MAS511 2020 Spring Homework 5

Problem 1

(Horseshoe Lemma) Suppose
$$0 \to M' \to M \to M'' \to 0$$
is a short exact sequence of R-modules. Suppose there are projective resolutions $P'_\bullet \to M'$ and $P''_\bullet \to M''$.

Prove that there is a projective resolution $P_\bullet \to M$ such that it fits into a short exact sequence of complexes:
$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

Lemmata

Lemma 1. *Let F_1, F_2 be free R-modules. $F_1 \oplus F_2$ is free.*

Proof. Note: $F_1 \oplus F_2$ is a product and a coproduct of F_1 and F_2. Disjoint union is a coproduct in the Sets category **Sets**.

Let F_1 is a free R-module on X and F_2 is a free R-module on Y.

Let M be an arbitrary R-module and there is an injective map $e : X_1 \sqcup X_2 \to M$.

$$\begin{tikzcd}
& M & \\
F_1 \arrow[r, hook] \arrow[ur, dashed, "\exists! f_1"] & F_1 \oplus F_2 \arrow[u, dashed, "f"] & F_2 \arrow[l, hook'] \arrow[ul, dashed, "\exists! f_2"'] \\
X \arrow[u, "i_1"] \arrow[r, "\iota_1"'] \arrow[uur, dashed, bend left=60, "e_1"] & X \sqcup Y \arrow[u, dashed, "\exists! h"] & Y \arrow[u, "i_2"'] \arrow[l, "\iota_2"] \arrow[uul, dashed, bend right=60, "e_2"']
\end{tikzcd}$$

Every element of $X \sqcup Y$ is in X or Y. Each $e_1 = e|_X$ and $e_2 = e|_Y$ are injective maps. Since F_1 and F_2 are free, there is unique R-module homomorrphisms $f_1 : F_1 \to M$ and $f_2 : F_2 \to M$ such that $e_1 = f_1 \circ i_1$ and $e_2 = f_2 \circ i_2$.

Let $j_1 : F_1 \to F_1 \oplus F_2$ and $j_2 : F_2 \to F_1 \oplus F_2$ be injective R-module homomorphisms (which consist of the coproduct). Then, take $h_1 = j_1 \circ \iota_1$ and $h_2 = j_2 \circ \iota_2$. In this case, there is a unique $h : X \sqcup Y \to F_1 \oplus F_2$ such that $h \circ \iota_1 = h_1 = j_1 \circ i_1$ and $h \circ \iota_2 = h_2 = j_2 \circ i_2$. This h is injective. ($\because$ Note that the only element of images of j_1 and j_2 is $(0,0)$, because $j_1(x) = (x, 0)$ and $j_2(y) = (0, y)$. Suppose that h is not injective. Then, there is $a, b \in X \sqcup Y$ such that $h(a) = h(b)$. If $a, b \in X$, it's impossible since h_1 is injective. If $a, b \in Y$, it's impossible since h_2 is injective. WLOG, let's assume $a \in X$ and $B \in Y$. Then, $h_1(x) = (i_1(x), 0) = (0, i_2(y)) = h_2(y)$. It means $i_1(x) = 0 = i_2(y)$. However, as we shown in the Homework #3 Problem 2, the generator of free modules must no contain 0. Thus, It's a contradiction.)

Also, since $F_1 \oplus F_2$ is a coproduct and there is a function $f_1 : F_1 \to M$ and $f_2 : F_2 \to M$, there is a unique map $f : F_1 \oplus F_2 \to M$ such that $f_1 = f \circ j_1$ and $f_2 = f \circ j_2$. Because f_1, f_2 exists uniquely when e is determined, f exists uniquely for e. In addition, as the above diagram, $e = f \circ h$.

Therefore, for $X \sqcup Y$, there exists $h : X \sqcup Y \to F_1 \oplus F_2$, and if $e : X \sqcup Y \to M$ was given, there is a unique map $f : F_1 \oplus F_2 \to M$ such that $e = f \circ h$.

Therefore, $F_1 \oplus F_2$ is free over $X \sqcup Y$. □

Proof

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1' \arrow[r, "\iota_1"] \arrow[d, "g_1"] & P_1 = P_1' \oplus P_1'' \arrow[r, "\pi_1"] \arrow[d, dashed, "f_1"] & P_1'' \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "\iota_0"] \arrow[d, "g_0"] & P_0 = P_0' \oplus P_0'' \arrow[r, "\pi_0"] \arrow[d, dashed, "f_0"] & P_0'' \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d] & M \arrow[r, "p"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $M, M', M'', P_\bullet', P_\bullet'', g_\bullet, h_\bullet, i, p$ are given as above. $P_\bullet' \to M'$, $P_\bullet'' \to M''$ are projective resolutions, $0 \to M' \to M \to M'' \to 0$ is exact.

Note that $0 \to P_k' \to P_k' \oplus P_k'' \to P_k'' \to 0$ is a split exact sequence with natural homomorphisms for every $k \in \mathbb{Z}^{\geq 0}$. Let $P_k = P_k' \oplus P_k''$ for each $k \in \mathbb{Z}^{\geq 0}$.

Note that each P_k are projective. Because, since P_k' and P_k'' are projective, there are R-modules Q' and Q'' such that $P_k' \oplus Q'$ and $P_k'' \oplus Q''$ are free. By Lemma **??**,

$$P_k' \oplus Q' \oplus P_k'' \oplus Q'' = (P_k' \oplus P_k'') \oplus (Q' \oplus Q'') = P_k \oplus (Q' \oplus Q'')$$

is free. Therefore, P_k is projective.

Thus, it's enough to find f_k which make $P_\bullet \to M$ as a projective resolution.

First, because p is surjective and P_0'' is projective, there is $h_0' : P_0'' \to M$ such that $p \circ h_0' = h_0$. Also, we have $g_0' : P_0' \to M$ such that $g_0' = i \circ g_0$. In this case, we have a homomorphism $f_0 : P_0 \to M$ such that

$$f_0(x, y) = g_0'(x) + h_0'(y)$$

. This is surjective, because, for $m \in M$, since $h_0 \circ \pi_0$ is surjective, there is $m' \in P_0$ such that $h_0(\pi_0(m')) = p(m)$ Then, $p(m - f_0(m')) = p(m) - p(f_0(m')) = p(m) - h_0(\pi_0(m')) = 0$. Since $m - f_0(m') \in \ker p = \operatorname{im} i$, Then, there is $a \in M'$ such that $i(a) = m - f_0(m')$. Since g_0 is surjective, there is $a' \in P_0'$ such that $i(g_0(a')) = m - f_0(m')$. Let $n = \iota_0(a') + m'$, then, $f_0(n) = f_0(\iota_0(a')) + f_0(m') = m$.

