The Theory of the Casimir Force

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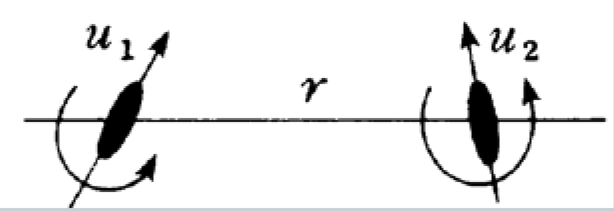
DECEMBER 19, 2012

GROUP MEETING

Classical E&M: Intermolecular Forces

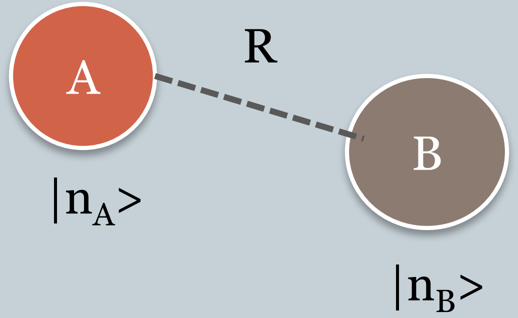


- F = - dE/dr ; F is observable.
- Charge-charge: $E \sim r^{-1}$
- Fixed permanent dipole – charge: $E \sim r^{-2}$
- Two fixed permanent dipoles: $E \sim r^{-3}$
- Free permanent dipole – charge: $E \sim r^{-4}$
- Van der Waals Forces (neutral bodies) $E \sim r^{-6}$:
  - Two free permanent dipoles (Keesom)
  - Permanent dipole – induced dipole (Debye)
  - Two induced dipoles (London) – NOT classical

weaker

London Dispersion Force (1936)

- Does not exist classically – quantum fluctuations
- 2^{nd} Order (time indep.) perturbation theory



$H = H_o + H'$ (($H' << H_o$))

$H_o = H_A + H_B$

$H_o |n_A \times n_B\rangle = (E_A + E_B)|n_A \times n_B\rangle$

$H' \propto u_A u_B / R^3$

$E^{(1)} = \langle 0,0| H' |0,0\rangle = 0$ – no permanent dipole

$$E^{(2)} = \sum_{n_A, n_B \neq (0_A, 0_B)} \frac{\langle 0_A^{(0)} 0_B^{(0)} | \hat{H}^{(1)} | n_A^{(0)} n_B^{(0)} \rangle \langle n_A^{(0)} n_B^{(0)} | \hat{H}^{(1)} | 0_A^{(0)} 0_B^{(0)} \rangle}{E^{(0)}_{0_A 0_B} - E^{(0)}_{n_A n_B}}$$

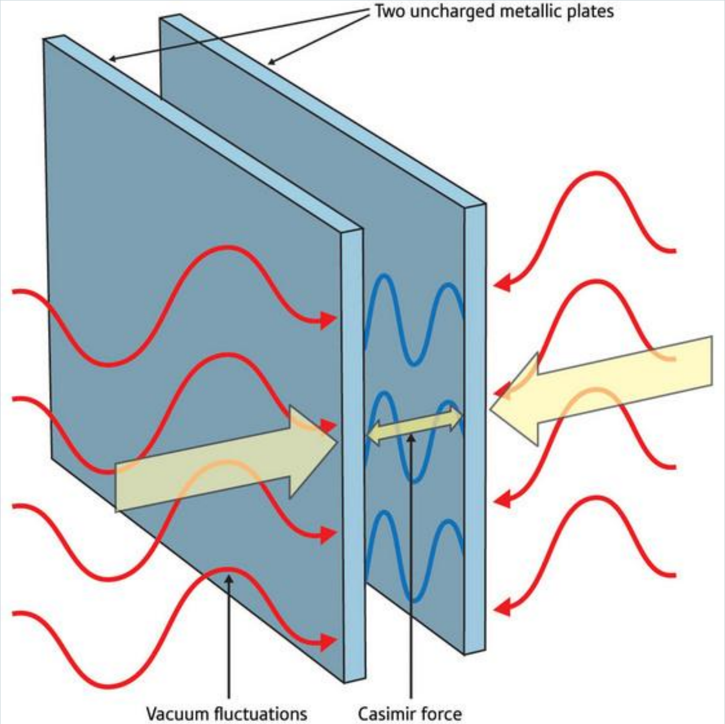
$E^{(2)} \propto 1/R^6$ result of quantum fluctuations between $|0,0\rangle$ and higher $|n\rangle$ states

A Different Force?

