Quantum Error Correction: Surface Codes

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Challenges of QEC

Error correction codes are algorithms that detect and correct errors which occur when data is transmitted or stored. Problems arise when we try to adapt existing classical methods to quantum error correction (QEC):

- **Continuum of errors**: errors can take a qubit to any point on the Bloch sphere.
- **No-cloning theorem**: it is impossible to construct a unitary operator U_{clone} such that
$$U_{\text{clone}}(|\psi\rangle \otimes |0\rangle) \to |\psi\rangle \otimes |\psi\rangle$$
- **Wavefunction collapse**: detecting an error through a measurement can corrupt information.
- There are **phase-flip errors** in addition to bit-flip errors.

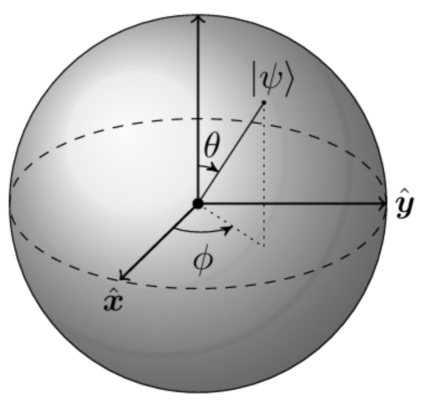

Figure 1. A qubit is a point on the Bloch sphere: $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

Errors and Encoding

⋆ A single qubit coherent error is represented by a unitary operation E. The state evolves:
$$E\,|\psi\rangle = \cos\frac{\theta+\delta\theta}{2}\,|0\rangle + e^{i(\phi+\delta\phi)}\sin\frac{\theta+\delta\theta}{2}\,|1\rangle$$
This can be written as a sum from the set of Pauli matrices $\{I, X, Z, XZ\}$:
$$E\,|\psi\rangle = \alpha_1\,I|\psi\rangle + \alpha_2\,X|\psi\rangle + \alpha_3\,Z|\psi\rangle + \alpha_4\,XZ|\psi\rangle$$
X-type errors are bit-flips: $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$, and **Z-type errors are phase flips**: $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$.

⋆ We **encode the state** through CNOT gates. For example, using Figure 2:
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \xrightarrow{\text{encoding}} |\psi\rangle_L = \alpha\,|000\rangle + \beta\,|111\rangle \equiv \alpha\,|0\rangle_L + \beta\,|1\rangle_L$$
States $|0\rangle_L$ and $|1\rangle_L$ are called logical qubits. Encoding distributes the quantum information across multiple qubits. This is allowed, since it is consistent with the no-cloning theorem.

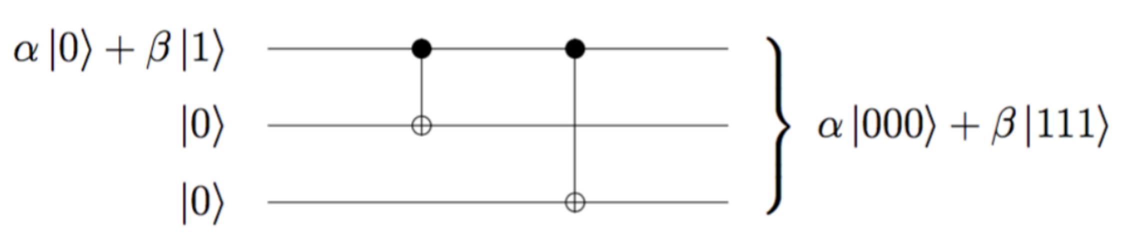

Figure 2. Using three CNOT gates to encode a three-qubit logical state

An Example of Syndrome Extraction

Consider a two-qubit encoding
$$|\psi\rangle_L = \alpha\,|00\rangle + \beta\,|11\rangle = \alpha\,|0\rangle_L + \beta\,|1\rangle_L$$
The state is subject to a single flip error X_1 acting on the first qubit
$$X_1\,|\,\psi\rangle_L = \alpha\,|10\rangle + \beta\,|01\rangle$$
Using the **servus qubits** $|0\rangle_S$ and $|1\rangle_S$, we make the transformation
$$X_1\,|\,\psi\rangle_L \;\to\; \frac{1}{2}(I_1I_2 + Z_1Z_2)\,X_1\,|\psi\rangle_L\,|0\rangle_S + \frac{1}{2}(I_1I_2 - Z_1Z_2)\,X_1\,|\psi\rangle_L\,|1\rangle_S = X_1\,|\psi\rangle_L\,|1\rangle_S$$
The servus qubit will measure to be 1 with certainty, and this will not affect $X_1|\psi\rangle_L$.