Then, by Snake Lemma (See Problem 7), $\ker g_0 \to \ker f_0 \to \ker h_0$ is exact. Also, since ι_0 is injective, natural homomorphism from $\ker g_0$ to $\ker f_0$ induced from ι_0 is also injective. Also, since $\operatorname{coker} g_0 = M'/\operatorname{im} g_0 = 0$ as $\operatorname{im} g_0 = M'$, $0 \to \ker g_0 \to \ker f_0 \to \ker h_0 \to 0$ is exact. Also, note that the image of g_1 and h_1 are $\ker g_0$ and $\ker h_0$. Then, we can build the below diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & P_1' \arrow[r, "\iota_1"] \arrow[d, "g_1"] & P_1 = P_1' \oplus P_1'' \arrow[r, "\pi_1"] \arrow[d, dashed, "f_1"] & P_1'' \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & \ker g_0 \arrow[r, "\tilde{\iota}_0"] \arrow[d] & \ker f_0 \arrow[r, "\tilde{\pi}_0"] \arrow[d] & \ker h_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Each rows are exact, the first and the third columns also exact.

In this case, we can construct f_1 repeating the above steps by considering each $\ker f_0$, $\ker g_0$, $\ker h_0$ as M, M', M'' and using the projectivity of P_1''. In the same way we did above, f_1 is also surjective ($\because$ $\operatorname{coker} g_1 = \ker f_0 / \operatorname{im} g_1 = 0$ as we consider the codomain of g_1 is restricted to $\ker f_0$), $0 \to \ker g_1 \to \ker f_1 \to \ker h_1 \to 0$ is exact, then, construct f_2 which is surjective, $\cdots$.

Then, because each $\operatorname{im} f_k$ is $\ker f_{k-1}$ for $k \in \mathbb{N}$, if we consider each f_k is a homomorphism from P_k to P_{k-1}, $P_\bullet$ be a projective resolution of M. $\square$

Failed Proof 1

Let the below is a short exact sequence:

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

and there are projective resolutions $P_\bullet' \to M'$ and $P_\bullet'' \to M''$.

Let's take $P_k = P_k' \oplus P_k''$ for $k \in \mathbb{Z}^{\geq 0}$.

First, note that:

$$0 \longrightarrow P_k' \xrightarrow{\iota_k} P_k = P_k' \oplus P_k'' \xrightarrow{\pi_k} P_k'' \longrightarrow 0$$

is a split short exact sequence, where ι_k is an embedding such tha $\iota_k : x \mapsto (x, 0)$ and π_k is a projection such that $\pi_k : (x, y) \mapsto y$.

Therefore, if we show that $P_0, P_1, \cdots$ make a projective resolution of M, M has a projective resolution $P_\bullet$ which makes each row and column of the below diagram exact.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1' \arrow[r, "\iota_1"] \arrow[d, "g_1"] & P_1 = P_1' \oplus P_1'' \arrow[r, "\pi_1"] \arrow[d, "f_1"] & P_1'' \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "\iota_0"] \arrow[d, "g_0"] & P_0 = P_0' \oplus P_0'' \arrow[r, "\pi_0"] \arrow[d, "f_0"] & P_0'' \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d] & M \arrow[r, "p"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

First, $P_k = P'_k \oplus P''_k$ is projective. Because, since P'_k and P''_k are projective, there are R-modules Q' and Q'' such that $P'_k \oplus Q'$ and $P''_k \oplus Q''$ are free. By Lemma **??**,

$$P'_k \oplus Q' \oplus P''_k \oplus Q'' = (P'_k \oplus P''_k) \oplus (Q' \oplus Q'') = P_k \oplus (Q' \oplus Q'')$$

is free. Therefore, P_k is projective.

Suppose that, for $k \in \mathbb{N}$, $0 \to P'_{k-1} \to P_{k-1} \to P''_{k-1} \to 0$ is exact and split, $0 \to P'_k \to P_k = P'_k \oplus P''_k \to P''_k \to 0$ is exact, and there is R-module homomorphisms $g_k : P'_k \to P'_{k-1}$ and $h_k : P''_k \to P''_{k-1}$.

$$\begin{tikzcd}
0 \arrow[r] & P'_k \arrow[r, "\iota_k"] \arrow[d, "g_k"] & P_k = P'_k \oplus P''_k \arrow[r, "\pi_k"] \arrow[d, dashed, "\exists f_k"] & P''_k \arrow[r] \arrow[d, "h_k"] & 0 \\
0 \arrow[r] & P'_{k-1} \arrow[r, "\iota_{k-1}"] \arrow[d, "g_{k-1}"] & P_{k-1} \simeq P'_{k-1} \oplus P''_{k-1} \arrow[r, "\pi_{k-1}"] \arrow[d, dashed, "\exists f_{k-1}"] & P''_{k-1} \arrow[r] \arrow[d, "h_{k-1}"] & 0 \\
0 \arrow[r] & P'_{k-2} \arrow[r, "\iota_{k-2}"] & P_{k-2} \simeq P'_{k-2} \oplus P''_{k-2} \arrow[r, "\pi_{k-2}"] & P''_{k-2} \arrow[r] & 0
\end{tikzcd}$$

Because the 2nd row is split, there is $\varphi : P''_{k-1} \to P'_{k-1} \oplus P''_{k-1}$ such that $\pi_{k-1} \circ \varphi = \mathrm{Id}_{P''_{k-1}}$. Then, let $f_k : P_k \to P_{k-1}$ be a R-module homomorphism such that

$$f_k(x, y) = \iota_{k-1}(g_k(x)) + \varphi(h_k(y))$$

Note that f_k is a R-module homomorphism, because each $\iota_{k-1}, \varphi, g_k, h_k$ are R-module homomorphisms.

In this way, we can construct R-module homomorphisms $f_1 : P_1 \to P_0, \cdots$.

Let's show $P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} P_{k-2}$ is exact for $k \in \mathbb{Z}^{\geq 2}$.

