MAS341 2020 Spring Midterm

June 4, 2022

Problem 1

Prove that $\sum_{n=1}^{\infty} e^{-n^2} z^n$ is an entire function.

Proof

Let $f(z) = \sum_{n=1}^{\infty} e^{-n^2} z^n$.

Let $L = \limsup_{n\to\infty} |e^{-n^2}|^{1/n}$. Since e^x is strictly increasing positive function and $-n$ is strictly decreasing as n increases, $|e^{-n^2}|^{1/n} = e^{-n^2/n} = e^{-n}$ and $0 < e^{-(n+1)} < e^{-n}$ for every $n \in \mathbb{N}$. It implies $L = \limsup_{n\to\infty} |e^{-n^2}|^{1/n} = \lim_{n\to\infty} e^{-n} = 0$. By Theorem 2.8, the given power series $f(z)$ converges everywhere in $\mathbb{C}$.

By Corollary 2.10, $f(z)$ is infinitely differentiable its convergence domain. And since $f(z)$ converges in $\mathbb{C}$, $f(z)$ is infinitely differentiable in every point of $\mathbb{C}$. Thus $f(z)$ is entire. $\square$

Problem 2

Find all entire functions f such that $f(n\pi) = 0$ for any $n \in \mathbb{Z}$ and $|f(x+iy)| \le Ce^{|y|} < \infty$, $x, y \in \mathbb{R}$ for some $C > 0$.

Answer

$$\{z \mapsto K \sin z \mid K \in \mathbb{C}\}$$

Proof

First of all, note that $f = K \sin$ for $K \in \mathbb{C}$ satisfies all conditions in the problem:

- sin is entire. Thus $K \sin$ is also entire.
- $K \sin(n\pi) = K \cdot 0 = 0$ for every $n \in \mathbb{Z}$.
- Let $C = \max(1, 2|K|) > 0$. Then, for $x, y \in \mathbb{R}$,

$$\begin{aligned} |K \sin(x+iy)| &= |K| \cdot \left| \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i} \right| \\ &\le |K| \cdot \left| e^{-y}e^{ix} - e^{y}e^{-ix} \right| \\ &\le |K| \cdot \left(\left| e^{-y}e^{ix} \right| + \left| e^{y}e^{-ix} \right| \right) \\ &\le |K| \cdot \left(\left| e^{-y} \right| + \left| e^{y} \right| \right) \\ &= |K|(e^{y} + e^{-y}) \le 2|K|e^{|y|} \le Ce^{|y|} \end{aligned}$$

Suppose that f is some function satisfies all conditions given in the problem: f is entire, $\forall n \in \mathbb{Z} : f(n\pi) = 0$, there is $C > 0$, which satisfies $\forall x, y \in \mathbb{R} : |f(x+iy)| \le Ce^{|y|} < \infty$.

First of all, let's consider

$$\frac{f(z)}{\sin z}$$

It's well-defined only for $\mathbb{C} \backslash \pi\mathbb{Z}$, where $\sin z$ is non-zero. ($\pi\mathbb{Z} = \{\pi n \mid b \in \mathbb{Z}\}$) We know that $\sin z$ is expanded as the Taylor series:

$$\begin{aligned} \sin z &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \end{aligned}$$

And since $\sin(z - n\pi) = (-1)^n \sin(z)$ for $n \in \mathbb{Z}$ ($\because \sin(z - n\pi) = (e^{i(z-n\pi)} - e^{-i(z-n\pi)})/(2i) = (e^{iz}e^{-n\pi i} - e^{-iz}e^{n\pi i})/(2i) = ((-1)^n e^{iz} - (-1)^n e^{-iz})/(2i) = (-1)^n (e^{iz} - e^{-iz})/(2i) = (-1)^n \sin z$)

$$\begin{aligned} \sin z &= (-1)^n \sin(z - n\pi) \\ &= (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{(z-n\pi)^{2k+1}}{(2k+1)!} \\ &= (-1)^n \left((z-n\pi) - \frac{(z-n\pi)^3}{3!} + \frac{(z-n\pi)^5}{5!} - \cdots \right) \end{aligned}$$

Because f is entire, we can expand f as:

$$f(z) = \sum_{k=0}^{\infty} b_{n,k} (z - n\pi)^k$$

Since $f(0\pi) = f(0) = 0$, $b_{n,0} = 0$. Thus,

$$f(z) = \sum_{k=1}^{\infty} b_{n,k}(z - n\pi)^k$$

and,

Note that for an entire function ϕ, for every $a \in \mathbb{C}$,

$$\psi(z) = \begin{cases} \frac{\phi(z)-\phi(a)}{z-a} & (z \neq a) \\ \phi'(a) & (z = a) \end{cases}$$

is entire by Proposition 5.8, continuous and converges at every points. Thus,

$$\begin{aligned}
\lim_{z\to n\pi} \frac{f(z)}{\sin z} &= \lim_{z\to n\pi} \frac{\sum_{k=1}^{\infty} b_{n,k}(z-n\pi)^k}{(-1)^n\left((z-n\pi) - \frac{(z-n\pi)^3}{3!} + \frac{(z-n\pi)^5}{5!} - \cdots\right)} \\
&= \lim_{z\to n\pi} \frac{\frac{\sum_{k=1}^{\infty} b_{n,k}(z-n\pi)^k}{z-n\pi}}{(-1)^n \frac{(z-n\pi) - \frac{(z-n\pi)^3}{3!} + \frac{(z-n\pi)^5}{5!} - \cdots}{z-n\pi}} \\
&= \lim_{z\to n\pi} (-1)^n \frac{\sum_{k=1}^{\infty} b_{n,k}(z-n\pi)^{k-1}}{1 - \frac{(z-n\pi)^2}{3!} + \frac{(z-n\pi)^4}{5!} - \cdots} \\
&= (-1)^n \frac{\lim_{z\to n\pi} \sum_{k=1}^{\infty} b_{n,k}(z-n\pi)^{k-1}}{\lim_{z\to n\pi}\left(1 - \frac{(z-n\pi)^2}{3!} + \frac{(z-n\pi)^4}{5!} - \cdots\right)} \\
&= (-1)^n \frac{\sum_{k=1}^{\infty} b_{n,k}(z-n\pi)^{k-1}\Big|_{z=n\pi}}{1 - \frac{(z-n\pi)^2}{3!} + \frac{(z-n\pi)^4}{5!} - \cdots\Big|_{z=n\pi}} = (-1)^n b_{n,1}
\end{aligned}$$

Thus, if we take

$$g(z) = \begin{cases} (-1)^n b_{n,1} & (z = n\pi \in \pi\mathbb{Z}) \\ f(z)/\sin z & (z \in \mathbb{C}\backslash\pi\mathbb{Z}) \end{cases}$$

g is continuous.

Since f, $\sin$ are entire and $\sin$ is non-zero at $\mathbb{C}\backslash\pi\mathbb{Z}$, g is analytic in $\mathbb{C}\backslash\pi\mathbb{Z}$. For each $n\pi \in \pi\mathbb{Z}$, we can take $D(n\pi; 1/2)$, and $D(n\pi; 1/2) \cap \{k\pi\}_{k\in\mathbb{Z}} = \{n\pi\}$ holds. Thus, g is analytic in $D(n\pi; 1/2)$ exception $n\pi$. Let $L = \{x + iy \mid y = 0, x \in [n\pi - 1/4, n\pi + 1/4]\}$. Then, g is continuous in $\mathbb{C}$, analytic in the open set $D(n\pi; 1/2)$ except the line segment L. By Theorem 7.7, g is analytic in $D(n\pi; 1/2)$. This implies that g is differentiable in each $n\pi \in \pi\mathbb{Z}$. Therefore, g is differentiable in $(\mathbb{C}\backslash\pi\mathbb{Z}) \cup (\pi\mathbb{Z}) = \mathbb{C}$, and g is entire.

Let's show that g is bounded.

Let $z = x + iy$ where $x, y \in \mathbb{R}$,

$$\begin{aligned}
|\sin(z)| = |\sin(x+iy)| &= \frac{1}{2}\left|e^{ix-y} - e^{-ix+y}\right| \\
&= \frac{1}{2}\left|e^{-y}e^{ix} - e^{y}e^{-ix}\right| \\
&\geq \frac{1}{2}\left||e^{-y}e^{ix}| - |e^{y}e^{-ix}|\right| \\
&\geq \frac{1}{2}\left|e^{-y} - e^{y}\right|
\end{aligned}$$

and, for $y \neq 0$,

$$\frac{1}{|\sin(z)|} \leq \frac{2}{|e^{-y} - e^{y}|}$$

Suppose that $|y| \geq 1$. Note that $e^{-2|y|} \leq e^{-2} \simeq 0.135 < 0.5$.

$$\frac{1}{|\sin(z)|} \leq \frac{2}{|e^{-y} - e^{y}|} = \frac{2}{e^{|y|} - e^{-|y|}} \leq \frac{2}{e^{|y|}(1 - e^{-2|y|})} \leq \frac{2}{e^{|y|}(1 - e^{-2})} \leq \frac{2}{0.5e^{|y|}} = 4e^{-|y|}$$

And,

$$\begin{aligned} |g(z)| &= \left| \frac{f(x+iy)}{\sin(x+iy)} \right| \\ &= |f(x+iy)| \left| \frac{1}{\sin(x+iy)} \right| \\ &\leq Ce^{|y|} \cdot 4e^{-|y|} = 4C \end{aligned}$$

and $g(z)$ is bounded by $4C$ for $z \in \{x + iy \mid x, y \in \mathbb{R}, |y| \geq 1\}$.