Similarly, the servus measures to 1 with error X_2, while I and X_1X_2 give 0. The outcome of the servus qubit measurement is called a **syndrome**.

This process is an example of a **stabilizer code**. The operator Z_1Z_2 is said to stabilize the logical state since it leaves it unchanged $Z_1Z_2|\psi\rangle = |\psi\rangle$.

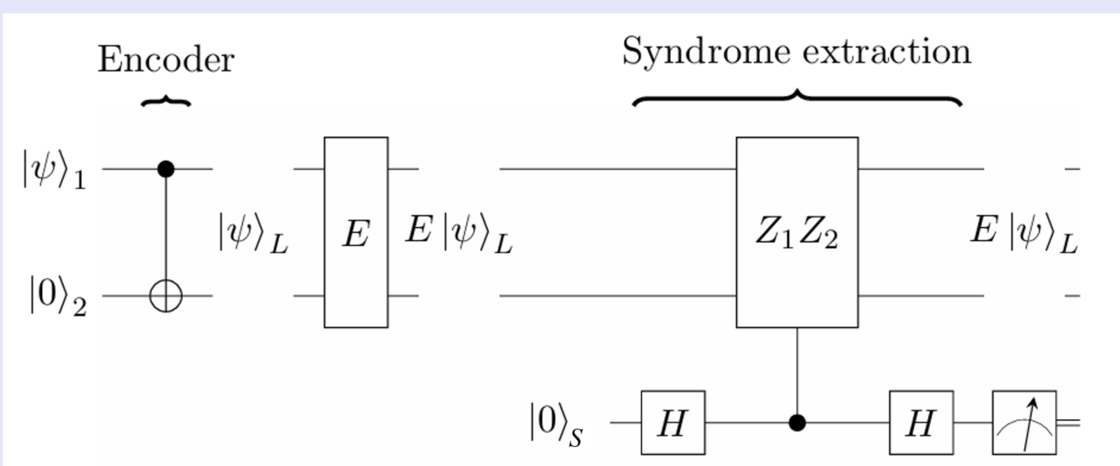

Figure 3. A circuit diagram summarising the QEC code of the two qubit

Stabilizer Codes

In the example, the code is unable to detect the presence of single-qubit Z-errors. More complex codes can be created to measure both X and Z errors:

$[[n,k,d]]$ stabiliser codes, with n the total number of qubits, k the number of logical qubits and d the code distance.

There are $m = n - k$ stabilizer measurements P_i, which gives the syndrome extraction:
$$E|\psi\rangle_L|0\rangle_{S_i} \;\longrightarrow\; \frac{1}{2}\left(I^{\otimes n} + P_i\right)E\,|\psi\rangle_L\,|0\rangle_{S_i} + \frac{1}{2}\left(I^{\otimes n} - P_i\right)E\,|\psi\rangle_L\,|0\rangle_{S_i}$$
If P_i commutes with E the syndrome is 0. If P_i anti-commutes with E the syndrome is 1.

A QEC code therefore involves **finding stabilizers that anti-commute with the errors to be detected**.

Error	Syndrome, S	Error	Syndrome, S
X_1	10	Z_1	01
X_2	10	Z_2	01
X_3	10	Z_3	01
X_4	10	Z_4	01

Figure 4. An example of a syndrome table for the $[[4,2,2]]$ stabiliser code

Surface codes

Pauli operators **commute** if they intersect non-trivially on an **even** number of qubits, and **anti-commute** if they intersect on an **odd** number of qubits.

