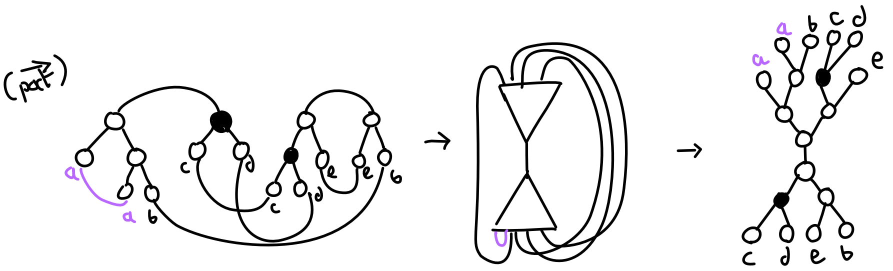
Combinator Arithmetization

Canonical bipartite binary tree representation
Any combinator network can be written as the interaction of two trees by merging parallel linear matches.
This distinguishes two types of path, internal semi-cycles (purple) and external cycles (black).



Code:
((a (a b)) (•(c d) e)) = (•(c d) (e b))

Cardinality of binary tree is given by Catalan numbers (for n+1 leaf ports):

$$|T_n| = C(n) = \frac{\binom{2n}{n}}{(n+1)} = \frac{(2n)!}{(n+1)!\,n!}$$

Cardinality of L labels:

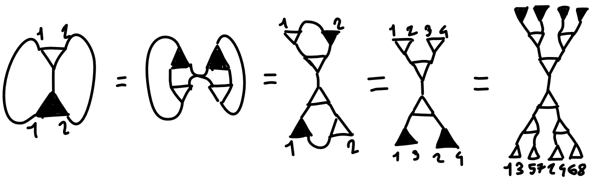
$$L^n$$

Cardinality of canonically-ordered 2-terms multinomial permutation
(injective into knots in Gauss code ignoring chirality):

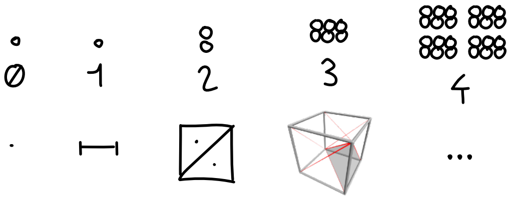
$$|S_n| = \frac{(2n)!}{2^n\, n!}$$

for $n \geqslant 6$ and $L = 2$. $\frac{2^n}{(n+1)!} < \frac{1}{2^n} \Rightarrow |T_n| \cdot |L^n| < |S_n|$

Thus the evaluation of combinatoric or Hopf algebraic construction yields a pair of sequences of multinomial permutations respectively representing computational state and structure (surjective).
In some circumstances, if no internal path is present or if the network can be arranged as such, computational state is a sequence of simple permutation. This is the case for a minimal exponential net in SIC.



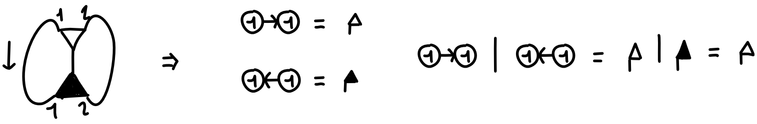
Thus there is a corresponding linear algebraic function or generating function describing the sequence as motion in factorial space.
(e.g. bijection within naturals and space-filling curve of simplicial tesselation of unit hypercubes with reflective symmetry)



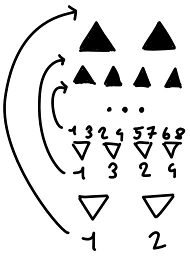
(Tangentially, token-level permutation independance in transformers may be achivable by reflection of simplicial permutation
where attention keys are interpreted as Turing machine program graph).

To find the arithmetic representation of permutation encoded by minimal exponential net, unfold equivalence transformations from total to partial
where maximum influence and self-division of permutation dimensions are represented by a complete multipartite graph.

An unfolded segment has either one input and one output wires, or none. This representation allows the evaluation of partial states at arbitrary evaluation depth,
in addition to subsequent optimization, such as elimination of cyclic permutation. Applying initial permutation to secondary nodes of first-order path gives:



Thus the unfolded view relative to each output defines the sequence generated by minimal exponential net by recursive application
of SIC's fundamental permutation (1 3 2 4).



1 3 5 7 2 4 6 8
(1 3) (2 4) (5 7) (6 8) (9 11) (10 12) (13 15) (14 16)
⇒
(1 2) → (1 3 2 4) → (1 3 5 7 2 4 6 8) → (1 3 9 11 2 4 10 12 5 7 13 15 6 8 14 16) → ...

Conjecture
All interaction combinators can be arithmetized, i.e. expressing combinatorial structure in terms of arithmetic operations. Nonetheless, since the behavior of a combinator subnet is Turing complete, the partial homoiconicity between state and structure may induce undecidability respectively indicated by the halting problem
(wrt state space, albeit limited to a diagonalization argument), and buzy beavers (non-computable structural space which seeks the graph of size n
with maximum reduction steps before halting). Thus while interaction combinators can be arithmetized in terms of structure, their termination behaviors sometimes can not.
Consequently, the set of halting graphs being non-computable implies that equivalence classes will be generally incomplete.