Note for Complex Analysis

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1 The Complex Numbers

2 Functions of the Complex Variable z

3 Analytic Functions

4 Line Integrals and Entire Functions

5 Properties of Entire Functions

Theorem 1. *(5.10; Liouville's Theorem) A bounded entire function is constant.*

Theorem 2. *(5.11; The ExtendedLiouville's Theorem) If f is entire and if, for some integer $k \geq 0$, there exist positive constants A and B such that*

$$f(z) \leq A + B|z|^k$$

then f is a polynomial of degree at most k.

Theorem 3. *(5.12; Fundamental Theorem of Algebra)*

$$\mathbb{C} = \overline{\mathbb{C}}$$

6 Properties of Analytic Functions

Property 1. *(6.8) If f is analytic at α, so is*

$$g(z) = \begin{cases} \frac{f(z)-f(\alpha)}{z-\alpha} \\ f'(\alpha) \end{cases}$$

Property 2. *(6.9; Uniqueness Theorem) Suppose that f is analytic in a region D and that $f(z_n) = 0$ where $\{z_n\}$ is a sequence of distinct points and $z_n \to z_0 \in D$. Then $f \equiv 0$ in D.*

Corollary 1. *(6.10) If two functions f and g, analytic in D, agree at a set of points with an accumulation point in D, then $f \equiv g$ in D.*

Theorem 4. *(6.11) If f is entire and if $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial.*

Theorem 5. *(6.12; Mean Value Theorem) Let f be analytic in D and $\alpha \in D$. For $r > 0$ such that $D(\alpha; r) \subseteq D$,*

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta$$

Theorem 6. *(6.13; Maximum-Modulus Theorem) Let f be a non-constant analytic function in a region D. For each $z \in D$ and $\delta > 0$, there exists some $\omega \in D(z; \delta) \cup D$ such that $|f(\omega)| > |f(z)|$.*

Corollary 2. *(6.14; Minimum-Modulus Theorem) Let f be a non-constant analytic function in a region D. $z \in D$ is a relative minimum of f iff $f(z) = 0$.*

Theorem 7. *(6.15) Suppose f is nonconstant and analytic on the closed disc D. Assume that f's maximum modulus at boundary point z_0. Then, $f'(z_0) \neq 0$.*

Definition 1. (6.16) z_0 is a *saddle point* of an analytic function f if it's a saddle point of $|f|$. i.e. $|f|_x(z_0) = |f|_y(z_0) = 0$ but z_0 is neither a local maximum nor a local minimum.

Theorem 8. *(6.17) z_0 is a saddle point of analytic f iff $f'(z_0) = 0$ and $f(z_0) \neq 0$.*

7 Further Properties of Analytic Functions

Theorem 9. *(7.1; Open Mapping Theorem) A non-constant analytic function is an open function.*

Theorem 10. *(7.2; Schwarz' Lemma) Suppose that f is analytic in the unit disc, the $|f| < 1$ there and that $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.*

In addition, $|f(z)| = |z|$, $|f'(0)| = 1$ and $f(z) = e^{i\theta} z$ are equivalent.

Proposition 1. *(7.3) If f is entire and satisfies*

$$|f(z)| \leq 1/|\operatorname{Im} z|$$

for all z, then $f \equiv 0$.

Theorem 11. *(7.4; Morera's Theorem) Let f be a continuous function on an open set D. If*

$$\int_\Gamma f(z) dz = 0$$

whenever Γ is the boundary of a closed rectangle in D, then F is analytic on D.

