MAS511 2020Spring Homework#06

Problem 1

(Injective horseshoe lemma) Suppose $0 \to N_1 \to N_2 \to N_3 \to 0$ is a s.e.s of R-modules.

Suppose there are injective resolutions $N_1 \to I_1^\bullet$ and $N_3 \to I_3^\bullet$.

Then prove that there is an injective resolution $N_2 \to I_2^\bullet$ that fits into a short exact sequence of complexes

$$0 \to I_1^\bullet \to I_2^\bullet \to I_3^\bullet \to 0$$

compatible with $0 \to N_1 \to N_2 \to N_3 \to 0$.

Lemmata

Lemma 1. *Let I_1, I_2 be injective R-modules. $I_1 \oplus I_2$ is injective.*

Proof. Let $0 \to M \xrightarrow{f} N$ be an exact sequence of R-modules. And suppose that there is a map $g : M \to I_1 \oplus I_2$. Let $\pi_k : I_1 \oplus I_2 \to I_k$ be a natural projection for $k = 1, 2$. Let $g_k = \pi_k \circ g$ for $k = 1, 2$. Since each I_k are injective, there is an extension $h_k : N \to I_k$ such that $g_k = h_k \circ f$. Then, let $h : N \to I_1 \oplus I_2 : x \mapsto (h_1(x), h_2(x))$. Trivially, it's a homomorphism as a composition of homomorphisms. h is an extension of g, because for $m \in M$,

$$h(f(m)) = (h_1(f(m)), h_2(f(m))) = (g_1(m), g_2(m)) = g(m)$$

. Therefore, $I_1 \oplus I_2$ is also injective. ∎

Proof

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, "g_0"] & M \arrow[r, "p"] \arrow[d, dashed, "f_0"] & M'' \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & I'^0 \arrow[r, "\iota_0"] \arrow[d, "g_1"] & I^0 = I'^0 \oplus I''^0 \arrow[r, "\pi_0"] \arrow[d, dashed, "f_1"] & I''^0 \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & I'^1 \arrow[r, "\iota_1"] \arrow[d, "g_2"] & I^1 = I'^1 \oplus I''^1 \arrow[r, "\pi_1"] \arrow[d, dashed, "f_2"] & I''^1 \arrow[r] \arrow[d, "h_2"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Suppose that the exact sequence $0 \to M' \to M \to M'' \to 0$ with i, p, an injective resolution $M' \to I'^\bullet$, $M'' \to I''^\bullet$ with g_k, h_k were given as above diagram.

As the Problem 1, take $I^k = I'^k \oplus I''^k$ for $k \in \mathbb{Z}^{\geq 0}$. Each I^k is injective by Lemma **??**. Let $\iota_k : I'^k \to I^k$ and $\pi_k : I^k \to I''^k$ be natural homomorphisms.

Let's define $f_k : I^{k-1} \to I^k$.

First, note that i, ι_0 and g_0 are injective because of exact sequences. Because I'^0 is injective, there is an extension $\alpha_1 : M \to I'^0$ of g_0 such that $g_0 = i \circ \alpha_1$. Let $f_0 : M \to I^0$ be $f_0 : m \mapsto (\alpha_1(m), h_0(p(m)))$.

Then, $0 \to M \xrightarrow{f_0} I^0$ is exact. If $f_0(m) = (0,0)$ for some $m \in M$, $\alpha_1(m) = 0$ and $h_0(p(m)) = 0$. Since h_0 is injective, $p(m) = 0$, and $m \in \ker p = \operatorname{im} i$. Let $x \in M'$ such that $i(x) = m$. $0 = f_0(m) = f_0(i(x)) = \iota_0(g_0(x))$. Since ι_0 is injective, $g_0(x) = 0$. Since g_0 is injective, $x = 0$. Therefore, $m = i(x) = 0$. This shows $\ker f_0 = \{0\}$ and f_0 is injective.

By Snake Lemma (See Problem 7), $\operatorname{coker} g_0 \xrightarrow{\tilde{\iota}_0} \operatorname{coker} f_0 \to \operatorname{coker} \tilde{\pi}_0 h_0$ is exact, where $\tilde{\iota}_0$ and $\tilde{\pi}_0$ are naturally induced from ι_0 and π_0. Also, $\tilde{\pi}_0$ is surjective as π_0 is surjective. Also, since $\ker h_0 = \operatorname{im} 0 = 0$, $0 = \ker h_0 \to \operatorname{coker} g_0 \to \operatorname{coker} f_0$ is exact. Because $\operatorname{im} g_0 = \ker g_1$ and $\operatorname{im} h_0 = \ker h_1$, we can build the below diagram with the induced injective homomorphisms $\tilde{g}_1 : \operatorname{coker} g_0 = I'^0/\ker g_1 \to I'^1$ and $\tilde{h}_1 : \operatorname{coker} h_0 = I''^0/\ker h_1 \to I''^1$ which obtained from g_1 and h_1:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{coker} g_0 \arrow[r, "\tilde{\iota}_0"] \arrow[d, "\tilde{g}_1"] & \operatorname{coker} f_0 \arrow[r, "\tilde{\pi}_0"] \arrow[d, dashed, "\tilde{f}_1"] & \operatorname{coker} h_0 \arrow[r] \arrow[d, "\tilde{h}_1"] & 0 \\
0 \arrow[r] & I'^1 \arrow[r, "\iota_1"] \arrow[d, "g_2"] & I^1 = I'^1 \oplus I''^1 \arrow[r, "\pi_1"] & I''^1 \arrow[r] \arrow[d, "h_2"] & 0 \\
 & \vdots & & \vdots &
\end{tikzcd}
$$

Note that each rows and the first and third column of above diagram is exact. In this case, we can repeat what we did above, just considering $\operatorname{coker} f_0$, $\operatorname{coker} g_0$, $\operatorname{coker} h_0$ as M, M', M''. Then, we can construct $\tilde{f}_1 : \operatorname{coker} f_0 \to I^1$ using the injectivity of I'^1, and an exact sequcne $0 \to \operatorname{coker} g_1 \to \operatorname{coker} f_1 \to \operatorname{coker} h_1 \to 0$, and $\tilde{f}_2 : \operatorname{coker} f_1 \to I^2, \cdots$.

Then, let $f_k : I^{k-1} \to I^k$ such as $f_k(x) = \tilde{f}_k(x + \operatorname{im} f_{k-1})$ for each $k \in \mathbb{Z}^{\geq 0}$ (Let's assume $I^{-1} = M$, $I^{-2} = 0$ and $f_{-2} = 0$ for convenience). Then, since each $\tilde{f}_k$ is injective, $\tilde{f}_k(x) = 0$ iff $x = \operatorname{im} f_{k-1}$. It implies $\ker f_k = \operatorname{im} f_{k-1}$.

Therefore, $I^\bullet$ is an injective resolution of M. $\square$

Problem 2

Prove that when

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is a s.e.s of cohomological complexes in degree ≥ 0, we have the associated long exact sequence of the cohomology modules.

Proof

Use Snake Lemma.

