VV256/MATH2560J Honors Calculus IV
Recitation Class Material

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Given an n-th order linear homogeneous ODE

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + a_{n-2}(t)\frac{\mathrm{d}^{n-2} y}{\mathrm{d}t^{n-2}} + \cdots + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)y = 0$$

we can define

$$x_1 := y, \quad x_2 = \frac{\mathrm{d}y}{\mathrm{d}t}, \quad \ldots, \quad x_n = \frac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}}$$

and let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$, then we have a first order matrix ODE in the vector form as $\dot{x} = A(t)x$, i.e.,

$$\dot{x} = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(n)} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{k-1}(t) \end{bmatrix}}_{A(t)} x$$

$\dot{x} = Ax$

$$y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y^{(1)} + a_0(t)\, y = 0$$

Let $x_1 = y$, $x_2 = y^{(1)}$, $\cdots\cdots$ $x_n = y^{(n-1)}$,

$X \leftarrow$ vector $= (x_1, x_2, \cdots, x_n)^T$

$$\underbrace{\frac{dX}{dt} = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}}_{X'} = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}}_{X}$$

$$x_{n+1} = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n$$

$$y^{(n)} = -a_0 y - a_1 y^{(1)} - \cdots - a_{n-1} y^{(n-1)}$$

Motivation

We want to solve this matrix ODE in this part, but we shall review some of the linear algebra.

Definition (Determinant). For a matrix $A \in M_n$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

We denote the **Determinant** of A as det(A), which is

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

The determinant can be viewed as a mapping $M_n \to R$ but note that it is **NOT** a linear mapping.

Definition of Determinant

Actually, the basic definition of the determinant has something related to the **number of inversions**, but for solving the exam problem easily, and you have mastered the second and third order determinant. We will introduce the **Expansion by Row(Column)** directly.

Theorem (Expansion by Row). Given a nxn determinant det(A)

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

it can be calculated as

$$\det(A) = a_{i1}A_{i1} + \ldots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij}$$

which the Mij is the Algebraic cofactor of the a_{ij}

Cofactor

In the determinant of order n, the determinant of order n-1 formed by deleting the elements in row i and column j where element a_{ij} is located, is called the cofactor of element a_{ij}.

Property of Determinant

1. $det(A^T) = det(A)$
2. We can extract the common factor of certain row or column

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

3. decomposition of determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_m \end{vmatrix} \text{(the } ith \text{ row)}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

4. interchange between row

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Property of Determinant

5.The sum of the product of the elements in ith row of determinant |A| and the algebraic cofactor of corresponding elements in kth row equals to zero. (Same as column). i.e $a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots a_{in}A_{kn} = 0$

Proof: You shall construct the following determinant.

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{nn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Since there are two same row in |B|, the value shall be 0, but you can perform the Laplace Expansion on the kth row, you will easily get the property 5.

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{array}{l} \text{1st row} \\ \text{2st row} \\ \\ \\ \end{array}$$

$$\det(A) = \sum_{j=1}^{n} \underline{a_{1j}} \, \underline{A_{1j}}.$$

$$\sum_{j=1}^{n} a_{1j} \, \underline{A_{2j}} = 0.$$

Given matrix $A = (a_{ij})_{s\times n}$, $B = (b_{ij})_{n\times s}$

- if $s > n$,then $|AB| = 0$
- else $|AB| = \sum_{1\leq<v_1<\cdots<v_s\leq n} A\begin{pmatrix} 1, & 2, & \cdots & s \\ v_1, & v_2, & \cdots & v_s \end{pmatrix} \cdot B\begin{pmatrix} v_1, & v_2, & \cdots & v_s \\ 1, & 2, & \cdots & s \end{pmatrix}$

The proof of this theorem is very complex, thus omitted here.

