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THE EFFECT OF NON-NORMAL DISTRIBUTIONS ON THE CONTROL LIMITS OF X-BAR CHART

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Abstract

Shewhart control charts are often used by industry organizations for quality control. The most common diagram used is the X-bar chart, which monitors the progress of the mean of parameter over time. For the use of control charts, practitioners a priori assume that the data has a normal distribution. The normality of data can be verified by various tests, but in practice, such verification is often circumvented. If the data does not have a normal distribution, it will of course also affect the values of the control limits. This article aims to investigate the effect of selected types of distributions on the resulting control limits in X-bar charts. The most common types of continuous distributions are analyzed and random data is used. The results point to the robustness of the control limits for some types of distributions and also to varying impact on varying sub-group ranges and varying sample sizes.

Keywords: *X-bar control charts, Contol Limits, Normality, Non-normality, Distributions.*

JEL Classification: *L15, C15, C18, C46*

1 Introduction

Control charts are currently the frequently used tools for statistical process control in industrial organizations (Human, Chakraborti & Smit, 2010). They were first introduced by Walter A. Shewhart in the 20s of the last century (Hrnčiar, 2014). Since then, they have been enforced and perfected many times. At present, control charts can be considered as the basic tool of statistical process control – ie process control in such way, that the outcome of the process is within the tolerance limits (Chen & Yeh, 2010). Control charts were standardized to ISO 8258 in 1991 and are currently indispensable tools for quality management – finally control charts are one of the seven basic quality tools (Djekic, Tomasevic, Zivkovic & Radovanovic, 2013).

1.1 Types of control charts

Control charts can be of several types depending on the type of variable that is monitored and the situation of the process itself. For continuous data, control charts I-mR, Xbar-R and Xbar-S can be used. If the amount of data is low and one measuring attempt is considered as one case or one time period, it is recommended to use the Individual moving range chart (I-mR). If more than one measurement attempt is used (maximum of 8), it is recommended to use the Xbar-R chart. If it is more, the most appropriate is the Xbar-S chart (Chowdhury, Mukherjee & Chakraborti, 2013).

In the case of discrete variables, the most frequent observed variable is occurrence of non-conforming products. If there is a fixed opportunity for an error, it is recommended to use the np-chart. If the opportunity is variable, the p-chart is more appropriate. The subject of monitoring can also be the number of non-conformities on the product – in this case, the c-chart is used in fixed opportunity and the u-chart is used in variable opportunity (Skinner, Montgomery & Runger, 2004).

Each control diagram has three levels in basic graphical expression. The first level is CL, the central line - it is the average value of the parameter or range. The Upper Control Limit (UCL) represents the upper natural boundary, and the Lower Control Limit (LCL) represents the lower natural boundary of the process. Below are the formulas to calculate these three levels in Xbar chart.

$$CL = \bar{X} = \frac{+}{N}\sum_{i=1}^{N}\bar{X}_i \qquad (1)$$

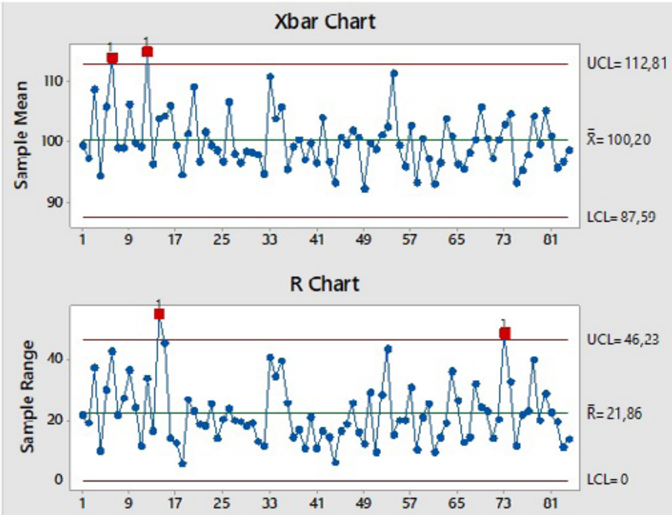
$$LCL = X - A_2\bar{R} \qquad (2)$$

$$UCL = X + A_2\bar{R} \qquad (3)$$

where X is parameter, N is number of subgroups, A_2 is constant and R is range between subgroup.