Suppose that $|y| \leq 1$.

If $x = n\pi + \frac{\pi}{2}$ for $n \in \mathbb{N}$, $e^{ix} = -e^{-ix}$ because $(ix) - (-ix) = 2n\pi + \pi$, and

$$|\sin(z)| = \frac{1}{2}\left|e^{-y}e^{ix} - e^{y}e^{-ix}\right| = \frac{1}{2}\left|(e^{-y} + e^{y})e^{ix}\right| = \frac{|e^{-y} + e^{y}|}{2} = \frac{e^{-y} + e^{y}}{2} \geq e^{-|y|}$$

And for this z,

$$|g(z)| = \frac{|f(z)|}{|\sin(z)|} \leq \frac{Ce^{|y|}}{e^{-|y|}} = Ce^{2|y|} \leq e^{2}C$$

If $|y| = 1$, as we shown above, $g(z)$ is bounded by $4C$.

Take $D_n = (\frac{\pi}{2} - n\pi, \frac{\pi}{2} + n\pi) \times (-1, 1)$ in the complex plane for $n \geq 2$. Since $(\frac{\pi}{2} - n\pi, \frac{\pi}{2} + n\pi), (-1, 1)$ are non-empty open interval, D_n is open and connected. Then, $z \in \partial D_n$ satisfies Re $z = \frac{\pi}{2} \pm n\pi$ or Im $z = 1$. Thus, $|g(z)| \leq \max(4C, e^{2}C) = e^{2}C$ for $z \in \partial D_n$. Let $M = \max_{z \in \partial D_n} |g(z)|$. ($M$ exists because g and $|\cdot|$ are continuous, ∂D_n is non-empty, bounded, closed and compact, continuous image of compact set is compact, and a non-empty compact set has maximum.) Then, $M \leq e^{2}C$. Then, by Maximum-Modulus Theorem, every $z \in D_n$, $|g(z)|$ cannot be a maximum of $\overline{D_n}$ and $|g(z)| \leq M \leq e^{2}C$.

Let $z = x + iy \in \mathbb{C}$ such that $|y| \leq 1$. And let $n = 2 + \lceil \frac{|x|}{\pi} \rceil$. Then, $z \in D_n$ holds and $|g(z)| \leq e^{2}C$ as we shown above. Therefore, modulus of g is bounded by $e^{2}C$ for $\{z = x + iy \mid x, y \in \mathbb{R}, |y| \leq 1\}$.

Lastly, let $U = \max(4C, e^{2}C) = e^{2}C$. For z such that $|\text{Im } z| \geq 1$, $|g(z)| \leq 4C \leq U$. For z such that $|\text{Im } z| \leq 1$, $|g(z)| \leq e^{2}C \leq U$. Thus, modulus of g is bounded by U in $\mathbb{C}$.

Because g is entire and bounded, by Liouville's Theorem, g is a constant. Let $K \in \mathbb{C}$ such that

$$g(z) = K$$

Then, for $z \notin \pi\mathbb{Z}$,

$$f(z) = K \sin z$$

Let $h(z) = f(z) - K\sin z$. Since $f(z) = K \sin z$ at $z \notin \pi\mathbb{Z}$, $h(z) = 0$ for $z \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. Since $[\frac{\pi}{4}, \frac{3\pi}{4}]$ is compact, it has a limit point in $\mathbb{C}$, such as $\frac{\pi}{2}$. Since f and sin are entire, h is entire. Thus, by Uniqueness Theorem, $h(z) \equiv 0$ and $f(z) = K\sin z$ for $z \in \mathbb{C}$.

In conclusion, if f is entire, $\forall n \in \mathbb{Z} : f(n\pi) = 0$, and $\forall x, y \in \mathbb{R} : |f(x + iy)| \leq Ce^{|y|} < \infty$ for some fixed $C > 0$, $f = K \sin z$ for some $K \in \mathbb{R}$. It means, $\{z \mapsto K \sin z \mid K \in \mathbb{C}\}$ is a set consisting of all functions which satisfies every condition in the problem. $\square$

Problem 3

Let a power series $f = \sum_{n=1}^{\infty} a_n z^n$ be uniformly convergent on any compact subset of $\mathbb{C}$. Let C be a close curve in $\mathbb{C}$.

(1) Prove that for any nonnegative integer n, $\int_C z^n dz = 0$.

(2) Using the fact in (1), prove that $\int_C f(z)dz = 0$.

Proof of (1)

Let $n \in \mathbb{Z}^{\geq 0}$. Since z^n is an analytic polynomial, it's analytic everywhere in $\mathbb{C}$ (i.e. entire). Thus, there are antiderivatives of z^n. For example, $\frac{1}{n+1} z^{n+1}$ is an entire function ($\because$ it's a analytic polynomial) and the derivative is z^n. Let $\sigma : [0, 1] \to \mathbb{C}$ be a function of the closed curve C. Then, $\sigma(0) = \sigma(1)$ by closedness of C and,

$$\begin{aligned}\int_C z^n dz &= \int_0^1 \sigma(t)^n \sigma'(t)dt \\ &= \left[\frac{1}{n+1}\sigma(t)^{n+1}\right]_{t=0}^1 = \frac{1}{n+1}(\sigma(1)^{n+1} - \sigma(0)^{n+1}) = \frac{1}{n+1}(\sigma(0)^{n+1} - \sigma(0)^{n+1}) = 0\end{aligned}$$

by Proposition 4.12. □

Proof of (2)

Let $f_m(z) = \sum_{n=1}^{m} a_n z^n$. Because C is a closed curve in $\mathbb{C}$, there is a compact subset $D \subseteq \mathbb{C}$ such that $C \subseteq D \subsetneq \mathbb{C}$. (e.g. $D = C$. $\because$ C is a closed curve implies that there is a continuous function $\sigma : [0, 1] \to \mathbb{C}$ such that $C = \sigma([0, 1])$. Since $[0, 1]$ is closed, bounded, and compact, the continuous image C of σ is also compact.) By the assumption in the problem, $f_m \to f$ uniformly in the compact domain D. Since each f_m is continuous ($\because$ it's a polynomial) and $f_m \to f$ uniformly in the compact set D, by Proposition 4.11,

$$\int_C f(z)dz = \lim_{m\to\infty} \int_C f_m(z)dz$$

holds. Note that,

$$\begin{aligned}\int_C f_m(z)dz &= \int_C \sum_{n=1}^{m} a_n z^m dz \\ &= \sum_{n=1}^{m} a_n \int_C z^m dz = \sum_{n=1}^{m} a_n \cdot 0 = 0\end{aligned}$$

because we can reorder finite sum and integration, and because of the fact we proved in (1). Therefore,

$$\begin{aligned}\int_C f(z)dz &= \lim_{m\to\infty} \int_C f_m(z)dz \\ &= \lim_{m\to\infty} 0 = 0\end{aligned}$$

□

Problem 4

Find all entire functions f which satisfies the property that for R, $C > 0$, and nonnegative integer n, $|f(z)| \geq \frac{C}{|z|^n}$ when $|z| \geq R$.

Note: Two constants $R > 0$, $C > 0$, and $n \in \mathbb{Z}^{\geq 0}$ are fixed.

Answer

$$\left\{ z \mapsto s(z - t_1) \cdots (z - t_m) \mid m \in \mathbb{Z}^{\geq 0}; s \in \mathbb{C} \backslash \{0\}; t_1, \cdots, t_m \in D(0; R); |s| \geq \max_{|\zeta|=R} \left| \frac{C}{R^n (\zeta - t_1) \cdots (\zeta - t_n)} \right| \right\}$$

(for $m = 0$, $(z - t_1) \cdots (z - t_m) = 1$.)

Note that $\max_{|\zeta|=R} \left| \frac{C}{R^n(\zeta - t_1) \cdots (\zeta - t_n)} \right|$ exists, because $\{z \in \mathbb{C} \mid |z| = R\}$ is non-empty and compact ($\because$ $\{z \in \mathbb{C} \mid |z| = R\}$ is bounded and closed), $\frac{C}{R^n(z-t_1)\cdots(z-t_n)}$ is continuous in $\{z \in \mathbb{C} \mid |z| = R\}$ ($\because$ $t_1, \cdots, t_m \in D(0; R)$, each of $z - t_k$ is non-zero and analytic for $z \in \mathbb{C}$ such that $|z| = R$, $\frac{C}{R^n(z-t_1)\cdots(z-t_n)}$ is analytic and continuous in $\{z \in \mathbb{C} \mid |z| = R\}$), $|\cdot|$ is continuous, continuous image of compact set is compact, and non-empty compact set has maximum.

Proof

Let $R > 0$, $C > 0$, $n \in \mathbb{Z}^{\geq 0}$ be fixed.