Let $(x, y) \in \operatorname{im} f_k \subseteq P_{k-1}$. Then, there is $(x', y') = f_k(x, y)$ for some $x' \in P'_k$ and $y' \in P''_k$. Then, $x = g_k(x')$ and $y = h_k(y')$. Thus, $x \in \operatorname{im} g_k = \ker g_{k-1}$ and $y \in \operatorname{im} h_k = \ker h_{k-1}$. Then, $f_{k-1}(x, y) = (g_{k-1}(x), h_{k-1}(y)) = (0, 0)$. Therefore, since $0 \to P'_{k-2} \to P_{k-2} \to P''_{k-2} \to 0$ is split, there is $\varphi : P''_{k-2} \to P_{k-2}$ such that $\pi_{k-2} \circ \varphi = \mathrm{Id}_{P''_{k-2}}$.

$$f_{k-1}(n_1, n_2) = \iota_{k-2}(g_{k-1}(n_1)) + \varphi(h_{k-1}(n_2)) = \iota_{k-2}(0) + \varphi(0) = 0$$

Thus, $(n_1, n_2) \in \ker f_{k-1}$ and $\operatorname{im} f_k \subseteq \ker f_{k-1}$.

Let $(x, y) \in \ker f_{k-1} \subseteq P_{k-1}$. Since $f_{k-1}(x, y) = 0$, its isomorphic image in $P'_{k-2} \oplus P''_{k-2}$ is $(0, 0)$ and each isomorphic images of $g_{k-1}(x)$ and $h_{k-1}(y)$ are zero. Thus, $x \in \ker g_{k-1} = \operatorname{im} g_k$ and $y \in \ker h_{k-1} = \operatorname{im} h_k$. Then, there is $x' \in P'_k$ and $y' \in P''_k$ such that $g_k(x') = x$ and $h_k(y') = y$. Then, $f_k(x', y') = (x, y)$ and $(x, y) \in \operatorname{im} f_k$. Therefore, $\ker f_{k-1} \subseteq \operatorname{im} f_k$.

This shows $\cdots \to P_2 \to P_1 \to P_0$ is exact.

Let $p_1 : P_0 \to P'_0$ and $p_2 : P''_0 \to P_0$ such that $p_1 \circ \iota_0 = \mathrm{Id}_{P'_0}$ and $\pi_0 \circ p_2 = \mathrm{Id}_{P''_0}$. Then, let $\alpha_1 : P_0 \to M$ such that $\alpha_1 = i \circ g_0 \circ p_1$. Since h_0 is surjective (because of exact sequence), and $p : M \to M''$ is surjective because of an exact sequence and P_0 is projective, there is $\gamma : P''_0 \to M$ such that $p \circ \gamma = h_0$. Let $f_0(x, y) = i(g_0(x)) + \gamma(y)$. Since f_0 is a sum of compositions of homomorphisms, f_0 is a homomorphism.

f_0 is surjective. Let $m \in M$. Since $h_0 \circ \pi_0$ is surjective, there is $m' \in P_0$ such that $h_0(\pi_0(m')) =$

$p(m)$. Then, $p(m-f_0(m')) = p(m)-p(f_0(m')) = p(m)-h_0(\pi_0(m')) = 0$. Since $m-f_0(m') \in \ker p = \operatorname{im} i$, Then, there is $a \in M'$ such that $i(a) = m - f_0(m')$. Since g_0 is surjective, there is $a' \in P_0'$ such that $i(g_0(a')) = m - f_0(m')$. Let $n = \iota_0(a') + m'$. Then,

$$\begin{aligned} f_0(n) &= f_0(\iota_0(a')) + f_0(m') \\ &= m - f_0(m') + f_0(m') = m \end{aligned}$$

Thus, f_0 is surjective.

$\operatorname{im} f_1 \supseteq \ker f_0$. Let $(x, y) \in \ker f_0$. Then, $f_0(x, y) = 0$. $h_0(y) = p(f_0(x, y)) = p(0) = 0$. Then, $y \in \ker h_0 = \operatorname{im} h_1$. And $\gamma(y) \in \ker p = \operatorname{im} i$. Then, $i(g_0(x)) = f_0(x, 0) = f_0(x, y) - \gamma(y) = \gamma(-y)$. $g_0(x) = i^{-1}\gamma(-y)$.

Since i is injective, $g_0(x) = 0$. Thus, $x \in \ker g_0 = \operatorname{im} g_1$. Therefore, there is $x' \in P_1'$ and $y' \in P_1''$ such that $g_0(x') = x$ and $h_0(y') = y$. And, $f_1(x', y') = (x, y)$. Thus, $(x, y) \in \operatorname{im} f_1$ and $\ker f_0 \subseteq \operatorname{im} f_1$.

$\operatorname{im} f_1 \subseteq \ker f_0$. Let $(x, y) \in \operatorname{im} f_1$. Then, there is $(x', y') \in P_1$, such that $f_1(x', y') = (x, y)$. Then, $g_1(x') = x$ and $h_1(y') = y$. Since $x \in \operatorname{im} g_1 = \ker g_0$ and $y \in \operatorname{im} h_1 = \ker h_0$, $g_0(x) = 0 = h_0(y)$. Then, $i(g_0(x)) = 0$. And, $\gamma(y) \in \ker p = \operatorname{im} i$. Let $z \in P_0'$ such that $\gamma(y) = i(g_0(z)) = f(z, 0)$. Then, $f(-z, y) = -i(g_0(z)) + \gamma(y) = 0$. It means $(-z, y) \in \ker f_0$. As we shown $\ker f_0 \subseteq \operatorname{im} f_1$ above, $(-z, y) \in \operatorname{im} f_1$. Then, there is $z' \in P_1'$ such that $g_1(z') = z$. It means, $z \in \operatorname{im} g_1 = \ker g_0$. Therefore, $\gamma(y) = i(g_0(z)) = i(0) = 0$. Therefore, $f(x, y) = 0$. It shows $(x, y) \in \ker f$ and $\operatorname{im} f_1 \subseteq \ker f_0$.

Therefore, $P_\bullet$ is a projective resolution of M. $\square$

Problem 2

Let $M_\bullet$ be a bounded above complex of R-modules. Then give a rigorous proof that there is a projective resolution $P_\bullet \to M_\bullet$.

(Hint: Use the existence for a single module case and break down $M_\bullet$ into short exact seqnuences. If needed, use the above horseshoe lemma.)

(Note, bounded above complex $K^-(R)$, such that $\cdots \to M_2 \to M_1 \to M_0 \to \cdots$ holds $M_n = 0$ for every $n > N$ for some $N \in \mathbb{N}$)

Solution

Let a bounded above chain complex $M_\bullet$ of R-modules is given. And let $f_\bullet$ be differentials of $M_\bullet$. Let $N \in \mathbb{Z}$ such that $M_n = 0$ for every integer $n \geq N$.

