EXAME DE 1^a ÉPOCA DE ÁLGEBRA LINEAR - LEEC

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JUSTIFIQUE TODAS AS RESPOSTAS

1) (3.5) Considere a transformação linear $T : \mathbb{R}^3 \to \mathbb{R}^2$ tal que

$$M\left(T; B_c^3; B_c^2\right) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Determine a expressão geral de T.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y \\ y+2z \end{bmatrix} \to T(x,y,z) = \begin{bmatrix} x+2y \\ y+2z \end{bmatrix}$$

2) (3.5) Considere a transformação linear $T : \mathbb{R}^3 \to \mathbb{R}^3$ tal que

$$T(0,1,0) = (2,1,5), \qquad T(0,0,1) = (1,4,1), \qquad T(2,0,1) = (3,10,3).$$

Determine $M(T; B_c^3; B_c^3)$.

↳ 2ª coluna ↳ 3ª coluna

$$\left[\frac{(3,10,3) - T(0,0,1)}{2}\right] = T(1,0,0)$$

↳ 1ª coluna

$$\frac{1}{2}(2,6,2) = (1,3,1)$$

$$M\left(T; B_c^3; B_c^3\right) = T(v) = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 1 & 5 & 1 \end{bmatrix}$$

3) (3.0) Determine uma base ortogonal de $\mathbb{R}^3$ que inclua o vector: $(1,1,1)$.

$\langle (1,1,1), (1,-1,0) \rangle = 0$

$\underbrace{}_{\text{ortogonais}}$

$$\mathcal{N}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix} = L\left\{\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right\}$$

$x = y = -\frac{z}{2}$

$z = -2y$

$y = -\frac{1}{2}z$

Base ortogonal: $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right\}$

4) (3.0) Seja $U = \{(x,y,z) \in \mathbb{R}^3 : x + z = 0 \text{ e } y + z = 0\}$. Calcule $d((0,0,1), U)$.

$x = -z \qquad y = -z$

$$U = \mathcal{N}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = L\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$x = -z$

$y = -z$

$$d\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, U\right) = \left\|(0,0,1) - \text{proj}_{(-1,-1,1)}(0,0,1)\right\| = \left\|(0,0,1) - \left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)\right\| = \sqrt{2\cdot\frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$$

$\text{proj}_v u = \frac{\langle v, u\rangle}{\|v\|^2} v$

$\text{proj}_{(-1,-1,1)}(0,0,1) = \frac{1}{3}(-1,-1,1)$

5) (2.0) Seja $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$. Determine uma factorização QR para A. Isto é, determine uma matriz ortogonal Q e uma matriz triangular superior R tais que $A = QR$.

$R = \begin{bmatrix} \|w_1\| & \langle u_1, v_2\rangle \\ 0 & \|w_2\| \end{bmatrix}$

$A = QR$

Base ortogonal $\{w_1, w_2\}$ | Base o.n. $\{v_1, v_2\}$

Gram-Schmidt:

Seja $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = w_1 \to \| \ \| = \sqrt{2}$

$v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad w_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ \frac{1}{2} \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\to \| \ \| = \sqrt{2 \cdot \frac{1}{4} + 1} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

$u_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$

$\|w_1\| = \sqrt{2}$ $\quad \langle v_1, v_2 \rangle = \frac{\sqrt{2}}{2}$

$\|w_2\| = \frac{\sqrt{6}}{2}$

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{bmatrix}$$

6) (2.0) Seja $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. Determine um escalar λ e projecções ortogonais P_1 e P_2 tais que

$$A = \underbrace{\lambda}_{-2} P_1 + 4P_2. \qquad A = PDP^T$$

$\lambda = \frac{2 \pm \sqrt{\underbrace{4+32}_{6}}}{2} = 1 \pm 3 \to \lambda = 4 \lor \lambda = -2$

$\mathcal{N}(A - 4I) = \mathcal{N}\begin{bmatrix} -3 & 3 \\ \cancel{3} & \cancel{-3} \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & -1 \end{bmatrix} = L\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$x = y$

$\lambda = -2$

$P_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

$\mathcal{N}(A + 2I) = \mathcal{N}\begin{bmatrix} 3 & 3 \\ \cancel{3} & \cancel{3} \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & 1 \end{bmatrix} = L\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$x = -y$

$P_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

A é simétrica logo $\mathcal{N}(A - 4I) \perp \mathcal{N}(A + 2I)$

7) (1.0) Seja $A \in \mathcal{M}_{2\times 3}(\mathbb{R})$ tal que $0 \notin \sigma_{AA^T}$. Classifique a forma quadrática $Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} AA^T \begin{bmatrix} x \\ y \end{bmatrix}$.