First, for each integer k, we have,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^k & \longrightarrow & B^k & \longrightarrow & C^k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^k} & & \downarrow{\scriptstyle g^k} & & \downarrow{\scriptstyle h^k} & & \\ 0 & \longrightarrow & A^{k+1} & \longrightarrow & B^{k+1} & \longrightarrow & C^{k+1} & \longrightarrow & 0 \end{array}$$

By Snake Lemma, we obtain the below exact sequence:

$$0 \longrightarrow \ker f^k \longrightarrow \ker g^k \longrightarrow \ker h^k \xrightarrow{\partial} \operatorname{coker} f^k \longrightarrow \operatorname{coker} g^k \longrightarrow \operatorname{coker} h^k \longrightarrow 0$$

Note that $\operatorname{coker} f^k = A^{k+1}/\operatorname{im} f^k$.

In this case, we can define $\tilde{f}^k : A^k/\operatorname{im} f^{k-1} \to \ker f^{k+1}$ as $\tilde{f}^k(x + \operatorname{im} f^{k-1}) = f^k(x)$. It's well-defined because, if $x + \operatorname{im} f^{k-1} = y + \operatorname{im} f^{k-1}$, $x - y \in \operatorname{im} f^{k-1} \subseteq \ker f^k$ thus $f^k(x-y) = 0$ and $f^k(x) = f^k(x) - f^k(x-y) = f^k(x - x + y) = f^k(y)$. Also, because $\operatorname{im} f^k \subseteq \ker f^{k+1}$, the image of $\tilde{f}^k$ is in $\ker f^{k+1}$.

Then, we have:

$$\begin{array}{ccccccc} & & \operatorname{coker} f^{k-1} & \longrightarrow & \operatorname{coker} g^{k-1} & \longrightarrow & \operatorname{coker} h^{k-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}^k} & & \downarrow{\scriptstyle \tilde{g}^k} & & \downarrow{\scriptstyle \tilde{h}^k} & & \\ 0 & \longrightarrow & \ker f^{k+1} & \longrightarrow & \ker g^{k+1} & \longrightarrow & \ker h^{k+1} & & \end{array}$$

and each rows of above diagram are exact because of the result of Snake Lemma.

Suppose that $\tilde{f}^k(x + \operatorname{im} f^{k-1}) = 0$. Then, $x \in \ker f^k$, and $x + \operatorname{im} f^{k-1} \in \ker f^k/\operatorname{im} f^{k-1}$. The converse is also true, because $x + \operatorname{im} f^{k-1} \in \ker f^k/\operatorname{im} f^{k-1}$ implies $x \in \ker f^k$ and $\tilde{f}^k(x + \operatorname{im} f^{k-1}) = f^k(x) = 0$. Thus, $\ker \tilde{f}^k = \ker f^k/\operatorname{im} f^{k-1} = H^k(A^\bullet)$.

Also, $\operatorname{coker} \tilde{f}^k = \operatorname{cod}(\tilde{f}^k)/\operatorname{im} \tilde{f}^k = \ker f^{k+1}/\operatorname{im} f^k = H^{k+1}(A^\bullet)$.

In the same way, we can show that $H^k(B^\bullet) = \ker \tilde{g}^k = \operatorname{coker} \tilde{g}^{k-1}$, $H^k(C^\bullet) = \ker \tilde{h}^k = \operatorname{coker} \tilde{h}^{k-1}$.

Thus, by Snake Lemma,

$$H^k(A^\bullet) \longrightarrow H^k(B^\bullet) \longrightarrow H^k(C^\bullet) \xrightarrow{\partial} H^{k+1}(A^\bullet) \longrightarrow H^{k+1}(B^\bullet) \longrightarrow H^{k+1}(C^\bullet)$$

Repeating this for every integer k, we obtain:

$$\cdots \longrightarrow H^{k-1}(C^\bullet) \longrightarrow H^k(A^\bullet) \longrightarrow H^k(B^\bullet) \longrightarrow H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet) \longrightarrow \cdots$$

□

Note: ∂ is defined as below (the proof is in HW5 Problem 7) Let $x \in \ker \operatorname{coker} h^{k-1} = \operatorname{im} \operatorname{coker} g^{k-1}$. Then, there is x' which is mapped to x by $\operatorname{coker} g^{k-1} \to \operatorname{coker} h^{k-1}$. Then, let y' be an image of x'

into $\ker g^{k+1}$. Then, let $y \in \ker f^{k+1}$ which is mapped to y' in $\ker g^{k+1}$. $\partial(x) = y + \mathrm{im}\,(\mathrm{coker}\, f^{k-1} \to \ker f^{k+1})$.

Problem 3

Let M, N be R-modules. And let $P_\bullet \xrightarrow{\epsilon} M$ be a projective resolution and $N \xrightarrow{\iota} I^\bullet$ be an injective resolution

Let $T^\bullet := \operatorname{Tot}(\operatorname{Hom}_R(P_\bullet, I^\bullet)) = \{T^n\}_{n\geq 0}$ with $T^n = \operatorname{Tot}(\operatorname{Hom}_R(P_\bullet, I^\bullet))) = \bigoplus_{i+j=n} \operatorname{Hom}_R(P_i, I^j)$. We note that we have two natural morphisms of complexes:

$$\begin{tikzcd} & \operatorname{Hom}_R(P_\bullet, N) \arrow[d, "\iota_*"] \\ \operatorname{Hom}_R(M, I^\bullet) \arrow[r, "\epsilon^*"] & T^* \end{tikzcd}$$

Prove that the above ι_* and ϵ^* are both quasi-isomorphisms.

(Hint: Show that $\operatorname{Hom}_R(P_\bullet, -)$ preserves quasi-isomorphisms between cohomological complexes and $\operatorname{Hom}_R(-, I^\bullet)$ preserved quasi-isomorphisms between homological complexes.)

Lemmata

Lemma 2. *Let F be a covariant exact functor. F preserves (1) kernels; (2) images; (3) cokernels; (4) (co)homology; (5) quasi-isomorphisms.*

Proof. (1) Let $f : A \to B$ be a homomorphism. Let's show $F(\ker f) = \ker F(f)$. Take $0 \to \ker f \to A \xrightarrow{f} \operatorname{im} f \to 0$. This is exact. We obtain $0 \to F(\ker f) \to F(A) \xrightarrow{F(f)} F(\operatorname{im} f) \to 0$. Because F is an exact functor, that is exact. Thus, $\ker F(f) = \operatorname{im}(F(\ker f) \to F(A)) = F(\ker f)$. It means F preserves kernels.

(2) Let $f : A \to B$ be a homomorphism. $\operatorname{im} F(f) = F(f)(F(A)) = F(f(A)) = \operatorname{im} F(f)$. It means F preserves images.

(3) Let $f : A \to B$ be a homomorphism. Then, we can construct an exact sequence $0 \to \operatorname{coim} f \xrightarrow{f} B \to \operatorname{coker} f \to 0$. By f, we obtain an exact sequence $0 \to F(\operatorname{coim} f) \xrightarrow{F(f)} F(B) \to F(\operatorname{coker} f) \to 0$. and, $F(\operatorname{coker} f) \simeq F(B)/\operatorname{im} F(f) = \operatorname{coker} F(f)$.