Note that 0 is a projective module. Let all of $\cdots, P_{N+2}, P_{N+1}, P_N$ are zero modules. Since $\cdots, M_{N+2}, M_{N+1}, M_N$ are zero modules, $H_k(M_\bullet) = \ker 0/\operatorname{im} 0 = 0$ and $H_k(P_\bullet) = \ker 0/\operatorname{im} 0 = 0$ for integer $k > N$. Thus, $H_k(M_\bullet) \simeq H_k(P_\bullet)$ for every $k > N$.

As the induction step, for $n < N$, suppose that we found projective R-modules $\cdots, P_{n+3}, P_{n+2}, P_{n+1}$, with differentials $g_\bullet$, such that $H_k(P_\bullet) \simeq H_k(M_\bullet)$ for every integer $k \geq n+1$, and $h_\bullet : M_\bullet \to P_\bullet$ be a quasi-isomorphism.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+2} \arrow[r, "f_{n+2}"] \arrow[d, "h_{n+2}"] & M_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "h_{n+2}"] & M_n \arrow[r] \arrow[d, dashed, "h_n"] & \cdots \\
\cdots \arrow[r] & P_{n+2} \arrow[r, "g_{n+2}"] & P_{n+1} \arrow[r, dashed, "g_{n+1}"] & P_n &
\end{tikzcd}$$

Since $0 \to M_n \to M_n \oplus P_{n+1} \to P_{n+1} \to 0$ is exact, there is a projective resolution $Q_\bullet \to M_n$, $R_\bullet \to P_{n+1}$ such that $0 \to Q_\bullet \to Q_\bullet \oplus R_\bullet \to R_\bullet \to 0$ is exact.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Q_1 \arrow[r] \arrow[d, two heads] & Q_1 \oplus R_1 \arrow[r] \arrow[d, two heads] & R_1 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \ker f \arrow[r] \arrow[d, tail] & \ker h \arrow[r] \arrow[d, tail] & \ker g \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & Q_0 \arrow[r] \arrow[d, "f"] & Q_0 \oplus R_0 \arrow[r] \arrow[d, "h"] & R_0 \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M_n \arrow[r] \arrow[d] & M_n \oplus P_{n+1} \arrow[r] \arrow[d] & P_{n+1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Take $P_n = Q_1 \oplus R_1$. In this case,

Problem 3

Give a rigorous proof of the below theorem:

Theorem 1. *(Pseudo-universal property) Let M, N be R-modules, and let $f^{-1} : M \to N$ be a R-module homomorphism. Let $M \to E^\bullet$ and $N \to I^\bullet$ be injective resolutions. Then there is a chain map $f^\bullet : E^\bullet \to I^\bullet$ that extends f^{-1}. Furthermore, this $f^\bullet$ is unique up to chain homotopy.*

Proof

The below diagram shows the ingrediants given in the problem. Dashed arrows are what we should construct.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "g^{-1}"] \arrow[d, "f^{-1}"] & E^0 \arrow[r, "g^0"] \arrow[d, dashed, "f^0"] & E^1 \arrow[r] \arrow[d, dashed, "f^1"] & \cdots \\
0 \arrow[r] & N \arrow[r, "h^{-1}"] & I^0 \arrow[r, "h^0"] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

We'll denote $E^{-1} = M$, $I^{-1} = N$, $E^{-2} = I^{-2} = 0$, $g^{-2} : 0 \to M, h^{-2} : 0 \to N$ are zero functions.

Existence of $f^\bullet$

Suppose that we already have $f^{-1}, \cdots, f^k$ for some $k \in \mathbb{Z}^{\geq -1}$, such that $f^{j+1}g^j = h^j f^j$ for every $j \in \{-1, 0, \cdots, k-1\}$ (i.e. they commute as the above diagram).

Then, we can make induced homomorphisms $\tilde{g}^k : \operatorname{coker} g^{k-1} = E^k / \operatorname{im} g^{k-1} \to E^{k+1}$ such that $\tilde{g}^k(x + \operatorname{im} g^{k-1}) = g^k(x)$, $\tilde{h}^k : \operatorname{coker} h^{k-1} = I^k / \operatorname{im} h^{k-1} \to I^{k+1}$ such that $\tilde{h}^k(x + \operatorname{im} h^{k-1}) = h^k(x)$, and $\tilde{f}^k : \operatorname{coker} g^{k-1} \to \operatorname{coker} h^{k-1}$ such that $\tilde{f}^k(x + \operatorname{im} g^{k-1}) = f^k(x) + \operatorname{im} h^{k-1}$.

Note that above isomorphisms are well defined Because if $x + \operatorname{im} g^{k-1} = y + \operatorname{im} g^{k-1}$, then $x - y \in \operatorname{im} g^{k-1} = \ker g^k$, and $g^k(x) - g^k(y) = g^k(x-y) = 0$. In the same way, $\tilde{h}^k$ is well-defined. If $x + \operatorname{im} g^{k-1} = y + \operatorname{im} g^{k-1}$, $x - y \in \operatorname{im} g^{k-1}$. Let $t \in E^{k-1}$ such that $g^{k-1}(t) = x - y$. $f^k(x-y) = h^{k-1}(f^{k-1}(t))$. Thus, $f^k(x-y) \in \operatorname{im} h^{k-1}$. This shows $\tilde{f}^k$ is well-defined.

$\tilde{g}^k$ is injective. Because $\ker g^k = \operatorname{im} g^{k-1}$ because of exactness of seqneuce, if $x + \operatorname{im} g^{k-1} \in \ker \tilde{g}^k$, $g^k(x) = \tilde{g}^k(x + \operatorname{im} g^{k-1}) = 0$ and $x \in \ker g^k = \operatorname{im} g^{k-1}$. But in this case, $x + \operatorname{im} g^{k-1} = \operatorname{im} g^{k-1}$, which is an additive identity of $\operatorname{coker} g^{k-1}$. This shows that $\ker \tilde{g}^k$ contains only an additive identity and $\tilde{g}^k$ is injective. In the same way, we can show that $\tilde{h}^k$ is also injective.

Then, rows of the below diagram are exact.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} g^{k-1} \arrow[r, "\tilde{g}^k"] \arrow[d, "\tilde{f}^k"] & E^{k+1} \arrow[r] \arrow[d, dashed, "f^{k+1}"] & \cdots \\
0 \arrow[r] & \operatorname{coker} h^{k-1} \arrow[r, "\tilde{h}^k"] & I^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Since $\tilde{g}^{k-1}$ is injective, by injectivity of I^{k+1}, we can extend $\tilde{h}^k \circ \tilde{f}^k$ to $f^{k+1} : E^{k+1} \to I^{k+1}$. Then, because f^{k+1} is an extension, $\tilde{h}_k \circ \tilde{f}^k = f^{k+1} \circ \tilde{g}_k$.

