Paradox in Cantor's diagonalization

1. Point: Constructive enumeration

Consider the set T of all infinite sequences of binary digits. If $s_0, s_1, ..., s_n$ is any enumeration of elements from T, then an element s of T can be constructed that doesn't correspond to any s_n in the enumeration. (Cantor 1890)

$$
\begin{aligned}
s_0 &= 0000...\\
s_1 &= 0101...\\
s_2 &= 1011...\\
s_3 &= 1010...\\
&\vdots\\
s &= 1001...
\end{aligned}
$$

Figure 1: Diagonal argument in base 2

1.a. Counter-point: Dimensional space-filling curve

Objects in constructivism must be defined through explicit steps or algorithms that preserve linear definability within the referential space encompassing all possible constructed and constructive expressions. If I is a set of inputs and O a set of outputs, then the information complexity, or entropy, of all functional mappings from I to O, including injective or surjective correspondence, is given by the repeated Cartesian product for each input onto each output.

$$F = \{f : I \to O\} \Leftrightarrow F = O^I$$

Therefor, any arithmetical information structure, characterized by hierarchical and existential axioms or by their superposition, has a sequential representation corresponding to a dimensional space-filling curve through all possible states at infinity. In this sense, T is the set of all functions from natural numbers to boolean values, information-equivalent to power set $\mathcal{P}(\mathbb{N})$.

$$T = \{0, 1\} \times \{0, 1\} \times ... = 2^{\mathbb{N}}$$

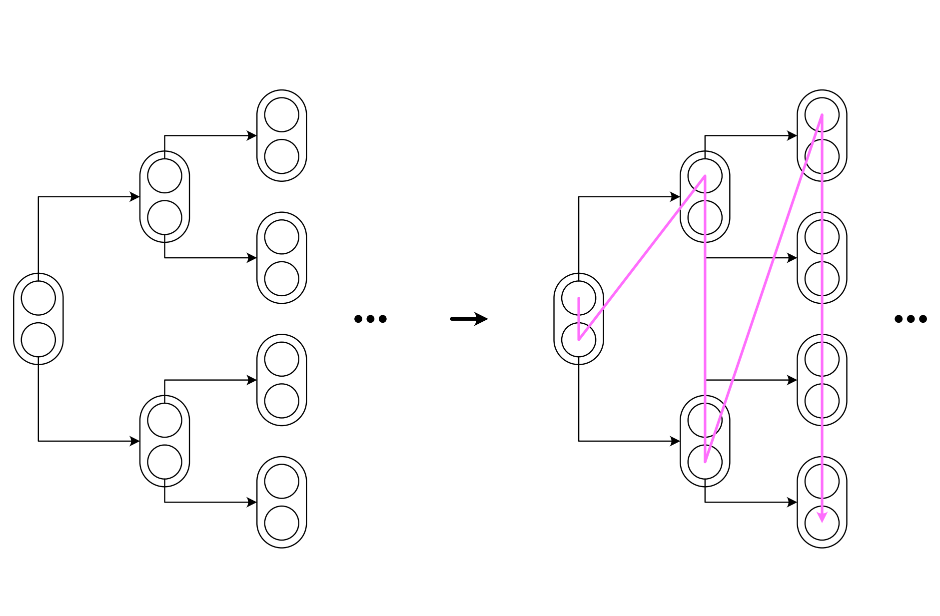

Figure 2: Cartesian filling

This property is maintained in circular, higher inductive, or dependent type space isomorphic to higher-order hypergraph, where the space-filling curve may expand onto the combinatorial or hyper-exponential growth of search itself.

$$T = B^{B^{\cdot^{\cdot^{D}}}}$$

For instance, first-order self-referential or infinite structures in both dimensional and basal junctions will be represented by expanding depth and width of search respectively.

$$T = B^D$$

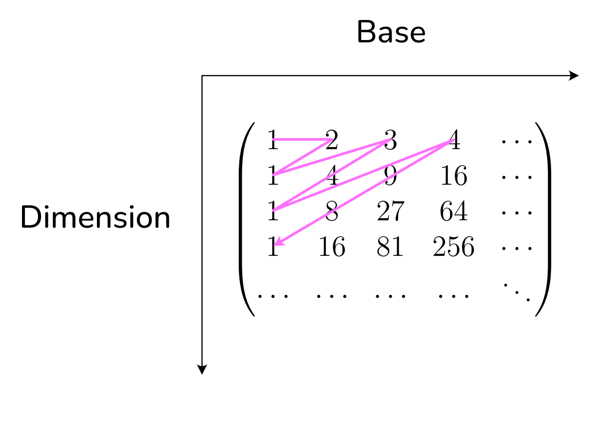

Figure 3: Higher-order space-filling curve

The symmetrical weight of evidence may not conclusively demonstrate the impossibility of constructing a complete infinite enumeration of elements nor its incompleteness in one instance. Nonetheless, the conflict itself suggests a limitation or ambiguity of provability in infinite objects.