- Casimir Force is often described as a property of the vacuum with no reference to atoms or dipoles of real materials.

  $F = - \pi^2 \hbar c A / 240R^4$

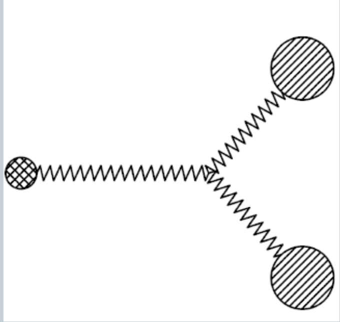
  No: fine structure constant,
  dipole moment, etc



- Mode-counting exercise
- "Evidence" of vacuum energy density?

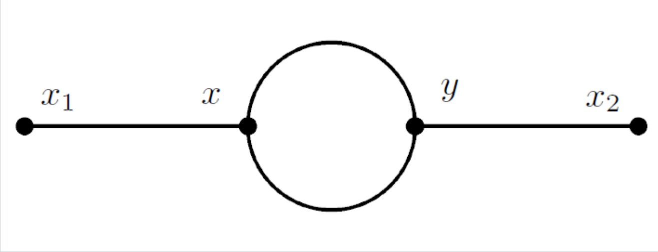
Some Background: Quantum vs. Classical

$$[x,p_x]=i\hbar$$



Classical fields proliferate

Quantum fields may contract



=> Creation/annihilation of virtual particles

Example: Rutherford Scattering (QED *)

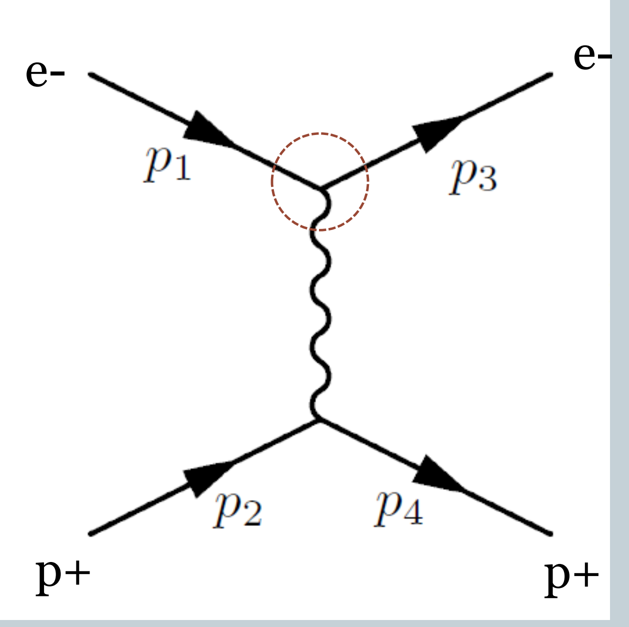
- e- + p+ -> e- + p+

$$p_e^{\mu} = (E, 0, 0, p)$$

$$p_e^{\mu\prime} = (E, psin\theta, 0, pcos\theta)$$

Off-shell photon Energy Momentum Conservation:

$$k_\gamma^{\mu} = (0, -psin\theta, 0, p[1 - cos\theta])$$

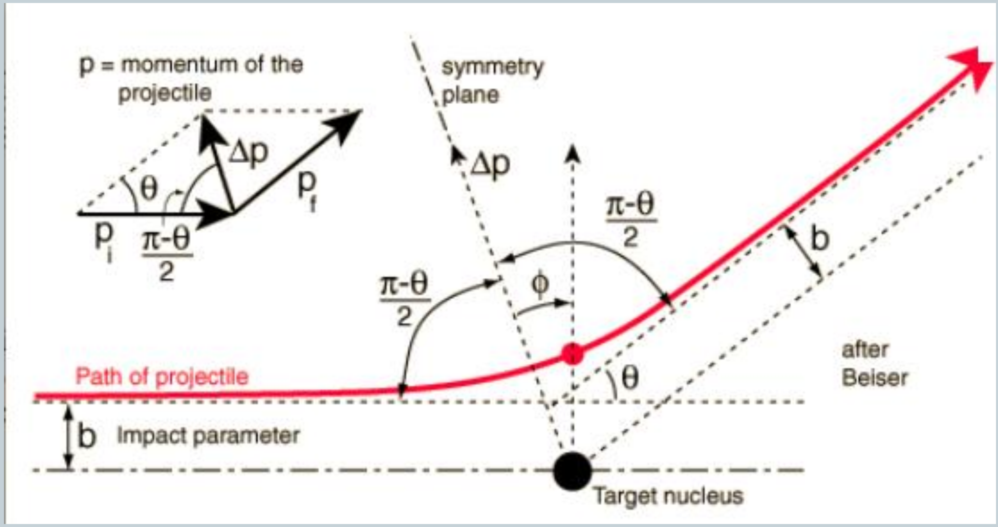


$$\frac{d\sigma}{d\Omega} = \frac{4Z^2e^4}{4\pi^2}\frac{m_\alpha^2}{k^4}$$

$$k^4 = 16p^4\sin^4\frac{\theta}{2}$$

Example: Rutherford Scattering (Classical)

- $V(r) = 2(Ze^2)/r$



Energy Momentum Conservation, etc

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \cdot \frac{2\pi b db}{d\cos\theta} = \frac{Z^2 e^4}{64\pi^2 \varepsilon_0^2 m^2 v^4 \sin^4(\theta/2)}$$

- We see 2 equivalent descriptions:
  - Virtual photon exchange (focus on mediator, local)
  - Classical electrostatic forces (focus on materials, non-local)

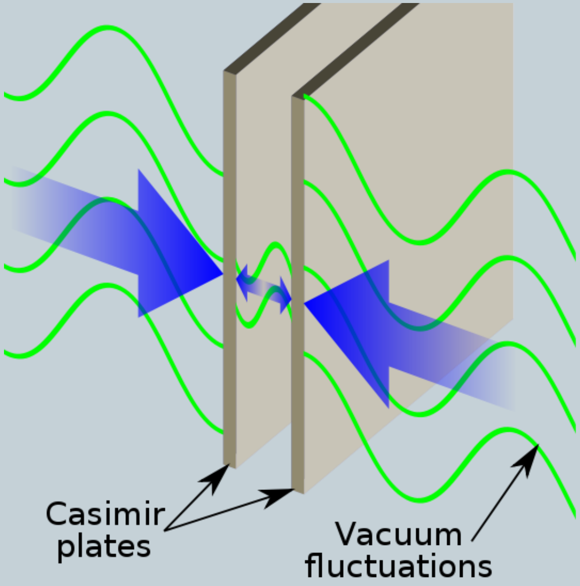
Casimir Force (PEC plates)

- Canonical Quantization picture (in vacuum, 1-D):

$$H = \int \frac{dk}{2\pi} w_k (a_k^\dagger a_k + \frac{1}{2})$$

E = <0|H|0> is divergent

- Add Boundary Condition: $|\psi>$ = 0 at x=0, r (Dirichlet)

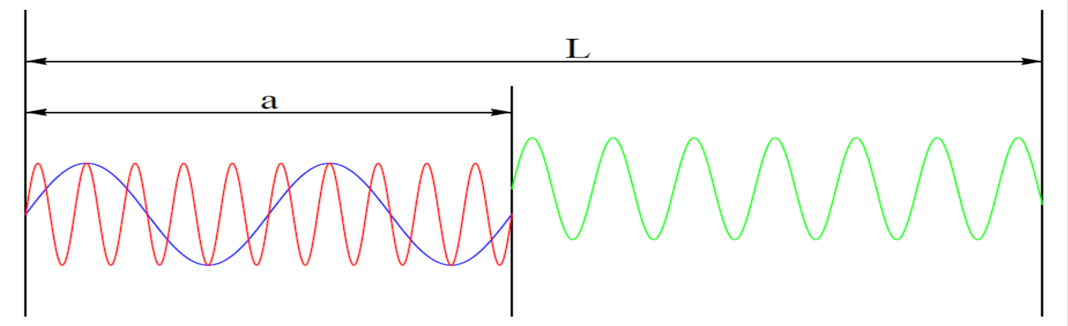


$$E(r) = \langle 0 | H | 0 \rangle = \sum_n \frac{\omega_n}{2}, \quad \omega_n = \frac{\pi}{r} n$$

Still infinite.
Need to be more careful with our infinities.