$s=2 \quad n=3$ $\qquad\qquad$ $n=3 \quad s=2$

$A \in \mathrm{Mat}_{2\times 3}$ $\qquad\qquad$ $B \in \mathrm{Mat}_{3\times 2}.$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad\qquad \begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} \qquad\qquad \begin{pmatrix} 6 & 5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ 2 & 1 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \cdot \begin{vmatrix} 6 & 5 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \times \begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix}$$

Exercise 6.1

Given $A' \in \mathrm{Mat}_{(n-1)\times n}(\mathbb{C})$, $B' \in \mathrm{Mat}_{n\times(n-1)}(\mathbb{C})$, show that

$$\det\left(A'B'\right) = \sum_{j=1}^{n} \det\left(M_j\right)\det\left(N_j\right)$$

where $M_j \in \mathrm{Mat}_{n-1}(\mathbb{C})$ is A' with j-th column deleted, and $N_j \in \mathrm{Mat}_{n-1}(\mathbb{C})$ is B' with j-th row deleted.

You will find the M_j, N_j are just all the minor of A' and B'. Thus directly proved. You can just directly use this theorem in the exam. But I have provided a detailed proof based on Chunked Matrix to TA HYC, and he will transfer it into latex after the due time.

Example 2

Given a matrix $A \in \mathrm{Mat}_n(\mathbb{C})$, the adjugate of A denoted by adj $A \in M_n(\mathbb{C})$ is given by $(\operatorname{adj} A)_{ij} := (-1)^{i+j} \det \left(a_{mk}\right)_{m \neq j, k \neq i}$ Show that
Let $A, B \in \mathrm{Mat}_n(\mathbb{C})$, then $\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A)$. I will first show you what is the adjugate matrix in a way that human can understand and denote adj(A) as a^*

Solution

Proof: let $\boldsymbol{C} = \boldsymbol{AB}$. denote M_{ij}, N_{ij}, P_{ij} as the cofactor of the (i,j)th element in $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. A_{ij}, B_{ij}, C_{ij} are respectively the algebraic cofator of the (i,j) th element in $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$.
We find that

$$\boldsymbol{A}^* = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}, \quad \boldsymbol{B}^* = \begin{pmatrix} B_{11} & B_{21} & \cdots & B_{n1} \\ B_{12} & B_{22} & \cdots & B_{n2} \\ \vdots & \vdots & & \vdots \\ B_{1n} & B_{2n} & \cdots & B_{nn} \end{pmatrix},$$

B^*A^* 's (i,j) th element be

$$\sum_{k=1}^{n} B_{ki}A_{jk}.\textbf{Directly by matrix multiplication}$$

and the (i,j) th element C^* is $C_{ji} = (-1)^{j+i}P_{ji}$. Using Cauchy-Binet theorem we get

Solution

$$
\begin{aligned}
C_{ji} &= (-1)^{j+i} P_{ji} = (-1)^{j+i} \sum_{k=1}^{n} M_{jk} N_{ki} \\
&= \sum_{k=1}^{n} (-1)^{j+k} M_{jk} (-1)^{i+k} N_{ki} = \sum_{k=1}^{n} A_{jk} B_{ki},
\end{aligned}
$$

The conclusion follows

Derivative of Determinant

Given a matrix function $A(\cdot) : \mathbb{R} \to \mathrm{Mat}_n(\mathbb{R}), t \mapsto A(t) = (a_{ij}(t))$ with smooth component a_{ij} 's, its determinant is just a single variable function $\det \circ A : \mathbb{R} \to \mathbb{R}, t \mapsto \det A(t)$. We can of course differentiate this function, only that we want a nice formula.

Explanation the matrix function $A(\cdot)$ is a matrix that contains a lot of functions $(a_{ij}(t))$, if you apply the matrix function A on the real number t, it will map the t from the real number field to the field of $M_n(R)$. And the small circle in the third line means **composition**, the det function maps $M_n(R)$ to $\mathbb{R}$, the matrix function A maps $\mathbb{R}$ to $M_n(R)$, so the composition of the two function is just a function mapping R to R.