Theorem 12. *(7.6) Suppose $\{f_n\}$ represents a sequence of functions, analytic in an open domain D and such that $f_n \to f$ uniformly on compacta. Then f is analytic in D.*

Theorem 13. *(7.7) Suppose f is continuous in an open set D and analytic there except possibly at the poitns of a line segment L. Then f is analytic throughout D.*

8 Simply Connected Domain

Definition 2. (8.1) A region D is *simply connected* if its complement is connected within $\epsilon to \infty$ (i.e. for any $z_0 \in \mathbb{C} \setminus D$, there is a curve γ such that $d(\gamma(t), \mathbb{C} \setminus D) < \epsilon$ for all $t \geq 0$, $\gamma(0) = z_0$, $\lim_{t \to \infty} \gamma(t) = \infty$))a

Definition 3. (8.2) The *number of levels of* Γ is the number of different imaginary part of points in horizontal lines in the polygonal path Γ.

Theorem 14. *(8.4) For every function which is analytic in a simply connected domain, there is a primitive function.*

Theorem 15. *(8.5; General Closed Curve Theorem) Suppose that f is analytic in a simply connected region D and that C is a smooth closed curve contained in D. Then,*

$$\int_C f = 0$$

Theorem 16. *(8.8) Suppose that D is simply connected and that $0 \notin D$. Choose $z_0 \in D$, fix a value of $\log z_0$ and set*

$$f(z) = \int_{z_0}^{z} \frac{d\zeta}{\zeta} + \log z_0$$

Then f is an analytic branch of $\log z$ in D.

9 Isolated Singularities of an Analytic Function

Theorem 17. *(9.3-9.5) Suppose f is analytic in a deleted neighborhood of z_0 and z_0 is a singularity of f. Then,*

1. *If $\lim_{z\to z_0}(z - z_0)f(z) = 0$, z_0 is a removable singularity.*
2. *If f is bounded in a deleted neighborhood, z_0 is a removable singularity.*
3. *If there is a posigive integer k such that*

$$\lim_{z\to z_0}(z - z_0)^k f(z) \neq 0, \lim_{z\to z_0}(z - z_0)^{k+1} f(z) = 0$$

then, z_0 is a pole of order k.

Theorem 18. *(9.6; Casorati-Weierstrass Theorem) If f has an essential singularity at z_0 and if D is a deleted neighborhood of z_0, then the range $R = \{f(z) \mid z \in D\}$ is dense in the complex plane.*

Theorem 19. *(9.10) Every function which is analytic in some annulus has a unique representation as a Laurent Series, where the coefficient of z^k is*

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{k+1}}$$

Definition 4. (9.12) Every terms of Laurent series with degress lower then 0 is a *principal part*, and other terms is an *analytic part.*

10 The Residue Theorem

Theorem 20. *Let z_0 be a simple pole of f.*

$$\mathrm{Res}(f; z_0) = \lim_{z\to z_0} f(z)$$

Let z_0 be an order $k \geq 1$ pole of f.

$$\mathrm{Res}(f; z_0) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z - z_0)^k f(z)$$

Definition 5. The winding number of γ around a is

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

This is an integer.

Theorem 21. *(10.5; Cauchy's Residue Theorem) Suppose f is analytic in a simply connected domain D except for isolated singularities at $z_1, \cdots, z_m$. Let γ be a closed curve not intersecting any of the singularities. Then,*

$$\int_\gamma f = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k) \operatorname{Res}(f; z_k)$$

Definition 6. (10.7) f is *meromorphic* if f is analytic there except at isolated poles.

Theorem 22. *(10.8) For regular curve γ and f which is analytic in and on γ, where every zero and pole of f is not on γ,*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \mathbb{Z} - \mathbb{P}$$

where $\mathbb{Z}/\mathbb{P}$ is the number of zeros/poles of f inside γ.

Corollary 3. *(10.9; Argument Principle) If f analytic inside and on a regular closed curve γ where zeros of f are not on γ, then $\mathbb{Z} = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f}$.*

If $\gamma : z(t)$ where $t \in [0, 1]$,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \frac{\log f(z(1)) - \log f(z(0))}{2\pi i} = \frac{1}{2\pi} \Delta \operatorname{Arg} f(z)$$

Theorem 23. *(10.10; Rouch́s Theorem) Suppose that f and g are analytic inside and on a regular closed curve γ and that $|f(z)| > |g(z)|$ for all $z \in \gamma$. Then*

$$\mathbb{Z}(f+g) = \mathbb{Z}(f)$$

inside γ.