Suppose that f is entire and $|f(z)| \geq \frac{C}{|z|^n}$ for every $z \in \mathbb{C}$ such that $|z| \geq R$.

First of all, $f \not\equiv 0$. ($\because$ If $f \equiv 0$, $0 = |f(R)| \geq \frac{C}{R^n} > 0$ and it's a contradiction.) Thus, let's assume that f is not zero.

For $|z| \geq R$, we obtain

$$|f(z)| \geq \frac{C}{|z|^n}$$
$$|z^{n+1} f(z)| = |z|^{n+1} \cdot |f(z)| \geq C|z|$$

Then, if $|z| = r \geq R$, $|z^{n+1}f(z)| \geq C|z| = Cr$. Thus, $\lim_{|z|\to\infty} |z^{n+1}f(z)| \geq \lim_{|z|\to\infty} C|z| = \infty$. As f is entire and z^{n+1} is entire ($\because$ z^{n+1} is an analytic polynomial since $n + 1 \geq 1$), $z^{n+1}f(z)$ is entire. By Theorem 6.11, $z^{n+1}f(z)$ is polynomial.

Since $z^{n+1}f(z)$ is a polynomial, let $z^{n+1}f(z) = \sum_{k=0}^{m'} a_k z^k$ where $m' \in \mathbb{Z}^{\geq 0}$ and $a_0, \cdots, a_{m'} \in \mathbb{C}$ and $a_{m'} \neq 0$. ($\because$ we assumed that f is not a zero function.) Then, for $z \neq 0$, by dividing by z^{n+1},

$$f(z) = \sum_{k=0}^{m'} a_k z^{k-n-1}$$

But if one of $a_0, \cdots, a_n$ is non-zero, $\lim_{x \in \mathbb{R}, x \to 0} f(x)$ diverges. ($\because$ if a_j is non-zero with minimum index such that $j \leq n$, for x with enough small absolute value, $|f(x)| = |a_j x^{j-n-1}| + \epsilon(x)$ and $|f(x)| \to \infty$ as $x \to 0$.) Then, $f(z)$ cannot be continuous, and cannot be entire. Thus, $a_0 = \cdots = a_n = 0$.

Let's reindex $f(z)$ as,

$$f(z) = \sum_{k=0}^{m} b_k z^k$$

where $m = m' - n - 1$, $b_k = a_{k+n+1}$ for each $k = 0, \cdots, m$.

Because $b_m = a_{m'} \neq 0$, $f(z)$ is degree m polynomials. By Fundamental Theorem of Algebra, $f(z)$ has m roots in $\mathbb{C}$. Let $t_1, \cdots, t_m$ be roots of $f(z)$ (allowing repetition). Then, $f(t_k) = 0$ for each $k = 1, \cdots, m$. However, if $|t_k| \geq R$, $0 = |f(t_k)| \geq \frac{C}{|t_k|^n} > 0$ and it's a contradiction. Therefore, $t_1, \cdots, t_m$ must be in the open set $D(0; R)$.

Let $s = b_m$. Then,

$$f(z) = s(z - t_1) \cdots (z - t_m)$$

(If $m = 0$, $f(z) = s$.)

Since $z - t_1, \cdots, z - t_m$ are continuous and non-zero at z where $|z| = R$, $|\cdot|$ is continuous, $\{z \in \mathbb{C} \mid |z| = R\}$ is compact as closed and bounded, $M = \max_{|\zeta|=R} \frac{C}{R^n|\zeta - t_1| \cdots |\zeta - t_m|}$ exists. ($\because$ continuous image of compact set is compact and non-empty compact real-value set contains the maximum value.)

Because, for each $z \in \mathbb{C}$ such that $|z| = R$,

$$|s| \cdot |z - t_1| \cdots |z - t_m| = |s(z - t_1) \cdots (z - t_m)| = |f(z)| \geq \frac{C}{|z|^n} = CR^{-n}$$

and since $t_1, \cdots, t_m \in D(0; R)$ and $|z| = R$, $|z - t_1|, \cdots, |z - t_m|$ are non-zero, and,

$$|s| \geq \frac{C}{R^n|z - t_1| \cdots |z - t_m|}$$

Thus,

$$|s| \geq M = \max_{|\zeta|=R} \frac{C}{R^n|\zeta - t_1| \cdots |\zeta - t_m|}$$

In conclusion at this point, if f is entire, and $|f(z)| \geq \frac{C}{|z|^n}$ for every $z \in \mathbb{C}$ such that $|z| \geq R$, at least the below conditions are satisfied:

- $f(z) = s(z - t_1) \cdots (z - t_m)$ for some $m \in \mathbb{Z}^{\geq 0}$, $s \in \mathbb{C}\backslash\{0\}$, $t_1, \cdots, t_m \in D(0; R)$.
- $|s| \geq \max_{|\zeta|=R} \frac{C}{R^n|\zeta - t_1| \cdots |\zeta - t_m|} = \max_{|\zeta|=R} \left| \frac{C}{R^n(\zeta - t_1) \cdots (\zeta - t_m)} \right|$

Conversely, let f be a function satisfying the above two conditions.

First, since f is a non-zero polynomial, f is entire.

Note that for $z \in \mathbb{C}$ such that $|z| \geq R$, $|z - t_k| \neq 0$ for each $k = 1, \cdots, m$ because $|t_k| < R$.

Also, for $z \in \mathbb{C}$ such that $|z| = R$,

$$\begin{aligned}
|f(z)| &= |s| \cdot |z - t_1| \cdots |z - t_m| \\
&\geq \left(\max_{|\zeta|=R} \frac{C}{R^n|\zeta - t_1| \cdots |\zeta - t_m|} \right) \cdot |z - t_1| \cdots |z - t_m| \\
&\geq \frac{C}{R^n|z - t_1| \cdots |z - t_m|} \cdot |z - t_1| \cdots |z - t_m| \\
&= \frac{C}{R^n} = \frac{C}{|z|^n}
\end{aligned}$$

Let $z \in \mathbb{C}$ such that $|z| > R$. Then, let $z_0 = z \cdot \frac{R}{|z|}$. Note that $|z_0| = R$, and z_0 is the nearest point from z in $\partial D(0; R)$. For every $t \in D(0; R)$, $|z - t| > |z_0 - t|$. ($\because$ Let $a = \overline{z}/|z|$. Then, multiplicating by a preserves modulus. Let $z' = az = |z|$, $z'_0 = az_0 = R$, $t' = at = x + iy$ for some $x, y \in \mathbb{R}$. Then, $|z - t| = |z' - t'| = \sqrt{(|z| - x)^2 + y^2}$ and $|z_0 - t| = |z'_0 - t'| = \sqrt{(R - x)^2 + y^2}$. Since $|z| > R$, $|z| - x > R - x$ and $|z - t| = \sqrt{(|z| - x)^2 + y^2} > \sqrt{(R - x)^2 + y^2} = |z_0 - t|$.) Since $t_1, \cdots, t_m \in D(0; R)$, $|z - t_k| \geq |z_0 - t_k|$ for each $k = 1, \cdots, m$. Thus,

$$\begin{aligned}
|f(z)| &= |s| \cdot |z - t_1| \cdots |z - t_m| \\
&\geq |s| \cdot |z_0 - t_1| \cdots |z_0 - t_m| \\
&\geq \left(\max_{|\zeta|=R} \frac{C}{R^n|\zeta - t_1| \cdots |\zeta - t_m|} \right) \cdot |z_0 - t_1| \cdots |z_0 - t_m| \\
&\geq \frac{C}{R^n|z_0 - t_1| \cdots |z_0 - t_m|} \cdot |z_0 - t_1| \cdots |z_0 - t_m| \\
&= \frac{C}{R^n} \geq \frac{C}{|z|^n}
\end{aligned}$$

Therefore, for every $z \in \mathbb{C}$ such that $|z| \geq R$, $|f(z)| \geq \frac{C}{|z|^n}$.

The conclusion. Let $R > 0$, $C > 0$, $n \in \mathbb{Z}^{\geq 0}$ are fixed. If f is entire and $|f(z)| \geq \frac{C}{|z|^n}$ for every $z \in \mathbb{C}$ such that $|z| \geq R$, then, below holds:

- $f(z) = s(z - t_1) \cdots (z - t_m)$ for some $m \in \mathbb{Z}^{\geq 0}$, $s \in \mathbb{C} \backslash \{0\}$, $t_1, \cdots, t_m \in D(0; R)$.
- $|s| \geq \max_{|\zeta|=R} \frac{C}{R^n |\zeta - t_1| \cdots |\zeta - t_m|} = \max_{|\zeta|=R} \left| \frac{C}{R^n (\zeta - t_1) \cdots (\zeta - t_m)} \right|$

Also, if f satisfies above two properties, f is entire and $|f(z)| \geq \frac{C}{|z|^n}$ for every $z \in \mathbb{C}$ such that $|z| \geq R$. Therefore,

$$\left\{ z \mapsto s(z - t_1) \cdots (z - t_m) \mid m \in \mathbb{Z}^{\geq 0}; s \in \mathbb{C} \backslash \{0\}; t_1, \cdots, t_m \in D(0; R); |s| \geq \max_{|\zeta|=R} \left| \frac{C}{R^n (\zeta - t_1) \cdots (\zeta - t_n)} \right| \right\}$$

is a set consisting of all functions which satisfies all properties in the problem. $\square$

Problem 5

Find all entire functions f satisfies $\limsup_{|x+iy|\to\infty} |f(x+iy)| e^{x^3-3xy^2} < \infty$.

