# Lifecycle Investment Model Specification

Random Walk Interest Rate Case ($\phi = 1$)

FINC450

# 1 Data Generating Process

This document describes the lifecycle investment model with **random walk interest rates** ($\phi = 1$). Under this specification, rates follow a pure random walk with no mean reversion.

## 1.1 Interest Rate Process (Random Walk)

The short-term interest rate evolves as:

$$r_{t+1} = r_t + \sigma_r \varepsilon_t^r, \quad \varepsilon_t^r \sim N(0,1) \tag{1}$$

where:

- $\sigma_r$ = interest rate volatility
- No drift term (expected change is zero)
- No mean reversion ($\phi = 1$ implies persistence = 1)
- Floor at $r_{\text{floor}}$ to prevent negative rates

**Key implication:** With $\phi = 1$, the effective Macaulay duration equals maturity, and bond pricing simplifies to standard discounting. Modified duration is then maturity divided by $(1 + r)$.

## 1.2 Stock Returns

Stock returns follow:

$$R_t^s = r_t + \mu_s + \sigma_s \varepsilon_t^s \tag{2}$$

where:

- $\mu_s$ = equity risk premium (excess return over short rate)
- $\sigma_s$ = stock volatility
- $\text{Corr}(\varepsilon_t^r, \varepsilon_t^s) = \rho$ (typically small or zero)

### 1.3 Bond Returns (Modified Duration Approximation)

For a bond with modified duration $D^{\text{mod}}$, the return approximation is:

$$R_t^b \approx r_t + \mu_b - D^{\text{mod}} \cdot \Delta r_{t+1} \tag{3}$$

where:

- $D^{\text{mod}}$ = modified duration (not Macaulay duration)
- $\mu_b$ = bond risk premium (term spread over short rate)
- $\Delta r_{t+1} = r_{t+1} - r_t = \sigma_r \varepsilon_t^r$

Substituting the rate change:

$$R_t^b \approx r_t + \mu_b - D^{\text{mod}} \cdot \sigma_r \varepsilon_t^r \tag{4}$$

**Note:** Modified duration $D^{\text{mod}} = D^{\text{Mac}}/(1+r)$ gives the percentage price change per unit change in yield. Using modified duration directly in the return equation is correct; using Macaulay duration would overstate interest rate sensitivity.

**Intuition:** When rates rise ($\varepsilon^r > 0$), bond prices fall, generating a negative capital gain proportional to modified duration.

## 2 Portfolio Allocation Rule

### 2.1 Human Capital

Human Capital (HC) is the present value of future labor earnings, discounted at the current rate:

$$HC_t = \sum_{s=t}^{T_R-1} \frac{Y_s}{(1+r_t)^{s-t}} \tag{5}$$

where:

- $Y_s$ = earnings at age $s$
- $T_R$ = retirement age
- $r_t$ = current interest rate

After retirement ($t \geq T_R$): $HC_t = 0$.

### 2.2 Present Value of Expenses (Liability)

The present value of future subsistence expenses:

$$L_t = PV_t(\text{expenses}) = \sum_{s=t}^{T-1} \frac{E_s}{(1+r_t)^{s-t}} \tag{6}$$

where:

- $E_s$ = subsistence expenses at age $s$
- $T$ = end age (death)
- $r_t$ = current interest rate

### 2.3 Total Wealth (Net Worth)

Total wealth is defined as assets minus liabilities:

$$TW_t = FW_t + HC_t - L_t \tag{7}$$

where:

- $FW_t$ = Financial Wealth (liquid portfolio)
- $HC_t$ = Human Capital (implicit bond-like asset)
- $L_t$ = Present value of future expenses (liability)

This is equivalent to **net worth**—the amount available for discretionary consumption beyond subsistence.

### 2.4 Target Total Allocation (Mean-Variance Optimization)

The optimal portfolio weights from mean-variance optimization:

$$\mathbf{w}^* = \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{\mu} \tag{8}$$

where:

- $\gamma$ = risk aversion coefficient
- $\Sigma$ = covariance matrix of asset returns
- $\boldsymbol{\mu}$ = vector of excess returns

This yields target weights $(w_s^*, w_b^*, w_c^*)$ for stocks, bonds, and cash.

### 2.5 Modified Duration of Human Capital and Expenses

HC and expenses have interest rate sensitivity (duration). We use **modified duration** throughout for consistency with the bond return approximation.

**Modified Duration of Human Capital:**

$$D_t^{HC,\text{mod}} = \frac{1}{1+r_t} \cdot \frac{\sum_{s=t}^{T_R-1}(s-t)\cdot\frac{Y_s}{(1+r_t)^{s-t}}}{HC_t} \tag{9}$$

**Modified Duration of Expenses:**

$$D_t^{L,\text{mod}} = \frac{1}{1+r_t} \cdot \frac{\sum_{s=t}^{T-1}(s-t)\cdot\frac{E_s}{(1+r_t)^{s-t}}}{L_t} \tag{10}$$

**Note:** The summation term is the Macaulay duration (weighted average time to cash flows). Dividing by $(1+r_t)$ converts to modified duration, which measures the percentage change in present value for a unit change in yield.

