Lifecycle Investment Model Specification

Random Walk Interest Rate Case ($\phi = 1$)

FINC450

1 Data Generating Process

This document describes the lifecycle investment model with **random walk interest rates** ($\phi = 1$). Under this specification, rates follow a pure random walk with no mean reversion.

Notation. The model uses continuous compounding throughout. Zero-coupon bond prices, present values, and durations are computed using exponential discounting $e^{-\tau r}$ rather than discrete discounting $(1+r)^{-\tau}$.

1.1 Interest Rate Process (Random Walk)

The short-term interest rate evolves as:

$$r_{t+1} = r_t + \sigma_r \varepsilon^r_{t+1}, \quad \varepsilon^r_{t+1} \sim N(0,1) \tag{1}$$

where:

- σ_r = interest rate volatility
- No drift term (expected change is zero)
- No mean reversion ($\phi = 1$ implies persistence = 1)
- Rates are unconstrained (no floor or ceiling)

General AR(1) case. The code supports mean-reverting rates $r_{t+1} = \bar{r} + \phi(r_t - \bar{r}) + \sigma_r \varepsilon^r_{t+1}$ with $0 < \phi < 1$, but the default is $\phi = 1$.

Key implication: With $\phi = 1$, the effective duration of a τ-year zero-coupon bond equals τ (maturity), and the zero-coupon price simplifies to $P(\tau) = e^{-\tau r}$.

1.2 Stock Returns

Stock returns from t to $t+1$:

$$R^s_{t+1} = r_t + \mu_s + \sigma_s \varepsilon^s_{t+1} \tag{2}$$

where:

- μ_s = equity risk premium (excess return over short rate)
- σ_s = stock volatility
- $\text{Corr}(\varepsilon^r_{t+1}, \varepsilon^s_{t+1}) = \rho$ (default 0)

1.3 Bond Returns (Duration Approximation)

For a bond portfolio with duration D, the return from t to $t+1$ is:

$$R_{t+1}^b \approx r_t + \mu_b - D \cdot \Delta r_{t+1} \tag{3}$$

where:

- D = duration of the bond portfolio (a fixed parameter)
- μ_b = bond risk premium (term spread over short rate)
- $\Delta r_{t+1} = r_{t+1} - r_t = \sigma_r \varepsilon_{t+1}^r$

Substituting the rate change:

$$R_{t+1}^b \approx r_t + \mu_b - D \cdot \sigma_r \varepsilon_{t+1}^r \tag{4}$$

Intuition: When rates rise ($\varepsilon_{t+1}^r > 0$), bond prices fall, generating a negative capital gain proportional to duration.

Bond risk premium. The bond excess return is parameterized via the Bond Sharpe ratio SR_b:

$$\mu_b = \mathrm{SR}_b \cdot D \cdot \sigma_r \tag{5}$$

The bond return volatility is $\sigma_b = D \cdot \sigma_r$, so $\mu_b = \mathrm{SR}_b \cdot \sigma_b$. The default is $\mathrm{SR}_b = 0$ (expectations hypothesis: no term premium).

2 Earnings and Expense Profiles

2.1 Earnings Profile

Earnings follow a hump-shaped profile over the working life ($t_0 \le t < T_R$):

$$Y_t = \begin{cases} Y_0(1+g_Y)^{t-t_0} & \text{if } t \le t_{\text{peak}} \\ Y_{\text{peak}}(1-d_Y)^{t-t_{\text{peak}}} & \text{if } t > t_{\text{peak}} \end{cases} \tag{6}$$

where:

- Y_0 = initial annual earnings
- g_Y = real earnings growth rate (default 0, flat)
- t_{peak} = age at peak earnings (default = T_R, so earnings are flat)
- d_Y = decline rate after peak (default 0)

After retirement ($t \ge T_R$): $Y_t = 0$.