Given a matrix $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{R})$, the determinant $\det A$ admits the Laplace/cofactor expansion by any row or column, that is, for each $i = 1 \ldots, n$,

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

where M_{ij} is the is the determinant of the submatrix obtained by removing the i th row and the j th column of A. The term $(-1)^{i+j}M_{ij}$ is called the cofactor of a_{ij} in A. Therefore consider $\det : \text{Mat } n(\mathbb{R}) \to \mathbb{R}$, we have

$$\frac{\partial \det A}{\partial a_{ij}} = (-1)^{i+j}M_{ij}$$

(note that M_{ij} does not contain a_{ij})
Explanation this is the expansion of a determinant respect to the rth row. And you will find that the cofactor of the element in the whole row will not contain any of the element in this row, so you can regard the cofactor as a constant when you perform the derivative. So when you perform the derivative respect to a_{ij}, only the term $(-1)^{i+j}a_{ij}M_{ij}$ will become $(-1)^{i+j}M_{ij}$ and other terms of this row will certainly vanish.

$$\frac{\partial \det A}{\partial a_{ij}} = \frac{\partial \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}}{\partial a_{ij}}$$

except a_{ij} itself

nothing of M_{ij} contain a_{ij}

Now by chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A(t) = \sum_{i,j=1}^{n} \frac{\partial \det A}{\partial a_{ij}} \frac{\mathrm{d}a_{ij}}{\mathrm{d}t} = \sum_{i,j=1}^{n} (-1)^{i+j} M_{ij}\dot{a}_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (-1)^{i+j} M_{ij}\dot{a}_{ij} = \underbrace{\sum_{j=1}^{n}\sum_{i=1}^{n} (-1)^{i+j} M_{ij}\dot{a}_{ij}}_{\lambda}$$

In the matrix form, you will find it just equal to you derivative each row(column) and add them.

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A(t) = \det\left(\begin{bmatrix} \dot{a}_{11} & \dot{a}_{12} & \cdots & \dot{a}_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right) + \det\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \dot{a}_{21} & \dot{a}_{22} & \cdots & \dot{a}_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right) + \cdots + \det\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \dot{a}_{n1} & \dot{a}_{n2} & \cdots & \dot{a}_{nn} \end{bmatrix}\right)$$

Note that, meanwhile, the determinant det : $\mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$ can also be regarded as an n-linear function

$$\det : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$$

The $\times$ is the Cartesian Product. You can view each of the $\mathbb{R}$ as a n dimensional column vector $(a_1, a_2, a_3, \cdots, a_n)$ and the Cartesian Product will form the n by n matrix, and the determinant function map the matrix to a real number. And for more detail about the determinant, please refer to the group theory (Symmetric Group) which will be taught in VE203.

Directional Derivative of multivariable function

Given a multivariable function $f(x, y) = 3x^4 + xy + y^3$, find the directional derivative of the direction (1,2).

$\frac{1}{\sqrt{5}}$ $\frac{2}{\sqrt{5}}$

Definition Let $f: U \subset \mathbf{E} \to \mathbf{F}$, where $\mathbf{E}$ and $\mathbf{F}$ are vector spaces. Let $u \in U$, then if f is differentiable at u, then the directional derivative of f in the direction $e \in \mathbf{E}$ at u is given by

$$\mathbf{D}f(u) \cdot e := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(u + te)$$

function direction

$$\underbrace{D\, f(u) \cdot e}_{} = \left.\frac{d}{dt}\right|_{t=0} f(u + te)$$

Examples

We list a few examples as follows, DER1 Let $q : M_n(\mathbb{C}) \to M_n(\mathbb{C}), X \mapsto X^2$, be the map of squaring of a square matrix, then the directional derivative of q in the direction $B \in M_n(\mathbb{C})$ at $A \in M_n(\mathbb{C})$ is given by