Lastly, let's check $h_k \circ f^k = f^{k+1} \circ g_k$. Note that $g^{k-1}(x) = \tilde{g}^{k-1}(x + \operatorname{im} g^k)$ and $h^{k-1}(x) = \tilde{h}^{k-1}(x + \operatorname{im} h^k)$. In other words, $(f^{k+1} \circ \tilde{g}^k)(x + \operatorname{im} g^k) = (f^{k+1} \circ g^k)(x)$, $(\tilde{h}^k \circ \tilde{f}^k)(x + \operatorname{im} g^k) = \tilde{h}^k(f^k(x) + \operatorname{im} h^k) = (h^k \circ f^k)(x)$ thus,

$$(h^k \circ f^k)(x) = (\tilde{h}^k \circ \tilde{f}^k)(x + \operatorname{im} g^k) = (f^{k+1} \circ \tilde{g}^k)(x + \operatorname{im} g^k) = (f^{k+1} \circ g^k)(x)$$

Therefore, $h^k \circ f^k = f^{k+1} \circ g^k$.

The above process constructs f^{k+1} from f^k such that $h^k \circ f^k = f^{k+1} \circ g^k$ holds. Because f^{-1} was given, we can construct $f^\bullet$ by repeating above process. $\square$

Uniqueness of $f^\bullet$

Suppose that there are two extensions $f^\bullet$ and $f'^\bullet$ from f. Let $\varphi^\bullet = f^\bullet - f'^\bullet$. Note that φ^{-1} is a zero map.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "g^{-1}"] \arrow[d, "\varphi^{-1}=0"'] & E^0 \arrow[r, "g^0"] \arrow[d, "\varphi^0"] \arrow[dl, dashed, "s^0"] & E^1 \arrow[r] \arrow[dl, dashed, "s^1"] & \cdots \\
0 \arrow[r] & N \arrow[r, "h^{-1}"'] & I^0 \arrow[r] & \cdots &
\end{tikzcd}$$

If we find $s^\bullet : E^\bullet \to I^{\bullet -1}$ such that $\varphi^k = s^{k+1}g^k + h^k s^k$, f and f' are chain homotopy equivalence by s.

First, let $\cdots, s^{-2}, s^{-1}$ are zero maps.

Let $s^0 = 0$. Then, $0 = \varphi^{-1} = h^{-2}s^{-1} + s^0 g^{-1} = 0 + 0 = 0$ holds.

Suppose that for $k \in \mathbb{N}$, for every integer $n < k$, $\varphi^n = h^{n-1}s^n + s^{n+1}g^n$ holds.

Let $\psi^k = \varphi^k - h^{k-1}s^k$

Let $x \in \ker g^k = \operatorname{im} g^{k-1}$. Then, there is $y \in E^{k-1}$ such that $g^{k-1}(y) = x$. Then, because $h^{k-1} \circ h^{k-2} = 0$,

$$\begin{aligned}
\psi^k(x) = (\varphi^k - h^{k-1}s^k)(x) &= (\varphi^k g^{k-1} - h^{k-1}s^k g^{k-1})(y) \\
&= (h^{k-1}\varphi^{k-1} - h^{k-1}s^k g^{k-1})(y) \\
&= (h^{k-1}(h^{k-2}s^{k-1} + s^k g^{k-1}) - h^{k-1}s^k g^{k-1})(y) \\
&= (h^{k-1}h^{k-2}s^{k-1} + h^{k-1}s^k g^{k-1} - h^{k-1}s^k g^{k-1})(y) \\
&= (h^{k-1}h^{k-2}s^{k-1})(y) \\
&= (0 \circ s^{k-1})(y) = 0
\end{aligned}$$

Therefore, $\ker g^k \subseteq \ker \psi^k$.

In this case, if $x + \operatorname{im} g^{k-1}, y + \operatorname{im} g^{k-1} \in \operatorname{coker} g^{k-1}$ satisfies $x + \operatorname{im} g^{k-1} = y + \operatorname{im} g^{k-1}$, $x - y \in \operatorname{im} g^{k-1} = \ker g^k$ and $\psi^k(x-y) = 0$. Therefore, if we make an induced homomorphism $\tilde{\psi}^k : \operatorname{coker} g^{k-1} \to I^k$ such that $\tilde{\psi}^k(x + \operatorname{im} g^{k-1}) = \psi^k(x)$, it's well-defined.

Also, note that we showed that there is an induced injective homomorphism $\tilde{g}^{k-1} : \operatorname{coker} g^{k-1} \to E^{k+1}$ in the proof of "Existence of $f^\bullet$".

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} g^{k-1} \arrow[r, "\tilde{g}^k"] \arrow[d, "\tilde{\psi}^k"'] & E^{k+1} \arrow[dl, dashed, "s^{k+1}"] \\
& I^k &
\end{tikzcd}$$

Therefore, by injectivity of I^k, we can extend $\tilde{\psi}^k$ to s^{k+1} such that $\tilde{\psi}^k = s^{k+1} \circ \tilde{g}^k$.

Let's check $\psi^k = s^{k+1} \circ g^k$. Because $\psi^k(x) = \tilde{\psi}^k(x + \operatorname{im} g^{k-1})$ and $g^k(x) = \tilde{g}^k(x + \operatorname{im} g^{k-1})$, for any $x \in E^k$,

$$\psi^k(x) = \tilde{\psi}^k(x + \operatorname{im} g^{k-1}) = s^{k+1}(\tilde{g}^k(x + \operatorname{im} g^{k-1})) = s^{k+1}(g^k(x))$$

.

Therefore, we obtained s^{k+1} such that

$$\psi^k = \varphi^k - h^{k-1}s^k = s^{k+1}g^k$$

Then, by induction, we can find $s^\bullet$ which satisfies above equation. In other words, for every $k \in \mathbb{Z}^{\geq 0}$,

$$f^k - f'^k = \varphi^k = h^{k-1}s^k + s^{k+1}g^k$$

holds. Thus, we can construct a chain homotopy for two extensions $f^\bullet$ and $f'^\bullet$ which extended from f^{-1}.