2.2 Risky Human Capital (Wage Shocks)

When $\beta_{HC} > 0$, earnings are subject to permanent shocks correlated with stock returns. The log wage level follows:

$$\ln W_t = \ln W_{t-1} + \beta_{HC} \cdot \sigma_s \cdot \varepsilon_{t-1}^s \tag{7}$$

where $W_0 = 1$ (no initial shock). Actual earnings are:

$$\tilde{Y}_t = Y_t \cdot W_t = Y_t \cdot \exp\left(\sum_{k=0}^{t-1} \beta_{HC} \cdot \sigma_s \cdot \varepsilon_k^s \right) \tag{8}$$

This captures the idea that human capital has systematic risk: when stock markets do well, wages tend to rise, and vice versa. The shock is permanent (a random walk in log wages), creating path dependence in lifetime earnings.

2.3 Expense Profile

Subsistence expenses are:

$$E_t = \begin{cases} E_0(1+g_E)^{t-t_0} & \text{if } t < T_R \\ E_R & \text{if } t \geq T_R \end{cases} \tag{9}$$

where E_0 = base working expenses, g_E = real expense growth (default 0), and E_R = fixed retirement expenses.

3 Present Values and Durations

3.1 Zero-Coupon Bond Pricing

Under the term structure model with $\phi = 1$, the price of a τ-year zero-coupon bond is:

$$P(\tau, r_t) = e^{-\tau r_t} \tag{10}$$

This is standard continuous-time discounting at the current short rate.

3.2 Human Capital

Human Capital (HC) is the present value of future labor earnings, discounted at the CAPM-adjusted rate:

$$HC_t = \sum_{s=t}^{T_R-1} \tilde{Y}_s \cdot e^{-(s-t)(r_t + \beta_{HC}\mu_s)} \tag{11}$$

where $\beta_{HC}\mu_s$ is the CAPM spread: human capital with stock-market exposure commands a higher discount rate, reducing its present value.

When $\beta_{HC} = 0$ (riskless HC), this simplifies to $HC_t = \sum_{s=t}^{T_R-1} Y_s \, e^{-(s-t)r_t}$.

After retirement ($t \geq T_R$): $HC_t = 0$.

3.3 Present Value of Expenses (Liability)

The present value of future subsistence expenses:

$$L_t = \sum_{s=t}^{T-1} E_s \cdot e^{-(s-t)r_t} \tag{12}$$

3.4 Duration of HC and Expenses

Duration measures the sensitivity of present value to changes in the short rate. With $\phi = 1$, the duration of a cashflow stream is the present-value-weighted average maturity:

$$D_t^{HC} = \frac{\sum_{s=t}^{T_R-1} (s-t) \cdot \tilde{Y}_s \cdot e^{-(s-t)(r_t + \beta_{HC}\mu_s)}}{HC_t} \tag{13}$$

$$D_t^{L} = \frac{\sum_{s=t}^{T-1} (s-t) \cdot E_s \cdot e^{-(s-t)r_t}}{L_t} \tag{14}$$

Duration cap. An optional parameter $D_{\max}$ caps computed durations: $D \leftarrow \min(D, D_{\max})$. This prevents bond-equivalent fractions from exceeding 1 when HC or expense duration exceeds bond duration.

4 Portfolio Allocation Rule

4.1 Net Worth

Net worth is defined as assets minus liabilities:

$$NW_t = FW_t + HC_t - L_t \tag{15}$$

where:

- FW_t = Financial Wealth (liquid portfolio)
- HC_t = Human Capital (implicit asset)
- L_t = Present value of future expenses (liability)

This is the amount available for discretionary consumption beyond subsistence.

4.2 Target Total Allocation (Mean-Variance Optimization)

The optimal portfolio weights from mean-variance optimization:

$$\mathbf{w}^* = \frac{1}{\gamma} \Sigma^{-1} \boldsymbol{\mu} \tag{16}$$

Explicit 2-asset system. Cash earns the risk-free rate, so we solve for stock and bond weights over cash. The excess return vector is:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_s \\ \mu_b \end{pmatrix} \tag{17}$$

The covariance matrix of risky asset returns uses $\sigma_b = D \cdot \sigma_r$ for bond volatility:

$$\Sigma = \begin{pmatrix} \sigma_s^2 & -D\,\sigma_s\,\sigma_r\,\rho \\ -D\,\sigma_s\,\sigma_r\,\rho & (D\,\sigma_r)^2 \end{pmatrix} \tag{18}$$

The off-diagonal term reflects $\mathrm{Cov}(R^s, R^b) = -D\,\sigma_s\,\sigma_r\,\rho$: the negative sign arises because rising rates hurt bond returns.