$A = U\Sigma V^T \iff A^T = V\Sigma^T U^T$

$AA^T = U\Sigma \underbrace{V^T V}_{I} \Sigma^T U^T$

8) (2.0) Sejam $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$, matrizes diagonalizáveis ortogonalmente, com $\sigma_A \subset \mathbb{R}^+$. Mostre que AB é diagonalizável.

Nesta folha pode completar as respostas.

Ficha 10

6. Base o.n. que inclua:

a) $(1,0,1)$ e $(1,0,-1)$ $\rightarrow$ $\langle u,v\rangle = 0$ logo $u \perp v$

$$U^\perp = \mathcal{N}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$x = 0$
$z = 0$

Base o.n. $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

b) $(1,2,-1)$

$\hookrightarrow \langle \underbrace{(1,2,-1)}_{\perp}, \underbrace{(1,0,1)}_{\perp} \rangle = 0$

$$U^\perp = \mathcal{N}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix} = L\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$x = -z$
$y = z$

Base o.n.: $\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$

c) $(1,2,4)$

$\hookrightarrow \perp (-2,1,0)$

Base o.n. $\{ \quad \}$

$$U^\perp = \mathcal{N}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix} = L\left\{\begin{bmatrix} \frac{4}{5} \\ -\frac{8}{5} \\ 1 \end{bmatrix}\right\}$$

$x + 2y + 4z = 0$

$5y + 8z = 0$

$y = -\frac{8}{5}z$

$x = -\frac{16}{5}z + \frac{20}{5}z = \frac{4}{5}z$

7. Seja $U = \{(x,y,z) \in \mathbb{R}^3 : x+y+2z = 0 \;\text{ e }\; x+2z = 0\}$.

a) Determine uma base ortogonal de $U^\perp$.

b) Determine $u \in U$ e $v \in U^\perp$ tais que $(1,2,3) = u + v$.

a) $U^\perp = \mathcal{L}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} = \left[\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{matrix}\right]>^\perp$, Logo $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right\}$ é Base o.n de $U^\perp$

b) $\dim U = 1 < \dim U^\perp$

$u = P_U(1,2,3) = \text{proj}_{(-2,0,1)}(1,2,3) = \frac{1}{5}(-2,0,1) = \left(-\frac{2}{5}, 0, \frac{1}{5}\right)$

$\hookrightarrow \|\,\|^2 = 5$

$(1,2,3) = u + v \iff v = (1,2,3) - \left(-\frac{2}{5}, 0, \frac{1}{5}\right)$

$v = \left(\frac{7}{5}, 2, \frac{14}{5}\right)$

$U = \mathcal{N}\begin{bmatrix} 0 & 1 & 0 \\ \ldots \end{bmatrix} = L\left\{\begin{bmatrix} -2 \\ 0 \\ \ldots \end{bmatrix}\right\}$

$$U = \mathcal{N}\left[\begin{matrix} 1 & 0 & 2 \end{matrix}\right] = L\left\{\left[\begin{matrix} \\ 1 \end{matrix}\right]\right\}$$

$y = 0$

$x = -2z$

c) Calcule $d\left((1,0,1), U\right)$.

$$d((1,0,1), U) = \|(1,0,1) - P_U(1,0,1)\| = \|(1,0,1) - \mathrm{proj}_{(-2,0,1)}(1,0,1)\| =$$

$$\frac{-1}{5}(-2,0,1)$$

$$= \|(1,0,1) + (-\tfrac{2}{5}, 0, \tfrac{1}{5})\|$$

$$= \sqrt{\frac{9}{25} + \frac{36}{25}} = \frac{\sqrt{45}}{5} = \frac{3\sqrt{5}}{5}$$

17. Determine uma factorização QR de cada uma das matrizes:

$$\textbf{a)} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \textbf{b)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \textbf{c)} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad \textbf{d)} \begin{bmatrix} -4 & 4 \\ 3 & 3 \end{bmatrix}.$$

a) $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -2 \end{bmatrix} = A$

Base ortogonal $\{w_1, w_2\}$

Seja $v_1 = (1, -1, 0)$

$w_1 = v_1 = (1, -1, 0) \rightarrow \|\ \| = \sqrt{2}$

$v_2 = (2, 0, -2)$

$w_2 = (2,0,-2) - \mathrm{proj}_{(1,-1,0)}(2,0,-2) = (1, 1, -2)$

$\frac{\cancel{2}}{\cancel{2}}(1,-1,0)$

$\|\ \| = \sqrt{2+4} = \sqrt{6}$

Base o.n. $\{u_1, u_2\}$

$u_1 = \frac{1}{\|w_1\|} w_1 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$

