\frac{d}{dt}\det \phi(t) = \sum_{i=1}^{n} \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{in} \\ \vdots & & & \\ y'_{ii} & \cdots & & y'_{in} \\ \vdots & & & \\ y_{n1} & & \cdots & y_{nn} \end{vmatrix}$$

$$\begin{aligned}
\phi(t) &= \begin{vmatrix} y_{11}(t) & \cdots & y_{1n}(t) \\ \vdots & & \vdots \\ y_{n1}(t) & \cdots & y_{nn(t)} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \sum_{j=1}^{n} a_{ij}y_{j1} & \cdots & \cdots \\ y_{n1} & \cdots & y_{n_n} \end{vmatrix} \\
&= \sum_{i=1}^{n} a_{ii}\det \phi(t) \\
&= \operatorname{tr}(A(t))\det \phi(t) \\
\frac{1}{\det \phi(t)} & d(\det \phi(t)) = \operatorname{tr}(A(t))dt \\
\ln \det \phi(t) &= \int \operatorname{tr}(A(s))ds
\end{aligned}$$

$$\begin{aligned}
\det \phi(t) &= det(\phi(t_0)e^{\int_{t_0}^{t} trA(s)ds} \\
&= det(\phi(t_0)e^{\int_0^t trA(s)ds} \text{ proved}
\end{aligned}$$

You can also refer to my RC in which I use Der 3 to prove this by chain rule.

# Exercise 6.5

Given a smooth vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, suppose $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, (x,t) \mapsto u(x,t)$ satisfies

$$\frac{\partial}{\partial t}u(x,t) = f(u(x,t),t), \quad u(x,0) = x$$

Let the Jacobian determinant be given by

$$J(x,t) = \det\left(\frac{\partial u}{\partial x}\right) = \det\left(\frac{\partial u_i}{\partial x_j}\right) = \det\left(\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{bmatrix}\right)$$

and the divergence of $f$ in the variable $u$ is given by

$$\operatorname{div} f(u,t) = \sum_{k=1}^{n} \frac{\partial f_k}{\partial u_k}(u,t)$$

show that

$$\frac{\partial}{\partial t}J(x,t) = \operatorname{div} f(u,t) \cdot J(x,t)$$

**Answer:**

$$\frac{\partial}{\partial t}J(x,t) = \frac{\partial}{\partial t}\det\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \ddots \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}$$

$$= \sum_{i=1}^{n} \det\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial u_i}{\partial x_1 \partial t} & \cdots & \frac{\partial u_i}{\partial x_n \partial t} \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}$$

$$= \sum_{i=1}^{n} \det\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f(u_i,t)}{\partial x_1} & \cdots & \frac{\partial f(u_i,t)}{\partial x_n} \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}$$

$$= \sum_{i=1}^{n} \det\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f(u_i,t)}{\partial x_1} & \cdots & \frac{\partial f(u_i,t)}{\partial x_n} \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}$$

$$= \sum_{i=1}^{n} \frac{\partial f\left(u_{i,t}\right)}{\partial u_i} \det(\cdot)$$

$$= \sum_{i=1}^{n} \frac{\partial f\left(u_{i,t}\right)}{\partial u_i} J(x,t) \text{ proved}$$

# Exercise 6.6

Let $A \in \mathrm{Mat}_n(\mathbb{R})$, let $f_k(A) = A^k, k \in \mathbb{N}$. Show that

$$\mathbf{D}\,\mathrm{tr}\,(f_k(A)) \cdot H = k\,\mathrm{tr}\left(A^{k-1}H\right)$$

**Answer:**

$$\begin{aligned}
\mathrm{tr}\,(Df_k(A)H) &= \mathrm{tr}\left(\left.\frac{d}{dt}\right|_{t=0}(A+tH)^k\right) \\
&= \mathrm{tr}\left(\sum_{m=0}^{k-1} A^m H A^{k-1-m}\right) \\
&= \mathrm{tr}\left(\sum_{m=0}^{k-1} A^{k-1} H\right) \\
&= k\,\mathrm{tr}\left(A^{k-1}H\right)
\end{aligned}$$

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0}\mathrm{tr}\,(f_k(A+tH)) \\
=&\left.\frac{d}{dt}\right|_{t=0}\mathrm{tr}((A+tH))^k \\
=&\left.\frac{d}{dt}\right|_{t=0}\sum_{m=0}^{k-1}\mathrm{tr}\left(A^m H A^{k-1-m}\right) \\
=&\left.\frac{d}{dt}\right|_{t=0}\mathrm{tr}\left(\sum_{m=0}^{k-1} A^m H A^{k-1-m}\right) \\
=&k\,\mathrm{tr}\left(A^{k-1}H\right)
\end{aligned}$$

# Exercise 6.7

Recall that

$$\exp(A) := \sum_{n\geq 0}\frac{A^n}{n!}$$

for a square matrix $A$.

