MAS511 2020Spring Homework#09

Problem 1

Theorem 1. *Let G be a finite group. Let M, N be two finite dimensional representations of G. Let χ, ψ be their characters. Then, $M \simeq N$ as $\mathbb{C}G$-modules if and only if $\chi = \psi$.*

Since $M_1, \cdots, M_r$ are all of the in-equivalent irreducible $\mathbb{C}G$-modules, we can write

$$\begin{cases} M \simeq M_1^{\bigoplus a_1} \oplus \cdots \oplus M_r^{\bigoplus a_r} \\ N \simeq M_1^{\bigoplus b_1} \oplus \cdots \oplus M_r^{\bigoplus b_r} \end{cases}$$

Thus, $\chi = \sum_i a_i \chi_i$ and $\psi = \sum_i b_i \chi_i$.

Since

$$\mathbb{C}G \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

for $1 \le j \le r$, we have $e_j \in \mathbb{C}G$ such that

$$e_j M_i = \delta_{ij} M_i$$

where δ_{ij} is the Kronecker delta.

Prove rigorously that such $e_j \in \mathbb{C}G$ exists.

Proof

Let $\eta : \mathbb{C}G \to M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ be an isomorphism.

For each $i \in \{1, \cdots, r\}$, since $M_{n_i}(\mathbb{C})$ is a set of (dimension of n_i) matrices over $\mathbb{C}$, it contains a zero matrix 0_{n_i} and an identity matrix I_{n_i}. Then, let

$$E_j = (0_{n_1}, \cdots, 0_{n_{j-1}}, I_{n_j}, 0_{n_{j+1}}, \cdots, 0_{n_r})$$

This E_j is idempotent, since

$$\begin{aligned} E_j E_j &= (0_{n_1}, \cdots, 0_{n_{j-1}}, I_{n_j}, 0_{n_{j+1}}, \cdots, 0_{n_r})(0_{n_1}, \cdots, 0_{n_{j-1}}, I_{n_j}, 0_{n_{j+1}}, \cdots, 0_{n_r}) \\ &= (0_{n_1}0_{n_1}, \cdots, 0_{n_{j-1}}0_{n_{j-1}}, I_{n_j}I_{n_j}, 0_{n_{j+1}}0_{n_{j+1}}, \cdots, 0_{n_r}0_{n_r}) \\ &= (0_{n_1}, \cdots, 0_{n_{j-1}}, I_{n_j}, 0_{n_{j+1}}, \cdots, 0_{n_r}) = E_j \end{aligned}$$

Also, if $i < j$,

$$\begin{aligned} E_i E_j &= (0_{n_1}, \cdots, 0_{n_{i-1}}, I_{n_i}, 0_{n_{i+1}}, \cdots, 0_{n_r})(0_{n_1}, \cdots, 0_{n_{j-1}}, I_{n_j}, 0_{n_{j+1}}, \cdots, 0_{n_r}) \\ &= (0_{n_1}0_{n_1}, \cdots, I_{n_i}0_{n_i}, \cdots, 0_{n_j}I_{n_j}, \cdots, 0_{n_r}0_{n_r}) \\ &= 0 \end{aligned}$$

Thus $E_iE_j = \delta_{ij}E_i$. In the same way, $E_jE_i = \delta_{ij}E_j$. Then, let $e_i = \eta^{-1}(E_i)$ for each $i = 1, \cdots, r$. They satisfy $e_ie_j = \delta_{ij}e_i$. Also, since each e_k is an isomorphic image of non-zero element, e_k is non-zero. Also, since $\sum_{i=1}^r E_i = I$, $\sum_{i=1}^r e_i = 1_G$.

Let $\phi : G \to \mathrm{GL}(M)$ be the representation given in the problem. Since $M \simeq M_1^{\bigoplus a_1} \oplus \cdots \oplus M_r^{\bigoplus a_r}$, there are representations ϕ_k which is correspond to M_k. And, in this case, $\phi = \phi_1^{\bigoplus a_1} \oplus \cdots \oplus \phi_r^{\bigoplus a_r}$.