$(A+\varepsilon B)(A+\varepsilon B)$

$$\mathbf{D}q(A)\cdot B := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} (A+\varepsilon B)^2 = AB + BA$$

$= \cancel{A^2} + \varepsilon BA + \varepsilon AB + \cancel{\varepsilon^2 B^2}$

DER2 Let $\iota : M_n(\mathbb{C}) \to M_n(\mathbb{C}), X \mapsto X^{-1}$, be the map of inversion of a square matrix, then the directional derivative of ι in the direction $B \in M_n(\mathbb{C})$ at $A \in M_n(\mathbb{C})$, A invertible, is given by

$$\mathbf{D}\iota(A)\cdot B := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} (A+\varepsilon B)^{-1} = -A^{-1}BA^{-1}$$

DER3 Consider $\det : M_n(\mathbb{C}) \to \mathbb{R}$, then the directional derivative of det in the direction $B \in M_n(\mathbb{C})$ at $A \in M_n(\mathbb{C})$, is given by

$\det(A)\, \mathrm{tr}(A^{-1}B)$

$$\mathbf{D}\det(A)\cdot B := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \det(A+\varepsilon B) = \mathrm{tr}[(\mathrm{adj}\, A)B]$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A^* = \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ A_{12} & \cdots & A_{n2} \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot A^* = \operatorname{adj}(A)$$

Remark

Compared with the directional derivative of multivariable function. You will see that the "directional matrix" we used was not normalized to "1". This is because in this course we haven't learnt about the **matrix norm**. If you feel interested, you can search for the $\mathbb{L}_2$ norm of a square matrix and discuss with me.

We know the Lemma 4.8 that

$A^{-1}I$ $A^{-1}H$

$$D(det(I_n))H = tr(H), H \in \mathbb{M}_n$$

Try to prove DER3

$D(\det(A))\cdot H$ $\qquad\qquad$ $\det(AB) = \det(A)\cdot\det(B)$

$$= \frac{d}{dt}\Big|_{t=0} \det(A+tH)$$

$$= \frac{d}{dt}\Big|_{t=0} \det(A(I_n + A^{-1}tH))$$

$$= \frac{d}{dt}\Big|_{t=0} \det(A)\cdot\det(I_n + A^{-1}tH)$$

$$= \det(A)\ \frac{d}{dt}\Big|_{t=0} \boxed{\det(I_n + tA^{-1}H)}$$

$$\downarrow$$

$$\boxed{D\det(I_n)\cdot(A^{-1}H)}$$

direction.

$$\downarrow$$

$$\frac{d}{dt}\Big|_{t=0} \det(I_n + tA^{-1}H)$$

$$= \det(A)\cdot D\det(I_n)(A^{-1}H)$$

$$= \det(A)\ \mathrm{tr}(A^{-1}H)$$

$$= \mathrm{tr}(\det(A)A^{-1}H)$$

$$= \mathrm{tr}(\mathrm{adj}(A)H) \quad \text{proved.}$$

Derivative of the n-linear mapping in a certain direction

Proposition . Given vector spaces $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_n$, and $\mathbf{F}$, (the F is a field) let

$$f \colon \mathbf{E}_1 \times \cdots \times \mathbf{E}_n \to \mathbf{F}$$

be a continuous n-linear mapping, then the derivative of f at $(a_1, \dots, a_n)$ in the direction $(h_1, \dots, h_n)$ is given by

$$\boxed{\begin{aligned} \mathbf{D}f(a_1, \dots, a_n) \cdot (h_1, \dots, h_n) &= \sum_{k=1}^{n} f(a_1, \dots, a_{k-1}, h_k, a_{k+1}, \dots, a_n) \\ &= f(h_1, a_2, \dots, a_n) + f(a_1, h_2, \dots, a_n) + \cdots + f(a_1, \dots, a_{n-1}, h_n) \end{aligned}}$$

where $\mathbf{D}f(a_1, \dots, a_n) : \mathbf{E}_1 \times \cdots \times \mathbf{E}_n \to \mathbf{F}$ is a linear map.