Casimir Force (PEC plates)

- L >> a
- Will take L -> ∞



$$E_{\text{tot}}(a) = E(a) + E(L-a) = \left(\frac{1}{a} + \frac{1}{L-a}\right)\frac{\pi}{2}\sum_{n=1}^{\infty} n$$

Regulator Λ: assume ∞ = ∞(Λ) + E_c (a)

$$E(r) = \frac{1}{2}\sum_n w_n$$

$$E(r) = \frac{1}{2}\sum_n \omega_n e^{-\omega_n/(\pi\Lambda)}$$

Casimir Force (PEC)

- Performing the sum

$$E(r) = \frac{1}{r}\frac{\pi}{2}\sum_{n=1}^{\infty} n e^{-n/(\Lambda r)}$$

$$E(r) = \frac{1}{r}\frac{\pi}{2}\left[\Lambda^2 r^2 - \frac{1}{12} + \frac{1}{240 r^2 \Lambda^2}\cdots\right] = \frac{\pi}{2} r \Lambda^2 - \frac{\pi}{24 r} + \cdots$$

- E = E(a) + E(L-a)

$$E = \frac{\pi}{2}L\Lambda^2 - \frac{\pi}{24a} - \frac{\pi}{24(L-a)} + \ldots$$

- Taking L $\rightarrow \infty$

$$E_c = -\frac{\pi \hbar c}{24a}$$

- In 3-D

$$E_c = -\frac{\pi^2 \hbar c A}{720 a^3}$$

Interpreting the Casimir Force

$$E_c = -\frac{\pi^2 \hbar c A}{720 a^3}$$

- No dependence on material properties
- A fundamental measure of vacuum energy density?
- # Modes outside > # Modes inside: attractive force?
- Not so fast: let's tweak the boundary conditions

Dirichlet

$|\psi\rangle = 0$ at $x = 0, r$

Mixed Dirichlet Neumann

$|\psi\rangle = 0$ at $x = 0$, $d/dx\,|\psi\rangle = 0$ at $x = r$

Casimir Force (mixed BC)

$$\omega_n = \frac{(2n+1)\pi}{2a}.$$

$$E(r) = \frac{\pi}{2r} e^{-\frac{1}{2r\Lambda}} \sum_n \left(n e^{-\frac{n}{r\Lambda}} + \frac{1}{2} e^{-\frac{n}{r\Lambda}} \right)$$

$$E(r) = \frac{\pi}{2r} \left(r^2 \Lambda^2 + \frac{1}{24} - \frac{7}{1920 r^2 \Lambda^2} \right)$$

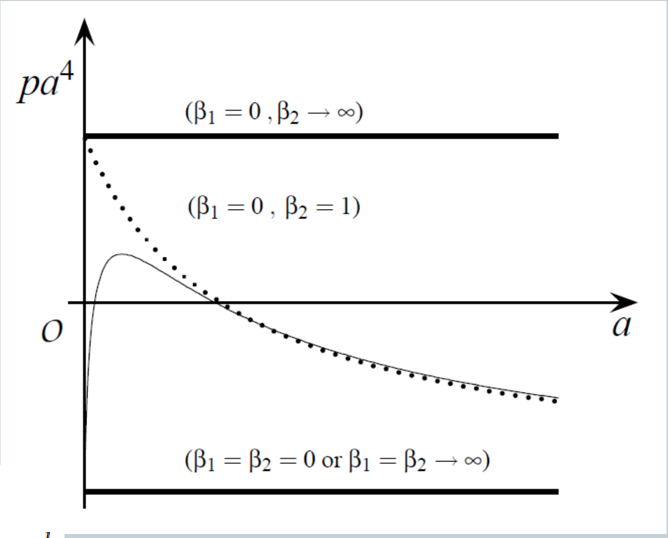
$$E_c = + \frac{\pi \hbar c}{48a}$$

Force is repulsive. What's going on

Casimir Force and Boundary Conditions

- No dependence on materials deceptive
  - Boundary conditions encode assumptions about material properties and response
  - Field is zero at plates -> atoms in plate polarized in a certain way to cancel field.
  - PEC assumption is $\alpha \to \infty$ limit, w_p and skin-depth both depend on α
- Mode-counting argument flawed
- Can get whole range of behaviors

