MAS511 Spring 2020 Homework #04

June 4, 2022

Problem 1

Determine whether $\mathbb{Q}$ satisfies each of the following properties as a $\mathbb{Z}$-module: (1) finitely generated, (2) free, (3) projective, (4) flat, (5) injective.

Solution

(1) Not finitely generated.

Suppose that $\mathbb{Q}$ is finitely generated. Then, there is $a_1, \cdots, a_n \in \mathbb{Q}$ which satisfies that for any given $q \in \mathbb{Q}$, there is $z_1, \cdots, z_n \in \mathbb{Z}$ such that

$$q = \sum_{k=1}^{n} z_k a_k$$

Let's assume that each $a_1, \cdots, a_n$ are non-zero. (If $a_i = 0$, $z_i a_i = 0$ for any $z_i \in \mathbb{Z}$. Thus, $\sum_{k=1}^{n} z_k a_k$ does not change even if we ignore the term of a_i.) Because every non-zero elemtent of $\mathbb{Q}$ can be expressed as $\frac{p}{q}$ where $p \in \mathbb{Z}\backslash\{0\}$, $q \in \mathbb{N}$ and p and q are relatively prime, let $p_1, \cdots, p_n \in \mathbb{Z}\backslash\{0\}$ and $q_1, \cdots, q_n \in \mathbb{N}$ such that $(p_k, q_k) = 1$ and $a_k = p_k/q_k$ for each $k \in \{1, 2, \cdots, n\}$.

Let

$$t = \frac{1}{2 \cdot q_1 \cdot q_2 \cdot \cdots \cdot q_n}$$

Trivially, it's a number obtained from 1 by dividing a natural number, and $t \in \mathbb{Q}$. Thus, there should be some $z_1, \cdots, z_n \in \mathbb{Z}$ such that,

$$t = \frac{1}{2 \cdot q_1 \cdot q_2 \cdot \cdots \cdot q_n} = \sum_{k=1}^{n} z_k a_k = \sum_{k=1}^{n} \frac{z_k p_k}{q_k}$$

By multiplicating $2 \cdot q_1 q_2 \cdots q_n$ in both sides,

$$1 = \sum_{k=1}^{n} t_k \text{ where } t_k = 2 z_k p_k \prod_{j \in \{1, \cdots, n\} \backslash \{k\}} q_j$$

Since z_k, p_k, q_j are integers, $\sum_{k=1}^{n} t_k$ should be an integer, which can be divided by 2. But, $\sum_{k=1}^{n} t_k = 1$ and it cannot be divided by 2. Thus, it's a contradiction.

Therefore, $\mathbb{Q}$ cannot be finitely generated as a $\mathbb{Z}$-module. $\square$

(2) Not free. It's because $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module. See (3). $\square$

(3) Not projective.

$$\begin{tikzcd}
\mathbb{Z}[x_1, x_2, \cdots] \arrow[r, two heads, "f"] & \mathbb{Q} \arrow[r] & 0 \\
& \mathbb{Q} \arrow[u, "\mathrm{Id}_{\mathbb{Q}}"'] \arrow[ul, dashed, "g"] &
\end{tikzcd}$$

Let f be a homomorphism such that $f(z) = z$ for every $z \in \mathbb{Z}$ and $f(x_k) = \frac{1}{k}$ for every $k \in \mathbb{N}$. Then, f is surjective since for any $q/p \in \mathbb{Q}$ where $q, p \in \mathbb{Z}$, $f(qx_p) = q/p$.

Suppose that there is a homomorphism $g : \mathbb{Q} \to \mathbb{Z}[x_1, x_2, \cdots]$ such that $\mathrm{Id}_{\mathbb{Q}} = f \circ g$. Let $q \in \mathbb{Q}\backslash\{0\}$. Then, $g(q) \neq 0$. (If not, $0 \neq q = \mathrm{Id}_{\mathbb{Q}}(q) = f(g(q)) = f(0) = 0$ and it's a contradiction.) Then, $g(q) = \sum_{j=0}^{n}\sum_{k=0}^{m_j} a_{j,k}x_j^k$ for some $n, m_k \in \mathbb{Z}^{\geq 0}, a_{j,k} \in \mathbb{Z}$ where $a_{n,m_n} \neq 0$. (For convinience, let $x_0 = 1$.) And, $f(\sum_{j=0}^{n}\sum_{k=0}^{m_j} a_{j,k}x_j^k) = q$. In the similar way, there are $p, q_k \in \mathbb{Z}^{\geq 0}, b_{j,k} \in \mathbb{Z}$ such that

$$g(\frac{q}{2a_{n,m_n}}) = \sum_{j=0}^{p}\sum_{k=0}^{q_j} b_{j,k}x_j^k$$

Then,

$$\sum_{j=0}^{n}\sum_{k=0}^{m_j} a_{j,k}x_j^k = g(q) = 2a_{n,m_n}g(\frac{q}{2a_{n,m_n}}) = \sum_{j=0}^{p}\sum_{k=0}^{q_j} 2a_{n,m_n}b_{j,k}x_j^k$$

Let α be the coefficient of x_j^k-term of $g(q)$ and β be the coefficient of x_j^k-term of $g(\frac{q}{2a_{n,m_n}})$. Then,

$$\alpha = a_{n,m_n}, \beta = \begin{cases} b_{n,m_n} & n \leq p, m_n \leq q_n \\ 0 & \text{Otherwise} \end{cases}$$

Note that $\alpha, \beta \in \mathbb{Z}$. Since $g(q) = 2a_{n,m_n}g(\frac{q}{2a_{n,m_n}})$, $\alpha = 2a_{n,m_n}\beta$. Then,

$$\beta = \frac{\alpha}{2a_{n,m_n}} = \frac{a_{n,m_n}}{2a_{n,m_n}} = \frac{1}{2}$$

It's a contradiction since $\beta \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$.