Therefore, f^k are unique up to chain homotopy. $\square$

Problem 4

Prove the injective resolution version of the Horseshoe Lemma:

Theorem 2. *(Horseshoe Lemma for injective modules) Suppose*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of R-modules. Suppose there are injective resolutions $M' \to I'^{\bullet}$ and $M'' \to I''^{\bullet}$. Then there is an injective resolution $M \to I^{\bullet}$ such that it fits into a short exact sequence of complexes:

$$0 \to I'_{\bullet} \to I_{\bullet} \to I''_{\bullet} \to 0$$

Lemmata

Lemma 2. *Let I_1, I_2 be injective R-modules. $I_1 \oplus I_2$ is injective.*

Proof. Let $0 \to M \xrightarrow{f} N$ be an exact sequence of R-modules. And suppose that there is a map $g : M \to I_1 \oplus I_2$. Let $\pi_k : I_1 \oplus I_2 \to I_k$ be a natural projection for $k = 1, 2$. Let $g_k = \pi_k \circ g$ for $k = 1, 2$. Since each I_k are injective, there is an extension $h_k : N \to I_k$ such that $g_k = h_k \circ f$. Then, let $h : N \to I_1 \oplus I_2 : x \mapsto (h_1(x), h_2(x))$. Trivially, it's a homomorphism as a composition of homomorphisms. h is an extension of g, because for $m \in M$,

$$h(f(m)) = (h_1(f(m)), h_2(f(m))) = (g_1(m), g_2(m)) = g(m)$$

. Therefore, $I_1 \oplus I_2$ is also injective. □

Proof

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, "g_0"] & M \arrow[r, "p"] \arrow[d, dashed, "f_0"] & M'' \arrow[r] \arrow[d, "h_0"] & 0 \\
0 \arrow[r] & I'^0 \arrow[r, "\iota_0"] \arrow[d, "g_1"] & I^0 = I'^0 \oplus I''^0 \arrow[r, "\pi_0"] \arrow[d, dashed, "f_1"] & I''^0 \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & I'^1 \arrow[r, "\iota_1"] \arrow[d, "g_2"] & I^1 = I'^1 \oplus I''^1 \arrow[r, "\pi_1"] \arrow[d, dashed, "f_2"] & I''^1 \arrow[r] \arrow[d, "h_2"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Suppose that the exact sequence $0 \to M' \to M \to M'' \to 0$ with i, p, an injective resolution $M' \to I'^{\bullet}$, $M'' \to I''^{\bullet}$ with g_k, h_k were given as above diagram.

As the Problem 1, take $I^k = I'^k \oplus I''^k$ for $k \in \mathbb{Z}^{\geq 0}$. Each I^k is injective by Lemma **??**. Let $\iota_k : I'^k \to I^k$ and $\pi_k : I^k \to I''^k$ be natural homomorphisms.

Let's define $f_k : I^{k-1} \to I^k$.

First, note that i, ι_0 and g_0 are injective because of exact sequences. Because I'^0 is injective, there is an extension $\alpha_1 : M \to I'^0$ of g_0 such that $g_0 = i \circ \alpha_1$. Let $f_0 : M \to I^0$ be $f_0 : m \mapsto (\alpha_1(m), h_0(p(m)))$.

Then, $0 \to M \xrightarrow{f_0} I^0$ is exact. If $f_0(m) = (0, 0)$ for some $m \in M$, $\alpha_1(m) = 0$ and $h_0(p(m)) = 0$. Since h_0 is injective, $p(m) = 0$, and $m \in \ker p = \operatorname{im} i$. Let $x \in M'$ such that $i(x) = m$. $0 =$

$f_0(m) = f_0(i(x)) = \iota_0(g_0(x))$. Since ι_0 is injective, $g_0(x) = 0$. Since g_0 is injective, $x = 0$. Therefore, $m = i(x) = 0$. This shows $\ker f_0 = \{0\}$ and f_0 is injective.

By Snake Lemma (See Problem 7), $\operatorname{coker} g_0 \xrightarrow{\tilde{\iota}_0} \operatorname{coker} f_0 \to \operatorname{coker} \tilde{\pi}_0 h_0$ is exact, where $\tilde{\iota}_0$ and $\tilde{\pi}_0$ are naturally induced from ι_0 and π_0. Also, $\tilde{\pi}_0$ is surjective as π_0 is surjective. Also, since $\ker h_0 = \operatorname{im} 0 = 0$, $0 = \ker h_0 \to \operatorname{coker} g_0 \to \operatorname{coker} f_0$ is exact. Because $\operatorname{im} g_0 = \ker g_1$ and $\operatorname{im} h_0 = \ker h_1$, we can build the below diagram with the induced injective homomorphisms $\tilde{g}_1 : \operatorname{coker} g_0 = I'^0/\ker g_1 \to I'^1$ and $\tilde{h}_1 : \operatorname{coker} h_0 = I''^0/\ker h_1 \to I''^1$ which obtained from g_1 and h_1:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{coker} g_0 \arrow[r, "\tilde{\iota}_0"] \arrow[d, "\tilde{g}_1"] & \operatorname{coker} f_0 \arrow[r, "\tilde{\pi}_0"] \arrow[d, dashed, "\tilde{f}_1"] & \operatorname{coker} h_0 \arrow[r] \arrow[d, "\tilde{h}_1"] & 0 \\
0 \arrow[r] & I'^1 \arrow[r, "\iota_1"] \arrow[d, "g_2"] & I^1 = I'^1 \oplus I''^1 \arrow[r, "\pi_1"] & I''^1 \arrow[r] \arrow[d, "h_2"] & 0 \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Note that each rows and the first and third column of above diagram is exact. In this case, we can repeat what we did above, just considering $\operatorname{coker} f_0$, $\operatorname{coker} g_0$, $\operatorname{coker} h_0$ as M, M', M''. Then, we can construct $\tilde{f}_1 : \operatorname{coker} f_0 \to I^1$ using the injectivity of I'^1, and an exact sequcne $0 \to \operatorname{coker} g_1 \to \operatorname{coker} f_1 \to \operatorname{coker} h_1 \to 0$, and $\tilde{f}_2 : \operatorname{coker} f_1 \to I^2$, $\cdots$.

Then, let $f_k : I^{k-1} \to I^k$ such as $f_k(x) = \tilde{f}_k(x + \operatorname{im} f_{k-1})$ for each $k \in \mathbb{Z}^{\geq 0}$ (Let's assume $I^{-1} = M$, $I^{-2} = 0$ and $f_{-2} = 0$ for convenience). Then, since each $\tilde{f}_k$ is injective, $\tilde{f}_k(x) = 0$ iff $x = \operatorname{im} f_{k-1}$. It implies $\ker f_k = \operatorname{im} f_{k-1}$.