Ideally, individual or average measured values move within these boundaries from LCL to UCL. Figure 1 is an example of the most commonly used control chart – Xbar and R chart.

Figure 1 **Xbar and R control charts**



1.2 Basic preconditions of control chart application

In order to create control charts, it is recommended to first determine the parameter to be monitored. Subsequently, individual measurements of this parameter are perfomed at certain time intervals (He, Grigoryan & Sigh, 2002). These intervals may vary depending on the nature of the process. Generally speaking, however, within a single interval, a certain number of measurement are made – eg, 5 products from production line are measured at 7:00, then another 5 products at 8:00, etc. The amount of these measurements within a single interval is called a subgroup. The subgroup size may vary from 2 to 25, but the most common subgroup size is from 4 to 6.

For the use of control chart, it is also assumed that the data has a normal distribution (Schoonhoven & Does, 2010). It is a distribution that is characteristic of the Gaussian curve and is one of the most frequent occurrences.

1.3 Normality in control charts

Normality of continuous data can be verified by a statistical tests. Depending on the sample size, it is possible to use the Ryan-Joiner test (up to 25 values) Kolmogorov-Smirnov (from 25 to 75 values) or Anderson-Darling test (more than 100 values) (Ghasemi & Zahediasl, 2012). In practice, however, it often happens that the normality test is not performed and control limits are calculated on the basis of an incorrect assumption (Lin & Chou, 2007). This can lead to incorrect values of control limits and to misinterpretation of process monitoring results. This article aims to examine the impact of three non-normal distributions on the values of control limits in the most frequently used control chart - Xbar chart. Beta-type distributions ($\alpha = 2$; $\beta = 5$), Uniform ($a = 82,5$; $b = 117,5$) and Weibull ($\lambda = 1.5$; $k = 3$) were examined. In addition to the effect of distribution types, the sample size and subgroups size was also tested. This objective reflects the open questions that were previously indicated by earlier studies (Quesenberry, 1992; Klein, 2000; Nedumaran & Pignatiello, 2001).

2 Data and methods

Random data was used to empirically verify the research aim. Because we also wanted to examine subgroups size of 2, 3, 4, 5, 6 and 7, the resulting number of cases had to be divisible by all of these numbers. The common denominator was 420 – this amount of data fields was generated for all three distributions examined (Beta, Uniform, Weibull) and for the reference (Normal) distribution.

By transforming the data across all distritutions, the average of the parameter is approximately 100 and the standard deviation is approximately 10. Basic characteristics of position, variability and shape are found in Table 1.

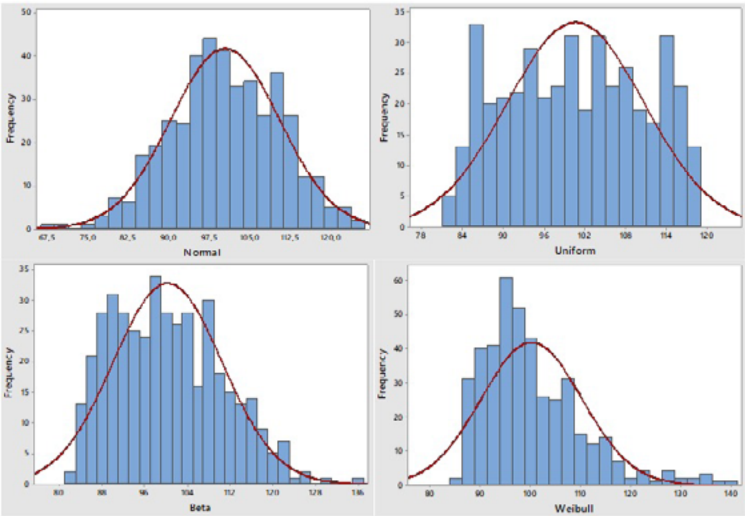
Table 1 **Main characteristics of distributions**

Distribution	N	Mean	SE Mean	StDev	Min	Q1	Median	Q3	Max
Normal	420	100,36	0,492	10,08	67,11	93,80	100,28	108,15	125,91
Uniform	420	100,59	0,491	10,06	82,59	91,90	100,56	108,92	117,45
Beta	420	100,26	0,499	10,22	82,62	91,69	99,27	108,04	135,02
Weibull	420	100,20	0,489	10,03	86,19	92,92	97,77	105,69	140,81

Source: Own elaboration.