## 2.6 Human Capital Decomposition

Assuming HC is $\beta_{HC}$ stock-like (correlated with market) and the rest is bond-like:

$$HC^{\text{stock}} = \beta_{HC} \cdot HC_t \tag{11}$$

$$HC^{\text{bond}} = (1 - \beta_{HC}) \cdot HC_t \cdot \frac{D_t^{HC,\text{mod}}}{D^{\text{mod}}} \tag{12}$$

$$HC^{\text{cash}} = (1 - \beta_{HC}) \cdot HC_t \cdot \left(1 - \frac{D_t^{HC,\text{mod}}}{D^{\text{mod}}}\right) \tag{13}$$

where $D^{\text{mod}}$ = modified duration of bonds in the financial portfolio.

## 2.7 Expense Liability Decomposition

The expense liability is decomposed by modified duration:

$$L^{\text{bond}} = L_t \cdot \frac{D_t^{L,\text{mod}}}{D^{\text{mod}}} \tag{14}$$

$$L^{\text{cash}} = L_t \cdot \left(1 - \frac{D_t^{L,\text{mod}}}{D^{\text{mod}}}\right) \tag{15}$$

## 2.8 Target Financial Holdings (LDI Adjustment)

The target allocation is applied to total wealth (net worth), $TW = FW + HC - L$. We then solve for the financial portfolio weights that achieve this target, accounting for implicit positions in HC and the need to hedge expenses.

$$w_{\text{stock}}^{\text{fin}} \cdot FW = w_{\text{stock}}^{*} \cdot TW - HC^{\text{stock}} \tag{16}$$

$$w_{\text{bond}}^{\text{fin}} \cdot FW = w_{\text{bond}}^{*} \cdot TW - HC^{\text{bond}} + L^{\text{bond}} \tag{17}$$

$$w_{\text{cash}}^{\text{fin}} \cdot FW = w_{\text{cash}}^{*} \cdot TW - HC^{\text{cash}} + L^{\text{cash}} \tag{18}$$

**Derivation:** The right-hand side represents the target dollar position in each asset class:

- $w_i^* \cdot TW$: target allocation of net worth to asset class $i$
- $-HC^{\text{stock/bond/cash}}$: subtract implicit exposure from human capital
- $+L^{\text{bond/cash}}$: add positions needed to hedge the expense liability

The left-hand side is the financial portfolio holding that achieves this net exposure.

## 2.9 Weight Truncation (No Leverage Constraint)

The LDI-adjusted weights can produce values outside $[0, 1]$, implying leverage or short positions. For example, when human capital is large relative to financial wealth (early career), the optimal stock weight in the financial portfolio may exceed 100%.

We impose a **no-leverage, no-short-selling constraint** by truncating weights:

**Step 1: Compute raw weights**

$$\tilde{w}_i = \frac{w_i^{\text{fin}} \cdot FW}{FW} = w_i^{\text{fin}}, \quad i \in \{\text{stock}, \text{bond}, \text{cash}\} \tag{19}$$

**Step 2: Truncate to** $[0, 1]$

$$\hat{w}_i = \max(0, \min(1, \tilde{w}_i)) \tag{20}$$

**Step 3: Renormalize to sum to 1**

$$w_i^{\text{final}} = \frac{\hat{w}_i}{\sum_j \hat{w}_j} \tag{21}$$

**Implications:**

- Early career: Stock weight is capped at 100%, forgoing the theoretically optimal leveraged position
- The constraint binds most when $HC/FW$ is high (young workers with little savings)
- As financial wealth grows and human capital depletes, the constraint relaxes
- This is a practical constraint reflecting real-world borrowing limitations

# 3 Consumption Rule

## 3.1 Consumption Function

Consumption depends on total wealth (net worth):

$$C_t = \underbrace{E_t}_{\text{subsistence}} + \underbrace{c \cdot \max(0, TW_t)}_{\text{variable}} \tag{22}$$

where:

- $E_t$ = subsistence expenses (baseline/floor)
- $c$ = consumption rate ($\approx$ expected portfolio return)
- Variable consumption is positive only if $TW > 0$

## 3.2 Constraints

**Working years ($t < T_R$):**

- Cannot consume more than earnings: $C_t \leq Y_t$

**Retirement ($t \geq T_R$):**

- Cannot consume more than financial wealth: $C_t \leq FW_t$
- If $FW_t \leq 0$: default (set $C_t = 0$)

### 3.3 Wealth Evolution

$$FW_{t+1} = FW_t \cdot (1 + R_t^p) + S_t \tag{23}$$

where:

- $R_t^p = w^s R_t^s + w^b R_t^b + w^c r_t$ (portfolio return)
- $S_t = Y_t - C_t$ (savings; can be negative in retirement)

# 4 Parameter Summary

| **Parameter** | **Symbol** | **Description** |
|---|---|---|
| *Economic Parameters* | | |
| $r_0$ | | Initial interest rate |
| $\sigma_r$ | | Interest rate volatility |
| $\phi$ | | Rate persistence (= 1 for random walk) |
| $\mu_s$ | | Equity risk premium |
| $\sigma_s$ | | Stock volatility |
| $\mu_b$ | | Bond risk premium |
| $D^{\text{mod}}$ | | Bond modified duration |
| $\rho$ | | Stock-rate correlation |
| *Lifecycle Parameters* | | |
| $T_R$ | | Retirement age |
| $T$ | | End age |
| $Y_t$ | | Earnings profile |
| $E_t$ | | Expense profile |
| $\beta_{HC}$ | | Stock beta of human capital |
| $\gamma$ | | Risk aversion |
| $c$ | | Consumption rate |