Therefore, there is no g such that $\mathrm{Id}_{\mathbb{Q}} = f \circ g$, and it shows that $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module. $\square$

(4) Flat.

To show that $\mathbb{Q}$ is flat, $\mathbb{Q} \otimes_{\mathbb{Z}} -$ and $- \otimes_{\mathbb{Z}} \mathbb{Q}$ are right exact fucntor.

Let's show that $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is right exact. Then, $- \otimes_{\mathbb{Z}} \mathbb{Q}$ can be shown that it's also right exact in the similar method.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, tail, "f"] \arrow[d, Rightarrow, "\mathbb{Q}\otimes_{\mathbb{Z}}-"] & N \\ 0 \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} M \arrow[r, "\varphi"] & \mathbb{Q} \otimes_{\mathbb{Z}} N \end{tikzcd}$$

Let $f : M \to N$ be an injective $\mathbb{Z}$-module homomorphism. Let $\varphi : \mathbb{Q} \otimes_{\mathbb{Z}} M \to \mathbb{Q} \otimes_{\mathbb{Z}} N$ such that $\varphi = \mathbb{Q} \otimes_{\mathbb{Z}} f$ (note, $\varphi(q \otimes m) = q \otimes f(m)$). Then, it's enough to show that φ is injective.

First, note that for arbitrary left $\mathbb{Z}$-module K:

- $0 \otimes 0$ is an (additive) identity of $\mathbb{Q} \otimes_{\mathbb{Z}} K$.
- $0 \otimes k = 0 \otimes 0$. It's because $0 \otimes k = (0 \cdot 0) \otimes k = 0 \otimes (0 \cdot k) = 0 \otimes 0$.
- If k is an (additive) identity of K, $k = 0 \cdot k$ and $q \otimes k = q \otimes (0 \cdot k) = (q \cdot 0) \otimes k = 0 \otimes k = 0 \otimes 0$ for any $q \in \mathbb{Q}$.
- For every $q \in \mathbb{Q}\backslash\{0\}$ and $k \in K$, there is $n \in \mathbb{N}$ such that $q \otimes k = \frac{1}{n} \otimes k'$ for some $k' \in K$. Because, let $q = \frac{m}{n}$ for some $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, $q \otimes k = (\frac{1}{n} \otimes m) \otimes k = \frac{1}{n} \otimes (m \cdot k)$
- Every non-zero element of $\mathbb{Q} \otimes_{\mathbb{Z}} K$ can be expressed as $\frac{1}{n} \otimes k$ for some $n \in \mathbb{N}$ and $k \in K$.

  Let $t \in \mathbb{Q} \otimes_{\mathbb{Z}} K$ is non-zero (i.e. not an additive identity). Then, there are $q_1, \cdots, q_m \in \mathbb{Q}$ and $k_1, \cdots, k_m \in K$ such that

  $$t = \sum_{i=1}^{m} q_i \otimes k_i$$

If some $q_i \otimes k_i$ are an additive identity, the sum does not change even if we ignore the terms. Thus, let's assume that every $q_i \otimes k_i$ is non-zero. (If every $q_i \otimes k_i$ are zero, t should be zero. But as we assumed that t is non-zero, t must be a sum of at least one non-zero $q_i \otimes k_i$.) But as we showed above, there are $n_1, \cdots, n_m \in \mathbb{N}$ and $k'_1, \cdots, k'_m \in K$ such that $q_i \otimes k_i = \frac{1}{n_i} \otimes k'_i$. Also, let $n = \prod_{i=1}^m n_i$ and $\kappa_i = \frac{n}{n_i} \cdot k'_i$. Then,

$$\frac{1}{n_i} \otimes k_i = \frac{1}{n} \otimes (\frac{n}{n_i} \cdot k_i) = \frac{1}{n} \otimes \kappa_i$$

and

$$t = \sum_{i=1}^m q_i \otimes k_i = \sum_{i=1}^m \frac{1}{n} \otimes \kappa_i = \frac{1}{n} \otimes (\sum_{i=1}^m \kappa_i)$$

Let $k = \sum_{i=1}^m \kappa_i$ then we obtain $t = \frac{1}{n} \otimes k$.

Let t is a non-zero element of $\mathbb{Q} \otimes_{\mathbb{Z}} M$, and suppose that $\varphi(t) = 0_{\mathbb{Q}\otimes_{\mathbb{Z}} N}$. As we showed above, there is $n \in \mathbb{N}$ and $m \in M$ such that $t = \frac{1}{n} \otimes m$. Then, $\varphi(t) = \varphi(\frac{1}{n} \otimes m) = \frac{1}{n} \otimes f(m) = 0_{\mathbb{Q}\otimes_{\mathbb{Z}} N}$. Then, since there is no $z \in \mathbb{Z}$ such that $\frac{z}{n} = 0$, there should be some $z \in \mathbb{Z}$ such that $zf(m) = f(zm) = 0$. Since f is injective, $zm = 0$. Then,

$$t = \frac{1}{n} \otimes m = \frac{1}{nz} \otimes zm = \frac{1}{nz} \otimes 0_M = 0_{\mathbb{Q}\otimes M}$$

It's a contradiction since we assumed that t is non-zero.

Thus, $\ker \varphi = \{0_{\mathbb{Q}\otimes_{\mathbb{Z}} M}\}$ and $\varphi = \mathbb{Q} \otimes_{\mathbb{Z}} f$ is injective. In the same way $f \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective for a $\mathbb{Z}$-module homomorphism $f : M \to N$ where M and N are right $\mathbb{Z}$-module homomorphism. Thus, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. □

(5) Injective

Use Baer Criterion. Since $\mathbb{Z}$ is a Euclidean domain, PID, every ideal of $\mathbb{Z}$ is principal. Let $n \in \mathbb{Z}$ and take an ideal (n). Let $g : (n) \to \mathbb{Q}$ be a $\mathbb{Z}$-module homomorphism.