$u_2 = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}\right)$

$$Q = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{6}}{3} \end{bmatrix}$$

$\langle u_1, v_1 \rangle = \|w_1\|$

$\langle u_2, v_2 \rangle = \|w_2\|$

$\langle u_1, v_2 \rangle = \sqrt{2}$

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} \end{bmatrix}$$

18. Considere em $\mathbb{R}^3$ o produto interno para o qual a base ordenada de $\mathbb{R}^3$:

$$\{(1,0,-1), (0,1,0), (-1,0,2)\}$$

é ortonormada. Verifique que esse produto interno é definido pela aplicação $\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ dada por:

$$\langle (x_1,x_2,x_3), (y_1,y_2,y_3) \rangle = 5x_1y_1 + 3x_1y_3 + x_2y_2 + 3x_3y_1 + 2x_3y_3$$

e determine, relativamente a ele, uma base ortonormada de $\left(L\left(\{(-2,0,3),(0,1,0)\}\right)\right)^{\perp}$.

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}_B \underbrace{G_B}_{\neq I} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_B$$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & x_1 \\ 0 & 1 & 0 & x_2 \\ -1 & 0 & 2 & x_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x_3 + 2x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 + x_1 \end{array}\right]$$

Logo $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = \begin{bmatrix} 2x_1 + x_3 & x_2 & x_1 + x_3 \end{bmatrix} \begin{bmatrix} 2y_1 + y_3 \\ y_2 \\ y_1 + y_3 \end{bmatrix}$

$= (2x_1 + x_3)(2y_1 + y_3) + x_2 y_2 + (x_1 + x_3)(y_1 + y_3)$

$= 4x_1 y_1 + 2x_1 y_3 + 2x_3 y_1 + x_3 y_3 + x_2 y_2 + x_1 y_1 + x_1 y_3 + x_3 y_1 + x_3 y_3$

$= 5x_1 y_1 + 3x_1 y_3 + 3x_3 y_1 + 2x_3 y_3 + x_2 y_2$

$$G_B = \begin{bmatrix} 5 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

$A^{+} = (A^T A)^{-1} A^T$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ v_1 \quad v_2 \end{array}$$

$w_1 = (1, 2) \rightarrow \| \ \| = \sqrt{5}$

$w_2 = (2, 1) - \text{proj}_{(1,2)} (2,1) = (2,1) - \left(\frac{4}{5}, \frac{8}{5}\right) = \left(\frac{6}{5}, -\frac{3}{5}\right)$

$\|(1,2)\|^2 = 5$

$\frac{4}{5}(1,2)$

$\| \ \| = \frac{\sqrt{45}}{5} = \frac{3\sqrt{5}}{5}$

$u_1 = \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$

$u_2 = \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = \left(\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$

$$Q = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}$$

$\langle u_1, v_2 \rangle = \frac{4\sqrt{5}}{5}$ $\quad R = \begin{bmatrix} \sqrt{5} & \frac{4\sqrt{5}}{5} \\ 0 & \frac{3\sqrt{5}}{5} \end{bmatrix}$

Fiche 9

2. Considere a transformação linear $T : \mathbb{R}^2 \to \mathbb{R}^3$ tal que

$$T(1,1) = (-2,0,3) \quad \text{e} \quad T(-3,-2) = (2,-2,-1).$$

Determine a expressão geral de T, isto é, T(x, y).

Seja $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \end{bmatrix} \right\}$

$$\left[\begin{array}{cc|c} 1 & -3 & x \\ 1 & -2 & y \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -2x+3y \\ 0 & 1 & -x+y \end{array}\right]$$

$$T(x,y) = T(B)\,[(x,y)]_B = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -2x+3y \\ -x+y \end{bmatrix}$$

$$= \begin{bmatrix} 4x-6y-2x+2y \\ 2x-2y \\ -6x+9y+x-y \end{bmatrix} = \begin{bmatrix} 2x-4y \\ 2x-2y \\ -5x+8y \end{bmatrix}$$

4. Seja $\mathcal{B} = \{(1,2),(-1,-3)\}$. Considere a transformação linear $T : \mathbb{R}^2 \to \mathbb{R}^2$ tal que

$$T(1,2) = (1,-2), \quad T(-1,-3) = (-3,1).$$

a) Determine a matriz $M(T; \mathcal{B}_c^2; \mathcal{B}_c^2)$.