(4) Let $A_\bullet$ be a chain complex with differentials $\partial_\bullet$. Take an exact sequence $0 \to \operatorname{im} \partial_{k-1} \xrightarrow{f} \ker \partial_k \to \operatorname{coker} f \to 0$ where f is an injection. Then, $0 \to \operatorname{im} F(\partial_{k-1}) = F(\operatorname{im} \partial_{k-1}) \xrightarrow{F(f)} \ker F(\partial_k) = F(\ker \partial_k) \to F(\operatorname{coker} f) \to 0$ is exact. Then, $\operatorname{coker} f = \ker \partial_k/\operatorname{im} f \simeq \ker \partial_k/\operatorname{im} \partial_{k-1} = H_k(A_\bullet)$, $F(\operatorname{coker} f) \simeq \operatorname{coker} F(f) = \ker F(\partial_k)/\operatorname{im} F(f) \simeq \ker F(\partial_k)/\operatorname{im} F(\partial_{k-1}) = H_k(F(A_\bullet))$. Thus, $F(H_k(A_\bullet)) = F(\operatorname{coker} f) = H_k(F(A_\bullet))$.

(5) Let $A_\bullet$, $B_\bullet$ be (co)chain complexes and $f_\bullet : A_\bullet \to B_\bullet$ be a quasi-isomorphism. Then, f_k restricted to $H_k(A_\bullet) \to H_k(B_\bullet)$ is an isomorphism (bijective homomorphism). In this case, $0 \to H_k(A_\bullet) \xrightarrow{f_k} H_k(B_\bullet) \to 0 \to 0$ is a short exact sequence. By F, $0 \to F(H_k(A_\bullet)) \xrightarrow{F(f_k)} F(H_k(B_\bullet)) \to 0 \to 0$ is exact. Since F preserve (co)homologies, we obtain an exact sequence $0 \to H_k(F(A_\bullet)) \xrightarrow{F(f_k)} H_k(F(B_\bullet)) \to 0 \to 0$. This shows, $F(f_\bullet)$ is a quasi-isomorphism from $F(A_\bullet)$ to $F(B_\bullet)$. ☐

Lemma 3. *Let F be a contravariant exact functor. F preserves a quasi-isomorphism.*

Proof. It's a dual of above lemma. ☐

Lemma 4. $\operatorname{Hom}_R(P_k, -)$ *preserves a quasi-isomorphism.* $\operatorname{Hom}_R(-, I^k)$ *preserves a quasi-isomorphism.*

Proof. Note that $\operatorname{Hom}_R(P_k, -)$ and $\operatorname{Hom}_R(-, I^k)$ are exact. ☐

Lemma 5. *Let $A_j^\bullet$ and $B_j^\bullet$ be cochain complexes for each $j \in \mathbb{N}$. And let $f_j^\bullet : A_j^\bullet \to B_j^\bullet$ be a quasi-isomorphism for each $j \in \mathbb{N}$. Let $A_\mathbb{N}^\bullet = \bigoplus_{j\in\mathbb{N}} A_j^\bullet$ $B_\mathbb{N}^\bullet = \bigoplus_{j\in\mathbb{N}} B_j^\bullet$ and $f_\mathbb{N}^\bullet = \bigoplus_{j\in\mathbb{N}} f_j^\bullet$. Then, $f_\mathbb{N}^\bullet : A_\mathbb{N}^\bullet \to B_\mathbb{N}^\bullet$ is a quasi-isomorphism.*

Proof. Let $\alpha_j^\bullet$ be a differential of $A_j^\bullet$ $\beta_j^\bullet$ be a differential of $B_j^\bullet$.

Note that $f_\mathbb{N}^\bullet$ maps j-th entry of $A_\mathbb{N}^\bullet$ which is an element of $A_j^\bullet$ to the j-th entry of $B_\mathbb{N}^\bullet$ which is an element of $B_j^\bullet$.

Since $f_\mathbb{N}^\bullet$ maps zero entries to zero entries, $f_\mathbb{N}^\bullet$ is a well-defined homomorphism even it's a direct sum of infinitely many modules.

Note that $\ker \alpha_\mathbb{N}^k = \bigoplus_{j\in\mathbb{N}} \ker \alpha_j^k$ for each $k \in \mathbb{Z}$, because for $a \in A_\mathbb{N}^k$, $\alpha_\mathbb{N}^k(a) = 0$ iff each j-th entry of a is in $\ker \alpha_j^k$.

Also, $\operatorname{im} \alpha_\mathbb{N}^k = \bigoplus_{j\in\mathbb{N}} \operatorname{im} \alpha_j^k$ for each $k \in \mathbb{Z}$, because for $b \in A_\mathbb{N}^{k+1}$, there is $a \in A_\mathbb{N}^k$ such that $\alpha_\mathbb{N}^k(a) = b$ iff each j-th entries of a are mapped to the j-th entry of b by f_j^k.

In the same way, $\ker \beta_\mathbb{N}^k = \bigoplus_{j\in\mathbb{N}} \ker \beta_j^k$ and $\operatorname{im} \beta_\mathbb{N}^k = \bigoplus_{j\in\mathbb{N}} \operatorname{im} \beta_j^k$.

Thus, $H^k(A_\mathbb{N}^\bullet) = \ker \alpha_\mathbb{N}^k / \operatorname{im} \alpha_\mathbb{N}^{k-1} \bigoplus_{j\in\mathbb{N}} \ker \alpha_j^k / \operatorname{im} \alpha_j^k$ and $H^k(B_\mathbb{N}^\bullet) = \ker \beta_\mathbb{N}^k / \operatorname{im} \beta_\mathbb{N}^{k-1} \bigoplus_{j\in\mathbb{N}} \ker \beta_j^k / \operatorname{im} \beta_j^k$

Because $f_\mathbb{N}^\bullet$ is an isomorphism between each j-th entry of $H^\bullet(A_\mathbb{N}^\bullet)$ and $H^\bullet(B_\mathbb{N}^\bullet)$. Therefore, $f_\mathbb{N}^\bullet$ is an isomorphism between $H^\bullet(A_\mathbb{N}^\bullet)$ and $H^\bullet(B_\mathbb{N}^\bullet)$.

This shows $f_\mathbb{N}^\bullet$ is a quasi-isomorphism. $\square$

Proof

Because $P_\bullet \to M$ is a projective resolution, we have a quasi isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\epsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Then, $\mathrm{Hom}_R(-, I^n)$ maps above as

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Hom}_R(M, I^n) \arrow[r] \arrow[d, "\mathrm{Hom}_R(\epsilon, I^n)"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{Hom}_R(P_0, I^n) \arrow[r] & \mathrm{Hom}_R(P_1, I^n) \arrow[r] & \mathrm{Hom}_R(P_2, I^n) \arrow[r] & \cdots
\end{tikzcd}$$

where the chain map between two rows is a quasi-isomorphism by Lemma **??**.

Let $A_n^{\bullet - n}$ be the above rows and $B_n^{\bullet - n}$ be the below rows of the above diagram. In other words,

$$A_n^k = \begin{cases} \mathrm{Hom}_R(M, I^k) & (k = n) \\ 0 & (k \neq n) \end{cases}, B_n^k = \begin{cases} \mathrm{Hom}_R(P_{k-n}, I^k) & (k \geq n) \\ 0 & (k < n) \end{cases} \quad \text{where } k \in \mathbb{Z}$$

Let $f_n^\bullet : A_n^\bullet \to B_n^\bullet$ be a quasi-isomorphism such that $f_n^n = \mathrm{Hom}_R(\epsilon, I^n)$, $f_n^k = 0$ for $k \neq n$.