If $n = 0$, $g(0) = 0$. (Otherwise, $2g(0) = g(2 \cdot 0) = g(0)$ should hold and $0 \neq g(0) = 2g(0) - g(0) = 0$. It's a contradiction.) Thus, if we define $\tilde{g} : \mathbb{Z} \to \mathbb{Q}$ as $\tilde{g} = 0$, it's a extension of g and a $\mathbb{Z}$-module homomorphism (more specifically, zero homomorphism).

For $n \neq 0$, every element m of $\mathbb{Z}$ can be expressed as $\frac{mn}{n}$. Then, let's define $\tilde{g} : \mathbb{Z} \to \mathbb{Q}$ as

$$\tilde{g}(m) = \frac{g(mn)}{n}$$

Then,

- $\tilde{g}$ is well-defined. For $m \in \mathbb{Z}$, $mn \in n\mathbb{Z} = (n) = \mathrm{Dom}(g)$.
- $\tilde{g}$ is an extension of g. Let $m \in (n)$. Then, $\tilde{g}(m) = \frac{g(mn)}{n} = \frac{ng(m)}{n} = g(m)$.
- $\tilde{g}$ is a $\mathbb{Z}$-module homomorphism. Let $x, y \in \mathbb{Z}$, then,

$$\tilde{g}(x + y) = \frac{g((x + y)n)}{n} = \frac{g(xn) + g(yn)}{n} = \tilde{g}(x) + \tilde{g}(y)$$

$$\tilde{g}(xy) = \frac{g(xyn)}{n} = \frac{xg(yn)}{n} = x\tilde{g}(y)$$

Thus, by Baer's Criterion, $\mathbb{Q}$ is a injective $\mathbb{Z}$-module. □

Problem 2

Let R be a ring and let

$$0 \to K_P \xrightarrow{f_1} P \xrightarrow{f_0} M \to 0$$
$$0 \to K_Q \xrightarrow{g_1} Q \xrightarrow{g_0} M \to 0$$

be exact sequences of left R-modules such that P and Q are projective. Prove that there exists an isomorphism of left R-modules

$$K_P \oplus Q \simeq K_Q \oplus P$$

Solution

$$\begin{tikzcd}
\ker f_0 \simeq \operatorname{im} f_1 \simeq K_P \arrow[r, tail, "f_1"] \arrow[dd, dashed, "\exists h"] & P \arrow[dr, "f_0"] \arrow[dd, dashed, shift left, "\exists \tilde{f_0}"] & \\
 & & M \\
\ker g_0 \simeq \operatorname{im} g_1 \simeq K_Q \arrow[r, tail, "g_1"'] & Q \arrow[ur, "g_0"'] \arrow[uu, dashed, shift left, "\exists \tilde{g_0}"] &
\end{tikzcd}$$

Since P and Q are projective, there are liftings $\tilde{f_0} : P \to Q$ and $\tilde{g_0} : Q \to P$. And, $f_0 = g_0 \circ \tilde{f_0}$ and $g_0 = f_0 \circ \tilde{g_0}$.

Since f_1 and g_1 are injective homomorphisms, $K_P \simeq \operatorname{im} f_1$ with an isomorphism f_1 and $K_Q \simeq \operatorname{im} g_1$ with an isomorphism g_1. Note that f_1 is an isomorphism by restricting its codomain to $\operatorname{im} f_1 = \ker f_0$ and g_1 is an isomorpihsm by restricting its codomain to $\operatorname{im} g_1 = \ker g_0$.

Let $x \in K_P$. Then, $0 = f_0(f_1(x)) = g_0((\tilde{f_0} \circ f_1)(x))$. Thus, $(\tilde{f_0} \circ f_1)(x) \in \ker g_0 = \operatorname{im} g_1$. Let $h : K_P \to K_Q$ be a map such that

$$h(x) = g_1^{-1}(\tilde{f_0}(f_1(x)))$$

Since $\operatorname{im}(\tilde{f_0} \circ f_1) \subseteq \operatorname{im} g_1$, it's well-defined, and it's a homomorphism since g_1 is an isomorphism and $\tilde{f_0} \circ f_1$ is a homomorphism from K_P to $\operatorname{im} g_1$.

Then, let $\varphi : K_P \to K_Q \oplus P$ such that $\varphi(k) = (h(k), f_1(k))$ for $k \in K_P$, and $\psi : K_Q \oplus P \to Q$ such that $\psi(k, p) = g_1(k) - \tilde{f_0}(p)$. Then,

- φ is injective. If $k \in \ker \varphi$, $\varphi(k) = (0, 0)$. Then, $f_1(k) = 0$ Since f_1 is injective, k should be 0.
- ψ is surjective. Let $q \in Q$. Then, $q - (\tilde{f_0} \circ \tilde{g_0})(q) \in \ker g_0 = \operatorname{im} g_1$, because $g_0(q - (\tilde{f_0} \circ \tilde{g_0})(q)) = g_0(q) - f_0(\tilde{g_0}(q)) = g_0(q) - g_0(q) = 0$. Take $k = g_1^{-1}(q - (\tilde{f_0} \circ \tilde{g_0})(q))$ and $p = -\tilde{g_0}(q)$. Then,

$$\psi(k, p) = g(g_1^{-1}(q - (\tilde{f_0} \circ \tilde{g_0})(q))) - (-\tilde{f_0}(\tilde{g_0}(q))) = q$$

- $\ker \psi = \operatorname{im} \varphi$. Let $k \in K_P$. Then,

$$\begin{aligned}
\psi(h(k), f_1(k)) &= g_1(h(k)) - \tilde{f_0}(f_1(k)) \\
&= g_1(g_1^{-1}(\tilde{f_0}(f_1(k)))) - \tilde{f_0}(f_1(k)) \\
&= \tilde{f_0}(f_1(k)) - \tilde{f_0}(f_1(k)) = 0
\end{aligned}$$

Thus, the below is a short exact sequence:

$$0 \longrightarrow K_P \xrightarrow{\varphi} K_Q \oplus P \xrightarrow{\psi} Q \longrightarrow 0$$