Therefore, $I^\bullet$ is an injective resolution of M. $\square$

Problem 5

Let $M^\bullet$ be a bounded below complex. Prove that there is an injective resolution of $M^\bullet$.

Solution

Problem 6

Let M and N be R-modules. Let $P_\bullet \to M$ and $Q_\bullet \to N$ be projective resolutions.

Note that

$$(*) : \cdots \to Q_i \otimes_R P_2 \to Q_i \otimes_R P_1 \to Q_i \otimes_R P_0 \to Q_i \otimes_R M$$

is exact for each $i \in \mathbb{Z}^{\geq 0}$.

Let $\mathrm{Tot}(Q \otimes P) = T_\bullet = \{T_n\}$ be a total complex, where $T_n = \bigoplus_{i+j=n} Q_i \otimes P_j$.

Using the exactness of $(*)$ for each i, prove that the natural map $T_\bullet \to Q_\bullet \otimes_R M$ is a quasi-isomorphism.

Solution

Problem 7

Prove the Snake Lemma:

Theorem 3. *(The Snake Lemma) Suppose we have a commutative diagram with exact rows:*

$$\begin{tikzcd} & A_1 \arrow[r, "\alpha"] \arrow[d, "f"] & A_2 \arrow[r, "\beta"] \arrow[d, "g"] & A_3 \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "\gamma"] & B_2 \arrow[r, "\delta"] & B_3 & \end{tikzcd}$$

Then there is a natural homomorphism ∂ and an exact sequence

$$\ker f \xrightarrow{\tilde{\alpha}} \ker g \xrightarrow{\tilde{\beta}} \ker h \xrightarrow{\partial} \operatorname{coker} f \xrightarrow{\tilde{\gamma}} \operatorname{coker} g \xrightarrow{\tilde{\delta}} \operatorname{coker} h$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are obtained from α and β by restricting each domains to $\ker f$ and $\ker g$, $\tilde{\gamma}$ and $\tilde{\delta}$ are module homomorphisms such that $\tilde{\gamma} : x + \operatorname{im} f \mapsto \gamma(x) + \operatorname{im} g$ and $\tilde{\delta} : x + \operatorname{im} g \mapsto \delta(x) + \operatorname{im} h$.

Proof

(1) The image of α from $\ker f$ is in $\ker g$. Because, if $x \in \ker f$,

$$0 = \gamma(0) = \gamma(f(x)) = g(\alpha(x))$$

holds, and $\alpha(x) \in \ker g$. In the same way, we can show that the image of $\tilde{\beta}$ from $\ker g$ is in $\ker h$.

(2) $\tilde{\gamma}$ is well-defined. If $x + \operatorname{im} f = x' + \operatorname{im} f$ for $x, x' \in B_1$, $x - x' \in \operatorname{im} f$. Thus, there is $d \in A_1$ such that $f(d) = x - x'$. Then,

$$\gamma(x) - \gamma(x') = \gamma(x - x') = \gamma(f(d)) = g(\alpha(d)) \in \operatorname{im} g$$

Therefore, $\gamma(x) + \operatorname{im} g = \gamma(x') + \operatorname{im} g$. In the same way, $\tilde{\delta}$ is well-defined.

(3) $\ker f \xrightarrow{\tilde{\alpha}} \ker g \xrightarrow{\tilde{\beta}} \ker h$ is exact. To show this, we need to check $\operatorname{im} \tilde{\alpha} = \ker \tilde{\beta}$.

Let $x \in \operatorname{im} \tilde{\alpha}$. Then, $x \in \operatorname{im} \alpha$. Since $A_1 \to A_2 \to A_3$ is exact, $x \in \ker \beta$. Then, $\tilde{\beta}(x) = \beta(x) = 0$ and $x \in \ker \tilde{\beta}$. This shows $\operatorname{im} \tilde{\alpha} \subseteq \ker \tilde{\beta}$.

Let $x \in \ker \tilde{\beta}$. Note that $g(x) = 0$ as $x \in \ker g$. Then, $\tilde{\beta}(x) = \beta(x) = 0$ and $x \in \ker \beta = \operatorname{im} \alpha$. Let $a \in A_1$ such that $\alpha(a) = x$. Then, $\gamma(f(a)) = g(\alpha(a)) = g(x) = 0$. Because γ is injective, $f(a) = 0$. Therefore, $a \in \ker f$ such that $\alpha(a) = \tilde{\alpha}(a) = x$. This shows $\ker \tilde{\beta} \subseteq \operatorname{im} \tilde{\alpha}$.

Thus, $\operatorname{im} \tilde{\alpha} = \ker \tilde{\beta}$.

(4) $\operatorname{coker} f \xrightarrow{\tilde{\gamma}} \operatorname{coker} g \xrightarrow{\tilde{\delta}} \operatorname{coker} h$ is exact. To show this, we need to check $\operatorname{im} \tilde{\gamma} = \ker \tilde{\delta}$.

Let $x + \operatorname{im} g \in \operatorname{im} \tilde{\gamma}$. Then, there is $y \in B_1$ such that $\gamma(y) + \operatorname{im} g = x + \operatorname{im} g$. This shows $x - \gamma(y) \in \operatorname{im} g$. Then, let $a \in A_2$ such that $g(a) = x - \gamma(y)$. Then, $\delta(x - \gamma(y)) = \delta(g(a)) = h(\beta(a)) \in \operatorname{im} h$. And, $\delta(x) - \delta(\gamma(y)) \in \operatorname{im} h$. Since $B_1 \to B_2 \to B_3$ is exact, $\delta \circ \gamma = 0$ and $\delta(\gamma(y)) = 0$. Therefore, $\delta(x) \in \operatorname{im} h$ and $\delta(x) + \operatorname{im} h = \operatorname{im} h$. It implies $\tilde{\delta}(x + \operatorname{im} g) = \delta(x) + \operatorname{im} h = \operatorname{im} h$ and $x + \operatorname{im} g \in \ker \tilde{\delta}$. This shows $\operatorname{im} \tilde{\gamma} \subseteq \ker \tilde{\delta}$.

Let $x + \operatorname{im} g \in \ker \tilde{\delta}$. $\delta(x) + \operatorname{im} h = \operatorname{im} h$ and $\delta(x) \in \operatorname{im} h$. Let $y \in A_3$ such that $h(y) = \delta(x)$. Since β is surjective, there is $y' \in A_2$ such that $\beta(y') = y$. Then, $\delta(x) = h(\beta(y')) = \delta(g(y'))$. Since δ is a homomorphism, $\delta(x - g(y')) = 0$. So, $x - g(y') \in \ker \delta = \operatorname{im} \gamma$. Let $x' \in B_1$ such that $\gamma(x') = x - g(y')$.

