# LESSON TITLE

Prepared By

Faculty Name

# GAUSSIAN ELIMINATION AND GAUSS JORDAN ELIMINATION

JOHN LOUIE S. MARASIGAN
Colegio de San Juan de Letran Calamba

# COURSE OBJECTIVES

**After completing this lesson, students must be able to:**

- **Determine the size of a matrix and write an augmented or**
- **coefficient matrix from a system of linear equations.**
- **Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.**
- **Use matrices and Gauss-Jordan elimination to solve a system of linear equations.**

# Matrices

$a_{ij}$
index
$i \rightarrow$ row subscript
"
$j$ - column

## Definition of a Matrix

If $m$ and $n$ are positive integers, an $m \times n$ (read "$m$ by $n$") matrix is a rectangular array

$$\begin{array}{cc}
 & \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array} \\
\begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} &
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\end{array}$$

in which each **entry,** $a_{ij}$, of the matrix is a number. An $m \times n$ matrix has $m$ rows and $n$ columns. Matrices are usually denoted by capital letters.

# Representation of Matrices

1. An uppercase letter such as $A$, $B$, or $C$
2. A representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$
3. A rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

# SIZE OF MATRICES

A matrix with $m$ rows and $n$ columns is said to be of **size** $m \times n$.

$m \times n$

$\begin{bmatrix} 2 \end{bmatrix}$ ✓

$1 \times 1$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $2 \times 2$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$2 \times 3$

# REPRESENTATION OF SYSTEMS OF LINEAR EQUATIONS

| System | Augmented Matrix | Coefficient Matrix |
|---|---|---|
| $\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \phantom{+3y} - 4z &= 6 \end{aligned}$ | $\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$ | $\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$ |

# ELEMENTARY ROW OPERATIONS

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new system of linear equations

## Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

$$x - 2y + 3z = 9 \rightarrow eq.1$$
$$-x + 3y = -4 \rightarrow eq.2$$
$$2x - 5y + 5z = 17 \rightarrow eq.3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \quad R_2 + R_1 \rightarrow R_{2N} \\ \rightarrow R_3 \quad R_3 + (-2)R_1 \rightarrow R_{3N} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \begin{array}{l} \rightarrow \\ \rightarrow R_{2N} \\ \rightarrow R_{3N} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \quad R_3 + R_2 \rightarrow R_{3N}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \quad \tfrac{1}{2}R_3 \rightarrow R_{3N}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

row-echelon form

$\longrightarrow$

$$x - 2y + 3 = 9$$
$$y + 3z = 5$$
$$z = 2$$

$z = 2$

$y = -1$

$x = 1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & n_1 \\ 0 & 1 & 0 & n_2 \\ 0 & 0 & 1 & n_3 \end{array}\right]$$

$$x = n_1$$
$$y = n_2$$
$$z = n_3$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

## Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ ✓

b. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$ ✗

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

d. $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ ✗

f. $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓

$y_1, y_2, y_3, y_4$ $x, y, z$

$x_1, x_2, x_3, x_4$

Solve the system

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \quad \begin{array}{l} R_3 + (-2)\cdot R_1 \rightarrow R_{3_N} \\ R_4 + (-1) R_1 \rightarrow R_{4_N} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right] \quad R_4 + 6R_2 \rightarrow R_{4N}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right] \quad R_4 + 6R_2 \rightarrow R_{4N}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right] \quad \begin{array}{l} \tfrac{1}{3}R_3 \rightarrow R_{3N} \\ -\tfrac{1}{13}R_4 \rightarrow R_{4N} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 + -2R_2 \rightarrow R_{1N}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & 8 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & 8 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 + 3R_3 \rightarrow R_{1N}$
$R_2 + (-1)R_3 \rightarrow R_{2N}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 + (-1)R_4 \rightarrow R_{1N}$
$R_2 + R_4 \rightarrow R_{2N}$
$R_3 + R_4 \rightarrow R_{3N}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]
\begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \\ \rightarrow R_4 \end{array}$$

# References:

The main references of this course are the following:


a. Anton, H., Rorres, C. (2010). Elementary Linear Algebra: Applications Version. United Kingdom: Wiley.

b. Anton. (2013). Elementary Linear Algebra: Applications Version, Tenth Edition Wiley E-Text Reg Card. (n.p.): John Wiley & Sons, Incorporated.

c. Larson, R. (2013). Elementary Linear Algebra. United Kingdom: Brooks/Cole, Cengage Learning.

d. Larson, C., Larson, R. (2017). Elementary Linear Algebra, International Metric Edition. United States: Brooks/Cole.

e. Hill, R. O. (2014). Elementary Linear Algebra. United Kingdom: Elsevier Science.