Since Q is projective, above sequence is split. Therefore, $K_Q \oplus P \simeq K_P \oplus Q$. $\square$

Problem 3

Let $n \geq 2$ be an integer. We define two complexes of $\mathbb{Z}$-modules $A_\bullet$, $B_\bullet$ by

$$A_\bullet = \left(\cdots \longrightarrow A_3 = 0 \longrightarrow A_2 = 0 \longrightarrow A_1 = \mathbb{Z} \xrightarrow{\times n} A_0 = \mathbb{Z} \longrightarrow A_{-1} = 0 \longrightarrow \cdots\right)$$

$$B_\bullet = (\cdots \longrightarrow B_3 = 0 \longrightarrow B_2 = 0 \longrightarrow B_1 = 0 \longrightarrow B_0 = \mathbb{Z}/n\mathbb{Z} \longrightarrow B_{-1} = 0 \longrightarrow \cdots)$$

Prove that $A_\bullet$ and $B_\bullet$ are quasi-isomorphic but *not* chain homotopy equivalent.

Solution

$$H_k(A_\bullet) = \frac{\ker f_k}{\mathrm{im}\, f_{k+1}} \text{ where } f_k : A_k \to A_{k-1}$$

Note that

$$\ker(0 \to M) = \mathrm{im}\,(M \to 0) = \mathrm{im}\,(0 \to M) = \{0\}$$

$$\ker(M \to 0) = M$$

$$\ker(\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}) = \{0\}$$

$$\mathrm{im}\,(\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}) = n\mathbb{Z}$$

where M is a $\mathbb{Z}$-module. And,

$$H_0(A_\bullet) = H_0(B_\bullet) = \mathbb{Z}_n$$

$$H_k(A_\bullet) = H_k(B_\bullet) = 0 \text{ for } k \in \mathbb{Z}\backslash\{0\}$$

Quasi-isomorphic

$$\begin{array}{ccccccccccc}
A_\bullet = & \cdots \longrightarrow & A_2 = 0 & \longrightarrow & A_1 = \mathbb{Z} & \xrightarrow{\times n} & A_0 = \mathbb{Z} & \longrightarrow & A_{-1} = 0 & \longrightarrow & \cdots \\
f_\bullet \uparrow & & f_2 \uparrow & & f_1 \uparrow & & f_0 \uparrow & & f_{-1} \uparrow & & \\
C_\bullet = & \cdots \longrightarrow & C_2 = 0 & \longrightarrow & C_1 = \mathbb{Z} & \xrightarrow{\times n} & C_0 = \mathbb{Z} & \longrightarrow & C_{-1} = 0 & \longrightarrow & \cdots \\
g_\bullet \downarrow & & g_2 \downarrow & & g_1 \downarrow & & g_0 \downarrow & & g_{-1} \downarrow & & \\
B_\bullet = & \cdots \longrightarrow & B_2 = 0 & \longrightarrow & B_1 = 0 & \longrightarrow & B_0 = \mathbb{Z}_n & \longrightarrow & B_{-1} = 0 & \longrightarrow & \cdots
\end{array}$$

Take f_k, g_k are identity maps except g_1 which is a zero map and g_0 which is a canonical injection ($g_0 : x \mapsto x + n\mathbb{Z}$).

Since all of $H_k(A_\bullet), H_k(B_\bullet), H_k(C_\bullet)$ are 0 for $k \in \mathbb{Z}\backslash\{0\}$, the homomorphism induced from f_k, g_k for this k is a zero map, which is an isomorphism between zero modules.

Since $H_0(A_\bullet) = H_0(C_\bullet) = \mathbb{Z}/n\mathbb{Z}$ and f_0 is an identity map, the homomorphism induced from f_0 is an identity map between $\mathbb{Z}/n\mathbb{Z}$. Thus it's an isomorphism between $H_0(A_\bullet) = H_0(C_\bullet) = \mathbb{Z}/n\mathbb{Z}$.

Since g_0 is a canonical injection from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$, it induces an identity map from $\mathbb{Z}/n\mathbb{Z}$ to itself. Thus, g_0 is also an isomorphism between $H_0(B_\bullet) = H_0(C_\bullet) = \mathbb{Z}/n\mathbb{Z}$.

Therefore, there is a $C_\bullet$ and chain maps from $C_\bullet$ to $A_\bullet$ and $B_\bullet$ which induces isomorphisms betwene homology groups. Thus, $A_\bullet$ and $B_\bullet$ are quasi-isomorphic. $\square$

Not chain homotopy equivalent

Suppose that there are chain maps $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to A_\bullet$ such that $f_\bullet \circ g_\bullet \sim \mathrm{Id}_{B_\bullet}$ and $g_\bullet \circ f_\bullet \sim \mathrm{Id}_{A_\bullet}$.

$$\begin{array}{ccccccccccc}
A_\bullet = & \cdots \longrightarrow & A_2 = 0 & \longrightarrow & A_1 = \mathbb{Z} & \longrightarrow & A_0 = \mathbb{Z} & \longrightarrow & A_{-1} = 0 & \longrightarrow & \cdots \\
g_\bullet \uparrow\downarrow f_\bullet & & g_2 \uparrow\downarrow f_2 & & g_1 \uparrow\downarrow f_1 & & g_0 \uparrow\downarrow f_0 & & g_{-1} \uparrow\downarrow f_{-1} & & \\
B_\bullet = & \cdots \longrightarrow & B_2 = 0 & \longrightarrow & B_1 = 0 & \longrightarrow & B_0 = \mathbb{Z}_n & \longrightarrow & B_{-1} = 0 & \longrightarrow & \cdots
\end{array}$$

Note that there are only one homomorphism from some $\mathbb{Z}$-module to 0, which is a zero map, and a zero map is the only homomorphism from 0 to some $\mathbb{Z}$-module. Thus, $f_k \equiv 0$ and $g_k \equiv 0$ for $k \in \mathbb{Z}\backslash\{0\}$.