1.b. Counter-point: Completeness of infinity

Any infinite abstraction fills the entirety of the space that shapes or constrains it. Let I be an infinite set, if $\exists x \in I.|I - \{x\}| < |I|$ then the set I can not be infinite, as the expression requires a collapsed, intrinsic or extrinsic limit to its expansion. This follows from the set-theoretical axiom of infinity capturing the essence of the paradox:

$$|I| = \infty \Leftrightarrow I \subsetneq I$$

Consequently, if the enumeration T is either finite or repeating, its incompleteness is trivial and does not imbue additional meaning necessitated by the conclusion drawn from the diagonalization argument, whereas a non-repeating infinite enumeration has a unique correspondence parity with a set structure which, by necessity, must be whole.

1.c. Counter-point: Representational discrepancy

In unlabeled structures, distinction between contained objects is intrinsic to the relation or order within the objects themselves. Thus in an infinite structure, there will exist a transformation that reverts any change in relative arrangement of information induced by the theoretical removal of a finite or infinite subset of its components, including if this operation is repeated unlimited times. The identity of a cloned or transported object becomes ambiguous whether considered self-contained and preserved into another expandable inner-relative structure, or devoid of intrinsic information outside its native original structure. This suggests a fundamental weakness in the expressiveness of construction of infinity derived as potential.

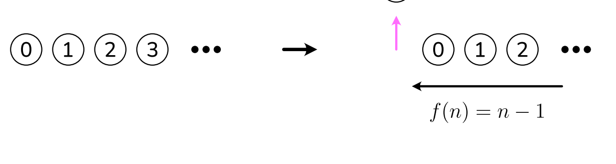

Figure 4: The infinite nuances of x may be meaningless outside of $\mathbb{N}$.

The original argument by Cantor is non-constructive, involving a closure from contradiction. However, the derivation to s from s_n being non-halting implies that the former solely exists as infinite potential, thus on unleveled abstraction that contrasts latter's definition as infinite actual. The existence of s is elusive. The heterogenous objects within the proof yields considerations parallelling implicit point-wise specialization of axiom of choice and decomposition into disjoint copies (Banach-Tarski 1924), which is broadly, succinctly summarized by:

$$\infty - \infty = \infty$$

In contexts where inference from constructive infinity is applicable, and manipulation of infinity as potential and actual is interchangeable, e.g. hypothetical hyperconstructivism, orthogonal objects can not be mutually constrained nor contradictory because they are intrinsically true by construction. Reflection precludes exclusive ownership of abstraction, including in self-reference.

2. Point: Contradiction of surjection

Suppose a surjective function $\exists f : S \to 2^S$. Define the diagonal subset $D = \{x \in S : x \notin f(x)\}$. Then, f is onto $\Rightarrow \exists x.D = f(x)$ where either $x \in D$ or $x \notin D$ contradicts the definition of D. Therefor, f can not be a surjection, followed by:

$$|S| < |2^S|$$

2.a. Counter-point: Inference-independence of paradox inherent to self-reference

Consider exclusively sets S with infinite cardinalities, ignoring trivial case where $|X| \in \mathbb{N} \Rightarrow |X| < |2^X|$. Define surjection notation $\forall X.\forall Y.(\exists f \in X \twoheadrightarrow Y) \Leftrightarrow (\exists f \in Y^X.\forall y \in Y.\exists x \in X.y = f(x))$.

$$P_0 \Leftrightarrow \forall S.\forall f \in S \twoheadrightarrow 2^S.\exists D \in 2^S.\forall x \in S.x \in D \oplus x \in f(x)$$

This is a specialization of the barber paradox. (Russell 1919) The symmetry suggests the following explanations:

- One or multiple quantified objects can not exist, such as surjection f (diagonal interpretation) or D.
- The resolution of this expression is divergent. Thus there are fundamental limitations associated with set-theoretical connectives parallel to the halting problem.