Theorem 24. *(10.11; Generalized Cauchy Integral Formula) For a function f analytic in a simply connected domain D,*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\omega)}{(\omega - z)^{k+1}} d\omega$$

Theorem 25. *(10.12) Suppose $\{f_n\}$ represents a sequence of functions, analytic in an open domain D and such that $f_n \to f$ uniformly on compacta. Then f is analytic in D, $f_n' \to f'$ uniformly on compacta.*

Theorem 26. *(10.13; Hurwitz's Theorem) Let $\{f_n\}$ be a sequence of non-vanishing analytic functions in a region D and suppose $f_n \to f$ uniformly on compacta of D. Then either $f \equiv 0$ in D or $f(z) \neq 0$ for all $z \in D$.*

Corollary 4. *(10.14) Suppose that f_n is a sequence of analytic function in a region D, that $f_n \to f$ uniformly on compact in D, and that $f_n \neq a$. Then either $f \equiv a$ or $f \neq a$ in D.*

Theorem 27. *(10.15) Suppose that f_n is a sequence of analytic function in a region D, that $f_n \to f$ uniformly on compact in D. If f_n is 1-1 in D for all n, then either f is constant or f is 1-1 in D.*

11 Applications of the Residue Theorem to the Evaluation of Integral of Sums

Proposition 2. *If P, Q are polynomials and* $\deg Q - \deg P > 1$,

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{P(z)}{Q(z)}dz = 0$$

where Γ_R is an arc of radius R.

Proposition 3. *If P, Q are polynomials and* $\deg Q > \deg P$,

$$\lim_{R\to\infty}\int_{\Gamma_R}e^{iz}\frac{P(z)}{Q(z)}dz = 0$$

where Γ_R is an arc of radius R.

Proposition 4. *Let P, Q be polynomials and* $\deg Q - \deg P \geq 2$.

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{P(z)}{Q(z)}\log z dz = 0$$

where Γ_R is an arc of radius R.

$$\lim_{r\to 0}\int_{\Gamma_r}\frac{P(z)}{Q(z)}\log z dz = 0$$

where Γ_r is an arc of radius R in $x \leq 0$ half-plane.

$$\int_0^\infty \frac{P(x)}{Q(x)}dx = -\sum_k \operatorname{Res}\left(\frac{P(z)}{Q(z)}\log z; z_k\right)$$

Proposition 5.

$$\sum_{n=-\infty}^{\infty} f(n) = \lim_{N\to\infty}\int_{C_N} f(z)\pi\cot\pi z dz = -\sum_k \operatorname{Res}(f(z)\pi\cot\pi z; z_k)$$

Proposition 6.

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = \lim_{N\to\infty}\int_{C_N} f(z)\pi\csc\pi z dz = -\sum_k \operatorname{Res}(f(z)\pi\csc\pi z; z_k)$$

Proposition 7.

$$\binom{n}{k} = \frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}$$

Proposition 8.

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \frac{1}{2\pi i}\int_C (1+z)^n\left(1+\frac{1}{z}\right)^n\frac{dz}{z}$$

12 Further Contour Integarl Techniques

Proposition 9.

$$\lim_{R\to\infty}\int_{C_R} e^z f(z)dz = 0$$

where the real part of C_R is bounded above and $|f|$ is bounded.

Proposition 10.

$$\int_{C_R} \frac{1}{az+\varepsilon(z)} \simeq \int_{C_R} \frac{1}{az}$$

for sufficient large R where the integral are equal for all sufficient large R.