Also, g_0 should be zero. (Because the order of $g_0(1)$ should be n since $ng_0(1) = g_0(n) = g_0(0) = 0$, but every element of $\mathbb{Z}$ has an infinite order.)

Then, $f_0 \circ g_0$ is a zero map. Also, since every f_k, g_k for $k \in \mathbb{Z}\backslash\{0\}$ are zero maps, $f \circ g$ is a zero chain map.

Suppose that $f \circ g$ is homotopic to Id_B. Then there is a maps $s_k : B_k \to B_{k+1}$ for each $k \in \mathbb{Z}$ such that $\mathrm{Id}_B - (f \circ g) = sd + ds$, where $d_k : B_k \to B_{k-1}$ are maps of chain complex $B_\bullet$. Let $h = \mathrm{Id}_B - (f \circ g)$ (i.e. $h_k = \mathrm{Id}_{B_k} - (f_k \circ g_k)$). Then, $h_k = \mathrm{Id}_{B_k}$ since $f_k \circ g_k = 0$ for each $k \in \mathbb{Z}$. And,

$$\begin{tikzcd}
\cdots \arrow[r] & B_1 = 0 \arrow[r, "d_1"] \arrow[d, "\mathrm{Id}_{B_1}"'] & B_0 = \mathbb{Z}_n \arrow[r, "d_0"] \arrow[d, "\mathrm{Id}_{B_0}"] \arrow[dl, "s_0"] & B_{-1} = 0 \arrow[r] \arrow[d, "\mathrm{Id}_{B_{-1}}"] \arrow[dl, "s_{-1}"] & \cdots \\
\cdots \arrow[r] & B_1 = 0 \arrow[r, "d_1"'] & B_0 = \mathbb{Z}_n \arrow[r, "d_0"'] & B_{-1} = 0 \arrow[r] & \cdots
\end{tikzcd}$$

Since s_0 and d_0 are homomorphisms to a zero module, $s_0 = d_0 = 0$. Since s_{-1} and d_1 are homomorphisms from a zero module, $s_{-1} = d_1 = 0$. Thus, $d_1 s_0 + s_{-1} d_0 = 0$. Since $\mathrm{Id}_{B_0} = \mathrm{Id}_{\mathbb{Z}_n} \neq 0$, $\mathrm{Id}_{\mathbb{Z}_n} = d_1 s_0 + s_{-1} d_0 = 0$ cannot hold. Thus, it's a contradiction.

Therefore, $f \circ g$ cannot be chain homotopic to Id_B. It implies that A and B cannot be chain homotopy equivalent. $\square$

Lemmata

Lemma 1. *Let G be an abelian group, and $K \leq G$ be a subgroup. Let $\varphi : K \to \mathbb{Q}/\mathbb{Z}$ is a group homomorphism. Then, there is a group homomorphism $\tilde{\varphi} : G \to \mathbb{Q}/\mathbb{Z}$ such that $\tilde{\varphi}|_K = f$.*

Proof. Let

$$\mathcal{P} = \{(H, f) \mid H \leq G, \text{homomorphism} f : H \to \mathbb{Q}/\mathbb{Z}\}$$

Note that $\mathcal{P}$ is non-empty since $(K, \varphi) \in \mathcal{P}$. Then, let $\prec$ be a relation of $\mathcal{P}$ such that $(H, f) \prec (H', f')$ if and only if $H \leq H'$ and $f'|_H = f$. It's a partial order since,

- Reflexive. Because $H \leq H$ and $f|_H = f$.
- Antisymmertic. If $(H, f) \prec (H', f')$ and $(H, f) \succ (H', f')$, $H \leq H' \leq H$ implies $H = H'$, and $f = f'|_H = f'|_{H'} = f'$ holds. Thus, $(H, f) = (H', f')$.
- Transitive. If $(H, f) \prec (H', f')$ and $(H', f') \prec (H'', f'')$, $H \leq H' \leq H''$ and $f = f'|_H = f''|_{H'}|_H = f''|_H$. Thus $(H, f) \prec (H'', f'')$.

If a chain of the relation $\prec$ is finite, there is a maximum element of the chain. Let $\{(H_\alpha, f_\alpha)\}_{\alpha \in \Lambda}$ be a infinitely long chain of the relation $\prec$. Then, we can construct a element $(M, g) \in \mathcal{P}$:

$$M = \bigcup_{\alpha \in \Lambda} H_\alpha$$

$$g(x) = f_\alpha(x) \text{ for some } \alpha \text{ such that } x \in H_\alpha$$

$M \subseteq G$, because each $H_\alpha \subseteq G$. M is a group, because, (1) Since each H_α contains 0_G, $0_G \in M$; (2) for $a, b \in M$, there is $\alpha, \beta \in \Lambda$ such that $a \in H_\alpha$ and $b \in H_\beta$, WLOG if we suppose that $(H_\alpha, f_\alpha) \prec (H_\beta, f_\beta)$, $a + b \in H_\beta \subseteq M$, thus M is closed under addition; (3) for $a, b, c \in M$, $(a + b) + c = a + (b + c)$ because there is $\alpha, \beta, \gamma \in \Lambda$ such that $a \in H_\alpha$, $b \in H_\beta$, and $c \in H_\gamma$, and WLOG if we assumed that $(H_\beta, f_\beta), (H_\gamma, f_\gamma) \prec (H_\alpha, f_\alpha)$, $(a + b) + c$ and $a + (b + c)$ are operations in H_α and they are equal since H_α is a group; (4) if $a \in M$, there is $\alpha \in \Lambda$ such that $a \in H_\alpha$, and since $a^{-1} \in H_\alpha$, $a^{-1} \in M$. Therefore, $M \leq G$. g is well-defined because if $x \in H_\alpha$ and $x \in H_\beta$ for some $\alpha, \beta \in \Lambda$, WLOG if we assumed that $(H_\alpha, f_\alpha) \prec (H_\beta, f_\beta)$, $f_\beta(x) = f_\beta|_{H_\alpha}(x) = f_\alpha(x)$ holds. g is a homomorphism from M to $\mathbb{Q}/\mathbb{Z}$, because if $x_\alpha \in H_\alpha, x_\beta \in H_\beta$, WLOG if we assumed that $(H_\alpha, f_\alpha) \prec (H_\beta, f_\beta)$, $g(x_\alpha + x_\beta) = f_\beta(x_\alpha + x_\beta) = f_\beta(x_\alpha) + f_\beta(x_\beta) = g(x_\alpha) + g(x_\beta)$. And (M, g) is the maximum of the chain since $H_\alpha \leq \bigcup_{\alpha \in \Lambda} H_\alpha = M$ for any $\alpha \in \Lambda$ and $g(x) = f_\alpha(x)$ for any $x \in H_\alpha$. It implies that (M, g) is the maximum element of the chain, which is contained in $\mathcal{P}$.