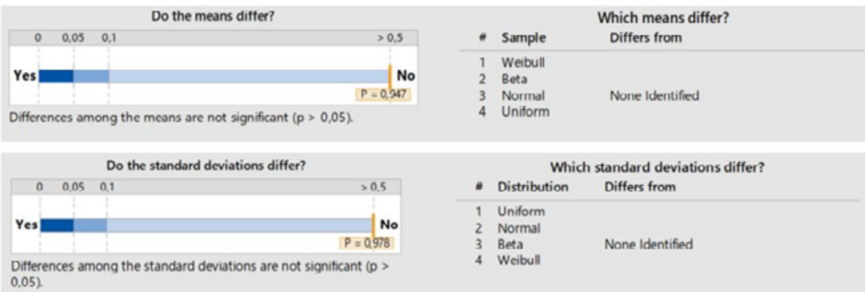
Histograms of random data are in Figure 2. In normality tests, it was explicitly proven that the three studied distributions are non-normal. Thus, the risk of data transformation in order to obtain the same average and standard deviation values could be that the data would begin to have a normal distribution. However, this risk was not confirmed by the test.

Figure 2 **Histograms of studied distributions**



At the same time, it was examined whether the average (approximately 100) and the standard deviation (approximately 10) could be considered the same in all types of distribution. One-way ANOVA was used to confirm that the mean and standard deviations did not differ – Figure 3.

Figure 3 **Testing of mean and standard deviation difference**



The basic assumptions for correct data analysis were therefore met. The following procedure was used to verify the effect of each type of distribution on control limits calculation:

1. First, the data (sample size = 420) was subdivided into subgroups of size 2
2. Subsequently, UCL and LCL were calculated
3. UCL and LCL values have been recorded
4. Subsequently, one subset was removed (sample size = 420-2 = 418)
5. Steps 2 through 4 have been repeated
6. After sample size = 0, the sample size again increased to 420 and the data was subdivided into subgroups of size 3 and the procedure was repeated

These steps were progressively performed for all three distributions and also for normal distribution. Individual datasets were encoded for easier interpetation. The coding consisted of three parts: (1) type of distribution (2) sample size and (3) type of control limit. For example, the "Weibull_4_UCL" signifies that this is data from Weibull's distribution with subgroup size of 4 and the data represents the calculated value of the Upper Control Limit.

3 Results

3.1 The effect of subgroup size

When the sample size remains constant (in our case 420), only one value for Upper and Lower Control Limit is calculated for each data file. However, it depends on which subgroup size will be used. Table 2 lists the control limits of each type of distribution with a changing subgroup size.

Table 2 **Upper and Lower Control Limits in different types of distributions and subgroup sizes**

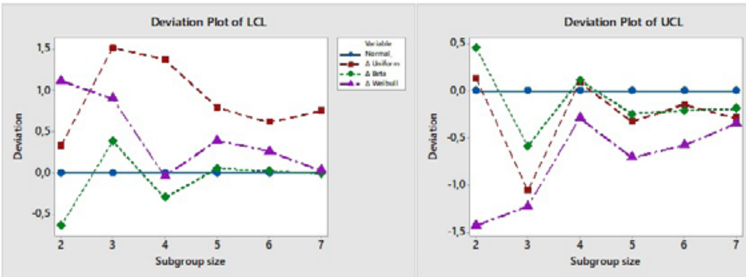
Distribution/Subgroup size	2	3	4	5	6	7
Normal (LCL)	78,88	82,38	85,44	86,5	87,83	88,69
Uniform (LCL)	79,21	83,89	86,81	87,29	88,44	89,44
Beta (LCL)	78,24	82,76	85,14	86,55	87,85	88,68
Weibull (LCL)	79,99	83,28	85,40	86,89	88,09	88,72
Normal (UCL)	121,84	118,35	115,28	114,22	112,89	112,03
Uniform (UCL)	121,97	117,29	115,37	113,89	112,74	111,74
Beta (UCL)	122,29	117,76	115,39	113,97	112,68	111,84

Distribution/Subgroup size	2	3	4	5	6	7
Weibull (UCL)	120,41	117,12	114,99	113,51	112,31	111,68

Source: Own elaboration.

From the table, at first glance, it may seem to be negligible differences, but in control charts, these differences may cause a different interpretation of process stability. If the differences are too large, it can result to not capturing the non-standard situations, respectively. incorrectly identifies them. The visualization of the deviations from the normal distribution is in Figure 4.