Because each $\phi(e_j)$ (Note: $\phi(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g \phi(g)$) is a linear transformation, the image of $\phi(e_j)$ is a submodule of M. In the same sense, the image of $\phi_k(e_j)$ is a submodule of M_k. Since each M_k are simple, M_k has only two submodule: 0 and M_k. Thus, $\phi_k(e_j)M_k$ should be 0 or M_k. Therefore, the image of $\phi(e_j)$ has a full of $M_k^{\bigoplus a_k}$ as a direct summand, or does not have any component M_k, for each k.

First, note that $\mathrm{Id} = \sum_{k=1}^r e_k$. Thus $(\sum_{k=1}^r e_k) \cdot M_j = \phi(\sum_{k=1}^r e_k)M_j = \sum_{k=1}^r \phi(e_k)M_j = M_j$. If $\phi(e_k)M_j = 0$ for all e_k, then $\sum_{k=1}^r \phi(e_k)M_j = 0$ and it's a contradiction. Thus, there must be some e_k such that $\phi(e_k)M_j \neq 0$. As we showed above, $\phi(e_k)M_j = 0$ or M_j. Thus, $\phi(e_k)M_j = M_j$ for some k. This shows, for every $j \in \{1, \cdots, r\}$ there is $k \in \{1, \cdots, r\}$ such that $e_k \cdot M_j = \phi(e_k)M_j = M_j$. $\cdots$ (1)

Suppose that $\phi_i(e_j)M_i \neq 0$ and $\phi_i(e_k)M_i \neq 0$ hold for some $i, j, k \in \{1, \cdots, r\}$ such that $j \neq k$. Then, $\phi_i(e_j)M_i = \phi_i(e_k)M_i = M_i$ by the simplicity of M_i. This implies $\phi_i(e_je_k)M_i = \phi_i(e_j)\phi_i(e_k)M_i = \phi_i(e_j)M_i = M_i$. But since $e_je_k = 0$, $\phi_i(e_je_k)M_i = 0$. Thus, it's a contradiction. Therefore, for each i, there are exactly one $j \in \{1, \cdots, r\}$ such that $\phi_i(e_j)M_i = e_j \cdot M_i$ is non-zero. $\cdots$ (2)

Now, we want to show that for each j, there is at most one k such that $\phi(e_j)M_k = M_k$.

Suppose that there is $i, j, k \in \{1, \cdots, r\}$ such that $j \neq k$ and $\phi_j(e_i)M_j$ and $\phi_k(e_i)M_k$ are non-zero. Thus, $\phi_j(e_i)M_j = M_j$, $\phi_k(e_i)M_k = M_k$ by the simplicity of M_j and M_k.

First, since each M_j is a non-zero module, there is a non-zero element m. In that case, we can consider $\mathbb{C}G \cdot m = \{g \cdot m \mid g \in \mathbb{C}G\}$., which is a generated set from $\mathbb{C}G$-action and m. Then, $\mathbb{C}G \cdot m$ is a submodule of M_j, since M_j is closed under $\mathbb{C}G$-action. Also, since $\mathbb{C}G$ contains 1, (more precisely, $1 \cdot 1_G$), $\mathbb{C}G \cdot m$ contains at least one non-zero element, m. However, since M_j is an irreducible $\mathbb{C}G$-module, there are no proper non-trivial submodule. Since $\mathbb{C}G \cdot m$ is non-trivial, $\mathbb{C}G \cdot m = M_j$.