Answer

$$\{z \mapsto Ke^{-z^3} \mid K \in \mathbb{C}\}$$

Proof

Suppose that f is entire and $\limsup_{|x+iy|\to\infty} |f(x+iy)| e^{x^3-3xy^2} < \infty$.

Let $z = x + iy$ for $x, y \in \mathbb{R}$.

$$z^3 = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

And Re $z^3 = x^3 - 3xy^2$.

For some $a, b \in \mathbb{R}$,

$$e^{a+ib} = e^a(\cos b + i \sin b)$$

and

$$\begin{aligned} \left|e^{a+ib}\right| &= |e^a| \cdot |\cos b + i \sin b| \\ &= |e^a| \end{aligned}$$

And for $a \in \mathbb{R}$, since exponential function is always positive,

$$|e^a| = e^a$$

Because $x^3 - 3xy^2 \in \mathbb{R}$ and $3x^2y - y^3 \in \mathbb{R}$,

$$e^{x^3-3xy^2} = \left|e^{x^3-3xy^2}\right| = \left|e^{(x^3-3xy^2)+i(3x^2y-y^3)}\right| = \left|e^{z^3}\right|$$

And,

$$|f(x+iy)| e^{x^3-3xy^2} = |f(z)| \cdot \left|e^{z^3}\right| = \left|f(z)e^{z^3}\right|$$

Thus,

$$\begin{aligned} L &= \limsup_{|z|\to\infty} \left|f(z)e^{z^3}\right| \\ &= \limsup_{|x+iy|\to\infty} |f(x+iy)| e^{x^3-3xy^2} < \infty \end{aligned}$$

Because e^z, z^3, $f(z)$ are entire, $f(z)e^{z^3}$ is entire and continuous.

Because $L = \limsup_{|z|\to\infty} \left|f(z)e^{z^3}\right|$, for every $\epsilon > 0$, there is $R > 0$ which satisfies for every $z \in \mathbb{C}$ such that $|z| > R$, $\left|f(z)e^{z^3}\right| < L + \epsilon$. Let $\epsilon = 1$ and obtain $R > 0$ such that for every $z \in \mathbb{C}$ such that $|z| > R$, $\left|f(z)e^{z^3}\right| < L + 1$. Then, in $\{z \in \mathbb{C} \mid |z| > R\}$, $f(z)e^{z^3}$ is bounded by $L + 1$.

Let $S = \overline{D(0;R)}$. Since S is non-empty and compact ($\because$ S is bounded, closed), and $\left|f(z)e^{z^3}\right|$ is continuous ($\because$ $f(z)e^{z^3}$ and $|\cdot|$ are continuous), $V = \left\{\left|f(z)e^{z^3}\right| \mid z \in S\right\}$ is non-empty and compact set of real values. ($\because$ continuous image of compact set is compact.) Therefore, $\max V$ exists. Then, $f(z)e^{z^3}$ is bounded by $\max V$ in S.

Let $M = 1 + \max(\max V, L + 1)$. Then, $\left|f(z)e^{z^3}\right| \le \max V < M$ for $z \in S$ (i.e. $|z| \le R$), and $\left|f(z)e^{z^3}\right| < L + 1 < M$ for $z \in \mathbb{C} \backslash S$ (i.e. $|z| > R$). Thus, for every $z \in \mathbb{C}$, $\left|f(z)e^{z^3}\right| < M$. In other words, $f(z)e^{z^3}$ is bounded in $\mathbb{C}$.

Since $f(z)e^{z^3}$ is entire and bounded, by Liouville's Theorem, $f(z)e^{z^3}$ is a constant function. In other words, there is $K \in \mathbb{C}$ such that $K = f(z)e^{z^3}$ for every $z \in \mathbb{C}$. Then, since e^{z^3} is nowhere zero, $f(z) = Ke^{-z^3}$. Therefore, if f is entire and $\limsup_{|x+iy|\to\infty} |f(x+iy)| e^{x^3-3xy^2} < \infty$, then f should be in a form of $f(z) = Ke^{-z^3}$ for some $K \in \mathbb{C}$.

Let $K \in \mathbb{C}$ and $f(z) = Ke^{-z^3}$. Since f is entire ($\because$ e^z and $-z^3$ are entire), it's enough to check that $\limsup_{|x+iy|\to\infty} |f(x+iy)| e^{x^3-3xy^2} < \infty$.

As we shown above, for $z = x + iy$,

$$\begin{aligned} |f(x+iy)| e^{x^3-3xy^2} &= \left| f(z)e^{z^3} \right| \\ &= \left| Ke^{-z^3} e^{z^3} \right| \\ &= |K| \end{aligned}$$

Then,

$$\begin{aligned} \limsup_{|x+iy|\to\infty} |f(x+iy)| e^{x^3-3xy^2} &= \limsup_{z\to\infty} |K| \\ &= |K| < \infty \end{aligned}$$

Thus, this $f(z)$ satisfies $\limsup_{|x+iy|\to\infty} |f(x+iy)| e^{x^3-3xy^2} < \infty$.

In conclusion, $\{z \mapsto Ke^{-z^3} \mid K \in \mathbb{C}\}$ is a set consisting of all functions satisfying all properties in the problem. $\square$

Problem 6

Let $f = u + iv$ be an entire function. Prove that

$$\det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = \det \begin{pmatrix} v_{xx} & v_{xy} \\ v_{yx} & v_{yy} \end{pmatrix} \leq 0$$

Lemmata

Note that if $f = u + iv$ is entire, $f_x = u_x + iv_x$, $f_y = u_y + iv_y$ exists by Proposition 3.1.

Lemma 1. *Let $f = u + iv$ be an entire function where $u, v : \mathbb{C} \to \mathbb{R}$. $f' = u_x + iv_x = v_y - iu_y$.*

Proof. Because f is analytic, f' is equal to every directional derivation. Thus,

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \in \mathbb{R}, h \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \in \mathbb{R}, h \to 0} \frac{u(z+h) - u(z) + i(v(z+h) - v(z))}{h} \\ &= u_x(z) + iv_x(z) \end{aligned}$$

And by Cauchy-Riemann Equation, $f' = u_x + iv_x = v_y - iu_y$. $\square$

Lemma 2. *Let $f = u + iv$ be an entire function where $u, v : \mathbb{C} \to \mathbb{R}$. Then, at least $u, v \in \mathcal{C}^2(\mathbb{C})$ (i.e. u, v are complex functions which have the second derivatives which are continuous).*

Proof. Since f is entire, $f' = u_x + iv_x$. Also, derivative of entire function is entire. Thus, f' is entire. Then, $f'' = u_{xx} + iv_{xx}$. In the similar way, there is $f^{(3)}$ with f''_x and f''_y. Because $f''_x = u_{xxx} + iv_{xxx}$ and $f''_y = u_{xxy} + iv_{xxy}$, $u_{xxx}, v_{xxx}, u_{xxy}, v_{xxy}$ exists. It means u_{xx} and v_{xx} are differentiable and continuous.

Because Cauchy-Riemann Equation holds as $if'_x = f'_y$, we can take $f'' = v_{xy} - iu_{xy}$ instead of $u_{xx} + iv_{xx}$. Then, in the similar way, we can conclude that v_{xy}, u_{xy} are differentiable and continuous.

Also, using Cauchy-Riemann Equation, for $f' = v_y - iu_y$ instead of $u_x + iv_x$, by following above processes, we can conclude that $v_{yx}, u_{yx}, v_{yy}, u_{yy}$ are differentiable and continuous.

Thus, $u, v \in \mathcal{C}^2(\mathbb{C})$. $\square$

Proof

Note that,

$$\det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = u_{xx}u_{yy} - u_{xy}u_{yx}, \quad \det \begin{pmatrix} v_{xx} & v_{xy} \\ v_{yx} & v_{yy} \end{pmatrix} = v_{xx}v_{yy} - v_{xy}v_{yx}$$

Since f is entire, $u_x = v_y$ and $u_y = -v_x$ by Cauchy-Riemann Equation. Thus,

$$\begin{aligned} u_{xx} &= \frac{\partial u_x}{\partial x} = \frac{\partial v_y}{\partial x} = v_{yx} \\ u_{xy} &= \frac{\partial u_x}{\partial y} = \frac{\partial v_y}{\partial y} = v_{yy} \\ u_{yx} &= \frac{\partial u_y}{\partial x} = \frac{\partial(-v_x)}{\partial x} = -v_{xx} \\ u_{yy} &= \frac{\partial u_y}{\partial y} = \frac{\partial(-v_x)}{\partial y} = -v_{xy} \end{aligned}$$

Therefore,

$$\det\begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = u_{xx}u_{yy} - u_{xy}u_{yx} = v_{yx}(-v_{xy}) - v_{yy}(-v_{xx})$$
$$= -v_{yx}v_{xy} + v_{yy}v_{xx} = v_{xx}v_{yy} - v_{xy}v_{yx} = \det\begin{pmatrix} v_{xx} & v_{xy} \\ v_{yx} & v_{yy} \end{pmatrix}$$

holds.