Let $A_\mathbb{N}^\bullet = \bigoplus_{j\in\mathbb{N}} A_j^\bullet$, $B_\mathbb{N}^\bullet = \bigoplus_{j\in\mathbb{N}} B_j^\bullet$, and $f_\mathbb{N}^\bullet = \bigoplus_{j\in\mathbb{N}} f_j^\bullet$.

Then,

$$A_\mathbb{N}^k = \begin{cases} \mathrm{Hom}_R(M, I^k) & (k \geq 0) \\ 0 & (k < 0) \end{cases}, B_\mathbb{N}^k = \begin{cases} \bigoplus_{i+j=k} \mathrm{Hom}_R(P_i, I^j) & (k \geq 0) \\ 0 & (k < 0) \end{cases} \quad \text{where } k \in \mathbb{Z}$$

Note that $A_\mathbb{N}^k = \mathrm{Hom}_R(M, I^k)$, and $B_\mathbb{N}^k = T^k$ since $P_\bullet, I^\bullet$ of negative degree are zero. In other words $f_\mathbb{N}^\bullet$ is a chain map from $\mathrm{Hom}_R(M, I^\bullet)$ to T^k.

By Lemma **??**, $f_\mathbb{N}^\bullet = \bigoplus_{j\in\mathbb{N}} f_j^\bullet : A_\mathbb{N}^\bullet \to B_\mathbb{N}^\bullet$ is a quasi-isomorphism. Let $\epsilon^* = f_\mathbb{N}^\bullet$.

In the same way, we have a quasi-isomorphism from an injective resolution:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

and we obtain the below diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_R(P_n, N) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(P_n,\iota)} & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_R(P_n, I^0) & \longrightarrow & \mathrm{Hom}_R(P_n, I^1) & \longrightarrow & \mathrm{Hom}_R(P_n, I^2) & \longrightarrow & \cdots \end{array}$$

where the chain map between two rows is a quasi-isomorphism by **??**.

Define $C_n^{\bullet-n}$, $D_n^{\bullet-n}$ as:

$$C_n^k = \begin{cases} \mathrm{Hom}_R(P_k, N) & (k = n) \\ 0 & (k \neq n) \end{cases}, D_n^k = \begin{cases} \mathrm{Hom}_R(P_k, I^{k-n}) & (k \geq n) \\ 0 & (k < n) \end{cases} \quad \text{where } k \in \mathbb{Z}$$

Let $g_n^\bullet : C_n^\bullet \to D_n^\bullet$ be a quasi-isomorphism such that $g_n^n = \mathrm{Hom}_R(P_n, \iota)$, $g_n^k = 0$ for $k \neq n$.

Let $C_{\mathbb{N}}^\bullet = \bigoplus_{j\in\mathbb{N}} C_j^\bullet$, $D_{\mathbb{N}}^\bullet = \bigoplus_{j\in\mathbb{N}} D_j^\bullet$, and $g_{\mathbb{N}}^\bullet = \bigoplus_{j\in\mathbb{N}} g_j^\bullet$.

Then,

$$C_{\mathbb{N}}^k = \begin{cases} \mathrm{Hom}_R(P_k, N) & (k \geq 0) \\ 0 & (k < 0) \end{cases}, D_{\mathbb{N}}^k = \begin{cases} \bigoplus_{i+j=k} \mathrm{Hom}_R(P_i, I^j) & (k \geq 0) \\ 0 & (k < 0) \end{cases} \quad \text{where } k \in \mathbb{Z}$$

Note that $C_{\mathbb{N}}^k = \mathrm{Hom}_R(P_k, N)$, $D_{\mathbb{N}}^k = T^k$. Also, by Lemma **??**, $g_{\mathbb{N}}^\bullet = \bigoplus_{j\in\mathbb{N}} g_j^\bullet : C_{\mathbb{N}}^\bullet \to D_{\mathbb{N}}^\bullet$ is a quasi-isomorphism. Let $\iota^* = g_{\mathbb{N}}^\bullet$.

Therefore, $\epsilon_* = f_{\mathbb{N}}^\bullet : \mathrm{Hom}_R(M, I^\bullet) \to T^\bullet$ and $\iota_* = g_{\mathbb{N}}^\bullet : \mathrm{Hom}_R(P_\bullet, N) \to T^\bullet$ are quasi-isomorphisms.
□

From the below problem, Hom without subscript means Hom_R.

Problem 4

Let R be a ring with a unity and let M, N be R-modules.

Note: $\mathrm{Ext}_R(M,N)$ is the set of equivalence classes of extensions of M by N (i.e. eq cls by $0 \to N \to T \to M \to 0$ and is there is a homo T to T')

For given $e := [0 \to N \to T \to M \to 0] \in \mathrm{Ext}_R(M,N)$, we can make $\mathrm{Ext}_R(M,-)$ long sequence:

$$0 \to \mathrm{Ext}^0_R(M,N) \to \mathrm{Ext}^0_R(M,T) \to \mathrm{Ext}^0_R(M,M) \xrightarrow{\partial} \mathrm{Ext}^1_R(M,N) \to \cdots$$

Note that $\mathrm{Id}_M \in \mathrm{Hom}_R(M,M) = \mathrm{Ext}^0_R(M,M)$.

Let $\delta : \mathrm{Ext}_R(M,N) \to \mathrm{Ext}^1_R(M,N)$. Then, let $\delta(e) = \partial(\mathrm{Id}_M)$.

We can find $0 \to K \xrightarrow{\alpha} P \to M \to 0$ where P is projective. Then, from long exact sequence, we obtain

$$\mathrm{Hom}_R(P,N) \to \mathrm{Hom}_R(K,N) \xrightarrow{\partial'} \mathrm{ext}^1_R(M,N) \to 0 = \mathrm{ext}^1_R(P,N)$$

Let $x \in \mathrm{Ext}^1_R(M,N) \simeq \mathrm{ext}^1_R(M,N)$. There is some $\beta \in \mathrm{Hom}_R(K,N)$ such that $\partial'(\beta) = x$.

Consider the push-out T of $N \xleftarrow{\beta} K \xrightarrow{\alpha} P$ which is $\mathrm{coker}\,(K \to P \oplus N)$ for the map $k \mapsto (\alpha(k), -\beta(k))$. Let $i : N \to T$ be the natural induced morphism. Also, because of surjection $\pi : P \to M$ and $\phi : P \to T$, we obtain $\psi : T \to M$ such that $\psi = \pi\phi^{-1}$.

In this case, we obtain the below diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "\beta"] & P \arrow[r, "\pi"] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "i"] & T \arrow[r, "\psi"] & M \arrow[r] & 0
\end{tikzcd}$$

Let $e = [0 \to N \to T \to M \to 0]$.

Prove that $\delta(e) = x$.

