Lifecycle Investment Model Specification

Random Walk Interest Rate Case ($\phi = 1$)

FINC450

1 Data Generating Process

This document describes the lifecycle investment model with **random walk interest rates** ($\phi = 1$). Under this specification, rates follow a pure random walk with no mean reversion.

1.1 Interest Rate Process (Random Walk)

The short-term interest rate evolves as:

$$r_{t+1} = r_t + \sigma_r \varepsilon^r_{t+1}, \quad \varepsilon^r_{t+1} \sim N(0,1) \tag{1}$$

where:

- σ_r = interest rate volatility
- No drift term (expected change is zero)
- No mean reversion ($\phi = 1$ implies persistence = 1)
- Floor at r_{floor} to prevent negative rates

Key implication: With $\phi = 1$, the effective Macaulay duration equals maturity, and bond pricing simplifies to standard discounting. Modified duration is then maturity divided by $(1 + r)$.

1.2 Stock Returns

Stock returns from t to $t + 1$:

$$R^s_{t+1} = r_t + \mu_s + \sigma_s \varepsilon^s_{t+1} \tag{2}$$

where:

- μ_s = equity risk premium (excess return over short rate)
- σ_s = stock volatility
- $\text{Corr}(\varepsilon^r_{t+1}, \varepsilon^s_{t+1}) = \rho$ (typically small or zero)

1.3 Bond Returns (Modified Duration Approximation)

For a bond with modified duration D^{mod}, the return from t to $t+1$ is:

$$R^b_{t+1} \approx r_t + \mu_b - D^{\text{mod}} \cdot \Delta r_{t+1} \tag{3}$$

where:

- D^{mod} = modified duration (not Macaulay duration)
- μ_b = bond risk premium (term spread over short rate)
- $\Delta r_{t+1} = r_{t+1} - r_t = \sigma_r \varepsilon^r_{t+1}$

Substituting the rate change:

$$R^b_{t+1} \approx r_t + \mu_b - D^{\text{mod}} \cdot \sigma_r \varepsilon^r_{t+1} \tag{4}$$

Note: Modified duration $D^{\text{mod}} = D^{\text{Mac}}/(1+r)$ gives the percentage price change per unit change in yield. Using modified duration directly in the return equation is correct; using Macaulay duration would overstate interest rate sensitivity.

Intuition: When rates rise ($\varepsilon^r_{t+1} > 0$), bond prices fall, generating a negative capital gain proportional to modified duration.

Bond risk premium. The bond excess return is parameterized via the Bond Sharpe ratio SR_b:

$$\mu_b = \text{SR}_b \cdot D^{\text{mod}} \cdot \sigma_r \tag{5}$$

The bond return volatility is $\sigma_b = D^{\text{mod}} \cdot \sigma_r$, so $\mu_b = \text{SR}_b \cdot \sigma_b$. The default is $\text{SR}_b = 0$ (expectations hypothesis: no term premium).

2 Portfolio Allocation Rule

2.1 Human Capital

Human Capital (HC) is the present value of future labor earnings, discounted at the current rate:

$$HC_t = \sum_{s=t}^{T_R - 1} \frac{Y_s}{(1+r_t)^{s-t}} \tag{6}$$

where:

- Y_s = earnings at age s
- T_R = retirement age
- r_t = current interest rate

After retirement ($t \geq T_R$): $HC_t = 0$.

2.2 Present Value of Expenses (Liability)

The present value of future subsistence expenses:

$$L_t = PV_t(\text{expenses}) = \sum_{s=t}^{T-1} \frac{E_s}{(1+r_t)^{s-t}} \tag{7}$$

where:

- E_s = subsistence expenses at age s
- T = end age (death)
- r_t = current interest rate

2.3 Total Wealth (Net Worth)

Total wealth is defined as assets minus liabilities:

$$TW_t = FW_t + HC_t - L_t \tag{8}$$

where:

- FW_t = Financial Wealth (liquid portfolio)
- HC_t = Human Capital (implicit bond-like asset)
- L_t = Present value of future expenses (liability)

This is equivalent to **net worth**—the amount available for discretionary consumption beyond subsistence.