Solving $\mathbf{w}^* = \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{\mu}$ gives the optimal stock and bond weights (w_s^*, w_b^*), with cash as the residual:

$$w_c^* = 1 - w_s^* - w_b^* \tag{19}$$

These MV weights are used *unconstrained* as target allocations. Constraints are applied only to the final dollar-level portfolio holdings (see Section 4.6).

4.3 Human Capital Decomposition

Assuming HC has stock exposure β_{HC} and the rest is bond-like:

$$HC^{\text{stock}} = \beta_{HC} \cdot HC_t \tag{20}$$

$$HC^{\text{bond}} = (1 - \beta_{HC}) \cdot HC_t \cdot \frac{D_t^{HC}}{D} \tag{21}$$

$$HC^{\text{cash}} = (1 - \beta_{HC}) \cdot HC_t \cdot \left(1 - \frac{D_t^{HC}}{D}\right) \tag{22}$$

where D = duration of bonds in the financial portfolio. The ratio D_t^{HC}/D determines how much of the non-stock HC is "bond-equivalent" vs "cash-equivalent" based on relative durations.

4.4 Expense Liability Decomposition

The expense liability is decomposed by duration matching:

$$L^{\text{bond}} = L_t \cdot \frac{D_t^L}{D} \tag{23}$$

$$L^{\text{cash}} = L_t \cdot \left(1 - \frac{D_t^L}{D}\right) \tag{24}$$

4.5 Target Financial Holdings (Surplus Optimization)

We define **surplus** as the positive part of net worth:

$$S_t = \max(0,\ NW_t) = \max(0,\ FW_t + HC_t - L_t) \tag{25}$$

The target allocation is applied to surplus rather than raw net worth. This prevents the optimizer from taking leveraged bets when the agent is "underwater" ($NW < 0$).

$$\text{target\_fin\_stock} = w_s^* \cdot S_t - HC^{\text{stock}} \tag{26}$$

$$\text{target\_fin\_bond} = w_b^* \cdot S_t - HC^{\text{bond}} + L^{\text{bond}} \tag{27}$$

$$\text{target\_fin\_cash} = w_c^* \cdot S_t - HC^{\text{cash}} + L^{\text{cash}} \tag{28}$$

Derivation: The right-hand side represents the target dollar position in each asset class:

- $w_i^* \cdot S_t$: target allocation of surplus to asset class i
- $-HC^{\text{stock/bond/cash}}$: subtract implicit exposure from human capital
- $+L^{\text{bond/cash}}$: add positions needed to hedge the expense liability

When $NW_t < 0$ (net worth is negative): surplus $S_t = 0$, so the stock target is $-HC^{\text{stock}} \leq 0$. After clipping negatives to zero (Section 4.6), the financial portfolio holds zero stocks and allocates entirely to bonds and cash to hedge the expense liability. This is the conservative "underwater" behavior: the agent cannot take equity risk until net worth recovers.

4.6 Weight Truncation (Leverage Constraint)

The target financial holdings can be negative (e.g., when human capital already provides more stock exposure than the optimum requires). We impose a **leverage cap** governed by the parameter $\lambda_{\max}$ (default 1.0):

Step 1: Clip negative risky-asset targets to zero (no short selling of stocks or bonds):

$$\hat{d}_{\text{stock}} = \max(0,\ \text{target\_fin\_stock}) \tag{29}$$

$$\hat{d}_{\text{bond}} = \max(0,\ \text{target\_fin\_bond}) \tag{30}$$

Step 2: Cap total long exposure

$$\text{If } \hat{d}_{\text{stock}} + \hat{d}_{\text{bond}} > \lambda_{\max} \cdot FW_t, \quad \text{scale both down proportionally} \tag{31}$$

Step 3: Cash is the residual

$$\hat{d}_{\text{cash}} = FW_t - \hat{d}_{\text{stock}} - \hat{d}_{\text{bond}} \tag{32}$$

Cash can be negative when $\lambda_{\max} > 1$, representing borrowing at the short rate.