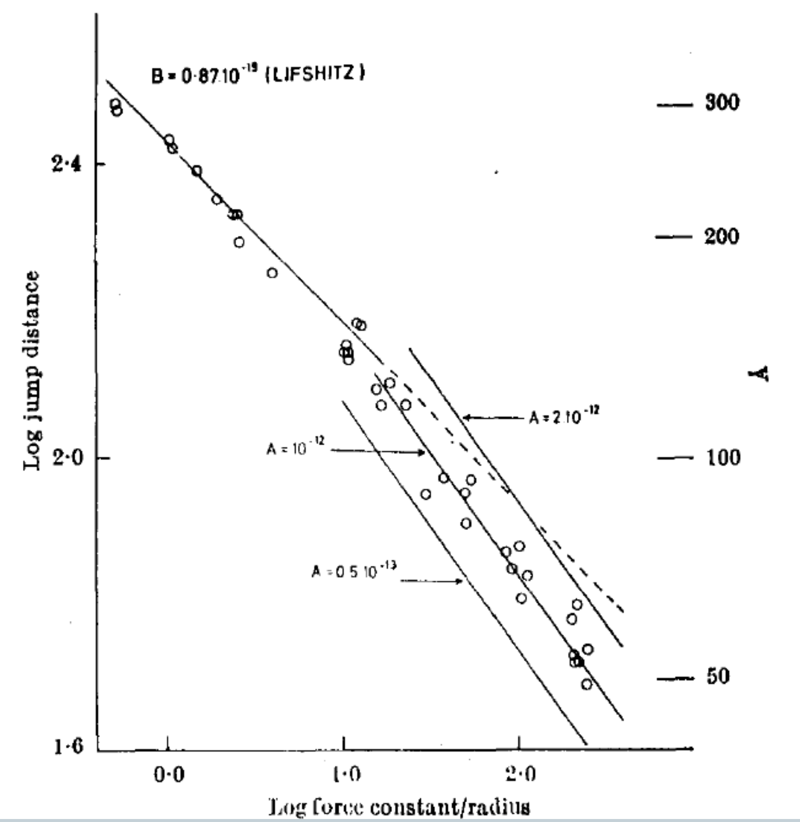
  By tweaking B.C. (right)
- Proof of vacuum energy density?
  - ??????????????????



$$\phi|_{bound.} = \beta \frac{\partial \phi}{\partial n}|_{bound.}$$

Casimir Force as Intermolecular Force

- Casimir's original goal: find the large r limit of London dispersion force – accounting for finite speed of light.



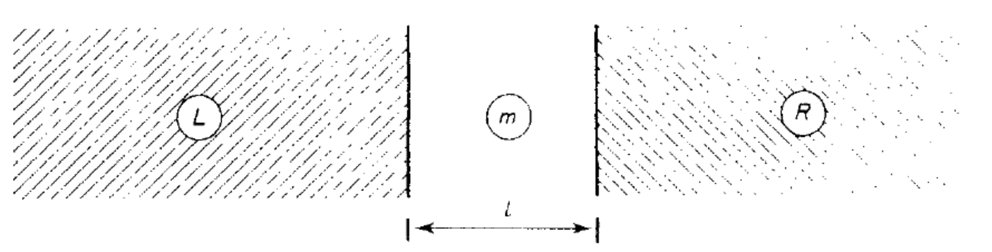
- LDF power law dependence changes at 200 A
  - Normal -> retarded van der Waals
- E ~ $R^{-6} \rightarrow R^{-7}$
- Casimir found very simple form (for 2 atoms):