Figure 4 **Upper and Lower Control Limits deviations from normal distribution**



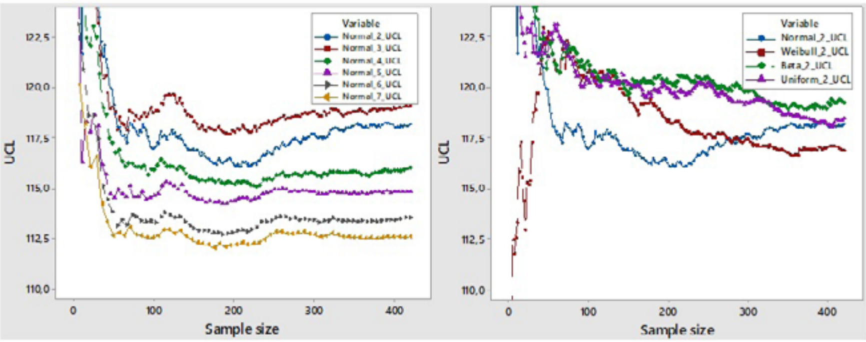
With the growing subgroup size, the difference in control limit values from the normal distribution decreases. Optically we could consider it a sufficient subgrop size of 5 or more. Lower subgroup sizes have a relatively high deviation from normal distribution and the risk of incorrect control chart interpretation is therefore higher.

3.2 The effect of sample size

The decreasing sample size reduces the accuracy of the control limits. Some authors' recommendations state that the sample should have at least 20 subgroups to allow for a relatively reliable calculation of control limits. For example, if there is a subgroup size of 4, we need to make at least 80 measurements (20x4). Random data for all types of distributions contained 420 theoretical values (measurements). For the highest subgroup size of 7 is the number of subgroups 60 (420/7). The range of data files is therefore sufficient.

What effect on the resulting control limit would have the gradual reduction of this range? As mentioned above, we have sequentially removed data fields, and the control limits have been calculated from decreasing data range. In Figure 5 there is a selection of comparison results for normal distribution with different subgrop sizes and for 4 types of distributions with subgroup size of 2. More results could be also displayed – the subgroup size ranged from 2 to 7 and 4 types of distribution (Normal, Beta, Uniform and Weibull). Only results for UCL are displayed, since formulas 1 and 3 shows that UCL and LCL is from scalar point of view identical.

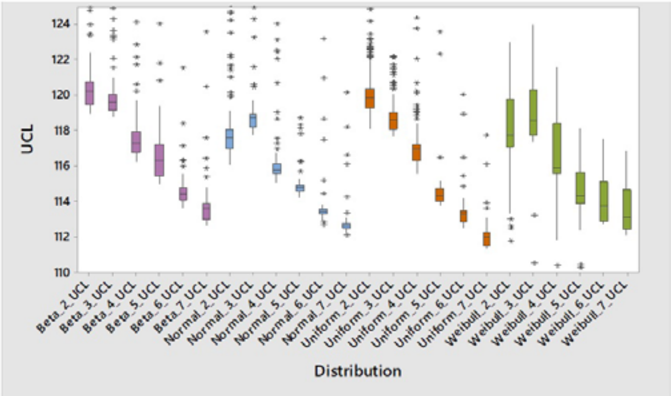
Figure 5 **Effect of sample size, subgroup size (left) and types of distribution (right) on control limit values**



In the left part of Figure 5, it can be seen that the increase of subgroup size will increase the strictness of control limits (this does not apply to the subgroup size of 2 and 3). It is also interesting to note that with the increasing sample size the control limits become more stable. In the right part of Figure 5 we can also find another interesting finding. While in the normal distribution, control limits stabilize by increasing the sample size, in other types of distributions, increase of sample size will result to decrease of control limits.

The variability of the individual control limits with respect to the subgroup size is shown in Figure 6. The greatest variability of the calculated control limits belongs to the normal distribution and the largest to Weibull distribution. For practical reasons, it is desirable that the calculated control limits be the same regardless of other factors such as, for example, distribution type, sample size, and subgroup size. From this point of view, it really appears that normal distribution results to the most robust calculation of control limits.