Since $\mathcal{P}$ is non-empty, by Zorn's Lemma, there is a maximal element of $\mathcal{P}$.

Let $(M, \tilde{\varphi})$ be a maximal element of (K, φ) in $\mathcal{P}$.

Suppose that $M \lneq G$. Then, there is $x \in G \backslash M$. In this case, there may be some $k \in \mathbb{N}$ such that $k \cdot x \in M$. If such k exists, let p be a minimum k and $z = \frac{\alpha(p \cdot x)}{p}$. If there is no such k, let $p = 0$ and $z = 0 + \mathbb{Z}$. And define $M' = \langle x \rangle + M$ and $\beta : M' \to \mathbb{Q}/\mathbb{Z}$ such that

$$\beta(k \cdot x + y) = kz + \tilde{\varphi}(y)$$

where $k \in \mathbb{Z}$, $y \in M$.

β is well-defined, because if $k \cdot x + y = k' \cdot x + y'$ for $k, k' \in \mathbb{Z}$ and $y, y', \in M$, $(k - k') \cdot x = y' - y$. Since $y' - y \in M$, $(k - k') \cdot x \in M$. Then, $\tilde{\varphi}(y') - \tilde{\varphi}(y) = \tilde{\varphi}((k - k') \cdot x)$ and $\tilde{\varphi}(y') = \tilde{\varphi}(y) + \tilde{\varphi}((k - k') \cdot x)$.

- If $p > 0$, $p \mid k - k'$. Let $q \in \mathbb{Z}$ such that $k - k' = pq$. Then,

$$\begin{aligned}
\beta(k' \cdot x + y') = k'z + \tilde{\varphi}(y') &= k'z + \tilde{\varphi}(y) + \tilde{\varphi}((k - k') \cdot x) \\
&= k'z + \tilde{\varphi}(y) + \tilde{\varphi}(qp \cdot x) \\
&= k'z + \tilde{\varphi}(y) + q\tilde{\varphi}(p \cdot x) \\
&= k'z + \tilde{\varphi}(y) + qpz \\
&= k'z + \tilde{\varphi}(y) + (k - k')z = kz + \tilde{\varphi}(y) = \beta(k \cdot x + y)
\end{aligned}$$

- If $p = 0$, $k - k'$ should be 0. Thus $(k - k') \cdot x = 0$ and $k = k'$. Then,

$$\begin{aligned}\beta(k' \cdot x + y') &= k'z + \tilde{\varphi}(y') \\ &= k'z + \tilde{\varphi}(y) + \tilde{\varphi}((k - k') \cdot x) \\ &= kz + \tilde{\varphi}(y) + \tilde{\varphi}(0 \cdot x) = kz + \tilde{\varphi}(y) = \beta(k \cdot x + y)\end{aligned}$$

Also, β is a homomorphism, because for $k, k' \in \mathbb{Z}$ and $y, y' \in M$,

$$\begin{aligned}\beta((k \cdot x + y) + (k' \cdot x + y')) &= \beta((k + k') \cdot x + (y + y')) \\ &= (k + k')z + \tilde{\varphi}(y + y') \\ &= kz + \tilde{\varphi}(y) + k'z + \tilde{\varphi}(y') = \beta(k \cdot x + y) + \beta(k' \cdot x + y')\end{aligned}$$

And, $\beta|_M = \tilde{\varphi}$ because for $y \in M$,

$$\beta(y) = 0z + \tilde{\varphi}(y) = \tilde{\varphi}(y)$$

However, it's a contradiction since we assumed that $(M, \tilde{\varphi})$ is maximal, but there is (M', β) such that $(M, \tilde{\varphi}) \prec (M'\beta)$ but $(M, \tilde{\varphi}) \neq (M', \beta)$.

Therefore, M should be G. In this case, $\tilde{\varphi} : G \to \mathbb{Q}/\mathbb{Z}$ is a homomorphism and,

$$\tilde{\varphi}(k) = \tilde{\varphi}|_K (k) = \varphi(k)$$

for every $k \in K$.

Thus, there is a homomorphism from G to $\mathbb{Q}/\mathbb{Z}$ which is extended from the given homomorphism φ. $\square$

Problem 4

Let R be a ring with unity and M be a left R-module. We equip the abelian group $M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ with right R-module structure via $(\varphi r)(m) := \varphi(rm)$ for $r \in R$, $\varphi \in M^*$ and $m \in M$.

(1) Let G be an abelian group. Prove that for any nonzero $g \in G$, there is a group homomorphism $\alpha : G \to \mathbb{Q}/\mathbb{Z}$ such that $\alpha(g) \neq 0$.

(2) Suppose that M^* is an injective right R-module. For any injective (in the set-theoretical sense) homomorphism of right R-modules $f : A \to B$, prove that

$$\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{-\circ(f \otimes_R \mathrm{Id}_M)} \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R M, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

is an exact sequence of abelian groups. Using this and (1), prove also that M is a flat left R-module.