(i) show that

$$\mathbf{D}\exp(A)\cdot H = \sum_{n\geq 0}\frac{1}{(n+1)!}\sum_{k=0}^{n} A^k H A^{n-k}$$

(ii) In particular, show that

$$\mathrm{tr}(\mathbf{D}\exp(A)\cdot H) = \mathrm{tr}(\exp(A)H)$$

(iii) Show that

$$\mathbf{D}\exp(A)\cdot H = \int_0^1 e^{(1-t)A}He^{tA}\ \mathrm{dt}$$

**Answer:**

(i)

$$\mathrm{Dexp}(A)\cdot H = \left.\frac{d}{dt}\right|_{t=0} \exp(A+tH)$$
$$= \left.\frac{d}{dt}\right|_{t=0} I + \frac{d}{dt}\frac{(A+tH)}{1} + \frac{d}{dt}\frac{(A+tH)^2}{2!} + \cdots + \frac{d}{dt}\frac{(A+tH)^{n+1}}{(n+1)!}$$

You perform derivative on $t = 0$, the only thing remain will be $t \cdot \sum_{k=0}^{n} A^k H A^{n-k}$ (something like binomial expansion) derivative it will remain $\sum_{k=0}^{n} A^k H A^{n-k}$, coefficient is easy to derive thus $\mathrm{Dexp}(A)H = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} A^k H A^{n-k}$ proved

(ii)

$$\exp(A)H = \sum_{n=0}^{\infty} \frac{A^n}{n!} H$$
$$= H + AH + \frac{1}{2}A^2H + \frac{1}{6}A^3H + \cdots$$
$$\mathrm{Dexp}(A)H = H + \frac{1}{2}[HA + AH] + \frac{1}{6}\left[A^2H + AHA + HA^2\right] + \cdots$$

You shall count how many terms and you will find you have proved this.

According to the property

$$\mathrm{tr}(A) + \mathrm{tr}(B) = \mathrm{tr}(A+B)$$

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

$$\begin{aligned}\mathrm{tr}\,\mathrm{Dexp}(A) &= \mathrm{tr}(H) + \frac{1}{2}[\mathrm{tr}(HA) + \mathrm{tr}(AH)] \\ &+ \frac{1}{6}\left[\mathrm{tr}\left(A^2H\right) + \mathrm{tr}(AHA) + \mathrm{tr}(HA^2)\right] + \\ &= \mathrm{tr(H)} + \mathrm{tr}(AH) + \frac{1}{2}\,\mathrm{tr}\left(A^2H\right) + \ldots \\ &= \mathrm{tr}\exp(A) \quad \text{proved.}\end{aligned}$$

$$
\begin{aligned}
&\int_0^1 e^{(1-t)A} H e^{tA} dt \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{[(1-t)A]^n}{n!} H \sum_{m=0}^{\infty} \frac{(tA)^m}{m!} \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(1-t)^n A^n}{n!} H \cdot \sum_{m=0}^{\infty} \frac{t^m A^m}{m!} dt \\
&= \sum_{m\geq 0} \sum_{n\geq 0} \frac{1}{m!n!} \int_0^1 t^m (1-t)^n dt A^m H A^n \\
&= \sum_{m\geq 0} \sum_{n\geq 0} \frac{1}{m!n!} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} A^m H A^n \\
&= \sum_{m\geq 0} \sum_{n\geq 0} \frac{1}{m!n!} \frac{m!n!}{(m+n+1)!} A^m H A^n \\
&= \sum_{m\geq 0} \sum_{n\geq 0} \frac{1}{(m+n+1)!} A^m H A^n \\
&= \sum_{m\geq 0} \frac{1}{(m+1)!} \sum_{n\geq 0}^{m} A^n H A^{m-n} \text{ proved}
\end{aligned}
$$

# Reference

1. Runze Cai, Notes on The Eigenvector-Eigenvalue Identity Paper (2021)
2. QiHong Xie, Advanced Algerbra Fudan University
3. Tongren Ding Ordinary Differential Equation Tutorial Beijing University