Explanation

Just like what you have done in VV255, let's compute the directional derivative together. And we take $n = 2$ as an example.

$$\mathbf{D}f(a_1, a_2) \cdot (h_1, h_2) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(a_1 + th_1, a_2 + th_2,)$$

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} f(a_1 + th_1, a_2 + th_2) \\
=& \frac{d}{dt}\Big|_{t=0} [f(a_1, a_2 + th_2) + f(th_1, a_2 + th_2)] \\
=& \frac{d}{dt}\Big|_{t=0} [f(a_1, a_2) + f(a_1, th_2) + f(th_1, a_2) + f(th_1, th_2)] \\
=& \frac{d}{dt}\Big|_{t=0} [f(a_1, a_2) + f(a_1, h_2) + f(h_1, a_2) + t^2 f(h_1, h_2)] \\
=& 0 + f(a_1, h_2) + f(h_1, a_2) + 0.
\end{aligned}$$

t t $2t$

Derivative of the n-linear mapping in a certain direction

Note that we have considered $\mathbf{E}_1 \times \cdots \times \mathbf{E}_n$ as a vector space, where

$$\begin{aligned}
(x_1, \dots, x_n) + (y_1, \dots, y_n) &:= (x_1 + y_1, \dots, x_n + y_n) \\
\lambda (x_1, \dots, x_n) &:= (\lambda x_1, \dots, \lambda x_n)
\end{aligned}$$

.

Hence

$$\begin{aligned}
&\mathbf{D}f(x_1, \dots, x_n) \cdot ((h_1, \dots, h_n) + (k_1, \dots, k_n)) \\
&\qquad = \mathbf{D}f(x_1, \dots, x_n) \cdot (h_1, \dots, h_n) + \mathbf{D}f(x_1, \dots, x_n) \cdot (k_1, \dots, k_n)
\end{aligned}$$

and

$$\mathbf{D}f(x_1, \dots, x_n) \cdot \lambda (h_1, \dots, h_n) = \lambda \mathbf{D}f(x_1, \dots, x_n) \cdot (h_1, \dots, h_n)$$

whereas

$$f(\lambda (x_1, \dots, x_n)) = f(\lambda x_1, \dots, \lambda x_n) = \lambda^n f(x_1, \dots, x_n)$$

Example of n-linear mapping

Example . Consider the matrix multiplication of three matrices as a trilinear mapping

$$
\begin{aligned}
f : \mathrm{Mat}_{m\times n}(\mathbb{R}) \times \mathrm{Mat}_{n\times p}(\mathbb{R}) \times \mathrm{Mat}_{p\times q}(\mathbb{R}) &\to \mathrm{Mat}_{m\times q}(\mathbb{R}) \\
(A_1, A_2, A_3) &\mapsto A_1 A_2 A_3
\end{aligned}
$$

then

$$
\mathrm{D}\ f(A_1, A_2, A_3) \cdot (H_1, H_2, H_3) = H_1 A_2 A_3 + A_1 H_2 A_3 + A_1 A_2 H_3
$$

Indeeed, since

$$
\begin{aligned}
\mathbf{D} f(A_1, A_2, A_3) \cdot (H_1, H_2, H_3) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(A_1 + tH_1, A_2 + tH_2, A_3 + tH_3) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (A_1 + tH_1)(A_2 + tH_2)(A_3 + tH_3) \\
&= H_1 A_2 A_3 + A_1 H_2 A_3 + A_1 A_2 H_3
\end{aligned}
$$

where $\mathbf{D} f(A_1, A_2, A_3) : \mathrm{Mat}_{m\times n}(\mathbb{R}) \times \mathrm{Mat}_{n\times p}(\mathbb{R}) \times \mathrm{Mat}_{p\times q}(\mathbb{R}) \to \mathrm{Mat}_{m\times q}(\mathbb{R})$ is a linear map.