Solution of (1)

Let G be an abelian group and $g \in G$ be a non-zero element.

First of all, we can easily make a homomorphism from $\langle g \rangle$ to $\mathbb{Q}/\mathbb{Z}$. Let $n = |\langle g \rangle|$ if $|\langle g \rangle| < \infty$, $n = 2$ otherwise. Then, let $f : \langle g \rangle \to \mathbb{Q}/\mathbb{Z}$ such that:

$$f(k \cdot g) = \frac{k}{n} + \mathbb{Z}$$

for every $k \in \mathbb{Z}$.

f is well-defined. Because if $k \cdot g = l \cdot g$ for $k \neq l$, $(l - k) \cdot g = 0$. If $|\langle g \rangle| = \infty$, it's not possible. Thus $n = |\langle g \rangle| < \infty$ and $n \mid l - k$. Then, $\frac{l-k}{n} \in \mathbb{Z}$ and

$$f(k \cdot g) = \frac{k}{n} + \mathbb{Z} = \frac{k}{n} + \frac{l-k}{n} + \mathbb{Z} = \frac{l}{n} + \mathbb{Z} = f(l \cdot g)$$

Also, f is a homomorphism since, for $k, l \in \mathbb{Z}$,

$$f(k \cdot g + l \cdot g) = f((k+l) \cdot g) = \frac{k+l}{n} + \mathbb{Z} = \left(\frac{k}{n} + \mathbb{Z}\right) + \left(\frac{l}{n} + \mathbb{Z}\right) = f(k \cdot g) + f(l \cdot g)$$

In addition $f(g) = \frac{1}{n} + \mathbb{Z} \neq 0 + \mathbb{Z}$. Thus, it maps g to some non-zero element.

Therefore, f is a homomorphism maps g to some non-zero element.

Since $\langle g \rangle \leq G$, by Lemma **??**, there is a group homomorphism $\alpha : G \to \mathbb{Q}/\mathbb{Z}$ such that $\alpha|_{\langle g \rangle} = f$. It implies, α is a homomorphism such that,

$$\alpha(g) = f(g) = \frac{1}{n} + \mathbb{Z} \neq 0 + \mathbb{Z}$$

Therefore, for an arbitrary abelian group G and some non-zero element $g \in G$, there is a group homomorphism α which maps g to a non-zero element. □

Solution of (2)

Suppose that M^* be an injective right R-module. And let $f : A \hookrightarrow B$ is an injective R-module homomorphism.

Then, we can construct an exact sequence:

$$0 \longrightarrow A \xrightarrow{f} B$$

Since M^* is injective, the contravariant functor $\mathrm{Hom}_R(-, M^*)$ is exact. Note that $\mathrm{Hom}_R(h, M) = - \circ h$ for a R-module homomorphism h. Thus,

$$\mathrm{Hom}_R(B, M^*) \xrightarrow{-\circ f} \mathrm{Hom}_R(A, M^*) \longrightarrow 0$$

is an exact sequence.

Since $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, by adjuction formula, for right R-module N,

$$\mathrm{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = \mathrm{Hom}_R(N, M^*)$$

Thus,

$$\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\varphi} \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R M, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

is exact, where φ was induced from f.

Let $\eta_N : \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z})$ be a natural isomorphism for a $\mathbb{R}$-module N, such that

$$\eta(f)(n \otimes m) = f(n)(m)$$

Then, $\varphi = \eta_A \circ (- \circ f) \circ \eta_B^{-1}$ and,

$$\begin{aligned}
\varphi(g)(a \otimes m) &= (\eta_A \circ (- \circ f) \circ \eta_B^{-1})(g)(a \otimes m) \\
&= \eta_A(\eta_B^{-1}(g) \circ f)(a \otimes m) \\
&= (\eta_B^{-1}(g) \circ f)(a)(m) \\
&= \eta_B^{-1}(g)(f(a))(m) \\
&= g(f(a) \otimes m)
\end{aligned}$$

for $g \in \mathrm{Hom}_{\mathbb{Z}}(B \otimes_R M, \mathbb{Q}/\mathbb{Z})$, $a \in A$ and $m \in M$, therefore,

$$\varphi = - \circ (f \otimes_R \mathrm{Id}_M)$$

(Since $A \otimes_R M$ is generated by $a \otimes m$ for $a \in A$ and $m \in M$, and φ is a homomorphism as a composition of homomorphisms, it's enough to check only for the case of generator $a \otimes m$.)

Therefore,

$$\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{-\circ(f \otimes_R \mathrm{Id}_M)} \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R M, \mathbb{Q}/\mathbb{Z}) \to 0$$

is exact. □

M is flat

Suppose that $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective.

To show that M is flat, it's enough to check that $- \otimes_R M$ is left exact.

In other words, for right R-modules A, B, C and $f : A \to B$ such that

$$0 \to A \xrightarrow{f} B \to C \to 0$$

the below should be exact:

$$0 \to A \otimes_R M \xrightarrow{f \otimes_R M} B \otimes_R M \to C \otimes_R M \to 0$$

Suppose that $\ker(f \otimes_R M) \neq \{0\}$. In other words, there is a non-zero $a \in \ker(f \otimes_R M) \subseteq \mathrm{Dom}(f \otimes_R M) = A \otimes_R M$. Since $A \otimes_R M$ is an abelian group, there is a group homomorphism $\alpha : A \otimes_R M \to \mathbb{Q}/\mathbb{Z}$ such that $\alpha(a) \neq 0$. Note that an abelian group is $\mathbb{Z}$-module and the group homomrphism α is a $\mathbb{Z}$-module homomorphism such that:

$$\alpha(n \cdot x) = n \cdot \alpha(x)$$

where $n \in \mathbb{Z}$, $x \in A \otimes_R M$. Thus, $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R M, \mathbb{Q}/\mathbb{Z})$.