Proof

Note that, if we apply Snake Lemma to below diagram with exact rows,

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B \arrow[r, "\beta'"] & C &
\end{tikzcd}$$

we obtain the exact sequence

$$\ker f \xrightarrow{\tilde{\alpha}} \ker g \xrightarrow{\tilde{\beta}} \ker h \xrightarrow{\partial} \mathrm{coker}\, f \xrightarrow{\tilde{\alpha}'} \mathrm{coker}\, g \xrightarrow{\tilde{\beta}'} \mathrm{coker}\, h$$

where the funcitons with tilde are induced from ones without tilde. Also ∂ is constructed as $\tilde{\alpha}'^{-1} \circ g \circ \tilde{\beta}^{-1}$ for the restricted domain $\ker h$.

Suppose that $0 \to N \xrightarrow{\alpha} T \xrightarrow{\pi} M \to 0$ is an exact sequence. Let $N \xrightarrow{f} I^\bullet_N$, $T \xrightarrow{g} I^\bullet_T$, $M \xrightarrow{h} I^\bullet_M$ be injective resolutions. We obtain Ext beginning from

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M, I_N^k) \arrow[r, "\mathrm{Hom}(M,\alpha^k)"] \arrow[d, "\mathrm{Hom}(M,f^k)"] & \mathrm{Hom}(M, I_T^k) \arrow[r, "\mathrm{Hom}(M,\pi^k)"] \arrow[d, "\mathrm{Hom}(M,g^k)"] & \mathrm{Hom}(M, I_M^k) \arrow[r] \arrow[d, "\mathrm{Hom}(M,h^k)"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(M, I_N^{k+1}) \arrow[r, "\mathrm{Hom}(M,\alpha^{k+1})"] & \mathrm{Hom}(M, I_T^{k+1}) \arrow[r, "\mathrm{Hom}(M,\pi^{k+1})"] & \mathrm{Hom}(M, I_M^{k+1}) \arrow[r] & 0
\end{tikzcd}$$

Note that Ext long exact sequence is obtained by applying the Snake Lemma twice to the above diagram. (See Problem 2)

At the first application of the Snake Lemma, we obtain an exact sequence

$$\ker \mathrm{Hom}(M, f^k) \xrightarrow{\sigma} \ker \mathrm{Hom}(M, g^k) \xrightarrow{\upsilon} \ker \mathrm{Hom}(M, h^k)$$
$$\xrightarrow{\partial} \operatorname{coker} \mathrm{Hom}(M, f^k) \xrightarrow{\sigma'} \operatorname{coker} \mathrm{Hom}(M, g^k) \xrightarrow{\upsilon'} \operatorname{coker} \mathrm{Hom}(M, h^k)$$

Note that each $\sigma, \upsilon, \sigma', \upsilon'$ is induced from $\mathrm{Hom}(M, \alpha^k), \mathrm{Hom}(M, \pi^k), \mathrm{Hom}(M, \alpha^{k+1}), \mathrm{Hom}(M, \pi^{k+1})$. Also, each $\mathrm{Hom}(M, f^k), \mathrm{Hom}(M, g^k), \mathrm{Hom}(M, h^k)$ induces to maps from cokernels to kernels.

And at the second application of the Snake Lemma, we obtain a map

$$\partial : \operatorname{coker} \mathrm{Hom}(M, h^{k-1})^* \to \ker \mathrm{Hom}(M, f^{k+1})^*$$

where $\mathrm{Hom}(M, h^{k-1})^*$ and $\mathrm{Hom}(M, f^{k+1})^*$ are induced one. Note that $\partial : \mathrm{Ext}_R^k(M, N) \to \mathrm{Ext}_R^{k+1}(M, N)$, and $\partial = (\sigma')^{-1} \circ \mathrm{Hom}(M, g^k)^* \circ (\upsilon)^{-1}$ for the restricted domain following applications of the Snake Lemma.

Note that the exact sequence

$$0 \to N \xrightarrow{\alpha} T \xrightarrow{\pi} M \to 0$$

was given. Note that we can induce ∂_j from above sequences. Then, for the injective resolutions $T \xrightarrow{g} I^\bullet$ and differentials $g^\bullet$ of $I^\bullet$,

$$\partial = (\mathrm{Hom}(M, \alpha^1)^*)^{-1} \circ \mathrm{Hom}(M, g^0)^* \circ (\mathrm{Hom}(M, \pi^0)^*)^{-1}$$

where each maps with a superscript $*$ are induced one in the diagram just before the second application of the Snake Lemma.

In the similar way, For the given exact sequence $0 \to K \xrightarrow{i} P \xrightarrow{\psi} M \to 0$,

$$\partial' : \mathrm{ext}_R^0(K, N) \to \mathrm{ext}_R^1(M, N)$$

obtained by ext long exact sequence is $\partial' = (\mathrm{Hom}(\psi_{k+1}, N)^*)^{-1} \circ \mathrm{Hom}(g_k, N)^* \circ (\mathrm{Hom}(i_k, N)^*)^{-1}$ where supserscript $*$ means 'induced', $0 \to P_{K\bullet} \xrightarrow{i_\bullet} P_{P\bullet} \xrightarrow{\psi_\bullet} P_{M\bullet} \to 0$ is exact, $P_{K\bullet} \xrightarrow{f} K$, $P_{P\bullet} \xrightarrow{g} P$, $P_{M\bullet} \xrightarrow{h} M$ are projective resolutions, and $g_\bullet$ are differentials of $P_{P\bullet}$.

Because $x = \partial'(\beta)$ and $\partial(\mathrm{Id}_M) = \delta(e)$, it's enough to show that $\partial(\mathrm{Id}_M) = \partial'(\beta)$.

Note that,

$$\partial(\mathrm{Id}_M) = ((\alpha^1)^*)^{-1} \circ (g^0)^* \circ ((\pi^0)^*)^{-1} \circ \mathrm{Id}^M = ((\alpha^1)^*)^{-1} \circ (g^0)^* \circ ((\pi^0)^*)^{-1}$$

$$\partial'(\beta) = \beta \circ ((i_0)^*)^{-1} \circ (g_0)^* \circ ((\psi_1')^*)^{-1}$$

Then, from the below given diagram,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "\beta"] & P \arrow[r, "\pi"] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "i"] & T \arrow[r, "\psi"] & M \arrow[r] & 0
\end{tikzcd}$$

We obtain the below commutative diagram:

$$\begin{tikzcd}
K \arrow[dd, "\beta"] & P \arrow[l, "\alpha^{\mathrm{op}}"'] & \\
 & & M \arrow[ul, "\pi^{\mathrm{op}}"'] \arrow[dl, blue, "\psi^{\mathrm{op}}"] \\
N & T \arrow[l, blue, "i^{\mathrm{op}}"] &
\end{tikzcd}$$

Since α^1, π^0, i_0, ψ_1 are induced by α, π, i, ψ, and $g_\bullet$ and $g^\bullet$ are transitions between resolutions, the composition of the above black arrows induces $\partial'(\beta)$ and the composition of the below blue arrows induces $\partial(\mathrm{Id}_M)$.

Therefore, $\partial(\mathrm{Id}_M) = \partial'(\beta)$. $\square$

Problem 5

(Continue from Problem 4)

We obtained a set map $\eta : \operatorname{Ext}_R^1(M,N) \to \operatorname{Ext}_R(M,N)$ which maps x to e.