Lemma 2 shows that $u, v \in \mathcal{C}^2(\mathbb{C})$. It means, we can change the order of partial derivatives like:

$$u_{xy} = u_{yx}, v_{xy} = v_{yx}$$

Thus,

$$\begin{aligned} u_{xx}u_{yy} - u_{xy}u_{yx} &= -v_{yx}v_{xy} - u_{xy}u_{yx} \\ &= -v_{xy}^2 - u_{xy}^2 \end{aligned}$$

Since u and v are real-valued functions, $v_{xy}^2 \geq 0$ and $u_{xy}^2 \geq 0$. So $-v_{xy}^2 - u_{xy}^2 \leq 0$.

Therefore,

$$\det\begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = \det\begin{pmatrix} v_{xx} & v_{xy} \\ v_{yx} & v_{yy} \end{pmatrix} = u_{xx}u_{yy} - u_{xy}u_{yx} = -v_{xy}^2 - u_{xy}^2 \leq 0$$

□

Problem 7

Note: For $n \in \mathbb{N}$, $f^n = f \cdot \overset{n}{\cdots} \cdot f$. i.e. for $z \in \mathrm{Dom}(f)$, $f^n(z) = f(z)^n$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Prove that

(1) f is entire if f^4 and f^7 are entire.

(2) f is entire if f^2 is entire and f is continuous.

Lemmata

Lemma 3. *Let $f : \mathbb{C} \to \mathbb{C}$ be functions. If f^m and f^n are entire for integers $0 < m < n$, f^{n-m} is entire too.*

Proof. If $f^m = 0$ or $f^n = 0$, then $f = 0$ because only 0 can be zero by powering by some integers. (If $z \neq 0$, then $|z| > 0$, $|z|^k > 0$ and $z^k \neq 0$ for $k \in \mathbb{N}$) Thus, in this case, f is entire.

If f^m is a constant $c \in \mathbb{C}\backslash\{0\}$, $f^n = f^m \cdot f^{n-m} = c \cdot f^{n-m}$ is entire. Thus f^{n-m} is entire.

Thus, let's consider that f^m is non-constant and f^n is non-zero.

Let

$$g(z) = \begin{cases} 0 & (f(z) = 0) \\ \frac{f^n(z)}{f^m(z)} & (f(z) \neq 0) \end{cases}$$

Note that this is well-defined, because if $f^m(z) = 0$ holds, $f(z)^m = 0$ and $f(z) = 0$. Also, $g(z) = f^{n-m}(z)$ because $g(z) = 0 = f(z)^{n-m}$ for $z \in \mathbb{C}$ such that $f(z) = 0$, and $g(z) = \frac{f(z)^n}{f(z)^m} = f(z)^{n-m}$ for $z \in \mathbb{C}$ such that $f(z) \neq 0$.

For $z_0 \in \mathbb{C}$ where $f(z_0) \neq 0$, g is differentiable at z_0. Because $g(z_0) = \frac{f^n(z_0)}{f^m(z_0)}$, $f^m(z_0) \neq 0$ and f^n, f^m are entire.

Let $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$. (Note that if f^n is a constant, there are no such case; because f^n is non-zero, if f^n is a constant, f^n is nowhere zero, and $0 \neq f(z_0)^n = 0^n = 0$ make a contradiction. Thus, if f^n is a constant, we can ignore this case. Also, let's assume that f^n is non-constant for this case.) Then, $f^n(z_0) = f^m(z_0) = 0$. In this case, since f^n and f^m are entire, there are $a_k, b_k \in \mathbb{C}$ such that:

$$f^n(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

$$f^m(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

If $a_0 \neq 0$, $0 = f_n(z_0) = a_0 \neq 0$ and it contradicts. Thus, $a_0 = 0$ holds, and $b_0 = 0$ also holds because of the same reason.

Because we assumed that f^n and f^m are non-constant, at least one a_j and b_k is non-zero for some $j, k \in \mathbb{N}$. Let $p \in \mathbb{N}$ be a minimum number which satisfies $a_p \neq 0$, and let $q \in \mathbb{N}$ be a minimum number which satisfies $b_q \neq 0$. Then, $a_0 = \cdots = a_{p-1} = 0$ and $b_0 = \cdots = b_{q-1} = 0$ holds. Let $c_k = a_{p+k}$, $d_k = b_{q+k}$ for every $k \in \mathbb{Z}^{\geq 0}$. We can rewrite the power series of f^n and f^m as:

$$f^n(z) = (z - z_0)^p \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

$$f^m(z) = (z - z_0)^q \sum_{k=0}^{\infty} d_k (z - z_0)^k$$

By assigning the above power series to f^{nm}, we obtain:

$$\begin{aligned}(z-z_0)^{pm}(\sum_{k=0}^{\infty} c_k(z-z_0)^k)^m &= (f^n(z))^m \\ &= f^{nm}(z) \\ &= (f^m(z))^n = (z-z_0)^{qn}(\sum_{k=0}^{\infty} d_k(z-z_0)^k)^n\end{aligned}$$

Since $c_0 \neq 0 \neq d_0$, and the non-zero coefficient lowest degree term of each of LHS and RHS are $c_0^m(z-z_0)^{pm}$ and $d_0^n(z-z_0)^{qn}$. So, $c_0^m = d_0^n$ and $pm = qn$. Then, because $n > m$,

$$p = \frac{pm}{m} = \frac{qn}{m} > \frac{qn}{n} = q$$

and since $p, q \in \mathbb{N}$, $p \geq q+1$.

Let $\phi_0 = f^n$

$$\phi_k(z) = \begin{cases} a_k & (z = z_0) \\ \frac{f^n(z)}{(z-z_0)^k} & (z \neq z_0) \end{cases}$$

for $k \in \{1, \cdots, p\}$. Note that $\frac{f^n(z)}{(z-z_0)^k} = \frac{\phi_{k-1}(z)}{z-z_0} = \frac{\phi_{k-1}(z)-0}{z-z_0} = \frac{\phi_{k-1}(z)-\phi_{k-1}(z_0)}{z-z_0}$ for $z \neq z_0$. As $a_1 = \phi_0'(z_0) = (f^n)'(z_0)$, by Proposition 5.8, ϕ_1 is entire. And for $k \leq p$, we can repeat this since if $\phi_0, \cdots, \phi_{k-1}$ are entire, $a_k = \phi_{k-1}'(z_0)$ and ϕ_k is entire by Proposition 5.8. It shows that $\phi_p(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ is entire. Since entire function is continuous,

$$\lim_{z \to z_0} \sum_{k=0}^{\infty} c_k(z-z_0)^k = c_0$$

In the same processes, we obtain

$$\lim_{z \to z_0} \sum_{k=0}^{\infty} d_k(z-z_0)^k = d_0$$

Thus,

$$\begin{aligned}\lim_{z \to z_0} \frac{g(z)}{z-z_0} &= \lim_{z \to z_0} \frac{1}{z-z_0} \frac{(z-z_0)^p \sum_{k=0}^{\infty} c_k(z-z_0)^k}{(z-z_0)^q \sum_{k=0}^{\infty} d_k(z-z_0)^k} \\ &= \lim_{z \to z_0} (z-z_0)^{p-q-1} \frac{\sum_{k=0}^{\infty} c_k(z-z_0)^k}{\sum_{k=0}^{\infty} d_k(z-z_0)^k}\end{aligned}$$

If $p - q - 1 = 0$,

$$\begin{aligned}\lim_{z \to z_0} \frac{g(z)}{z-z_0} &= \lim_{z \to z_0} \frac{\sum_{k=0}^{\infty} c_k(z-z_0)^k}{\sum_{k=0}^{\infty} d_k(z-z_0)^k} \\ &= \frac{c_0}{d_0}\end{aligned}$$

Otherwise,

$$\begin{aligned}\lim_{z \to z_0} \frac{g(z)}{z-z_0} &= \lim_{z \to z_0} (z-z_0)^{p-q-1} \frac{\sum_{k=0}^{\infty} c_k(z-z_0)^k}{\sum_{k=0}^{\infty} d_k(z-z_0)^k} \\ &= 0^{p-q-1} \cdot \frac{c_0}{d_0} \\ &= 0\end{aligned}$$

Therefore,

$$\lim_{h \to 0} \frac{g(z_0+h)-g(z_0)}{h} = \lim_{z \to z_0} \frac{g(z)}{z-z_0} = \begin{cases} c_0/d_0 & (p-q-1=0) \\ 0 & (p-q-1 \neq 0) \end{cases}$$

Thus, g is differentiable at z_0 where $f(z_0) = 0$.

Therefore, g is differentiable everywhere in $\mathbb{C}$, i.e. entire. Since $g = f^{n-m}$, f^{n-m} is entire. □

Proof of (1)

Since f^7 and f^4 are entire, $f^3 = f^{7-4}$ is entire by Lemma 3. Since f^4 and f^3 are entire, $f = f^{4-3}$ is entire by Lemma 3. $\square$

Proof of (2)

Let f^2 be entire and f continuous.