But, note that e_i is an idempotent element, which preserve M_j. Because $e_i \cdot M_j = \phi_j(e_i)M_j = M_j$, there is some $x \in M_j$ such that $e_i \cdot x = m$. Since $e_i \cdot 0 = 0$ and $m \neq 0$, $x \neq 0$. In this case, $M_j = \mathbb{C}G \cdot m = \mathbb{C}G \cdot (e_i \cdot x) = (\mathbb{C}Ge_i) \cdot x = \{(ge_i) \cdot x \mid g \in \mathbb{C}G\}$.

In this case, $- \cdot x$ gives a surjective $\mathbb{C}G$-module homomorphism from $\mathbb{C}Ge_i$ to $\mathbb{C}G \cdot m = M_j$. Let $\gamma = - \cdot x$. Then, $M_j \simeq \mathbb{C}Ge_i / \ker \gamma$.

Note that e_i is comes from E_i, which all entries are zero except the i-th entry, which is valued as an identity matrix. Since $\mathbb{C}G \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$,

$$\mathbb{C}Ge_i \simeq 0 \times \cdots \times 0 \times M_{n_i}(\mathbb{C}) \times 0 \times \cdots \times 0 \simeq M_{n_i}(\mathbb{C})$$

Also, since $\mathbb{C}$ is a division ring, $M_{n_i}(\mathbb{C})$ is a simple ring. Since $\mathbb{C}G$ contains an isomorphic image of $M_{n_i}(\mathbb{C})$, $M_{n_i}(\mathbb{C})$ is simple as a $\mathbb{C}G$-module. Therefore, $\mathbb{C}Ge_i \simeq M_{n_i}(\mathbb{C})$ has no proper non-trivial submodule.

This shows $\ker \gamma$, which is a submodule of $\mathbb{C}Ge_i$, should be 0 or $\mathbb{C}Ge_i$. Since M_j is non-zero, $\ker \gamma \neq \mathbb{C}Ge_i$. Thus $\ker \gamma = 0$ and γ is an isomorphism. And we can conclude that $M_j \simeq \mathbb{C}Ge_i$.

We can repeat exactly same process for M_k and we obtain $M_k \simeq \mathbb{C}Ge_i$. Thus, $M_k \simeq \mathbb{C}Ge_i \simeq M_j$. However, it's a contradiction, because the problem assumed that M_j and M_k are inequivalent where

$j \neq k$.

Therefore, such i, j, k cannot exist. It means, for every $i \in \{1, \cdots, r\}$, if $e_i \cdot M_j = M_j$ for some j, $e_i \cdot M_k = 0$ for all $k \neq j$. $\cdots$ (3)

Therefore, from (1), (2) and (3), we know that (1) for each $i \in \{1, \cdots, r\}$, there are at least one $j \in \{1, \cdots, r\}$ such that $\phi_i(e_j)M_i = M_i$; (2) for each $i \in \{1, \cdots, r\}$, there are at most one $j \in \{1, \cdots, r\}$ such that $\phi_i(e_j)M_i = M_i$ and $\phi_i(e_k)M_i = 0$ for $j \neq k$; (3) for each $i \in \{1, \cdots, r\}$, there are at most one $j \in \{1, \cdots, r\}$ such that $\phi_j(e_i)M_j = M_j$ and $\phi_k(e_i)M_k = 0$ for every $k \neq j$. Therefore, for each M_j there is exactly one e_i such that $e_i \cdot M_j = M_j$ and $e_i \cdot M_k = 0$ for every $k \neq j$, and for each e_i there is exactly one M_j such that $e_i \cdot M_j = M_j$ and $e_i \cdot M_k = 0$ for every $k \neq j$. Then, by reordering $\{e_i\}_{i=1}^r$, we have e_i and M_i such that $e_i \cdot M_i = M_i$ and $\forall j \neq i, e_i \cdot M_j = 0$. Thus, $e_i \cdot M_j = \delta_{ij} M_j$. Since each e_i is an idempotent element of $\mathbb{C}G$, these $\{e_i\}_{i=1}^r$ are the required ones in the problem. $\square$

Problem 2

Give the proof of the below theorem:

Theorem 2. *Let $\phi : G \to \mathrm{GL}(V)$ be a representation for a finite dimensoinal vector space V. Let $\{v_1, \cdots, v_n\}$ be a basis of V and let $\{v_1^*, \cdots, v_n^*\}$ be its dual basis.*

Then $\mathrm{Tr}\phi(g) = \sum_{i=1}^{n} v_i^*(g \cdot v_i)$.