Another Example

Example 2. Consider the mapping $s : \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R}), A \mapsto A^2$, by definition, we have

$$\mathbf{D}s(A) \cdot H = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (A + tH)^2 = AH + HA$$

Or equivalently observe that $s = b \circ (\ \mathrm{id}\ \times\ \mathrm{id})$, where

$$\begin{aligned} \mathrm{id}\ \times\ \mathrm{id}\ &: \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R}) \times \mathrm{Mat}_n(\mathbb{R}), \quad & A &\mapsto (A, A) \\ b &: \mathrm{Mat}_n(\mathbb{R}) \times \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R}), \quad & (A, B) &\mapsto AB \end{aligned}$$

thus by chain rule,

$$\begin{aligned} \mathbf{D}s(A) \cdot H &= \mathbf{D}(b(\mathrm{id}(A), \mathrm{id}(A))) \cdot H \\ &= \mathbf{D}b(A, A) \cdot \mathbf{D}(\mathrm{id} \times \mathrm{id})(A) \cdot H \\ &= \mathbf{D}b(A, A) \cdot (H, H) \\ &= HA + AH \end{aligned}$$

Liouville Formula

Theorem (Liouville's Formula). Consider the first order homogeneous matrix ODE

matrix

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A(t)\Phi, \qquad t \in I \subset \mathbb{R}$$

where the solution $\Phi(t) \in \mathrm{Mat}_n(\mathbb{R})$ for all $t \in I$. If $\operatorname{tr} A : I \subset \mathbb{R}$ continous, then

$$\det \Phi(t) = \det \Phi\left(t_0\right) \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s) \mathrm{d}s\right)$$

for all $t, t_0 \in I$. It is immediate that Abel's identity follows from Liouville's Formula. We will establish a version of Liouville's Formula when the matrix A is constant. The nonconstant case easily follows.

$$\frac{d\det(\phi(t))}{dt} \leftarrow \text{chain rule.}$$

$$= \underbrace{D\det\phi}_{A} \cdot \left(\frac{d\phi(t)}{dt}\right) \Rightarrow H \qquad \underbrace{D\det(A)\,H}_{} = \det(A)\operatorname{tr}(A^{-1}H)$$

$$= \det\phi \cdot \operatorname{tr}\left(\phi^{-1}\frac{d\phi(t)}{dt}\right)$$

$$= \det\phi\ \operatorname{tr}(\phi^{-1}A(t)\phi) \qquad \operatorname{tr}(AB) = \operatorname{tr}(BA)$$

$$= \det\phi\ \operatorname{tr}(A(t))$$

$$\frac{d\det\phi}{dt} = \det\phi\ \operatorname{tr}(A(t)),$$

Prove:

$$\frac{d}{dt}\left((A(t))^{-1}\right) = -A(t)^{-1}\frac{dA(t)}{dt}A(t)^{-1}$$

using: $DA^{-1}H = -A^{-1}HA^{-1}$

$$D(A)H = -A^{-1}HA^{-1}$$

$$\frac{d}{dt}\left((A(t))^{-1}\right) = \frac{d}{dt}\left(l_A(A(t))\right)$$

$$= Dl_A\left(\frac{dA(t)}{dt}\right) \quad = l-1$$

$$=$$

$$= -A(t)^{-1}\,\frac{dA(t)}{dt}\,A(t)^{-1}$$

Explanation

$$\det(\phi(t)) = \begin{array}{c} \begin{array}{ccc} y_1 & \quad\quad & y_n \end{array} \\ \begin{vmatrix} y_{11}(t) & \cdots & y_{1n}(t) \\ \vdots & & \vdots \\ y_{n1}(t) & \cdots & y_{nn(t)} \end{vmatrix} \end{array}$$