2.4 Target Total Allocation (Mean-Variance Optimization)

The optimal portfolio weights from mean-variance optimization:

$$\mathbf{w}^* = \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{\mu} \tag{9}$$

Explicit 2-asset system. Cash earns the risk-free rate, so we solve for stock and bond weights over cash. The excess return vector is:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_s \\ \mu_b \end{pmatrix} \tag{10}$$

The covariance matrix of risky asset returns uses $\sigma_b = D^{\text{mod}} \cdot \sigma_r$ for bond volatility:

$$\Sigma = \begin{pmatrix} \sigma_s^2 & -D^{\text{mod}}\sigma_s\sigma_r\rho \\ -D^{\text{mod}}\sigma_s\sigma_r\rho & (D^{\text{mod}}\sigma_r)^2 \end{pmatrix} \tag{11}$$

The off-diagonal term reflects $\text{Cov}(R^s_{t+1}, R^b_{t+1}) = -D^{\text{mod}}\sigma_s\sigma_r\rho$: the negative sign arises because rising rates ($\varepsilon^r_{t+1} > 0$) hurt bond returns.

Solving $\mathbf{w}^* = \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{\mu}$ gives the optimal stock and bond weights (w_s^*, w_b^*), with cash as the residual:

$$w_c^* = 1 - w_s^* - w_b^* \tag{12}$$

These MV weights are used *unconstrained* as target allocations. Constraints are applied only to the final dollar-level portfolio holdings (see Section 2.9).

2.5 Modified Duration of Human Capital and Expenses

HC and expenses have interest rate sensitivity (duration). We use **modified duration** throughout for consistency with the bond return approximation.

Modified Duration of Human Capital:

$$D_t^{HC,\text{mod}} = \frac{1}{1+r_t} \cdot \frac{\sum_{s=t}^{T_R-1} (s-t) \cdot \frac{Y_s}{(1+r_t)^{s-t}}}{HC_t} \tag{13}$$

Modified Duration of Expenses:

$$D_t^{L,\text{mod}} = \frac{1}{1+r_t} \cdot \frac{\sum_{s=t}^{T-1} (s-t) \cdot \frac{E_s}{(1+r_t)^{s-t}}}{L_t} \tag{14}$$

Note: The summation term is the Macaulay duration (weighted average time to cash flows). Dividing by $(1+r_t)$ converts to modified duration, which measures the percentage change in present value for a unit change in yield.

2.6 Human Capital Decomposition

Assuming HC is β_{HC} stock-like (correlated with market) and the rest is bond-like:

$$HC^{\text{stock}} = \beta_{HC} \cdot HC_t \tag{15}$$

$$HC^{\text{bond}} = (1-\beta_{HC}) \cdot HC_t \cdot \frac{D_t^{HC,\text{mod}}}{D^{\text{mod}}} \tag{16}$$

$$HC^{\text{cash}} = (1-\beta_{HC}) \cdot HC_t \cdot \left(1 - \frac{D_t^{HC,\text{mod}}}{D^{\text{mod}}}\right) \tag{17}$$

where D^{mod} = modified duration of bonds in the financial portfolio.

2.7 Expense Liability Decomposition

The expense liability is decomposed by modified duration:

$$L^{\text{bond}} = L_t \cdot \frac{D_t^{L,\text{mod}}}{D^{\text{mod}}} \tag{18}$$

$$L^{\text{cash}} = L_t \cdot \left(1 - \frac{D_t^{L,\text{mod}}}{D^{\text{mod}}}\right) \tag{19}$$

2.8 Target Financial Holdings (Surplus Optimization)

We define **surplus** as the positive part of net worth:

$$S_t = \max(0,\ TW_t) = \max(0,\ FW_t + HC_t - L_t) \tag{20}$$

The target allocation is applied to surplus rather than raw net worth. This prevents the optimizer from taking leveraged bets when the agent is “underwater” ($TW < 0$).

$$\text{target\_fin\_stock} = w^*_{\text{stock}} \cdot S_t - HC^{\text{stock}} \tag{21}$$

$$\text{target\_fin\_bond} = w^*_{\text{bond}} \cdot S_t - HC^{\text{bond}} + L^{\text{bond}} \tag{22}$$

$$\text{target\_fin\_cash} = w^*_{\text{cash}} \cdot S_t - HC^{\text{cash}} + L^{\text{cash}} \tag{23}$$

Derivation: The right-hand side represents the target dollar position in each asset class:

- $w^*_i \cdot S_t$: target allocation of surplus to asset class i
- $-HC^{\text{stock/bond/cash}}$: subtract implicit exposure from human capital
- $+L^{\text{bond/cash}}$: add positions needed to hedge the expense liability

When $TW_t < 0$ (net worth is negative): surplus $S_t = 0$, so the stock target is $-HC^{\text{stock}} \leq 0$. After clipping negatives to zero (Section 2.9), the financial portfolio holds zero stocks and allocates entirely to bonds and cash to hedge the expense liability. This is the conservative "underwater" behavior: the agent cannot take equity risk until net worth recovers.