Proof

Let V is a vector space over a field F, and let $n = \dim_F V$.

Note, the action of G on V is the representation ϕ. i.e. $g \cdot v_i = \phi(g)(v_i)$.

Let $\mathcal{B} = \{v_1, \cdots, v_n\}$. And let $[A]_{\mathcal{B}}$ be a matrix of a linear transformation $A \in \mathrm{End}(V)$ with respect to $\mathcal{B}$, and let $[v]_{\mathcal{B}}$ be a column matrix of $v \in V$ with respect to $\mathcal{B}$.

Let $(a_{i,j}) = [\phi(g)]_{\mathcal{B}}$. Note that $[v_i]_{\mathcal{B}} = e_i$, where e_i is an element of V such that all entries are zero except the i-th entry which is valued by 1_F. In this case,

$$[\phi(g)(v_i)]_{\mathcal{B}} = [\phi(g)]_{\mathcal{B}} e_i = \begin{pmatrix} a_{1,i} \\ \vdots \\ a_{n,i} \end{pmatrix}$$

Thus,

$$\phi(g)(v_i) = \sum_{j=1}^{n} a_{j,i} v_j$$

Then, by the definition of dual basis,

$$\begin{aligned} v_i^*(\phi(g)(v_i)) &= v_i^*(\sum_{j=1}^{n} a_{j,i} v_j) \\ &= \sum_{j=1}^{n} a_{j,i} \delta_{i,j} = a_{i,i} \end{aligned}$$

Thus,

$$\sum_{i=1}^{n} v_i^*(\phi(g)(v_i)) = \sum_{i=1}^{n} a_{i,i} = \mathrm{Tr}[\phi(g)]_{\mathcal{B}} = \mathrm{Tr}\phi(g)$$

since trace is not changed by change of basis (because change of basis give a similar linear transformation, and two similar linear transformations have same trace). $\square$

Problem 3

Theorem 3. *Let ψ_1, ψ_2 be characters of G. Then so is $\psi_1\psi_2$. In particular, $\mathcal{F}$ is closed under the product of class functions.*

More precisely, if $\psi_i = \mathrm{Tr}\phi_i$ for representations ϕ_i, then $\psi_1\psi_2 = \mathrm{Tr}\phi_1 \otimes \phi_2$.

One way to prove the above theorem is using the fact that

$$\mathrm{Tr}(T_1 \otimes T_2) = \mathrm{Tr}(T_1) \cdot \mathrm{Tr}(T_2)$$

where $T_1 \otimes T_2$ is the tensor product of linear transformations.

Give another proof of the above, using dual basis descrption of the characters.

Proof

Let F be a field, V_1, V_2 be finite dimension vector spaces over F, $m = \dim_F V_1$, $n = \dim_F V_2$, $\mathcal{A} = \{a_1, \cdots, a_m\}$ be a basis of V_1, $\mathcal{B} = \{b_1, \cdots, b_n\}$ be a basis of V_2, $\mathcal{A}^* = \{a_1^*, \cdots, a_m^*\}$ be a dual of $\mathcal{A}$, $\mathcal{B}^* = \{b_1^*, \cdots, b_n^*\}$ be a dual of $\mathcal{B}$.

Let $\phi_1 : G \to \mathrm{GL}(V_1)$ and $\phi_2 : G \to \mathrm{GL}(V_2)$ be linear representations. And let ψ_k is the character of ϕ_k for $k = 1, 2$.