As we showed above,

$$\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{-\circ(f \otimes_R \mathrm{Id}_M)} \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R M, \mathbb{Q}/\mathbb{Z}) \to 0$$

is exact. In other words, $- \circ (f \otimes_R \mathrm{Id}_M) = - \circ (f \otimes_R M)$ is surjective. Because $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R M, \mathbb{Q}/\mathbb{Z})$, there is $\beta \in \mathrm{Hom}_{\mathbb{Z}}(B \otimes_R M, \mathbb{Q}/\mathbb{Z})$ such that $\beta \circ (f \otimes_R \mathrm{Id}_M) = \alpha$. Then,

$$\begin{aligned} \alpha(a) &= \beta((f \otimes_R \mathrm{Id}_M)(a)) \\ &= \beta(0) = 0 \end{aligned}$$

since $a \in \ker(f \otimes_R \mathrm{Id}_M)$. However, it's a contradiction, because α is a homomorphism such that $\alpha(a) \neq 0$.

Thus, $\ker(f \otimes_R M)$ must be $\{0\}$, and M is flat. $\square$

Problem 5

Let R be a ring with unity and M be a left R-module. Prove that M is a flat left R-module if and only if for any right ideal I of R,

$$I \otimes_R M \to IM : \sum_j a_j \otimes m_j \mapsto \sum_j a_j m_j$$

is an isomorphism of abelian groups. (Hint: Try to use Baer's criterion.)

Solution

Note that $R \otimes_R M \simeq M$. because for every $r \in R$ and $m \in M$, $r \otimes m = 0_R \otimes (rm)$, and

$$\sum_j r_j \otimes m_j = \sum_j 1_R \otimes (r_j m_j) = 1_R \otimes \sum_j r_j m_j$$

It implies, every element of $R \otimes_R M$ can be expressed as $1_R \otimes m$ for some $m \in M$. Thus $R \otimes_R M \simeq M$, with an isomorphism $\psi : R \otimes_R M \to M$ such that $\psi(1_R \otimes m) = m$.

$(\Longrightarrow)$

Suppose that M is a flat left R-module.

Let I be an ideal of R.

Then, take a short exact sequence,

$$0 \longrightarrow I \xrightarrow{\iota} R \xrightarrow{\pi} R/I \longrightarrow 0$$

where $\iota : I \to R$ is an injection, and $\pi : R \to R/I$ be a R homomorphism such that $\pi : r \mapsto r + I$.

Since M is flat, the below is exact:

$$0 \longrightarrow I \otimes_R M \xrightarrow{\iota \otimes_R M} R \otimes_R M \xrightarrow{\pi \otimes_R M} R/I \otimes_R M \longrightarrow 0$$

Since $\iota \otimes_R M = \iota \otimes_R \mathrm{Id}_M$, it's a map from $I \otimes_R M$ to $R \otimes_R M \simeq M$. And because of the exactness, $\iota \otimes_R M$ is injective.

Since ψ we defined above is an isomorphism, $\eta = \psi \circ (\iota \otimes_R M)$ is an injective homomorphism, such that

$$\eta(\sum_j a_j \otimes m_j) = \psi(\sum_j a_j \otimes m_j) = \psi(1_R \otimes \sum_j a_j m_j) = \sum_j a_j m_j$$

where $a_j \in I$ and $m_j \in M$.

Also, because, for each $a_j \in I$ and $m_j \in M$, $a_j m_j \in IM$ and $\sum_j a_j m_j \in IM$, and $\eta(\sum_j a_j \otimes m_j) = \sum_j a_j m_j \in IM$. And, for each $\sum_j a_j m_j \in IM$, there is $\sum_j a_j \otimes m_j \in I \otimes_R M$ such that $\eta(\sum_j a_j \otimes m_j) = \sum_j a_j m_j$. It means, the image of η is IM.

Therefore, $\tilde{\eta} : I \otimes_R M \to IM$, which obtained from η by restricting the codomain to IM, which given in the problem, is an isomorphism. $\square$

$(\Longleftarrow)$

Suppose that for every right idel $I \subseteq R$,

$$I \otimes_R M \to IM : \sum_j a_j \otimes m_j \mapsto \sum_j a_j m_j$$

is an isomorphism of abelian groups. It means, $I \otimes_R M \simeq IM$ for every right ideal $I \subseteq R$.

Let $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ be R-module. Then, by adjunction formula and above isomorphism relations,

$$\mathrm{Hom}_R(I, M^*) = \mathrm{Hom}_R(I, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(I \otimes_R M, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(IM, \mathbb{Q}/\mathbb{Z})$$

$$\mathrm{Hom}_R(R, M^*) = \mathrm{Hom}_R(R, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R M, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

(Note: in this part, for any abelian group A, A is a $\mathbb{Z}$-module such that $n \cdot a$ is sum of a n-times. Then, group homomorphisms between abelian groups are $\mathbb{Z}$-module homomorphisms.)

Let $g \in \mathrm{Hom}_R(I, M^*)$. Then, there is $h \in \mathrm{Hom}_{\mathbb{Z}}(IM, \mathbb{Q}/\mathbb{Z})$, which is the isomorphic image of g. Since h is a group homomorphism from $IM \leq M$ to $\mathbb{Q}/\mathbb{Z}$, by Lemma **??**, there is an extension $\tilde{h} \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of h. Note that since M and $\mathbb{Q}/\mathbb{Z}$ are abelian groups, $\tilde{h}$ is a $\mathbb{Z}$-module homomorphism. And, let $\tilde{g} \in \mathrm{Hom}_R(R, M^*)$ be an isomorphic image of $\tilde{h}$. Then, $\tilde{g}$ is an extension of g.

In other words, for every R-module homomorphism $g : I \to M^*$, there is a homomorphism $\tilde{g} : R \to M^*$ which extends g. Thus, by Baer's Criterion, M^* is an injective module.

As we proved in Problem 4, since M^* is injective, M is flat $\square$