$$\frac{d}{dt}\det\phi(t) = \sum_{i=1}^{n} \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{in} \\ \vdots & & & \\ y'_{ii} & \cdots & & y'_{in} \\ \vdots & & & \\ y_{n1} & & \cdots & y_{nn} \end{vmatrix}$$

$$\frac{dx}{dt} = A(t)\,X.$$

$$\frac{dx_i}{dx} = \sum_{j=1}^{n} a_{ij}(t)\,x_j.$$

Explanation

$$= \sum_{i=1}^{n} \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \sum_{j=1}^{n} a_{ij} y_{j1} & \cdots & \cdots \\ y_{n1} & \cdots & y_{n_n} \end{vmatrix} ****$$

$$= \sum_{i=1}^{n} a_{ii} \det \phi(t)$$

$$= \mathrm{tr}(A(t)) \det \phi(t)$$

$$\frac{1}{\det \phi(t)} d(\det \phi(t)) = \mathrm{tr}(A(t)) dt$$

$$\ln \det \phi(t) = \int \mathrm{tr}(A(s)) ds$$

$$\det \phi(t) = e^{\int_{t_0}^{t} trA(s)ds}$$

$$= e^{\int_{0}^{t} trA(s)ds} \text{ proved}$$

Further explanation about ****

I guess you are quite confused with this line. So, I will provide detailed explanation.

$$\sum_{i=1}^{n} \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \sum_{j=1}^{n} a_{ij} y_{j1} & \cdots & \cdots \\ y_{n1} & \cdots & y_{n_n} \end{vmatrix}$$

Please look at my handwritten derivation.

$$\sum_{i=1}^{n}\begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \sum_{j=1}^{n} a_{ij}y_{j1} & \cdots & \cdots \\ y_{n1} & \cdots & y_{n_n} \end{vmatrix}$$

$$\sum_{i=1}^{n}\begin{vmatrix} y_{11} & \cdots & y_{12} & \cdots & y_{1n} \\ y_{21} & & y_{22} & & y_{2n} \\ a_{i1}y_{11}+a_{i2}y_{21}+\cdots & & a_{i1}y_{12}+a_{i1}y_{22} & & a_{i1}y_{1n} \end{vmatrix} \quad a_{i1} \quad a_{i2}$$

a_{i1}

$=$

$$\sum_{j=1}^{n} a_{ij}y_{j1} = a_{i1}y_{11} + a_{i2}y_{21} + a_{i3}y_{31} + \cdots$$

$$\begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ a_{i1}y_{11} & a_{i1}y_{12} & & a_{i1}y_{1n} \\ \vdots & & & \end{vmatrix} + \begin{vmatrix} \\ \\ \end{vmatrix}$$

$0.$ $\quad$ $\shortparallel$ $\quad$ $0.$

Handwritten

The Matrix Exponential

Given a matrix $A \in \mathrm{Mat}_n(\mathbb{R})$, we define the matrix exponential of A, denoted e^A or $\exp A$, in either of the following two equivalent forms

$$e^A := \sum_{k \geq 0} \frac{A^k}{k!}$$

$$e^A := \lim_{n \to \infty} \left(I_n + \frac{A}{n} \right)^n$$

where $A^0 = I_n$ is the $n \times n$ identity matrix. It can be verified that $x(t) = e^{tA} x_0$ is the solution to the IVP

$$\dot{x} = Ax, \quad x(0) = x_0$$

and $\Phi(t) = e^{tA}$ is the solution to the matrix ODE

$$\dot{\Phi} = A\Phi, \quad \Phi(0) = I_n$$

Note that for a constant $A \in \mathrm{Mat}_n(\mathbb{R})$, the Liouville's Formula becomes

$$\det e^{tA} = e^{t \operatorname{tr} A}$$

Some Important Property of the Matrix Exponential

1. $(e^A)^T = e^{A^T}$
2. If $AB = BA$ then $e^A * e^B = e^{(A+B)}$.
3. $(e^A)^{-1} = e^{-A}$
4. $det(e^A) = e^{tr(A)}$
5. For invertible matrix P, $P^{-1}e^A P = e^{P^{-1}AP}$, and actually, e^A is also defined by $Pdiag\{e^{J_1}, \cdot, e^{J_s}\}P^{-1}$, $P^{-1}AP = diag\{J_1, \cdots, J_s\} = J$, which is a Jordan Normal Form which will be taught in **Advanced Algebra** in detail.