Note that one of basis of $A \otimes B$ is $\mathcal{T} = \{a_i \otimes b_j \mid a_i \in \mathcal{A}, b_j \in \mathcal{B}\}$.

As the theorem in the Problem 2, for $g \in G$,

$$\psi_1(g) = \mathrm{Tr}(\phi_1(g)) = \sum_{i=1}^{m} a_i^*(\phi_1(g)(a_i))$$

$$\psi_2(g) = \mathrm{Tr}(\phi_2(g)) = \sum_{i=1}^{n} b_i^*(\phi_2(g)(b_i))$$

Some notes:

- Since we defined G acts on $V_1 \otimes V_2$ such as $g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2)$ in the lecture, $(\phi_1 \otimes \phi_2)(g)(v_1 \otimes v_2) = \phi_1(g)(v_1) \otimes \phi_2(g)(v_2)$.
- Each dual basis element $(a_i \otimes b_j)^*$ is defined as $(a_i \otimes b_j)^*(x)$ is 0 for all elements of $\mathcal{T}$ except $a_i \otimes b_j$ which is valued by 1_F. Since a_i^* makes all elements of $\mathcal{A}$ zero except a_i (which gives 1_F) and b_j^* makes all elements of $\mathcal{B}$ zero except b_j (which gives 1_F), $(a_i \otimes b_j)^*(a \otimes b) = a_i^*(a)b_j^*(b)$ for every $a \otimes b \in \mathcal{T}$. Because linear transformation is determined uniquely by the image of basis, $(a_i \otimes b_j)^*(a \otimes b) = a_i^*(a)b_j^*(b)$ holds for every $a \otimes b \in V_1 \otimes V_2$. (Note that we do not have to check for every element of $V_1 \otimes V_2$, which may have a form of $\sum_k a_k \otimes b_k$, because only the form of $a \otimes b$ appears in the below calculation.)

Then, for every $g \in G$,

$$\begin{aligned}
\mathrm{Tr}((\phi_1 \otimes \phi_2)(g)) &= \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i \otimes b_j)^* ((\phi_1(g) \otimes \phi_2(g))(a_i \otimes b_j)) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i \otimes b_j)^* (\phi_1(g)(a_i) \otimes \phi_2(g)(b_j)) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i^* (\phi_1(g)(a_i)) b_j^* (\phi_2(g)(b_j)) \\
&= \sum_{i=1}^{m} a_i^* (\phi_1(g)(a_i)) \sum_{j=1}^{n} b_j^* (\phi_2(g)(b_j)) \\
&= \left(\sum_{i=1}^{m} a_i^* (\phi_1(g)(a_i)) \right) \left(\sum_{j=1}^{n} b_j^* (\phi_2(g)(b_j)) \right) \\
&= \psi_1(g) \psi_2(g) \\
&= (\psi_1 \psi_2)(g)
\end{aligned}$$

This shows $\mathrm{Tr} \circ (\phi_1 \otimes \phi_2) = \psi_1 \psi_2$.

Therefore, the character of tensor product of representations is a multiplication of the characters of each representations. Thus, $\mathcal{F}$ is closed under the multiplication. $\square$

Problem 4

Prove below:

Theorem 4. *For a representation V of G, let χ be its character. Then the character for the dual representation V^* is the complex conjugate $\overline{\chi}$.*

Proof

(Since trace is well-defined only for finite dimension matrix, I'll assume that V is finite-dimensional.)

Let V be a n-dimensional vector space over $\mathbb{C}$. Let $\mathcal{B} = \{v_1, \cdots, v_n\}$ be a basis of V. Then, the dual space V^* exists and one of its basis is $\mathcal{B}^* = \{v_1^*, \cdots, v_n^*\}$, the dual basis of $\mathcal{B}$.