Let $\beta' \in \operatorname{Hom}_R(K,N)$ such that $\partial'(\beta') = x$. Since

$$\operatorname{Hom}_R(P,N) \xrightarrow{\alpha^*} \operatorname{Hom}_R(K,N) \to \operatorname{Ext}_1^R(M,N)$$

is exact, there is some $\gamma \in \operatorname{Hom}_R(P,N)$ such that $\beta' = \beta + \gamma \circ \alpha$

Let $e' = [0 \to N \to T' \to M \to 0]$ obtained by taking the push-out of α and β'.

Prove that e and e' are equivalent short exact sequences.

Proof

Let $f : K \to P \oplus N$ be a homomorphism $K \to P \oplus N$, such that $f(k) = (\alpha(k), -\beta(k))$. And, let $f' : K \to P \oplus N$ be a homomorphism $K \to P \oplus N$, such that $f(k) = (\alpha(k), -\beta'(k))$.

$$\begin{tikzcd}
0 \arrow[rr] & & N \arrow[rr, "i"] & & T \arrow[rr, "\psi"] \arrow[dd, dashed, "\tau"] & & M \arrow[r] \arrow[dd, equal] & 0 \\
0 \arrow[r] & K \arrow[ur, "\beta"] \arrow[rr, "\alpha"] \arrow[dr, "\beta'"'] & & P \arrow[ur, "\phi"] \arrow[rr, "\pi"] \arrow[dr, "\phi'"'] & & M \arrow[ur, equal] \arrow[dr, equal] \arrow[rr] & & 0 \\
0 \arrow[rr] & & N \arrow[rr, "i'"] & & T' \arrow[rr, "\psi'"] & & M \arrow[r] & 0
\end{tikzcd}$$

In this case, take $\tau : T \to T'$ as, $\tau((p,n) + \operatorname{im} f) = (p, n - \gamma(p)) + \operatorname{im} f'$.

First, this is well-defined. Let $p, p' \in P$, $n, n' \in N$ such that $(p,n) + \operatorname{im} f = (p', n') + \operatorname{im} f$. Then, $(p-p', n-n') \in \operatorname{im} f$. In other words, there is $k \in K$ such that $p - p' = \alpha(k)$, $n - n' = -\beta(k)$. Then, $-\beta'(k) = -\beta(k) - \gamma(\alpha(k)) = -\beta(k) - \gamma(p-p')$. Then, $p - p' = \alpha(k)$, and $n - n' - \gamma(p-p') = -\beta'(k)$. This shows that $(p-p', (n-\gamma(p)) - (n' - \gamma(p'))) = (p-p', n-n'-\gamma(p-p')) \in \operatorname{im} f'$, and $(p, n-\gamma(p)) + \operatorname{im} f' = (p', n' - \gamma(p')) + \operatorname{im} f'$. Therefore, τ is well-defined.

This is a homomorphism because it's a combination of homomorphisms.

Let $n \in N$. $\tau(i(n)) = \tau((0,n) + \operatorname{im} f) = (0, n - \gamma(0)) + \operatorname{im} f' = (0,n) + \operatorname{im} f' = i'(n)$. Therefore, $\tau \circ i = i'$. And by the definition of class of extensions, e and e' are equivalent. $\square$

Problem 6

Note: Let $e_i : 0 \to N \to T_i \to M \to 0$ be s.e.s for $i = 1, 2$.

Consider the pull-back T' of $T_1 \to M \leftarrow T_2$ i.e. $T' \subseteq T_1 \times T_2$ consisting of (t_1, t_2) whose images in M coincide. Let $D \subseteq T'$ be generated by $(-n, n)$ for $n \in N$ and let $T = T'/D$.

This gives a s.e.s $e : 0 \to N \to T \to M \to 0$, which is a Baer sum of e_1 and e_2.

For $e_1, e_2 \in \mathrm{Ext}_R(M, N)$, prove that defining $e_1 + e_2 = e$ gives an abelian group structure on $\mathrm{Ext}_R(M, N)$.

For $0 \to N \to T \to M \to 0$, what is the inverse in the group?

Proof

Let the below be the diagram of pullback T':

$$\begin{tikzcd} T' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & T_2 \arrow[d, "\mu_2"] \\ T_1 \arrow[r, "\mu_1"] & M \end{tikzcd}$$

Also, let $\nu_j : N \to T_j$ for $j = 1, 2, 3$.

Let $e = e_1 + e_2 : 0 \to N \xrightarrow{\nu} T \xrightarrow{\mu} M \to 0$ such that $\nu(n) = (\nu_1(n), 0) + D = (0, \nu_2(n)) + D$ and $\mu((t_1, t_2) + D) = \mu_1(t_1) = \mu_2(t_2)$. Note that they are well-defined. First, $(\nu_1(-n), \nu_2(n)) \in D$ for any $n \in N$. This shows that $(\nu_1(n), 0) + D = (0, \nu_2(n)) + D$. Second, for $t_1 \in T_1$ and $t_2 \in T_2$, $(t_1, t_2) \in T$ implies $\mu_1(t_1) = \mu_2(t_2)$. Also, if there are $t_1, t_1' \in T_1$ and $t_2, t_2' \in T_2$ such that $(t_1, t_2) + D = (t_1', t_2') + D$, $(t_1 - t_1', t_2 - t_2') \in D$. Then, there is $n \in N$ such that $\nu_1(n) = t_1 - t_1'$ and $\nu_2(n) = t_2 - t_2'$. Since $\mu_j \circ \nu_j = 0$, $\mu_1(t_1 - t_1') = \mu_2(t_2 - t_2') = 0$. This shows μ is well-defined.

First, $\mathrm{Ext}_R(M, N)$ is closed under $+$.

$(\mathrm{Ext}_R(M, N), +)$ is commutative. It's because $T_1 \times T_2 \simeq T_2 \times T_1$ and $T_1 \times T_2$ and $T_2 \times T_1$ gives isomorphic pullbacks. Let φ be an isomorphism from $T_1 \times T_2$ to $T_2 \times T_1$. Then, let $U = T_1 + T_2$ be T'/D obtained from $T_1 \times T_2$, and let $V = T_2 + T_1$ be T'/D obtained from $T_2 \times T_1$. Then, $U \simeq V$ by an isomorphism induced by φ. Therefore, $0 \to N \to U \to M \to 0$ and $0 \to N \to V \to M \to 0$ are equivalent by φ. This shows $T_1 + T_2 = T_2 + T_1$.