Exercise

Let $A \in \mathbb{M}_n$. Then the inverse of e^{At} (MATH2860J Mid 1, Fall 2022)

1. always exists, it is e^{-At}
2. does not always exist because the equation $e^{At}e^{Bt} = e^{(A+B)t}$ does not hold generally.
3. exists only if $\det(e^{A}(t)) \neq 0$ and this is not always the case.

choose 1,2,3.

Example

6.7 Recall that

$$\exp(A) := \sum_{n\geq 0} \frac{A^n}{n!}$$

for a square matrix A.

(i) show that

$$\mathbf{D}\exp(A)\cdot H = \sum_{n\geq 0} \frac{1}{(n+1)!} \sum_{k=0}^{n} A^k H A^{n-k}$$

(ii) In particular, show that

$$\operatorname{tr}(\mathbf{D}\exp(A)\cdot H) = \operatorname{tr}(\exp(A)H)$$

(iii) Show that

$$\mathbf{D}\exp(A)\cdot H = \int_0^1 e^{(1-t)A} H e^{tA}\,\mathrm{d}t$$

Solution

(iii)

$$
\begin{aligned}
&\int_0^1 e^{(1-t)A} H e^{tA} dt \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{[(1-t)A]^n}{n!} H \sum_{m=0}^{\infty} \frac{(tA)^m}{m!} \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(1-t)^n A^n}{n!} H \cdot \sum_{m=0}^{\infty} \frac{t^m A^m}{m!} dt \\
&= \underline{\underline{\sum_{m\geq 0} \sum_{n\geq 0}}} \frac{1}{m!n!} \underbrace{\int_0^1 t^m (1-t)^n dt}_{\mathrm{Beta}(m+1,\, n+1)} \underline{\underline{A^m H A^n}}
\end{aligned}
$$

$$
\beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}
$$

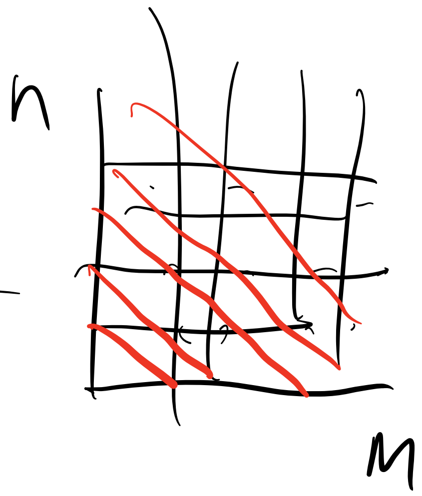
Solution

$$= \sum_{m\geq 0}\sum_{n\geq 0} \frac{1}{m!n!}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}A^m H A^n$$

$$= \sum_{m\geq 0}\sum_{n\geq 0} \frac{1}{\cancel{m!}n!}\frac{\cancel{m!}n!}{(m+n+1)!}A^m H A^n$$

$$= \sum_{m\geq 0}\sum_{n\geq 0} \frac{1}{(m+n+1)!}A^m H A^n$$

$$= \sum_{m\geq 0} \frac{1}{(m+1)!}\sum_{n\geq 0}^{m} A^n H A^{m-n}$$



proved

Remark

This Solution is provided by my friend in the game Osu! who is a math PHD researching on modern algebra in Beijing University. In this solution the skill of rearrangement of series and the convergent of matrix series e^A are used, but I think the convergence of matrix series is not in the scope of this course.