Let $\varphi : G \to \mathrm{GL}(V)$ be a representation V of G and χ be a character of the representation φ. Then, as the Problem 2,

$$\chi(g) = \mathrm{Tr}\varphi(g) = \sum_{i=1}^{n} v_i^*(\varphi(g)(v_i))$$

Also,

$$\chi^*(g) = \sum_{i=1}^{n} v_i^{**}(\varphi^*(g)(v_i^*))$$

where φ^* be the dual representation of φ, and χ^* is the character of φ^*. By the definition of φ^*,

$$\varphi^*(g)(f)(v) = (g \cdot f)(v) = f(g^{-1} \cdot v) = f(\varphi(g^{-1})(v))$$

for $f \in V^*$.

$$\chi^*(g) = \sum_{i=1}^{n} v_i^{**}(\varphi^*(g)(v_i^*)) = \sum_{i=1}^{n} v_i^{**}(v_i^* \circ \varphi(g^{-1}))$$

Let $(a_{i,j}) = [\varphi(g^{-1})]_{\mathcal{B}}$. Then, $\varphi(g^{-1})(v_i) = \sum_{k=1}^{n} a_{k,i}v_k$. Thus, $v_i^* \circ \varphi(g^{-1})$ maps v_j to $a_{i,j}$. In this case, we can denote $v_i^* \circ \varphi(g^{-1})$ as

$$v_i^* \circ \varphi(g^{-1}) = \sum_{j=1}^{n} a_{i,j} v_j^*$$

Then, by the linearity of dual basis,

$$\begin{aligned}
\chi^*(g) = \sum_{i=1}^{n} v_i^{**}(v_i^* \circ \varphi(g^{-1})) &= \sum_{i=1}^{n} v_i^{**}(\sum_{j=1}^{n} a_{i,j} v_j^*) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i^{**}(a_{i,j} v_j^*) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\delta_{i,j} \\
&= \sum_{i=1}^{n} a_{i,i} = \mathrm{Tr}\varphi(g^{-1}) = \chi(g^{-1})
\end{aligned}$$

We proved the below lemma in the lecture (Lemma (B)):

Lemma 1. *For a character $\psi : G \to \mathbb{C}$, $\psi(x^{-1}) = \overline{\psi(x)}$ for all $x \in G$.*

Thus,

$$\chi^*(g) = \chi(g^{-1}) = \overline{\chi(g)}$$

$\square$

Problem 5

Try $G = \{1, x, x^2\}$, a group of order 3. This is also cyclic. So $r = 3$ and all $d_i = 1$. We begin with

$$\begin{array}{c|ccc} & 1 & x & x^2 \\ & 1 & 1 & 1 \\ \hline \chi_1 & 1 & 1 & 1 \\ \chi_2 & 1 & ?_1 & ?_2 \\ \chi_3 & 1 & ?_3 & ?_4 \end{array}$$

Here, the rows and columns are orthogonal. From the 1st and the 2nd columns, we get $1+?_1+?_3 = 0$, and similarly $1+?_1+?_2 = 0$, etc. This shows $?_2 = ?_3 = -1-?_1$ and $?_4 = ?_1$.

Show that when ω is a primitive 3rd root of unity, $?_1 = ?_4 = \omega$, $?_2 = ?_3 = \omega^2$.

Proof

Between two different columns, we obtain

$$1+?_1+?_3 = 0 \tag{1}$$
$$1+?_2+?_4 = 0 \tag{2}$$
$$1+?_1\overline{?_2}+?_3\overline{?_4} = 0 \tag{3}$$

Between two different rows, we obtain

$$1+?_1+?_2 = 0 \tag{4}$$
$$1+?_3+?_4 = 0 \tag{5}$$
$$1+?_1\overline{?_3}+?_2\overline{?_4} = 0 \tag{6}$$

From (1) and (4), we obtain

$$?_2 = ?_3 = -1-?_1$$

In the similar way, we obtain

$$?_1 = ?_4 = -1-?_3$$

Let $z = ?_1 = ?_4$ and $w = ?_2 = ?_3$.