$$U = -(23\hbar c/4\pi)\,(\alpha_1\alpha_2/r^7)$$

- Can be (was) described entirely in molecular terms

Lifshitz Theory

- Addressed an additional difficulty (+ c):
  - Dispersion forces are NOT additive (non-linear)
  - Building up from single atoms very non-trivial
- Starting point instead: continuous slabs of constant ϵ
  - General expression for planar geometry



$$G(l, T) = \frac{kT}{8\pi l^2} \sum_{n=0}^{\infty}{}' \int_{r}^{\infty} x \{\ln [1 - \bar{\Delta}_{mL}\bar{\Delta}_{mR}\, \mathrm{e}^{-x}] + \ln [1 - \Delta_{mL}\Delta_{mR}\, \mathrm{e}^{-x}]\}\, \mathrm{d}x$$

  - Term with $-(\varepsilon_1 - \varepsilon_3)\,(\varepsilon_2 - \varepsilon_3)$, this interaction can change sign as well

Two Derivations - Two Interpretations

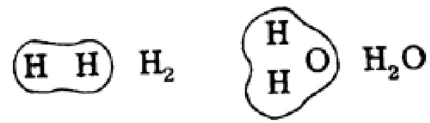
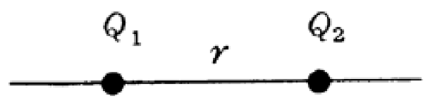
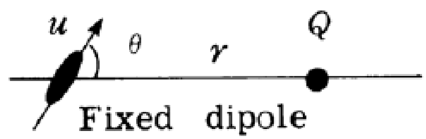
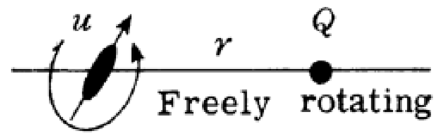
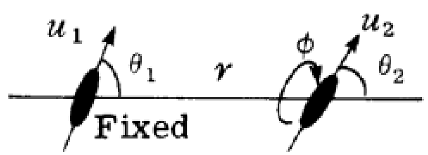
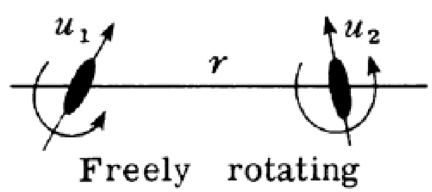
- Quantum mechanical (London dispersion)
  - Quantum fluctuations of dipoles interpretation + finite speed of light
- Quantumelectrodynamical (Virtual particles)
  - Vacuum fluctuations w/ B.C. interpretation
- But are they really different?
  - QFT is a relativistic , local formulation of quantum mechanics
  - Agreement should be expected
  - Similar to fields vs. charges interpretation in E&M.
- Reality of vacuum energy?
  - ?????????????????????????????

Casimir Force: What we know

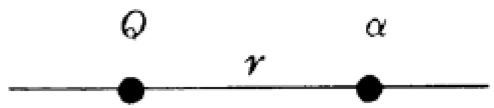
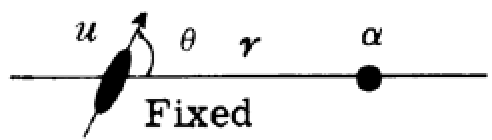
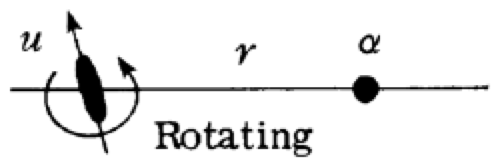
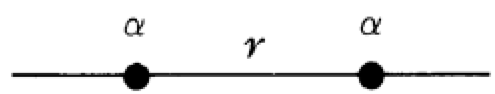
- Extremely unintuitive
  - Closed spherical shell in vacuum is repulsive (!) – (Boyle's PhD, Casimir's electron theory busted)
  - $E = +0.9 / 2a$
  - Bring together 2 halves of sphere (attractive -> repulsive??)
- Non-linear, temperature dependent (though non-zero at $T=0$), material-dependent, torques, etc.
- Area of very active theoretical and experimental research.
- Me: Contact-free measurement of repulsive slab – fluid – sphere geometry. (A Woolf, A Rodriguez)

HAVE A NICE BREAK!!!!