$(\mathrm{Ext}_R(M, N), +)$ is associative. Let we have three extensions of M by N, through T_1, T_2, T_3. Let $T_1' = \{(t_1, t_2) \mid \mu_1(t_1) = \mu_2(t_2)\} \subseteq T_1 \times T_2$. Let D_1 be generated by $(-n, n) \in T_1'$ for $n \in N$. Then, we obtain induced $\tilde{\mu}_1 : T_1'/D_1 \to T$ such that $\tilde{\mu}_1((t_1, t_2) + D_1) = \mu_1(t_1) = \mu_2(t_2)$, and $\tilde{\nu}_1(n) = (\nu_1(n), 0) + D_1 = (0, \nu_2(n)) + D_1$. It's well-defined because D_1 is a submodule which makes $\mu_1(t_1) = \mu_2(t_2) = 0$ because $\mu_k \circ \nu_k = 0$. Then, let $T_1'' = \{(t_1, t_3) \mid \tilde{\mu}_1(t_1) = \mu_3(t_3)\} \subseteq T_1'/D_1 \times T_3$. Let D_1' be generated by $(-n, n) \in T_1''$ for $n \in N$. In the same way, we can generate $T_2', D_2, \tilde{\mu}_2$ from T_2 and T_3, and we can generate T_2'', D_2' from T_2'. What we need to show is $T_1''/D_1' \simeq T_2''/D_2'$. Let $t_k \in T_k$ for $k = 1, 2, 3$. Then, let $\tau : T_1''/D_1' \to T_2''/D_2'$ such that

$$\tau : ((t_1, t_2) + D_1, t_3) + D_1' \mapsto (t_1, (t_2, t_3) + D_2) + D_2'$$

First, let's check it's well-defined. Suppose that $((t_1, t_2) + D_1, t_3) + D_1' = ((t_1', t_2') + D_1, t_3') + D_1' \in T_1''/D_1'$ for $t_k, t_k' \in T_k$ for $k = 1, 2, 3$. Let $d_k = t_k - t_k'$ for $k = 1, 2, 3$. Then, $0 = ((d_1, d_2) + D_1, d_3) + D_1'$. This shows $((d_1, d_2) + D_1, d_3) \in D_1'$, and there is $n \in N$ such that $d_3 = \nu_3(n)$ and $(d_1, d_2) + D_1 = \tilde{\nu}_1(n) = (\nu_1(n), 0) + D_1$. Then, there is $n' \in N$ such that $d_1 = \nu_1(n - n')$, $d_2 = \nu_2(n')$. Then, $(\nu_1(n - n'), (\nu_2(n'), \nu_3(n)) + D_2) + D_2' = (0, (\nu_2(n), \nu_3(n)) + D_2) + D_2' = (0, (0, 0) + D_2) + D_2'$. Thus, τ maps $((t_1, t_2) + D_1, t_3) + D_1'$ and $((t_1', t_2') + D_1, t_3') + D_1'$ to the same value if they are same. τ is a homomorphism, since it's a composition of homomorphism. Thus, because of the equivalence of extension and there is a homomorphism τ, $0 \to N \to T_1''/D_1' \to M \to 0$ and $0 \to N \to T_2''/D_2' \to M \to 0$ are equivalent. This shows that $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$.

$\mathrm{Ext}_R(M, N)$ contains an identity, $N \oplus M$. Suppose that $T_2 = N \oplus M$ and μ_2 be a canonical projection of $N \oplus M$ to M. Let $t \in T_1, n \in N, m \in M$. $(t, (n, m)) + D \in T'/D$ if $\mu_1(t) = \mu_2(n, m) = m$. Since, $(-n, n)$ for $n \in N$ is mapped to $(\nu_1(-n), (n, 0))$ by (ν_1, ν_2), and $\mu_1(\nu_1(-n)) = 0 = \mu_2(n, 0)$ because $\mu_1 \circ \nu_1 = \mu_2 \circ \nu_2 = 0$, $(\nu_1(-n), (n, 0)) \in T'$. Thus, $D = \{(\nu_1(-n), (n, 0)) \mid n \in N\}$. Let $\varphi : T_1 \to T'/D$ such that $\varphi : t \mapsto (t, (0, \mu_1(t))) + D$. This is injective, because if $(t, (0, \mu_1(t))) + D = D$, $(t, (0, \mu_1(t))) \in D$, and $t = \nu_1(0)$. Since ν_1 is injective, $t = 0$. This shows $\ker \varphi = \{0\}$. φ is surjective. If $(t, (n, m)) + D \in T'/D$, since $(\nu_1(-n), (n, 0)) \in D$, $(t, (n, m)) + D = (t + \nu_1(n), (0, m)) + D$. Note that $(t, (n, m)) \in T'$, $\mu_1(t) = \mu_2(n, m) = m$. Thus, $\mu_1(t) = \mu_1(t + \nu_1(n)) = \mu_2(0, m) = m$, as $\mu_1 \circ \nu_1 = 0$. Therefore, $t + \nu_1(n)$ is mapped to $(t, (n, m)) + D \in T'/D$ by φ and φ is surjective. Lastly, φ is a homomorphism because it's a composition of homomorphisms. Therefore, φ is an isomorphism and $T_1 \simeq T'/D$. This shows, $N \oplus M$ is the right identity. Since $\mathrm{Ext}_R(N, M)$ is commutative, $N \oplus M$ is also the left identity.

$\mathrm{Ext}_R(M, N)$ contains an inverse of arbitrary T_1. Let $T_2 = T_1$, $\nu_2 = -\nu_1$ and $\mu_2 = \mu_1$. Let's show that T_2 is an inverse of T_1. Then, $T' \subseteq T_1 \times T_2 = T_1^2$. And, $(t_1, t_2) \in T'$, $\mu_1(t_1) = \mu_2(t_2) = \mu_2(t_2)$. In other words, $T' = \{(t_1, t_2) \mid t_1 - t_2 \in \ker \mu_1 = \operatorname{im} \nu_1\}$. And $D = \{(\nu_1(-n), \nu_2(n)) \mid n \in N\} = \{(t, t) \mid n \in N, t = -\nu_1(t) = \nu_2(t)\}$. Then, let $\varphi : M \to T'/D$ such that $\varphi(m) = (t, t) + D$ where $\mu_1(t) = \mu_2(t) = m$. Let $t_1, t_2 \in T_1 = T_2$. If $\mu_1(t_1) = \mu_1(t_2) = m$, $\mu_1(t_1 - t_2) = 0$, $t_1 - t_2 \in \ker \mu_1 = \operatorname{im} \nu_1$. Then, there is $n \in N$ such that $\nu_1(n) = t_1 - t_2$. This shows $(t_1 - t_2, t_1 - t_2) \in D$, and $(t_1, t_1) + D = (t_2, t_2) + D$. Thus, φ is well-defined. And since it's an homomorphism because it's an inverse of homomorphism. Therefore, $0 \to N \to T'/D \to M \to 0$ is split. This shows $T'/D \simeq N \otimes M$, which is an identity of Baer sum. Therefore, the $T_2 = T_1$ with $\nu_2 = -\nu_1$ and $\mu_2 = \mu_1$ is an inverse of T_1.

Therefore, $(\mathrm{Ext}_R(M, N), +)$ is closed, associative, commutative, and contains an identity and all inverses of its elements. $\square$

Problem 7

With the above Baer sum group structure on $\mathrm{Ext}_R(M,N)$, prove that $\delta : \mathrm{Ext}_R(M,N) \to \mathrm{Ext}^1_R(M,N)$ is a group homomorphism.

Proof

Let $e_1 : [0 \to N \to T_1 \to M \to 0]$ and $e_2 : [0 \to N \to T_2 \to M \to 0]$. Then, $e_3 = e_1 + e_2 : [0 \to N \to (T_1 \times_M T_2)/D \to M \to 0]$.

$\delta(e_1) = \partial_1(\mathrm{Id}_M)$, $\delta(e_2) = \partial_2(\mathrm{Id}_M)$, $\delta(e_3) = \delta(e_1 + e_2) = \partial_3(\mathrm{Id}_M)$ where each ∂_1, ∂_2, ∂_3 are come from the Ext long exact sequence of e_1, e_2, $e_1 + e_2$.