If $f \equiv 0$, it's trivially entire. Thus let's assume that f is non-zero at some point. Then, there is some $z \in \mathbb{C}$ such that $f(z) \neq 0$, and $f(z)^2 \neq 0$ (Because $|f(z)^2| = |f(z)|^2 > 0$ for this z.) Therefore, f^2 is also non-zero at some point. (i.e. $f^2 \not\equiv 0$.)

Suppose that $f(z_0) \neq 0$ at $z_0 \in \mathbb{C}$.

$$\begin{aligned}(f^2)'(z_0) &= \lim_{z \to z_0} \frac{f(z)^2 - f(z_0)^2}{z - z_0} \\ &= \lim_{z \to z_0} \frac{(f(z) + f(z_0))(f(z) - f(z_0))}{z - z_0}\end{aligned}$$

converges. Also, because f is continuous and $f(z_0) \neq 0$,

$$\lim_{z \to z_0} \frac{1}{f(z) + f(z_0)} = \frac{1}{2f(z_0)}$$

converges. Thus,

$$\begin{aligned}\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \left(\lim_{z \to z_0} \frac{(f(z) + f(z_0))(f(z) - f(z_0))}{z - z_0} \right) \left(\lim_{z \to z_0} \frac{1}{f(z) + f(z_0)} \right) \\ &= \frac{(f^2)'(z_0)}{2f(z_0)}\end{aligned}$$

converges. Thus f is differentiable at every $z_0 \in \mathbb{C}$ where $f(z_0) \neq 0$.

Suppose that $f(z_0) = 0$. Then, there is $\rho > 0$ such that $f(z) \neq 0$ for every $z \in D(z_0; \rho) \backslash \{z_0\}$. ($\because$ If not, for every $n \in \mathbb{N}$, there is $z_n \in D(z_0; 1/n) \backslash \{z_0\}$ such that $f(z_n) = 0$. It implies $f(z_n)^2 = 0$ trivially. Then, $\{z_n\}_{n \in \mathbb{N}}$ is a subset of $\mathbb{C}$, where $f^2(z_n) = 0$, and it has a limit point, $z_0 = \lim_{n \to \infty} z_n$, which is in $\mathbb{C}$. Since f^2 is entire, by Uniqueness Theorem, $f^2 \equiv 0$. It's a contradiction because of our assumption that $f^2 \not\equiv 0$.) Let $D = D(z_0; \rho)$ and $L = \{x + iy \mid y = \text{Im } z_0, x \in [-\frac{\rho}{2} + \text{Re } z_0, \frac{\rho}{2} + \text{Re } z_0]\}$. As we proved that f is differentiable in every point $z \in \mathbb{C}$ where $f(z) \neq 0$ and every point $z \in D \backslash \{z_0\}$ satisfies $f(z) \neq 0$, f is analytic in D except the line segment L. By applying Theorem 7.7 with the continuity of f, f is analytic in D. Since $z_0 \in D$, f is differentiable at z_0.

Therefore, f is differentiable at every point of $\mathbb{C}$. It means f is entire. $\square$

Problem 8

Find all analytic functions f on the closed unit disc in $\mathbb{C}$ such that $|f(z)| \leq 1$ for $|z| \leq 1$, $|f(z)| = 1$ for $|z| = 1$, and f is injective.

Answer

$$\left\{ z \mapsto \frac{Kz + \alpha}{1 + \overline{\alpha} K z} \mid \alpha \in \mathbb{D}; K \in \mathbb{C}; |K| = 1 \right\}$$

Proof

I'll denote $\mathbb{D} = D(0; 1)$.

Abstract: First, I'll show that the function of f restricted by $\mathbb{D}$ is bijective and it's inverse is also analytic in $\mathbb{D}$. Then, using Blaschke Factor to make bijective analytic function g which satisfies every properties given in problem and $g(0) = 0$. Next, applying Schwarz's Lemma to g and g^{-1} and get $g(z) = Kz$ for some $K \in \mathbb{C}$ such that $|K| = 1$. Finally, transform $g(z)$ to $f(z)$.

Before the proof, note that we learned about the Blaschke Factor: The Blaschke Factor for $\alpha \in \mathbb{D}$ is:

$$B_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$

As we proved in the lecture, it has below properties:

- B_α is analytic in $\mathbb{D}$.
- B_α maps $\mathbb{D} \mapsto \mathbb{D}$, $\partial\mathbb{D} \mapsto \partial\mathbb{D}$, $0 \mapsto -\alpha$, and $\alpha \mapsto 0$.
- B_α is bijective with its inverse $B_\alpha^{-1} = B_{-\alpha}$

Also, B_α is continuous in $\overline{\mathbb{D}}$ because for $z \in \overline{\mathbb{D}}$, the denominator of $B_\alpha(z)$, $1 - \overline{\alpha} z$ cannot be zero since $|\alpha| < 1$ and denominator and numerator of B_α are continuous (as linear polynomial).

Suppose that f is analytic in $\overline{\mathbb{D}}$, $\forall z \in \overline{\mathbb{D}} : |f(z)| \leq 1$, $\forall z \in \partial\mathbb{D} : |f(z)| = 1$ and f is injective.

First of all, note that f cannot be constant. Because, if f is a constant function, $f(1) = f(-1)$ and it violates injectivity of f.

Also, note that f maps all elements of $\mathbb{D}$ into $\mathbb{D}$. Because, by Maximum-Modulus Theorem, every interier point of $\overline{\mathbb{D}}$ cannot has maximum modulus of f. Since f has maximum modulus, 1, at every points of $\partial\mathbb{D}$, $|f(z)| < 1$ for each element of $z \in \mathbb{D}$.

Let $h = f|_{\mathbb{D}}$, which is the function obtained from f by restricting the domain to $\mathbb{D}$. We will show that h is surjective.

Let t be an arbitrary element of $\mathbb{D}$. Then, we can take the Blaschke Factor of t. Let $F = B_t \circ f$. Then, $F(t) = 0$. Because $\overline{\mathbb{D}}$ is non-empty compact and $|F|$ is continuous ($\because$ $|\cdot|$, f and B_t are continuous in $\overline{\mathbb{D}}$), $S = |F|(\overline{\mathbb{D}}) = \{|F(z)| \mid z \in \overline{\mathbb{D}}\}$ is non-empty compact. Since S is a non-empty compact set of real values, S contains a minimum value. Let $v \in \overline{\mathbb{D}}$ and $|F(v)|$ is minimum.

Suppose that $v \in \partial\mathbb{D}$. $|f(v)| = 1$ because of the property of f, and $|F(v)| = |B_t(f(v))| = 1$ because of the property of B_t. Thus, $|F(v)| = 1$ is the minimum value of S. Since the image of F is a subset of $\overline{\mathbb{D}}$, $s \leq 1$ for every $s \in S$. Then, $1 = |F(v)| \leq s \leq 1$ and $s = 1$ for every $s \in S$. In other words, for every $z \in \overline{\mathbb{D}}$, $|F(z)| = 1$. But it's a contradiction because $0 \in \mathbb{D}$, but $|F(0)| = |B_t(f(0))| = 1$ implies $f(0) \in \partial\mathbb{D}$ and $0 \in \partial\mathbb{D}$.

Thus, $v \in \mathbb{D}$. But in this case, v is a minimum point of the open connected set $\mathbb{D}$ for the analytic function F. By Minimum-Modulus Theorem, $|F(v)| = 0$ and $F(v) = 0$. Because of the property of

B_t, $F(v) = B_t(f(v)) = 0$ means $f(v) = t$. Therefore, for every $t \in \mathbb{D}$, there is $v \in \mathbb{D}$ such that $f(v) = t$. And this shows that h, obtained from f by restricting the domain to $\mathbb{D}$, is surjective.

Because h is injective ($\because$ f is injective) and surjective, there is an inverse function h^{-1}. What we next to check is h^{-1} is analytic in $\mathbb{D}$.