b) Calcule $T(1,-3)$.

a) $T(1,2) + T(-1,-3) = T(0,-1) = (-2,-1)$

$\Leftrightarrow T(0,1) = (2,1)$

$T(1,2) = T(1,0) + 2T(0,1) \Leftrightarrow T(1,0) = (1,-2) - (4,2) = (-3,-4)$

Logo

$$M(T; B_c^2; B_c^2) = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

b) $T(1,-3) = T(1,0) - 3T(0,1) = (-3,-4) + (-6,-3) = (-9,-7)$

—//— ou

$$T(1,-3) = T(B)\,[(1,-3)]_B$$

$$\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & -3 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & -1 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 5 \end{array}\right]$$

$$= \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 6-15 \\ -12+5 \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \end{bmatrix}$$

12. Considere a transformação linear $T : \mathbb{R}^3 \to \mathbb{R}^3$ tal que $M\left(T; \mathcal{B}_c^3; \mathcal{B}_c^3\right) = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

a) Calcule $T(0,1,0)$ e $T(4,0,-2)$ e verifique se T é injectiva.

a) $\underbrace{T(0,1,0)}_{2^a \text{ coluna de } M} = (2,-2,0)$

$T(4,0,-2) = 4T(1,0,0) - 2T(0,0,1) = (8,0,4)-(6,2,4) = (2,-2,0)$

Como $(0,1,0) \neq (4,0,2)$ mas $T(0,1,0) = T(4,0,2)$,

$\to$ T não é injetiva

(ou)

$\text{nul}\ M(T; B_c^3; B_c^3) = \text{nul} \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \text{nul} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ \cancel{0} & \cancel{-2} & \cancel{1} \end{bmatrix} = 1 \neq 0$ Logo T não é injetiva

b) Determine uma base de $\mathcal{N}(T)$ e uma base de $\mathcal{I}(T)$.

$A = M(T; B_c^3; B_c^3) = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} \underline{1} & 0 & 2 \\ 0 & \underline{1} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

logo $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \right\}$ é base de $\mathcal{C}(A)$ $= \mathcal{I}(T)$

Como , $\mathcal{N}(T) = \mathcal{N}(A) = L\left\{ \begin{bmatrix} -2 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$

$x = -2z$
$y = \frac{1}{2}z$

Base de $\mathcal{N}(A) = \mathcal{N}(T)$

c) Verifique se a equação linear $T(x,y,z) = (0,0,1)$ tem solução e diga se T é sobrejectiva.

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ é impossível, logo T não é sobrejetiva

$(0,0,1) \notin \mathcal{I}(T)$

(ou) $\quad car\ A = car \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = 2 < 3$ (nº de linhas)

d) Calcule $T(1,-1,1)$ e resolva a equação linear $T(x,y,z) = (-1,-1,-1)$.

$$CS_{T(v)=b} = SP_{T(v)=b} + \underbrace{CS_{T(v)=0}}_{=\mathcal{N}(T)}$$

$$T(1,-1,1) = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-2+3 \\ 2+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$(-1,-1,-1) = -\frac{1}{3}(3,3,3) = -\frac{1}{3}T(1,-1,1) = T\left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$$

Logo $CS_{T(v)=(-1,-1,-1)} = \left\{\left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)\right\} + L\left\{\left(-2, \frac{1}{2}, 1\right)\right\}$

Ficha 11

5. Determine uma matriz simétrica $A \in \mathcal{M}_{3\times 3}(\mathbb{R})$ tal que $\operatorname{tr} A = 2$ e

$$\mathcal{N}(A-2I) = L(\{(1,0,1),(0,1,0)\}).$$

$A = PDP^T$

$\dim \mathcal{N}(A-2I) = m_g(2) = 2 \rightarrow$ raiz dupla

$\operatorname{tr} A = 2 = 2+2+\lambda_2 \Leftrightarrow \lambda = -2 \rightarrow \sigma_A = \{-2, 2\}$

Como A é simétrica, $\mathcal{N}(A+2I) = (\mathcal{N}(A-2I))^\perp = L\{(1,0,-1)\}$

$\|(1,0,1)\| = \sqrt{2}$
$\|(0,1,0)\| = 1$
$\|(1,0,-1)\| = \sqrt{2}$

$\rightarrow$ Base o.n. $\left\{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), (0,1,0), \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)\right\}$

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$A = PDP^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