Note that $\partial_1 : (\ker(\mathrm{Hom}(M,I^0) \to \mathrm{Hom}(M,I^1))/\mathrm{im}\,(\mathrm{Hom}(M,I^{-1}) \to \mathrm{Hom}(M,I^0))) \to (\ker(\mathrm{Hom}(N,I^1) \to \mathrm{Hom}(N,I^2))/\mathrm{im}\,(\mathrm{Hom}(N,I^0) \to \mathrm{Hom}(N,I^1)))$.

Note that, if we apply Snake Lemma to below diagram with exact rows,

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B \arrow[r, "\beta'"] & C & \end{tikzcd}$$

we obtain the exact sequence

$$\ker f \xrightarrow{\tilde{\alpha}} \ker g \xrightarrow{\tilde{\beta}} \ker h \xrightarrow{\partial} \operatorname{coker} f \xrightarrow{\tilde{\alpha}'} \operatorname{coker} g \xrightarrow{\tilde{\beta}'} \operatorname{coker} h$$

where the funcitons with tilde are induced from ones without tilde. Also ∂ is constructed as $\tilde{\alpha}'^{-1} \circ g \circ \tilde{\beta}^{-1}$ for the restricted domain $\ker h$.

Suppose that $0 \to N \xrightarrow{\nu} T \xrightarrow{\mu} M \to 0$ is an exact sequence. Let $N \xrightarrow{f} I_N^\bullet$, $T \xrightarrow{g} I_T^\bullet$, $M \xrightarrow{h} I_M^\bullet$ be injective resolutions. We obtain Ext beginning from

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(M, I_N^k) \arrow[r, "\mathrm{Hom}(M,\nu^k)"] \arrow[d, "\mathrm{Hom}(M,f^k)"] & \mathrm{Hom}(M, I_T^k) \arrow[r, "\mathrm{Hom}(M,\mu^k)"] \arrow[d, "\mathrm{Hom}(M,g^k)"] & \mathrm{Hom}(M, I_M^k) \arrow[r] \arrow[d, "\mathrm{Hom}(M,h^k)"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(M, I_N^{k+1}) \arrow[r, "\mathrm{Hom}(M,\nu^{k+1})"] & \mathrm{Hom}(M, I_T^{k+1}) \arrow[r, "\mathrm{Hom}(M,\mu^{k+1})"] & \mathrm{Hom}(M, I_M^{k+1}) \arrow[r] & 0 \end{tikzcd}$$

Note that Ext long exact sequence is obtained by applying the Snake Lemma twice to the above diagram. (See Problem 2)

At the first application of the Snake Lemma, we obtain an exact sequence

$$\ker \mathrm{Hom}(M, f^k) \xrightarrow{\phi} \ker \mathrm{Hom}(M, g^k) \xrightarrow{\psi} \ker \mathrm{Hom}(M, h^k)$$
$$\xrightarrow{\partial} \operatorname{coker} \mathrm{Hom}(M, f^k) \xrightarrow{\phi'} \operatorname{coker} \mathrm{Hom}(M, g^k) \xrightarrow{\psi'} \operatorname{coker} \mathrm{Hom}(M, h^k)$$

Note that each ϕ, ψ, ϕ', ψ' is induced from $\mathrm{Hom}(M,\nu^k), \mathrm{Hom}(M,\mu^k), \mathrm{Hom}(M,\nu^{k+1}), \mathrm{Hom}(M,\mu^{k+1})$. Also, each $\mathrm{Hom}(M,f^k), \mathrm{Hom}(M,g^k), \mathrm{Hom}(M,h^k)$ induces to maps from cokernels to kernels.

And at the second application of the Snake Lemma, we obtain a map

$$\partial : \operatorname{coker} \mathrm{Hom}(M, h^{k-1})^* \to \ker \mathrm{Hom}(M, f^{k+1})^*$$

where $\mathrm{Hom}(M,h^{k-1})^*$ and $\mathrm{Hom}(M,f^{k+1})^*$ are induced one. Note that $\partial : \mathrm{Ext}^k_R(M,N) \to \mathrm{Ext}^{k+1}_R(M,N)$, and $\partial = (\phi')^{-1} \circ \mathrm{Hom}(M, g^k)^* \circ (\psi)^{-1}$ for the restricted domain following applications of the Snake Lemma.

Let

$$0 \to N \xrightarrow{\nu_j} T_j \xrightarrow{\mu_j} M \to 0$$

be exact sequences for $j = 1, 2, 3$. Note that we can induce ∂_j from above sequences. Then, for each injective resolutions $N \xrightarrow{f^\bullet} I_N^\bullet$, $T_j \xrightarrow{g_j^\bullet} I_{Tj}^\bullet$, $M \xrightarrow{h^\bullet} I_M^\bullet$, exact sequence $0 \to I_N^\bullet \xrightarrow{\nu_j^\bullet} I_{Tj}^\bullet \xrightarrow{\mu_j^\bullet} I_M^\bullet \to 0$,

$$\partial_j = (\mathrm{Hom}(M, \nu_j^1)^*)^{-1} \circ \mathrm{Hom}(M, g_j^0)^* \circ (\mathrm{Hom}(M, \mu_j^0)^*)^{-1}$$

where each maps with a superscript $*$ are induced one in the diagram just before the second application of the Snake Lemma.

What we need to show is $\partial_3(\mathrm{Id}_M) = \partial_1(\mathrm{Id}_M) + \partial_2(\mathrm{Id}_M)$.

Note that

$$\begin{aligned}\partial_j(\mathrm{Id}_M) &= ((\nu_j^1)^*)^{-1} \circ (g_j^0)^* \circ ((\mu_j^0)^*)^{-1} \circ \mathrm{Id}_M \\ &= ((\nu_j^1)^*)^{-1} \circ (g_j^0)^* \circ ((\mu_j^0)^*)^{-1}\end{aligned}$$

which is a homomorphism from M to $\mathrm{Cod}(\nu_j^1)$.

Let $m \in \mathrm{Dom}(\operatorname{im} \partial_j)$. Let $\partial_j(m) = \overline{n_j}$ for $n_j \in N$.

As we noted at the Problem 6, $\mu_3(\overline{(t_1, t_2)}) = \mu_1(t_1) = \mu_2(t_2)$. And since Baer sum is defined as quotient of pullback, $\overline{(t_1, t_2)} = \overline{(\nu_1(n_1), \nu_2(n_2))} = \overline{(\nu_1(n_1), 0)} + \overline{(0, \nu_2(n_2))}$ And, $\nu_3^{-1}(\overline{(t_1, t_2)}) = n_1 + n_2$. Thus, for every m,

$$\partial_3(\mathrm{Id}_M)(m) = n_1 + n_2 = \partial_1(\mathrm{Id}_M)(m) + \partial_2(\mathrm{Id}_M)(m)$$

Therefore,

$$\delta(e_1 + e_2) = \delta(e_3) = \partial_3(\mathrm{Id}_M) = \partial_1(\mathrm{Id}_M) + \partial_2(\mathrm{Id}_M) = \delta(e_1) + \delta(e_2)$$

$\square$