Portfolio weights:

$$w_i^{\text{final}} = \hat{d}_i / FW_t \tag{33}$$

Key values of $\lambda_{\max}$:

- $\lambda_{\max} = 1$: No borrowing. Cash is always non-negative. This is the default.
- $\lambda_{\max} = 2$: Can borrow up to $1 \times FW$ (total risky exposure up to $2 \times FW$).
- $\lambda_{\max} = \infty$: Unconstrained (no leverage limit).

5 Consumption Rule

5.1 Consumption Function

Consumption depends on net worth:

$$C_t = \underbrace{E_t}_{\text{subsistence}} + \underbrace{c_t \cdot \max(0, NW_t)}_{\text{variable}} \tag{34}$$

The consumption rate c_t is computed dynamically at each time step from the current portfolio and market state.

Expected portfolio return (arithmetic mean, conditional on time-t information):

$$\mathbb{E}_t[R^p_{t+1}] = w_s(r_t + \mu_s) + w_b(r_t + \mu_b) + w_c \cdot r_t \tag{35}$$

Portfolio variance (using realized post-constraint weights w_s, w_b, w_c):

$$\text{Var}_t(R^p_{t+1}) = w_s^2 \sigma_s^2 + w_b^2 (D\,\sigma_r)^2 + 2 w_s w_b (-D\,\sigma_s\,\sigma_r\,\rho) \tag{36}$$

Consumption rate (certainty-equivalent return):

$$c_t = \mathbb{E}_t[R^p_{t+1}] - \tfrac{1}{2}\text{Var}_t(R^p_{t+1}) + \delta \tag{37}$$

where δ is an optional consumption boost parameter (default 0).

Key features:

- **Jensen's correction** ($-\frac{1}{2}\text{Var}_t$): converts the arithmetic mean return to the median (geometric) return, which governs long-run wealth growth. Without this correction, the consumption rate would be too high and wealth would deplete on average.
- **Dynamic rate**: uses the current short rate r_t (not the long-run mean $\bar{r}$), so consumption responds to the prevailing interest rate environment.
- **Realized weights**: uses the post-constraint portfolio weights w_s, w_b, w_c, not the unconstrained MV targets. This ensures the variance correction matches the actual portfolio risk.

5.2 Constraints

Working years ($t < T_R$):

- Cannot consume more than earnings: $C_t \leq Y_t$

Retirement ($t \geq T_R$):

- Cannot consume more than financial wealth: $C_t \leq FW_t$
- If $FW_t \leq 0$: default (set $C_t = 0$ for all future periods)

5.3 Wealth Evolution

$$FW_{t+1} = FW_t \cdot (1 + R^p_{t+1}) + S_t \tag{38}$$

where:

- $R^p_{t+1} = w_s R^s_{t+1} + w_b R^b_{t+1} + w_c \, r_t$ (portfolio return using realized asset returns)
- $S_t = Y_t - C_t$ (savings; negative in retirement when consuming from wealth)

6 Simulation Architecture

6.1 Unified Engine

The model uses a single simulation engine for both deterministic and stochastic analysis:

- **Zero-shock path** (called "median path" in the code): Pass $\varepsilon^r = \varepsilon^s = 0$ for all t.
- **Monte Carlo**: Pass random shocks drawn from $N(0, 1)$ with correlation ρ. Produces stochastic paths with dynamic revaluation of HC and L at each step.

6.2 Jensen's Correction: Consistent Geometric Returns

Both the consumption rate and wealth evolution use the **geometric (median) return**:

$$R^p_{\text{geom}} = \mathbb{E}_t[R^p_{t+1}] - \tfrac{1}{2}\text{Var}_t(R^p_{t+1})$$

where

$$\mathbb{E}_t[R^p_{t+1}] = w_s(r + \mu_s) + w_b(r + \mu_b) + w_c \cdot r, \qquad \text{Var}_t(R^p_{t+1}) = w_s^2\sigma_s^2 + w_b^2(D\sigma_r)^2 + 2w_s w_b \text{Cov}(R^s, R^b)$$

The Jensen's correction $\frac{1}{2}\text{Var}(R^p)$ varies at each time step because portfolio weights change over the lifecycle (stock-heavy when young due to HC, bond/cash-heavy when old).