In this case,
$$\tilde{\gamma}(x' + \operatorname{im} f) = \gamma(x') + \operatorname{im} g = x - g(y') + \operatorname{im} g = x + \operatorname{im} g$$
. Therefore, $x + \operatorname{im} g \in \operatorname{im} \tilde{\gamma}$. This shows $\ker \tilde{\delta} \subseteq \operatorname{im} \tilde{\gamma}$.

Therefore, $\operatorname{im} \tilde{\gamma} = \ker \tilde{\delta}$.

(5) Define $\partial : \ker h \to \operatorname{coker} f$ as:

Let $x \in \ker h$. Since β is surjective, let $x_2 \in A_2$ such that $\beta(x_2) = x$. Since $x \in \ker h$, $\delta(g(x_2)) = h(\beta(x_2)) = h(x) = 0$, and $g(x_2) \in \ker \delta = \operatorname{im} \gamma$. For $y_1 \in B_1$ such that $\gamma(y_1) = g(x_2)$, let $\partial(x) = y_1 + \operatorname{im} f$.

(6) ∂ is well-defined.

First, since γ is injective, for given $y_2 \in \operatorname{im} \gamma$, there is a unique $y_1 \in B_1$ such that $\gamma(y_1) = y_2$.

Therefore, the only pary which can violates well-definedness is choosing x_2 from x_1.

Suppose that there is $x_2' \in A_2$ such that $\beta(x_2') = x$. Then, there is a unique y_1' such that $\gamma(y_1') = g(x_2')$.

As $\beta(x_2) - \beta(x_2') = x - x = 0$, $x_2 - x_2' \in \ker \beta = \operatorname{im} \alpha$. Thus, there is $x_1 \in A_1$ such that $\alpha(x_1) = x_2 - x_2'$. Then,
$$\gamma(y_1 - y_1') = g(x_2 - x_2') = g(\alpha(x_1)) = \gamma(f(x_1))$$
Since γ is injective, $y_1 - y_1' = f(x_1)$. Since $y_1 - y_1' \in \operatorname{im} f$, $y_1 + \operatorname{im} f = y_1' + \operatorname{im} f$. Therefore, for given x_3, $y_1 + \operatorname{im} f$ is a unique choice, and ∂ is well-defined.

(7) ∂ is a module homomorphism. Note that, since γ is an injective module homomorphism, it's an isomorphism between B_1 and its image. Therefore, for $t, t' \in \operatorname{im} \gamma$, $g^{-1}(t) + g^{-1}(t') = g^{-1}(t + t')$.

Let $x, x' \in B_3$. Then, there are some $x_2, x_2' \in B_2$ such that $\beta(x_2) = x$, $\gamma(\partial(x)) = g(x_2)$, $\beta(x_2') = x'$, $\gamma(\partial(x')) = g(x_2')$. Since β is homomorphism, $\beta(x_2 + x_2') = \beta(x_2) + \beta(x_2') = x + x'$. Then, as γ^{-1} and g are homomorphisms,
$$\begin{aligned}\partial(x) + \partial(x') = \gamma^{-1}(g(x_2)) + \gamma^{-1}(g(x_2')) &= \gamma^{-1}(g(x_2 + x_2')) \\ &= \partial(x + x')\end{aligned}$$
holds.

In the similar way, for $r \in R$,
$$r\partial(x) = r\gamma^{-1}(g(x_2)) = \gamma^{-1}(g(rx_2)) = \partial(rx)$$
holds.

(8) $\ker g \xrightarrow{\tilde{\beta}} \ker h \xrightarrow{\partial} \operatorname{coker} f$ is exact.

Let $x \in \operatorname{im} \tilde{\beta} \subseteq \ker h \subseteq A_3$. Then, there is $x_2 \in \ker g \subseteq A_2$ such that $\tilde{\beta}(x_2) = x$. Then, $g(x_2) = 0$. Then, $\partial(x) = \gamma^{-1}(g(x_2)) = \gamma^{-1}(0) = 0$ by the definition. Thus, $x \in \ker \partial$ and $\operatorname{im} \tilde{\beta} \subseteq \ker \partial$.

Let $x \in \ker \partial \subseteq \ker h \subseteq A_3$. Let $x_2 \in A_2$ such that $\beta(x_2) = x$. Then, $g(x_2) = \gamma(\partial(x)) = 0$. This shows $x_2 \in \ker g$. In this case, $x = \beta(x_2) = \tilde{\beta}(x_2)$ and $x \in \operatorname{im} \tilde{\beta}$. Therefore, $\ker \partial \subseteq \operatorname{im} \tilde{\beta}$.

Thus, $\operatorname{im} \tilde{\beta} = \ker \partial$.

(9) $\ker h \xrightarrow{\partial} \operatorname{coker} f \xrightarrow{\tilde{\gamma}} \operatorname{coker} g$ is exact.

Let $y + \operatorname{im} f \in \operatorname{im} \partial \subseteq \operatorname{coker} f$ where $y \in B_1$. Then, there is $x \in A_3$ such that $\partial(x) = y + \operatorname{im} f$. Let $x_2 \in A_2$ such that $\beta(x_2) = x$. Then, $y' = \gamma^{-1}(g(x_2)) \subseteq y + \operatorname{im} f$ and, $\tilde{\gamma}(y + \operatorname{im} f) = \tilde{\gamma}(y' + \operatorname{im} f) =$

$\gamma(y') + \operatorname{im} g = g(x_2) + \operatorname{im} g = \operatorname{im} g$. Thus, $y + \operatorname{im} f \in \ker \tilde{\gamma}$ and $\operatorname{im} \partial \subseteq \ker \tilde{\gamma}$.

Let $y + \operatorname{im} f \in \ker \tilde{\gamma} \subseteq \operatorname{coker} f$ where $y \in B_1$. Then, $\tilde{\gamma}(y + \operatorname{im} f) = \gamma(y) + \operatorname{im} g = \operatorname{im} g$ and $\gamma(y) \in \operatorname{im} g$. Let $x_2 \in A_2$ such that $g(x_2) = \gamma(y)$ and $x \in A_3$ such that $\beta(x_2) = x$. Then, $\partial(x) = y + \operatorname{im} f$. This shows that $y + \operatorname{im} f \in \operatorname{im} \partial$ and $\ker \tilde{\gamma} \subseteq \operatorname{im} \partial$.

Thus, $\operatorname{im} \partial = \ker \tilde{\gamma}$. $\square$