Figure 6 **Boxplots of control limits variability under different conditions**



In addition to the graphical comparison, the numerical comparison of the results is better from the point of view of the report. The extent to which particular control limit values differ from each other has been verified through ANOVA. Normal distribution was considered a reference distribution. The results are in Table 3.

Table 3 **p-values of distributions comparison regarding to subgroup size**

Distribution/ Subgroup size	**Normal**					
	2	**3**	**4**	**5**	**6**	**7**
Beta (2, 3, 4, 5, 6, 7)	0,000	0,024	0,000	0,000	0,002	0,001
Uniform (2, 3, 4, 5, 6, 7)	0,000	0,523*	0,004	0,028	0,143*	0,000
Weibull (2, 3, 4, 5, 6, 7)	0,980*	0,226*	0,088*	0,136*	0,412*	0,025

Source: Own elaboration.

In the table, it can be seen which types of distributions differ significantly from the normal distribution. Individual cases are marked with "*". It can be seen from the results that the calculation of the control limits is not a risky for the Beta distribution and in part even for the Uniform distribution. In the case of data with Weibull distribution, the resulting control limits differ significantly from the values obtained from the normal distribution data.

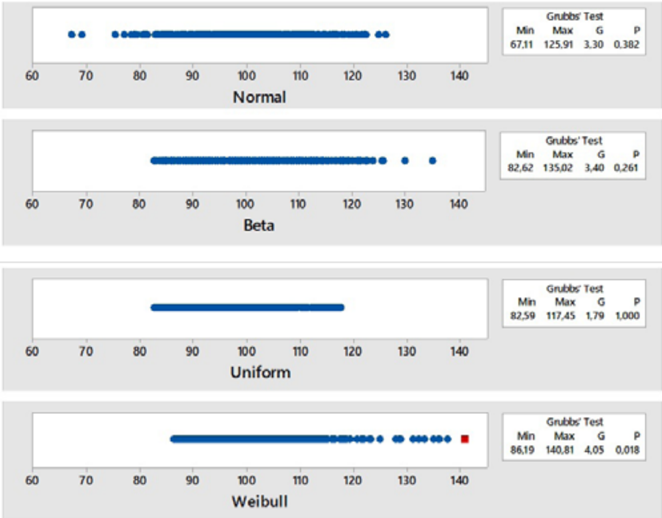
4 Discussion and conclusion

The creation of control charts also entails risks of methodological nature. This article aims to examine the impact of three non-normal distributions on the values of control limits in the most frequently used control chart - Xbar chart. Beta-type distributions ($\alpha = 2$; $\beta = 5$), Uniform ($a = 82{,}5$; $b = 117{,}5$) and Weibull ($\lambda = 1.5$; $k = 3$) were examined.

For sample size 4 and above, the control limits in the normal distribution tend to stabilize. However, if data are not normally distributed, control limits will tend to decrease. The resulting control chart will thus have more tightly set control intervals. In sample size 4 and above, it was also found that the differences between the control limits are not so significant. However, it should be noted that while the variability of the calculated control limits in the normal distribution is relatively low, in Weibull distribution is variability significantly higher. If the data for the Xbar control chart has a Weibull distribution, there is a higher risk that the resulting control limit values will be incorrect. The results of this paper can be contrasted with past studies focusing on the calculation of control limits (Nedumaran & Pignatiello, 2001; Chen & Yeh, 2010), non-normality effect (Schoonhoven & Does, 2010; Lin & Chou, 2007), sample size (Quesenberry, 1992), or subgroup size (Torng & Lee, 2009).

From the practical point of view, the results of this study have two implications. The first is a determining of subgroup size. In general, there is no unique tutorial that could provide guidance to a precise subgroup size determination in Xbar control chart. Although this study does not provide such a guideline, the knowledge that subgroup size 4 and more is sufficient for stability control limits can be relatively valuable. The second implication is that data that do not differ significantly from normal distribution do not tend to determine incorrect control limits. The greatest risk is also the data whose histogram has a long "tail". An example can be seen in Figure 7 where a simple oulier test is performed.

Figure 7 **Grubbs outlier test of distributions with**



The presented article offers partial theoretical and practical answers to how particular types of distributions, different subgroup size and sample sizes affect the resulting control limits. The results are a brief summary of some selected types of distributions, but the presented approach may indicate possible paths for further research. Questions still remain open about other types of distributions and subgroup size of 8 and more. These themes can represent the potential of further research.

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