As Proposition 3.5, if h is differentiable at $z_1 = h^{-1}(z_0)$ and $h'(z_1) \neq 0$, then h^{-1} is differentiable at z_0. Since h is analytic in $\mathbb{D}$ ($\because$ f is analytic in $\overline{\mathbb{D}}$), we know that $h'(z_0)$ exists for every $z_0 \in \mathbb{D}$. Suppose that $Z = \{z \in \mathbb{D} \mid h'(z) = 0\}$ is non-empty. Let z_0 be any element of Z. If $(D(z_0; r) \cap Z)\backslash\{z_0\} \neq \emptyset$ for every $r > 0$, it contradicts. ($\because$ If $(D(z_0; r) \cap Z)\backslash\{z_0\} \neq \emptyset$ for every $r > 0$, z_0 is a limit point of Z. Then, since Z is a zero of h', by Uniqueness Theorem, $h' \equiv 0$ in $\mathbb{D}$. Then, h should be constant function in $\mathbb{D}$, but it's a contradiction because $h(1/42) = h(0)$ violates the injectivity of h.) Therefore, there is $r > 0$ such that $D(z_0; r) \cap Z = \{z_0\}$. In other words, for $z \in D(z_0; r)\backslash\{z_0\}$, $h'(z) \neq 0$. Since h is analytic in $\mathbb{D}$, by Open Mapping Theorem, $T = h(D(z_0; r))$ is also open. Since $h(z_0)$ is an interier point of the open set T, take $s > 0$ such that $D(h(z_0); s) \subseteq T$. Because h' is non-zero for every points of $D(z_0; r)\backslash\{z_0\}$, h^{-1} is differentiable at every points of $T\backslash\{h(z_0)\}$, and h^{-1} is differentiable at every points of $D(h(z_0); s)\backslash\{h(z_0)\}$. Let $L = \{x + iy \mid y = \text{Im } h(z_0), x = [-\frac{s}{2} + \text{Re } h(z_0), \frac{s}{2} + \text{Re } h(z_0)]\}$. Since h^{-1} is continuous in $\mathbb{D}$ ($\because$ h is an open map because h is analytic in $\mathbb{D}$ and because of Open Mapping Theorem. Since h is an inverse map of h^{-1}, inverse map of h^{-1} is an open map in $\mathbb{D}$. Thus, $(h^{-1})^{-1}$ maps open sets to open sets and it implies h^{-1} is continuous in $\mathbb{D}$.) and analytic in $D(h(z_0), s)$ except the line segment L ($\because$ h^{-1} is differentiable in the open set $D(h(z_0), s)\backslash L$), by Theorem 7.7, h^{-1} is analytic throughout $D(h(z_0), s)$. Then, since $h(z_0) \in D(h(z_0), s)$, h^{-1} is differentiable at $h(z_0)$. However, it's a contradiction because if h^{-1} is differentiable at $h(z_0)$, since h is differentiable at z_0 and,

$$0 = (h^{-1})'(h(z_0)) \cdot 0 = (h^{-1})'(h(z_0))h'(z_0) \neq (h^{-1} \circ h)'(z_0) = \text{Id}'_{\mathbb{D}}(z_0) = 1$$

It violates Chain Rule. Thus, Z should be empty. And h' is nowhere zero and h^{-1} is analytic in $\mathbb{D}$.

So to sum up at this point, for $h = f|_{\mathbb{D}}$, $h : \mathbb{D} \to \mathbb{D}$ is bijective, continuous and analytic in $\mathbb{D}$ and so is $h^{-1} : \mathbb{D} \to \mathbb{D}$.

Next, we will show that $B_q \circ h$ is a linear function for some $q \in \mathbb{D}$.

Let $\alpha = h(0)$. Then, let $g : \mathbb{D} \to \mathbb{D}$ such that $g = B_\alpha \circ h$. Because of the properties of h and B_α,

- g is analytic in $\mathbb{D}$.
- g is continuous in $\mathbb{D}$.
- g is bijective in $\mathbb{D}$.
- $g(0) = B_\alpha(h(0)) = B_\alpha(\alpha) = 0$.
- g maps every element of $\mathbb{D}$ into $\mathbb{D}$. It's because B_α and h are automorphisms of $\mathbb{D}$. Thus, $|g(z)| < 1$.
- g' is nowhere zero in $\mathbb{D}$, and g^{-1} is continuous and analytic in $\mathbb{D}$. Because g is a bijective function in $\mathbb{D} \to \mathbb{D}$, we can show this continuity and differentiability in $\mathbb{D}$ in exactly same way to show that h^{-1} is analytic in $\mathbb{D}$ above.
- $g^{-1}(0) = 0$ and $g^{-1}(z) < 1$ for every $z \in \mathbb{D}$. It's because g is an automorphism of $\mathbb{D}$ and because of above properties.

Since g and g^{-1} are analytic in the unit disc $\mathbb{D}$, $g \ll 1$, $g^{-1} \ll 1$, $g(0) = g^{-1}(0) = 0$, we can apply Schwarz's Lemma to g and g^{-1}.

By applying Schwarz's Lemma to g^{-1}, we obtain $\left|(g^{-1})'(0)\right| \le 1$ and,

$$\frac{1}{|g'(0)|} = \left|(g^{-1})'(g(0))\right| = \left|(g^{-1})'(0)\right| \le 1$$
$$|g'(0)| \ge 1$$

And by applying Schwarz's Lemma to g, we obtain

$$|g'(0)| \le 1$$

By combining above two inequalities, we obtain

$$|g'(0)| = 1$$

and by Schwarz's Lemma, there is some $K \in \mathbb{C}$ such that $|K| = 1$ and

$$g(z) = Kz$$

Because $B_{-\alpha} = B_\alpha^{-1}$,

$$\begin{aligned} h(z) &= B_{-\alpha}(B_\alpha(h(z))) \\ &= B_{-\alpha}(g(z)) \\ &= B_{-\alpha}(Kz) \\ &= \frac{Kz - (-\alpha)}{1 - \overline{(-\alpha)}Kz} \\ &= \frac{Kz + \alpha}{1 + \overline{\alpha}Kz} \quad \cdots (*) \end{aligned}$$

Because $\alpha = h(0) \in \mathbb{D}$ (i.e. $|\alpha| < 1$) and $|K| = 1$, $|\overline{\alpha}Kz| < 1$ for $z \in \overline{\mathbb{D}}$. Thus, for every $z \in \overline{\mathbb{D}}$, $1 + \overline{\alpha}Kz \neq 0$. Also, since the numerator and denominator of (*) are entire as linear polynomial, (*) is analytic in $\overline{\mathbb{D}}$. Since $h(z) = f(z)$ for every $z \in \mathbb{D}$ and h is analytic, by Uniqueness Theorem,

$$f(z) = \frac{Kz + \alpha}{1 + \overline{\alpha}Kz}$$

for $\alpha = h(0) = f(0)$, $|K| = 1$.

Let $K \in \mathbb{C}$ such that $|K| = 1$ and $\alpha \in \mathbb{D}$. And let $f(z) = B_{-\alpha}(Kz) = \frac{Kz+\alpha}{1+\overline{\alpha}Kz}$. First, as we shown above, the numerator and denominator of $f(z)$ are entire as linear polynomial and $1 + \overline{\alpha}Kz$ cannot be zero for every $z \in \overline{\mathbb{D}}$ because $|\alpha| < 1$. Thus $f(z)$ is entire.

And let $g = B_\alpha \circ f$. Then,

$$\begin{aligned} g(z) &= B_\alpha(f(z)) \\ &= B_{-\alpha}(B_\alpha(Kz)) \\ &= Kz \end{aligned}$$

Since Kz is a linear polynomial as $|K| = 1 > 0$, Kz is bijective, and $g(z)$ is bijective since $B_\alpha, B_{-\alpha}, Kz$ are bijective. For $|z| \le 1$, $|g(z)| = |K| \cdot |z| = 1 \cdot |z| = |z| \le 1$. In other words, since B_α is a bijection which maps $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$, $|f(z)| \le 1$ for $|z| \le 1$. Also, for $|z| = 1$, $|g(z)| = |K| \cdot |z| = |z| = 1$, and since B_α is a bijection which maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$, $|f(z)| = 1$ for $|z| = 1$.

Thus, the $f(z) = \frac{Kz+\alpha}{1+\overline{\alpha}Kz}$ satisfies all condition given in the problem.

In conclusion. The set consisting of all analytic functions f on the closed unit disc in $\mathbb{C}$ such that $|f(z)| \le 1$ for $|z| \le 1$, $|f(z)| = 1$ for $|z| = 1$, and f is injective is:

$$\left\{ z \mapsto \frac{Kz + \alpha}{1 + \overline{\alpha}Kz} \mid \alpha \in \mathbb{D}; K \in \mathbb{C}; |K| = 1 \right\}$$

□

Problem 9

Let h be a continuous function on the boundary of the unit disc in $\mathbb{C}$. Let C be a curve given by $C = C(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. For $z \in \mathbb{C}$ with $|z| \neq 1$, we define

$$f(z) = \frac{1}{2\pi i}\int_C \frac{h(\zeta)}{\zeta - z}d\zeta$$

(1) Prove that f is analytic in $\mathbb{C}\backslash C$.

(2) Prove or disprove that for each $z_0 \in \mathbb{C}$ with $|z_0| = 1$, $\lim_{|z|<1, z\to z_0} f(z) = h(z_0)$.