The proposition asymptotically defines D as the symmetric difference of the pre-image set with itself:

$$D = D \oplus f^{-1}(D)$$

The solution of the pre-image $f^{-1}(D) = \emptyset$ suggests that the proposition implicitly excludes D from being image of f in itself. Thus, hypothesized partial correspondence of f is independent of P_0.

$$P_0 \Leftrightarrow \mathtt{I} \text{ (paradox)}$$

Nonetheless, interpretation of set behavior at a singularity may be incomplete if $\exists x.x \neq x$. To further demonstrate that the expression does not infer information about the underlying functional connectives, define the structurally-equivalent proposition in the absence of indefinite surjection with reversed domain and codomain:

$$P_1 \Leftrightarrow \forall S.\forall f \in S^{(2^S)}.\exists D \in 2^S.\forall x \in S.x \in D \oplus (\exists X.x = f(X \cup \{x\})) \Leftrightarrow \mathtt{I}$$

P_1 is metaphorically equivalent to P_0 and the barber interpretation, despite having no associated range uncertainty. Therefor, implicit self-reference of the original apparent contradiction may not directly infer a systemic conclusion.

2.b. Counter-point: Reverse diagonalization

Let $P = \{p_0, p_1, p_2, ...\}$ be the set of all prime numbers where p_i is the i-th prime, and does not assume axiom of choice.

- $|P|$ is infinite. (Euclid [no date])
- The number 0 is represented by $\prod p_i^0 - 1$.
- Each natural n is represented by a unique prime factorization $\prod p_i^{f_n(i)} - 1$ where $f_n \in \mathbb{N}^{\mathbb{N}}$.
- Then there exists a prime factorization that corresponds to each prime exponent.

$$\forall f \in \mathbb{N}^{\mathbb{N}}.\forall i \in \mathbb{N}.\exists g \in \mathbb{N}^{\mathbb{N}}.f(i) = \prod_{p_j \in P} p_j^{g(j)} - 1$$

- Construction of a recursive **anti-diagonal** of naturals induces the information equivalence:

$$|\mathbb{N}| = |\mathbb{N}^{\mathbb{N}}| = |{}^{\mathbb{N}}\mathbb{N}| = ... = |\mathbb{N} \uparrow^{\mathbb{N}} \mathbb{N}| \text{ (hyperoperation)}$$
$$\therefore \forall S.|S| = \infty \Leftrightarrow |S| = |2^S|$$

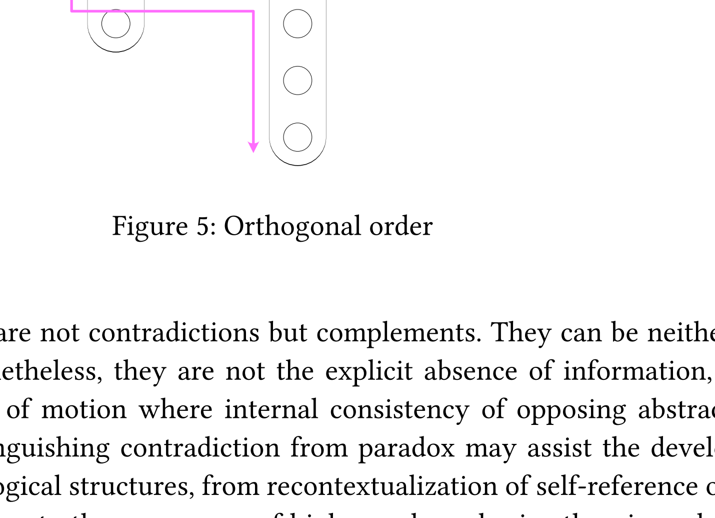

Figure 5: Orthogonal order

3. Conclusion

In metaphysics, paradoxes are not contradictions but complements. They can be neither definitively proven nor disproven. Nonetheless, they are not the explicit absence of information, as suggested by the dichotomy paradox of motion where internal consistency of opposing abstraction coexists rather than conflicts. Distinguishing contradiction from paradox may assist the development of incompleteness-minimizing logical structures, from recontextualization of self-reference of non-halting in-between information space to the emergence of higher-order cohesive theories, where ostensible absence of distinction juxtaposes with provisional tangibility.

Finally, intrinsically reproducible and self-contained information may be an effective focal point to formalism, in contexts where extrinsic interpretation of truth value interferes with minimization of information loss associated with overdetermined axiomatic constraints on epistemological search space. For example, continuum may be both and neither countable or uncountable. The hypothetical subsequent definition of reciprocal equivalence between point and space may generalize the notion of continuity to arithmetical structures.

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