Intermolecular Forces (1)

type of interaction		interaction energy w(r)
Covalent, metallic		Complicated, short range
Charge–charge		$Q_1Q_2/4\pi\varepsilon_0 r$ (Coulomb energy)
Charge–dipole		$-Qu\cos\theta/4\pi\varepsilon_0 r^2$
		$-Q^2u^2/6(4\pi\varepsilon_0)^2kTr^4$
Dipole–dipole		$-u_1u_2[2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\phi]/4\pi\varepsilon_0 r^3$
		$-u_1^2u_2^2/3(4\pi\varepsilon_0)^2kTr^6$ (Keesom energy)

Intermolecular Forces (2)

type of interaction		interaction energy w(r)
Charge–non-polar		$-Q^2\alpha/2(4\pi\varepsilon_0)^2 r^4$
		$-u^2\alpha(1+3\cos^2\theta)/2(4\pi\varepsilon_0)^2 r^6$
Dipole–non-dipolar		
		$-u^2\alpha/(4\pi\varepsilon_0)^2 r^6$ (Debye energy)
Two non-polar molecules		$-\frac{3}{4}\frac{h\nu\alpha^2}{(4\pi\varepsilon_0)^2 r^6}$ (London dispersion energy)

R- dependence of intermolecular forces (1)

$$V(r) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

$$V(r) = \frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \tilde{M} \cdot \hat{r}}{r^3} + \ldots$$

Dipole-dipole (fixed): $U = -\vec{p} \cdot \vec{E}$

$$\vec{E} = -\vec{\nabla} V(r)_{dipole} \propto \frac{1}{r^3}$$

Free-dipole – charge:

$$U = - < \vec{p} > \cdot \vec{E} = -|\vec{p}| |\frac{q^2}{r^2}| < cos(\phi) >$$

$$< cos\phi >= \int_{-1}^{1} p(cos\phi) cos\phi d(cos\phi) \propto \int_{-1}^{1} e^{-\frac{|\vec{p}|q^2 cos\phi}{r^2 kT}} cos\phi d(cos\phi) \approx$$
$$\int_{-1}^{1} (1 - \frac{|\vec{p}|q^2 cos\phi}{r^2 kT} + \ldots) cos\phi d(cos\phi) \approx \frac{2}{3} \frac{|\vec{p}|q^2 cos\phi}{r^2 kT}$$

$$U \propto \frac{1}{r^4}$$

R-dependence of intermolecular forces (2)

Dipole – induced dipole:

$$U = -\vec{p} \cdot \vec{E}$$

$$\vec{p} \propto \vec{E}$$

$$U \propto \frac{1}{r^6}$$

No classical description for induced-dipole induced-dipole

Alpha -> inf limit

skin depth, δ. $\omega_{\rm pl}$ characterizes the frequency above which the conductivity goes t
δ measures the distance that electromagnetic fields penetrate the metal. Both $\omega_{\rm pl}$
depend on the fine structure constant, α, and vanish as $\alpha \to 0$. In the Drude mode

$$\begin{aligned} \omega_{\rm pl}^2 &= \frac{4\pi e^2 n}{m} \\ \delta^{-2} &= \frac{2\pi\omega|\sigma|}{c^2} \text{ where } \sigma = \frac{ne^2}{m(\gamma_0 - i\omega)} \end{aligned}$$

where n is the total number of conduction electrons per unit volume, m is their ef
mass, and γ_0 is the damping parameter for the Drude oscillators. Typically the fre
cies of interest are much greater than γ_0, so $\delta \approx c/\sqrt{2}\omega_{\rm pl}$.

The frequencies that dominate the Casimir force are of order c/d[12]. So the p
conductor approximation is adequate if $c/d \ll \omega_{\rm pl}$, or

$$\alpha \gg \frac{mc}{4\pi\hbar n d^2}.$$

Typical Casimir force measurements are made at separations of order 0.5 micron
a good conductor like copper, eq. (5) requires α to be greater than about 10^{-5}, wh
amply satisfied by the physical value $\alpha \approx 1/137$. Thus the standard Casimir resu

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