Zero-shock path ("median path"). With $\varepsilon^r = \varepsilon^s = 0$:

- The consumption rate is $c_t = R^p_{\text{geom}} + \delta$ (Eq. 37)
- Wealth evolves at the same geometric return: $FW_{t+1} = FW_t(1 + R^p_{\text{geom}}) + Y_t - C_t$
- This produces a **true median trajectory**: the path that the median Monte Carlo simulation converges to as $N \to \infty$

Monte Carlo. With random shocks:

- Each asset earns a realized return that includes shocks
- Wealth evolves at the realized (stochastic) portfolio return
- The median across paths $\approx$ the zero-shock geometric path

The correction is largest early in life when the portfolio is stock-heavy. With default parameters and ∼100% stocks: correction $= \frac{1}{2}(0.18)^2 = 1.62$ pp/yr. At mid-career (∼69% stocks): correction ≈ 0.79 pp/yr.

6.3 Time Step Sequence

At each period t, the simulation proceeds in order:

1. Observe current state: FW_t, r_t, wage multiplier W_t
2. Compute HC_t, L_t, NW_t using current rate r_t
3. Decompose HC and L into stock/bond/cash components
4. Compute target financial holdings via surplus optimization
5. Normalize to portfolio weights (w_s, w_b, w_c) with leverage constraint
6. Compute dynamic consumption rate c_t using realized weights and Jensen's correction
7. Compute consumption C_t and apply constraints
8. Evolve wealth: $FW_{t+1} = FW_t(1 + R^p_{t+1}) + Y_t - C_t$. For zero-shock paths, $R^p_{t+1} = R^p_{\text{geom}}$; for MC, R^p_{t+1} is the realized stochastic return

The order is important: portfolio weights are determined *before* consumption, so the variance correction in the consumption rate uses the actual portfolio allocation.

7 Parameter Summary

Notes:

- The consumption rate c_t is derived dynamically (Eq. 37), not set as a parameter.
- With default parameters ($g_Y = d_Y = g_E = 0$), earnings are flat at \$200K and expenses are flat at \$100K.
- The bond risk premium is derived: $\mu_b = \text{SR}_b \cdot D \cdot \sigma_r$. With default $\text{SR}_b = 0$, bonds earn no excess return.

Table 1: Economic Parameters (`EconomicParams`)

Parameter	Symbol	Default	Description
`r_bar`	$\bar{r}$	0.02	Long-run mean interest rate (= initial rate r_0)
`phi`	ϕ	1.0	Rate persistence (1.0 = random walk)
`sigma_r`	σ_r	0.003	Interest rate volatility (0.3 pp)
`mu_excess`	μ_s	0.045	Equity risk premium (4.5 pp)
`sigma_s`	σ_s	0.18	Stock return volatility
`bond_sharpe`	SR_b	0.0	Bond Sharpe ratio ($\mu_b = \mathrm{SR}_b \cdot D \cdot \sigma_r$)
`bond_duration`	D	20.0	Duration of bond portfolio
`rho`	ρ	0.0	Correlation between rate and stock shocks
`max_duration`	$D_{\max}$	None	Optional cap on computed durations

Table 2: Lifecycle Parameters (`LifecycleParams`)

Parameter	Symbol	Default	Description
`start_age`	t_0	25	Career start age
`retirement_age`	T_R	65	Retirement age
`end_age`	T	95	Planning horizon / end of life
`initial_earnings`	Y_0	200	Initial annual earnings ($K)
`base_expenses`	E_0	100	Base annual expenses ($K)
`retirement_expenses`	E_R	100	Retirement subsistence expenses ($K)
`initial_wealth`	FW_0	100	Initial financial wealth ($K)
`gamma`	γ	2.0	Risk aversion coefficient
`stock_beta_human_capital`	β_{HC}	0.0	Stock beta of human capital
`max_leverage`	$\lambda_{\max}$	1.0	Max risky-asset exposure / FW
`consumption_boost`	δ	0.0	Consumption rate boost