Lemmata

Lemma 4. *Let $R \subseteq \mathbb{C}$ be a closed rectangle, which does not contains* 0. *Then,*

$$\int_{\partial R} \frac{1}{z}dz = 0$$

Proof. Let $R = [a, b] \times [c, d]$ in $\mathbb{C}$ for some $a, b, c, d \in \mathbb{R}$, $a < b$ and $c < d$. (If $a = b$ or $c = d$, $\int_{\partial R} \frac{1}{z}dz$ is just an integration following a line segment forward and backward. Thus it's trivially 0.) First, since R is non-empty, closed, bounded and compact, and modulus $|\cdot|$ is continuous, $\{|z| \mid z \in R\}$ has minimum. Let $r = \min\{|z| \mid z \in R\}$. Since $0 \notin R$, $r > 0$ and $|z| \geq r > 0$ for every $z \in R$. Then, let $t = \frac{r}{2\sqrt{2}}$. And split R into, $R_{jk} = [a_{j-1}, a_j] \times [c_{k-1}, c_k]$ in $\mathbb{C}$ where $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, $c = c_0 < c_1 < \cdots < c_{m-1} < c_m = d$, $a_k = a_{k-1} + t$ for integer $0 < k < n$, $c_k = c_{k-1} + t$ for integer $0 < k < m$ and n, m is an integer such that $b \leq a + nt < b + t$ and $d \leq c + mt < d + t$. ($n = \lceil \frac{b-a}{t} \rceil$, $m = \lceil \frac{d-c}{t} \rceil$) Then let $N_{jk} = D(\frac{(a_{j-1}+a_j)+i(c_{k-1}+c_k)}{2}; t)$. N_{jk} is a open neighborhood which containing R_{jk}. In this case, every N_{jk} does not contains 0. ($\because$ N_{jk} contains R_{jk}. For every $x \in R_{jk}$, $|x| > r = 2\sqrt{2}t$. Because $|y| \geq |x| - |x - y|$ every $y \in N_{jk}$, $|y| \geq |x| - |x - y| \geq 2\sqrt{2}t - 2t > 0$ and $|y| > 0$. Thus $y \neq 0$ for every $y \in N_{jk}$.)

$\frac{1}{z}$ is differentiable in $\mathbb{C}\backslash\{0\}$. ($\because$ For $z = x + iy \neq 0$, $\frac{\partial}{\partial x}\frac{1}{x+iy} = \frac{-1}{(x+iy)^2} = -z^{-2}$ and $\frac{\partial}{\partial y}\frac{1}{x+iy} = \frac{-i}{(x+iy)^2} = -iz^{-2}$ They are continuous in some neighborhood of z and Cauchy-Riemann Equation, $i\frac{\partial}{\partial x}\frac{1}{x+iy} = -iz^{-2} = \frac{\partial}{\partial y}\frac{1}{x+iy}$ holds.) Therefore, $\frac{1}{z}$ is analytic in the open set $\mathbb{C}\backslash\{0\}$, and analytic in each $R_{jk} \subseteq \mathbb{C}\backslash\{0\}$.

Then, by Theorem 6.2, 6.3,

$$\int_{\partial R_{jk}} \frac{1}{z}dz = 0$$

holds.

Since $\{R_{jk}\}_{1\leq j\leq n, 1\leq k\leq m}$ is a set of splitted rectangles of R,

$$\int_{\partial R} \frac{1}{z}dz = \sum_{j=1}^{n}\sum_{k=1}^{m}\int_{\partial R_{jk}} \frac{1}{z}dz = \sum_{j=1}^{n}\sum_{k=1}^{m} 0 = 0$$

□

Lemma 5. *If $D(c; r)$ does not contain 0, $\frac{1}{z}$ has an antiderivative in $D(c; r)$.*

Proof. Take $F(z) = \int_c^z \frac{1}{\zeta}d\zeta$. Then as the proof of Theorem 4.15 applying Lemma 4 as Retangle Theorem, we can show that $F'(z) = \frac{1}{z}$. □

Lemma 6. *Let $C : \sigma(t) = c + r \cdot e^{i\theta}$ where $t \in [0, 2\pi]$, $c \in \mathbb{C}$, $r > 0$, $|c| < r$. Then,*

$$\int_C \frac{1}{z}dz = 2\pi i$$

Proof. In this case, C is a circle centered at c with radius r and $|0-c| = |c| < r$. Thus, by Lemma 5.4,

$$\int_C \frac{dz}{z} = 2\pi i$$

□

Proof of (1)

Let $D_1 = \{z \in \mathbb{C} \mid |z| > 1\}$ and $D_2 = \{z \in \mathbb{C} \mid |z| < 1\}$. Since $C = \{z \in \mathbb{C} \mid |z| = 1\}$, $D_1 \sqcup D_2 = \mathbb{C} \backslash C$. Also, both of D_1 and D_2 are regions (open, connected).

First, note that $f(z)$ is continuous in $\mathbb{C} \backslash C$ because it's an integration of continuous function. ($\because$ h is continuous on C and $\zeta - z$ cannot be 0 for $z \in C$ because if not, $\zeta = z$ but $1 = |\zeta| = |z| \neq 1$.)

Let $\Gamma \subseteq D_1$ be a boundary of a closed rectangle R in D_1. Because $R \subseteq D_1$, for every points x of R, $|x| > 1$. Thus, for any $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$, $\zeta - x \neq 0$ for every $x \in R$. (If not, $\zeta = x$ for some x, but it contradicts since $1 = |\zeta| = |x| > 1$) It implies that $\zeta - R$ does not contains 0. Since f is continuous in D_1, f is Riemann-integrable and it's integration is continuous. Thus $\int_\Gamma \int_C \frac{h(\zeta)}{\zeta - z} d\zeta dz$ converges and we can change the order of integrations. Therefore, by Lemma 4,

$$\begin{aligned}
\int_\Gamma \frac{1}{2\pi i} \int_C \frac{h(\zeta)}{\zeta - z} d\zeta dz &= \frac{1}{2\pi i} \int_\Gamma \int_C \frac{h(\zeta)}{\zeta - z} dz d\zeta \\
&= \frac{1}{2\pi i} \int_C \int_\Gamma \frac{h(\zeta)}{\zeta - z} dz d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \int_\Gamma \frac{1}{\zeta - z} dz d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \left(- \int_{\zeta - \Gamma} \frac{1}{z} dz \right) d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \left(- \int_{\partial(\zeta - R)} \frac{1}{z} dz \right) d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \cdot 0 d\zeta = \frac{1}{2\pi i} \int_C 0 d\zeta = \frac{1}{2\pi i} \cdot 0 = 0
\end{aligned}$$

Thus, by Morera's Theorem, f is analytic on D_1.

Let Γ be a closed rectangle R in D_2. Because $R \subseteq D_2$, every point $x \in R$ satisfies $|x| < 1$. Then, for every $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$, $0 \notin \zeta - R$. (If not, there is some $x \in R$ such that $\zeta - x = 0$, but in this case, $\zeta = x$ and $1 = |\zeta| = |x| < 1$. It's a contradiction.) Then, as above,

$$\begin{aligned}
\int_\Gamma \frac{1}{2\pi i} \int_C \frac{h(\zeta)}{\zeta - z} d\zeta dz &= \frac{1}{2\pi i} \int_\Gamma \int_C \frac{h(\zeta)}{\zeta - z} dz d\zeta \\
&= \frac{1}{2\pi i} \int_C \int_\Gamma \frac{h(\zeta)}{\zeta - z} dz d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \int_\Gamma \frac{1}{\zeta - z} dz d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \left(- \int_{\zeta - \Gamma} \frac{1}{z} dz \right) d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \left(- \int_{\partial(\zeta - R)} \frac{1}{z} dz \right) d\zeta \\
&= \frac{1}{2\pi i} \int_C h(\zeta) \cdot 0 d\zeta = \frac{1}{2\pi i} \int_C 0 d\zeta = \frac{1}{2\pi i} \cdot 0 = 0
\end{aligned}$$

Thus, by Morera's Theorem, f is analytic on D_2, too.

In conclusion, f is analytic in $\mathbb{C} \backslash C = D_1 \sqcup D_2$ by Morera's Theorem. □

Answer of (2)

Let $h(z) = 1/z$. It's obvious that h is continuous in $C = \partial D(0;1)$ since z is a continuous as polynomial and $z \neq 0$ for $z \in \mathbb{C}\backslash\{0\}$. Then, for $|z| < 1$,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{h(\zeta)}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_C \frac{1}{\zeta(\zeta - z)}d\zeta \\
&= \frac{1}{2\pi i}\int_C \frac{-1}{z\zeta} + \frac{1}{z(\zeta - z)}d\zeta \\
&= \frac{1}{2\pi i}\frac{1}{z}\int_C \frac{-1}{\zeta} + \frac{1}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\frac{1}{z}\left(\int_C \frac{-1}{\zeta}d\zeta + \int_C \frac{1}{\zeta - z}d\zeta\right) \\
&= \frac{1}{2\pi i}\frac{1}{z}\left(\int_C \frac{-1}{\zeta}d\zeta + \int_{C-z} \frac{1}{\zeta}d\zeta\right)
\end{aligned}$$

Beceause C is a boundary of $D(0;1)$, $C - z$ is a boundary of $D(-z;1)$. And since $|-z-0| < 1$, $D(-z;1)$ contains 0. Also $D(0;1)$ contains 0 trivially. Thus, by Lemma 6,

$$\int_C \frac{1}{\zeta}d\zeta = 2\pi i = \int_{C-z} \frac{1}{\zeta}d\zeta$$

And,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\frac{1}{z}\left(\int_C \frac{-1}{\zeta}d\zeta + \int_{C-z} \frac{1}{\zeta}d\zeta\right) \\
&= \frac{1}{2\pi i}\frac{1}{z}\left(-2\pi i + 2\pi i\right) \\
&= \frac{1}{2\pi i}\frac{1}{z}\cdot 0 \\
&= 0
\end{aligned}$$

Therefore, $f(z) = 0$ for $h(z) = 1/z$ and $|z| < 1$. Then,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 0 = 0 \neq 1 = h(1)$$

So, $h(z) = 1/z$ is a counterexample of the problem (2). $\square$