Surface codes use this property and their topology to create an error-protected 'fabric'. It can be scaled in size while ensuring stabilizer commutativity.

The elementary blocks of surface codes are **four-cycle**: 2 logical and 2 servus qubits connected by edges which represent X and Z gates. E.g. in Figure 5, servus S_2 measures the $Z_{D_1}Z_{D_2}$ stabilizer.

Codes are then built by tiling four-cycles in a square lattice. Appropriate stabilizers can be read from the figures. By construction, they intersect an even number of times with each other and an odd number of times with an error on a qubit.

Decoding algorithms are used to correct errors once they are detected. The frequency with which the decoder fails (p_L) decreases as the code distance is increased. But this increases the rate of physical errors (p_{ph}).
Hence p_L is only decreased under a certain threshold on p_{th}. This threshold is high compared to other QEC codes: surface codes are promising!

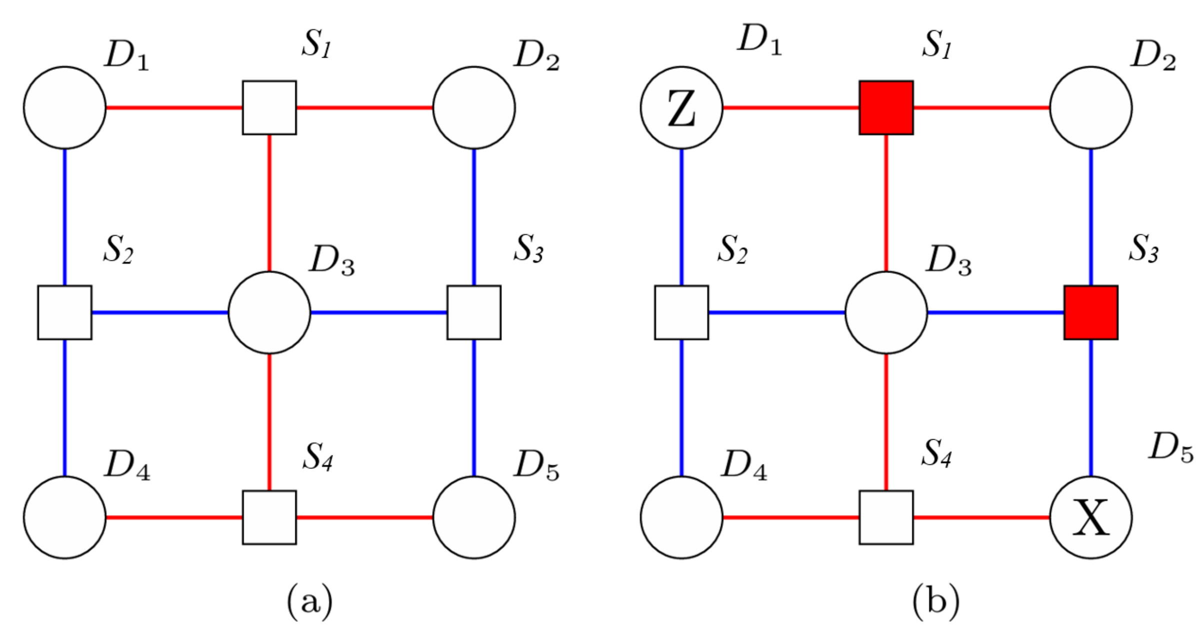

Figure 5. (a) A $[[5,1,2]]$ surface code: red edges are X-controlled gates, blue are Z-controlled. (b) Examples of errors: the red servus measures as 1.

Summary

- Any coherent error process can be decomposed into a sum from the Pauli set.
- The information is spread through the logical qubits without cloning.
- Introducing servus qubits allows measurement of the error without collapsing the logical qubit.
- Stabiliser codes can simultaneously detect X-type errors and Z-type errors.
- Stabilizer operators should anti-commute with errors.
- Pauli operators anti-commute if they intersect on an odd number of qubits.
- Surface codes use their topology to intersect the (Pauli) stabilizers with the (Pauli) error such that they anti-commute.
- The threshold on the rate of physical errors is high in surface codes.

References

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