9. Escreva A como combinação linear de duas projecções ortogonais.

a) $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

a) $\lambda = \dfrac{7 \pm \sqrt{\overset{25}{49-24}}}{2} \Leftrightarrow \lambda = 1 \vee \lambda = 6$

$\mathcal{N}(A-I) = \mathcal{N}\begin{bmatrix} 1 & 2 \\ \cancel{2} & \cancel{4} \end{bmatrix} = L\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ normalizado: $\left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$

$x = -2y$ $\qquad \|\ \| = \sqrt{5}$

$\mathcal{N}(A-6I) = \mathcal{N}\begin{bmatrix} -4 & 2 \\ \cancel{2} & \cancel{-1} \end{bmatrix} = L\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ $\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$

$y = 2x$

$\begin{bmatrix} -\frac{2\sqrt{5}}{5} \\ \dots \end{bmatrix} \begin{bmatrix} -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix} \quad \begin{bmatrix} \frac{\sqrt{5}}{5} \\ \dots \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}$

$$A = 1 \underbrace{\begin{bmatrix} \\ \frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \frac{\ }{5} & \frac{\ }{5} \end{bmatrix}}_{P_1} + 6 \underbrace{\begin{bmatrix} \\ \frac{2\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \frac{\ }{5} & \frac{\ }{5} \end{bmatrix}}_{P_2}$$

16. Classifique e diagonalize as seguintes formas quadráticas.

a) $Q(x, y) = xy$

$$Q(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & \underset{xy}{1/2} \\ \underset{yx}{1/2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\frac{1}{2}xy + \frac{1}{2}xy = xy$

Seja $A =$

$$\lambda = \frac{0 \pm \sqrt{0 + 1}}{2} = \pm \frac{1}{2}$$

logo a forma quadrática é indefinida

20. Determine uma decomposição em valores singulares para cada uma das matrizes:

a) $\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$, **b)** $\begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$, **c)** $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$, **d)** $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

b) car $A = 2$

$$A^T A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

$$\begin{array}{r} 36 \\ \times\ 4 \\ \hline 144 \end{array}$$

$$\lambda = \frac{13 \pm \sqrt{\overset{=5}{169 - 144}}}{2} \Leftrightarrow \lambda = 9 \lor \lambda = 4$$

$\sigma_1 = 3 \quad \sigma_2 = 2$

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$N(A^T A - 4I) = N \begin{bmatrix} 1 & -2 \\ \cancel{-2} & \cancel{4} \end{bmatrix} = L\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$x = 2y$

normalizar: $\left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right)$

$\| \ \| = \sqrt{5}$

$$N(A^T A - 9I) = N \begin{bmatrix} -4 & -2 \\ \cancel{-2} & \cancel{-1} \end{bmatrix} = L\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} = L\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

$y = -2x$

$\left(-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right)$

$$V = \begin{bmatrix} -\frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{bmatrix} = \frac{1}{3} \left(-\frac{6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}, 0 \right)$$

$$= \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 0\right)$$

$$u_2 = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix} = \frac{1}{2}\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}, 0\right) = \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, 0\right)$$

$$\mathcal{N}(A^T) = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix} = \mathcal{N}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = L\{(0,0,1)\}$$

$x = 0 \quad y = 0$

Como $\{u_1, u_2, (0,0,1)\}$ é o.n.

então $U = \begin{bmatrix} -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = U \Sigma V^T$$

$$U\Sigma V^T = \begin{bmatrix} -\frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix}^T = \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} = A.$$

[redacted]

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\lambda = \frac{4}{2} \pm \frac{\overbrace{\sqrt{16 - 0}}^{=4}}{2} = 2 \pm 2 \Leftrightarrow \lambda = 0 \lor \lambda = 4 \rightarrow \sigma_1 = 2,\ \sigma_2 = 0$$

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V \rightarrow \mathcal{N}(A^T A) = \mathcal{N}[1 \ 1] = L\{\underbrace{(1,-1)}_{w_2}\}$$

$$\mathcal{N}(A^T A - 4I) = \mathcal{N}[-1 \ 1] = L\{\underbrace{(1,1)}_{w_1}\}$$

Gram-shmidt

$w_1 \perp w_2$, logo $v_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\rightarrow$ $V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

$v_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2}(\sqrt{2}, \sqrt{2}, 0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

u_2 seria $\frac{1}{0}$ ...

Logo $u_2, u_3 \in N(A^T) = N \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \mathcal{L}\{(1, -1, 0), (0, 0, 1)\}$

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$