13 Introduction to Conformal Mapping

Theorem 28. *(13.4) If f is analytic at z_0 and has non-zero derivative f' at z_0, then f is conformal and locally 1-1 at z_0.*

Theorem 29. *(13.6) For integer k, z^k magnifies angles at 0 by a factor of k, and maps $D(0;r)$ onto $D(0;r^k)$.*

Theorem 30. *(13.7) Suppose f is analytic at z_0 with $f'(z_0)=0$. If f is non-constant, there is some small open neighborhood of z_0 where f is a k-to-1 mapping and f magnifies angles at z_0 by a factor of k, where k is the least positive integer of which $f^{(k)}(z_0) \neq 0$.*

Theorem 31. *(13.8) Suppsoe f is a 1-1 analytic function in a region D. Then f^{-1} exists an analytic in $f(D)$, and f and f^{-1} are conformal in D and $f(D)$ respectively.*

Theorem 32. *(13.9) Conformal equivalence (existency of conformal mapping between two region) is an equivalence relation.*

Definition 7. Bilinear transformation

$$\omega = \frac{az+b}{cz+d}$$

where $ad-bc \neq 0$. ($\because \omega' = \frac{ad-bc}{(cz+d)^2}$, which must be non-zero for conformal mapping)

Lemma 1. *(13.10) $1/z$ maps circle/line to circle/line.*

Theorem 33. *(13.11) The image of circle/line by a bilinear transformation is a circle/line.*

Theorem 34. *(13.13) For a given conformal mapping $f : D_1 \to D_2$,*

If there is a conformal mapping $h : D_1 \to D_2$, there is a conformal automorphism $g : D_1 \to D_1$ such that $h = g \circ f$. (which is $g = h \circ f^{-1}$).

If h is a conformal automorphism of D, $h = f^{-1} \circ g \circ f$ for some conformal automorphism g of D_2.

Lemma 2. *(13.14) The only automorphisms of the unit disc with $f(0)=0$ are given by $f(z) = e^{i\theta z}$.*

Theorem 35. *(13.15) The automorphisms of the unit disc are of the form $f(z) = e^{i\theta z}\left(\frac{z-\alpha}{1-\overline{\alpha}z}\right)$.*

Theorem 36. *(13.16) The conformal mappings h of the upper half plane onto the unit disc are of the form*

$$h(z) = e^{i\theta}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right)$$

where $\operatorname{Im}\alpha > 0$.

Theorem 37. *(13.17) The automorphisms of the upper half-plane of the form*

$$h(z) = \frac{az+b}{cz+d}$$

with $a,b,c,d \in \mathbb{R}$ and $ad-bc > 0$.

Theorem 38. *(13.19) A non-identity bilinear transformatoin has a tmost two fixed points.*

Lemma 3. *(13.20) The unique bilinear mapping sending z_1, z_2, z_3 into $\infty, 0, 1$ respectively, is given by*

$$T(z) = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$$

Definition 8. (13.21) The *cross-ratio* of four complex z_1, z_2, z_3, z_4 is

$$(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)}$$

Theorem 39. *(13.22) Bilinear transformation preserves cross-ratio*

Theorem 40. *(13.23) The unique bilinear transformation $\omega = f(z)$ mapping z_1, z_2, z_3 into $\omega_1, \omega_2, \omega_3$ respectively, is given by*

$$\frac{(\omega - \omega_2)(\omega_3 - \omega_1)}{(\omega - \omega_1)(\omega_3 - \omega_2)} = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$$

Proposition 11. *$f(z) = \sin z$ maps semi-infinite strip*

$$\frac{-\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}; \operatorname{Im} z > 0$$

conformally onto the upper half-plane by considering its behavior on the ractangle R:

$$\frac{-\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}; 0 \leq \operatorname{Im} z \leq N$$

Note

$$\sin z = \sin x \cosh N + i \cos x \sinh N$$

14 The Riemann Mapping Theorem

Theorem 41. *(Riemann Mapping Theorem) For any simply connected domain $R(\neq \mathbb{C})$ and $z_0 \in R$, there exists a unique conformal mapping φ of R onto U (unit disk) such that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$.*

Note,

$$\varphi(z) = c\frac{z - z_0}{1 - \overline{z_0} z}$$