2.9 Weight Truncation (No Leverage Constraint)

The target financial holdings can be negative (e.g., when human capital already provides more stock exposure than the optimum requires). We impose a **no-leverage, no-short-selling constraint**:

Step 1: Clip negative dollar targets to zero

$$\hat{d}_i = \max(0,\ \text{target\_fin}_i), \quad i \in \{\text{stock}, \text{bond}, \text{cash}\} \tag{24}$$

Step 2: Normalize to portfolio weights

$$w_i^{\text{final}} = \frac{\hat{d}_i}{\sum_j \hat{d}_j} \tag{25}$$

There is no separate $\min(1, \cdot)$ step—the upper bound is enforced automatically by normalization. The clipping operates on *dollar amounts* (not weights), preserving the relative sizing of positive positions.

Implications:

- Early career: Stock weight is capped at 100%, forgoing the theoretically optimal leveraged position
- The constraint binds most when HC/FW is high (young workers with little savings)
- As financial wealth grows and human capital depletes, the constraint relaxes
- This is a practical constraint reflecting real-world borrowing limitations

3 Consumption Rule

3.1 Consumption Function

Consumption depends on net worth:

$$C_t = \underbrace{E_t}_{\text{subsistence}} + \underbrace{c_t \cdot \max(0, TW_t)}_{\text{variable}} \tag{26}$$

The consumption rate c_t is computed dynamically at each time step from the current portfolio and market state.

Expected portfolio return (arithmetic mean, conditional on time-t information):

$$\mathbb{E}_t[R^p_{t+1}] = w_s(r_t + \mu_s) + w_b(r_t + \mu_b) + w_c \cdot r_t \tag{27}$$

Portfolio variance (using realized post-constraint weights w_s, w_b, w_c):

$$\mathrm{Var}_t(R^p_{t+1}) = w_s^2\sigma_s^2 + w_b^2(D^{\mathrm{mod}}\sigma_r)^2 + 2w_s w_b(-D^{\mathrm{mod}}\sigma_s\sigma_r\rho) \tag{28}$$

Consumption rate (certainty-equivalent return):

$$c_t = \mathbb{E}_t[R^p_{t+1}] - \tfrac{1}{2}\mathrm{Var}_t(R^p_{t+1}) + \delta \tag{29}$$

where δ is an optional consumption boost parameter (default 0).

Key features:

- **Jensen's correction** ($-\frac{1}{2}\mathrm{Var}_t$): converts the arithmetic mean return to the median (geometric) return, which governs long-run wealth growth. Without this correction, the consumption rate would be too high and wealth would deplete on average.
- **Dynamic rate**: uses the current short rate r_t (not the long-run mean $\bar{r}$), so consumption responds to the prevailing interest rate environment.
- **Realized weights**: uses the post-constraint portfolio weights w_s, w_b, w_c, not the unconstrained MV targets. This ensures the variance correction matches the actual portfolio risk.

3.2 Constraints

Working years ($t < T_R$):

- Cannot consume more than earnings: $C_t \leq Y_t$

Retirement ($t \geq T_R$):

- Cannot consume more than financial wealth: $C_t \leq FW_t$
- If $FW_t \leq 0$: default (set $C_t = 0$)

3.3 Wealth Evolution

$$FW_{t+1} = FW_t \cdot (1 + R^p_{t+1}) + S_t \tag{30}$$

where:

- $R^p_{t+1} = w_s R^s_{t+1} + w_b R^b_{t+1} + w_c\, r_t$ (portfolio return)
- $S_t = Y_t - C_t$ (savings; can be negative in retirement)

4 Parameter Summary

Parameter	Symbol	Description
Economic Parameters		
r_0		Initial interest rate
σ_r		Interest rate volatility
ϕ		Rate persistence (= 1 for random walk)
μ_s		Equity risk premium
σ_s		Stock volatility
SR_b		Bond Sharpe ratio (default 0; $\mu_b = \mathrm{SR}_b \cdot D^{\mathrm{mod}} \cdot \sigma_r$)
D^{mod}		Bond modified duration
ρ		Stock-rate correlation
Lifecycle Parameters		
T_R		Retirement age
T		End age
Y_t		Earnings profile
E_t		Expense profile
β_{HC}		Stock beta of human capital
γ		Risk aversion
δ		Consumption boost (default 0)
Note: consumption rate c_t is derived dynamically (Eq. 29)		