Since $x^{-1} = x^2$, $\chi_k(x^2) = \chi_k(x^{-1}) = \overline{\chi_k(x)}$ for each $k = 1, 2, 3$. Thus, $z = \overline{w}$. Then, from (4),

$$0 = 1 + z + w = 1 + z + \overline{z} = 1 + 2\operatorname{Re} z$$

$$\operatorname{Re} z = -\frac{1}{2}$$

From (3) using $?_2 = ?_3 = -1-?_1$, we obtain

$$1 - z - \overline{z} - 2z\overline{z} = 0$$

And we can simplify it such as:

$$2 - 2z\overline{z} = 0$$

$$z\overline{z} = 1$$

$$|z|^2 = 1$$

Thus $|z| = 1$. Since $\operatorname{Re} z = -1/2$,

$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$w = \overline{z} = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i$$

Note that $\omega = e^{\frac{2}{3}\pi i} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$. Thus, $?_1 = ?_4 = z = \omega$ and $?_2 = ?_3 = w = \omega^2$. $\square$
(Since the equations are symmetric, $?_1 = ?_4 = \omega^2$, $?_2 = ?_3 = \omega$ is also possible.)

Problem 6

Note:

Definition 1. For two functions $f_1, f_2 : G \to \mathbb{C}$, define the *convolution* to be a function $f_1 * f_2 : G \to \mathbb{C}$ given by,

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(gh^{-1}) f_2(h).$$

Definition 2. Let $f \in F(G, \mathbb{C})$ and let $\varphi : G \to \mathrm{GL}(V)$ be a representation. Then the *Fourier transform of f at φ* is defined to be

$$\widehat{f}(\varphi) := \sum_{g \in G} f(g)\varphi(g).$$

Let $f_1, f_2 \in F(G, \mathbb{C})$. Then prove that

$$\widehat{f_1 * f_2} = \widehat{f_1}\widehat{f_2}.$$

Proof

For any representation $\varphi : G \to \mathrm{GL}(V)$,

$$\begin{aligned}
\widehat{f_1 * f_2}(\varphi) &= \sum_{g \in G} (f_1 * f_2)(g)\varphi(g) \\
&= \sum_{g \in G} \left(\sum_{h \in G} f_1(gh^{-1}) f_2(h) \right) \varphi(g) \\
&= \sum_{g \in G} \left(\sum_{h \in G} f_1(gh^{-1}) f_2(h) \right) \varphi(gh^{-1})\varphi(h) \\
&= \sum_{g' \in Gh^{-1}} \left(\sum_{h \in G} f_1(g') f_2(h) \right) \varphi(g')\varphi(h) \\
&= \sum_{g' \in G} \left(\sum_{h \in G} f_1(g') f_2(h) \right) \varphi(g')\varphi(h) \\
&= \sum_{g' \in G} \sum_{h \in G} f_1(g')\varphi(g') f_2(h)\varphi(h) \\
&= \sum_{g' \in G} f_1(g')\varphi(g') \sum_{h \in G} f_2(h)\varphi(h) \\
&= \left(\sum_{g' \in G} f_1(g')\varphi(g') \right) \left(\sum_{h \in G} f_2(h)\varphi(h) \right) \\
&= \widehat{f_1}(\varphi)\widehat{f_2}(\varphi)
\end{aligned}$$

because $Ga = G$ for every group G and its element a. Note that beacuse f_k maps elements of G into the $\mathbb{C}$, the set of scalar values, it can commute to other scalar functions and linear transformations. However, $\varphi(gh^{-1})$ and $\varphi(h)$ may not commute each other, because their image is in $\mathrm{GL}(V)$ which is a set of some linear transformations.

Therefore, $\widehat{f_1 * f_2} = \widehat{f_1}\widehat{f_2}$ holds. □